Normal form and 2-dimensional chains of an elliptic CR surface in $\mathbb{C}^4$

Vladimir V. Ežov *
Gerd Schmalz **

*
Oklahoma State University
Department of Mathematics
College of Arts and Sciences
Stillwater, Oklahoma 74078-0613

USA

**
Mathematisches Institut der
Universität Bonn
Wegelerstraße 10
D-5300 Bonn 1

Germany

Max-Planck-Institut für Mathematik
Gottfried-Claren-Straße 26
D-5300 Bonn 3

Germany

MPI / 93-13

NORMAL FORM AND 2-DIMENSIONAL CHAINS OF AN ELLIPTIC CR SURFACE IN $\mathbb{C}^4$

VLADIMIR V. EŽOV AND GERD SCHMALZ

ABSTRACT. We suggest a construction of a normal form of a real analytic surface of codimension two in $\mathbb{C}^4$ with elliptic Levi form which generalizes Chern-Moser normal form of a hypersurface.

1. INTRODUCTION

We consider some properties of real generic CR surfaces of codimension 2 in $\mathbb{C}^4$ with nondegenerate Levi form. Locally, any C^2 smooth surface of such kind belongs to one of three types, corresponding to the type of its Levi form, namely elliptic, hyperbolic and parabolic. The elliptic and hyperbolic are the types of general position, the parabolic type is exceptional in many senses. A normal form of a hyperbolic surface was constructed by A. Loboda [Lob88]. We give a construction of a normal form of elliptic surfaces (Theorem 1). This normal form is analogous to Chern-Moser's normal form of a hypersurface [MS74]. For example, there are as many normal forms at a fixed point as there are isotropic automorphisms of the tangent elliptic quadric (Theorem 2) since the group of germs of "normalizations" of a hypersurface is isomorphic to the isotropy group of the tangent hyperquadric.

Normal forms use to be the most efficient tool in the proof of the extension of local holomorphic maps of hypersurfaces and in equivariant linearization of their automorphisms ([EKV84], [Vit85], [Ežo83], [Kru83]). All these results involve a family of Chern-Moser chains, - a special biholomorphically invariant dense family of curves on a surface. We introduce a biholomorphically invariant family of 2-dimensional surfaces on an elliptic surface which we also call chains. Geometrically, the chains are the solutions of certain systems of linear partial differential equations. Up to terms of higher order they admit an approximation by the chains on the tangent elliptic quadrics, being the intersections of the quadric with so called "matrix lines".

Another analogy with Chern Moser's normal form which might be very helpful in the analysis of local holomorphic maps is the fact that the chain-preserving nor-

Research of the first author was supported by Max-Planck-Institut Bonn.
Research of the second author was supported by Deutsche Forschungsgemeinschaft.

malizations of a hypersurface being already given in the normal form has to be a fractional linear transformation ultimately (Theorem 3). æ

2. Results

Let $z = (z^1, z^2)$, $w = (w^1 = u^1 + iv^1, w^2 = u^2 + iv^2)$ coordinates in $\mathbf{C}^4 = \mathbf{C}^2 \times \mathbf{C}^2$.
Let

$$\langle z, z\rangle = \begin{pmatrix} \langle z, z\rangle^1 = z'H^1\bar{z} \\ \langle z, z\rangle^2 = z'H^2\bar{z} \end{pmatrix} -$$

a nondegenerate $\mathbf{R}^2$-valued Hermitian form in $\mathbf{C}^2$. (z' is the transposed vector of z.)

There are only three types of nonequivalent Hermitian forms (up to the action of the group $G^{2,2} = \mathrm{GL}(2,\mathbf{C}) \times \mathrm{GL}(2,\mathbf{R})$ given by

$$(C,\rho) \circ \langle z, z\rangle = \rho\langle C^{-1}z, C^{-1}z\rangle).$$

These types are represented by

$$\begin{aligned} \langle z, z\rangle_1 &= \begin{pmatrix} \operatorname{Re} z^1\bar{z}^2 \\ \operatorname{Im} z^1\bar{z}^2 \end{pmatrix} \\ \langle z, z\rangle_2 &= \begin{pmatrix} |z^1|^2 \\ |z^2|^2 \end{pmatrix} \\ \langle z, z\rangle_3 &= \begin{pmatrix} |z^1|^2 \\ \operatorname{Re} z^1\bar{z}^2 \end{pmatrix}. \end{aligned}$$

These three forms are called elliptic, hyperbolic and parabolic, respectively, according to the distribution of the roots of the polynomial invariant

$$\mathcal{P}_k(t) = \det(H_k^1 + tH_k^2), \quad k = 1,2,3,$$

where H_k^j is the Hermitian matrix related to $\langle z, z\rangle_k^j$.

In particular, $\mathcal{P}_1(t) = t^2 + 1$, $\mathcal{P}_2(t) = t$, $\mathcal{P}_3(t) = t^2$ (up to the sign).

The stable cases are the first and the second so far.

A hyperbolic form is also called strictly pseudoconvex since it is the only one which admits a positive definite linear combination of $\langle z, z\rangle_2^1$ and $\langle z, z\rangle_2^2$.

Consider the group $\mathcal{O}_0$ of the germs of local biholomorphic isomorphisms of $\mathbf{C}^4$ preserving the origin and let us consider the orbit $\mathcal{O}_0(M)$ of a surface $M = M^6 \subset \mathbf{C}^4$ with nondegenerate Levi form.

Of course, $\mathcal{O}_0(M)$ has infinite dimension. The purpose of a normal form is to decompose the space $\mathcal{F}$ of formal vector power series of the form $\operatorname{Im} w - G(z,\bar{z},\operatorname{Re} w)$; $G(0) = 0$, $\frac{\partial G}{\partial(w,\bar{w}}|_0 = 0$ with nondegenerate Hermitian term in $(z,\bar{z})$ into the sum

$$\mathcal{F} = \mathcal{N} \oplus \mathcal{R},$$

where $\mathcal{O}_0(M) \cap \mathcal{N} \neq \emptyset$ has finite dimension for any M.

The normal form for strictly pseudoconvex surfaces was found by A. Loboda [Lob88]. V. Beloshapka [Bel89],[Bel90] suggested the way to define the $\mathcal{R}$ space in the general case. But it might lead to divergent normal forms if we do not care about the choice of $\mathcal{N}$, the direct complement to $\mathcal{R}$.

A convergence theorem can be proved if we explicitly define $\mathcal{N}$ and the projection $\pi : \mathcal{F} \longrightarrow \mathcal{N}$.

We now describe a normal form of real analytic surfaces M with elliptic Levi form.

It is convenient to write down the equation of M in coordinates $(z, V, \bar{V}, \eta, \bar{\eta})$, where $\omega^1 := w^1 + iw^2 = \eta + iV$, $\omega^2 := w^1 - iw^2 = \bar{\eta} + i\bar{V}$, where $\eta = u^1 + iu^2$, $V = v^1 + iv^2$. The equation of M takes then the form

$$V = z^1\bar{z}^2 + \dots \tag{1}$$

Let $\mathcal{K}_{kl}(z, \bar{z}, \eta, \bar{\eta})$ be a polynomial in $(z, \bar{z})$ of degree k, l respectively, with coefficients being formal power series in $(\eta, \bar{\eta})$, representing the right hand sides of equation (1) of elliptic surfaces.

We set

$$\begin{aligned}\mathcal{R} &= \Bigg\{R_{1,1} + \sum_{k=1}^{\infty} R_{k,0} + \sum_{k=1}^{\infty} R_{0,k} + \\ &+ \sum_{k=3}^{\infty} R_{k,1} + \sum_{k=3}^{\infty} R_{1,k} + R_{22} + R_{32} + R_{23} + R_{33}, \Bigg\}\end{aligned}$$

where $R_{k,0}$ ($R_{0,k}$) are arbitrary holomorphic (antiholomorphic) polynomials of degree k in z ($\bar{z}$), and

$$\begin{aligned}
R_{1,1}(z,\bar z,0,0) &= 0\\
R_{2,1} &= \phi_{21}(\eta,\bar\eta)z^1z^2\bar z^1 + R_{2,0}(z,\eta,\bar\eta)\bar z^2\\
R_{1,2} &= \phi_{12}(\eta,\bar\eta)z^2\bar z^1\bar z^2 + R_{0,2}(\bar z,\eta,\bar\eta)z^1\\
R_{k,1} &= R_{k0}(z,\eta,\bar\eta)\bar z^2, \quad k\ge 3\\
R_{1,k} &= z^1R_{0k}(\bar z,\eta,\bar\eta), \quad k\ge 3\\
R_{2,2} &= \phi_{22}(\eta,\bar\eta)|z^1|^2|z^2|^2 + \psi_{22}(\eta)(z^1\bar z^2)^2\\
R_{3,2} &= \phi_{32}(\eta)z^1(z^1\bar z^2)^2\\
R_{2,3} &= \phi_{23}(\eta)(z^1\bar z^2)^2\bar z^2\\
R_{3,3} &= \phi_{33}(\eta)(z^1\bar z^2)^3
\end{aligned}$$

Let $\nu = z^1\bar z^2$, $\frac{\partial}{\partial\nu} = \frac{\partial^2}{\partial z^1\partial\bar z^2}$. Then the space $\mathcal{N}$ is defined as

$$\mathcal{N} = \{z^1\bar z^2 + N_{21} + N_{12} + \sum(N_{k1}+N_{1k}) + \sum_{k,l\ge 2} N_{kl}\},$$

where

$$\begin{aligned}
\frac{\partial}{\partial\bar z^2}N_{21} &= \frac{\partial}{\partial z^1}N_{12} = 0\\
\frac{\partial^2}{\partial z^1\partial\bar\nu}N_{21} &= \frac{\partial^2}{\partial\bar z^2\partial\bar\nu}N_{12} = 0\\
\frac{\partial}{\partial\bar z^2}N_{k1} &= \frac{\partial}{\partial z^1}N_{1k} = 0, \quad k\ge 3\\
\frac{\partial^2}{\partial\nu\partial\bar\nu}N_{22} &= 0\\
\left.\frac{\partial^3}{(\partial\nu)^2\partial\eta}\right|_{\bar\eta=0} N_{22} &= 0\\
\left.\frac{\partial^4}{(\partial\nu)^2\partial z^1\partial\eta}\right|_{\bar\eta=0} N_{32} &= \left.\frac{\partial^4}{(\partial\nu)^2\partial\bar z^1\partial\eta}\right|_{\bar\eta=0} N_{23} = 0\\
\left.\frac{\partial^4}{(\partial\nu)^3\partial\eta}\right|_{\bar\eta=0} N_{33} &= 0
\end{aligned} \tag{2}$$

It follows that $\mathcal{F} = \mathcal{R}\oplus\mathcal{N}$.

Remark. Let $\mathcal{F}_p$ be the subspace of $\mathcal{F}$ consisting of the polynomials in $z,\bar z,\eta,\bar\eta$. Then in $\mathcal{F}_p$ a scalar product $(,)$ can be defined by the property that the monomials form an orthonormal basis in $\mathcal{F}_0$.

It follows that $\mathcal{N}\cap\mathcal{F}_p = (\mathcal{R}\cap\mathcal{F}_p)^\perp$ with respect to $(,)$.

Now let M be a real-analytic CR surface of codimension 2 in $\mathbb{C}^4$ with elliptic Levi form, passing through the origin.

Theorem 1. *In some neighbourhood of the origin there exist coordinates* (ω^j, z^j), $j = 1,2$ *such that the equation of* M *takes the form*

$$V = N(z, \bar{z}, \eta, \bar{\eta}), \tag{3}$$

where $N(z, \bar{z}, \eta, \bar{\eta}) \in \mathcal{N}$.

Equation (3) is called a normal form of M (at the origin).

Suppose that M is given in a normal form. How many different representations in normal form at the origin does M admit?

Let $\mathcal{O}_N(M)$ be the group of germs $\psi_N(M)$ of holomorphic transformations at the origin preserving the normal form of M.

We consider the isotropy group $I_0(Q_e)$ of the elliptic quadric

$$Q_e : V = z^1 \bar{z}^2.$$

It was shown in [ES92] that $I_0(Q_e)$ consists of fractional linear transformations having the form

$$\Phi = \Phi_0 \circ \Phi_1,$$

where Φ_0 is a (C, ρ)-transformation defined by

$$\begin{aligned} z^* &= e^{\lambda + i\phi} \begin{pmatrix} e^{\mu + i\theta} & 0 \\ 0 & e^{-(\mu + i\theta)} \end{pmatrix} z, \\ \omega^* &= \begin{pmatrix} e^{2(\lambda + i\theta)} & 0 \\ 0 & e^{2(\lambda - i\theta)} \end{pmatrix} \omega, \end{aligned}$$

where $\lambda, \mu, \phi, \theta \in \mathbb{R}$.

$$\begin{aligned} \Phi_1 : (z^*)^1 &= \frac{z^1 + a^1\omega^1}{1 - 2iz^1\bar{a}^2 - (r + ia^1\bar{a}^2)\omega^1} \\ (z^*)^2 &= \frac{z^2 + a^2\omega^2}{1 - 2iz^2\bar{a}^1 - (\bar{r} + ia^2\bar{a}^1)\omega^2} \\ (\omega^*)^1 &= \frac{\omega^1}{1 - 2iz^1\bar{a}^2 - (r + ia^1\bar{a}^2)\omega^1} \\ (\omega^*)^2 &= \frac{\omega^2}{1 - 2iz^2\bar{a}^1 - (\bar{r} + ia^2\bar{a}^1)\omega^2}, \end{aligned} \tag{4}$$

where $a^1, a^2, r \in \mathbb{C}$.

Hence, $I_0(Q_e) \cong \mathbb{R}^4 \ltimes \mathbb{C}^2 \ltimes \mathbb{C}$.

Theorem 2. *The group $\mathcal{O}_N(M)$ is isomorphic to $I_0(Q_e) \cong \mathbb{R}^4 \ltimes \mathbb{C}^2 \ltimes \mathbb{C}$. This isomorphism is given as follows: Let $\psi_N(M): z^* = f(z,\omega), \quad \omega^* = g(z,\omega)$, then*

$$
\begin{aligned}
e^{\lambda+i\phi}\begin{pmatrix} e^{\mu+i\theta} & 0 \\ 0 & e^{-(\mu+i\theta)} \end{pmatrix} &= \left.\frac{\partial f}{\partial z}\right|_0 \\
a^1 &= \left.\frac{\partial f^1}{\partial \omega^1}\right|_0 \\
a^2 &= \left.\frac{\partial f^2}{\partial \omega^2}\right|_0 \\
r &= \frac{1}{2}\left(\left.\frac{\partial^2 g^1}{(\partial\omega^1)^2}\right|_0 + /overline \left.\frac{\partial^2 g^2}{(\partial\omega^2)^2}\right|_0 \right)
\end{aligned} \tag{5}
$$

If M is already given in normal form then it is easy to see that the 2-dimensional $\mathbb{R}$-plane $\Gamma_0 : z = 0, V = 0$ belongs to M. This observation provides us with a notion of chains:

A 2-dimensional real submanifold $\Gamma \subset M$ is called a chain if there is a transformation $\psi_N \in \mathcal{O}_N(M)$ such that $\psi_N(\Gamma) \subset \Gamma_0$. Such ψ_N we call a normalization of M straightening Γ or a normalization of M along Γ.

The following questions occur: How many chains pass through a fixed point on M? What are the chain-preserving normalizations of M?

Let $\xi \in M$ and Γ_ξ be the family of chains passing through ξ.

Theorem 3. *(i.) Γ_ξ is a $\mathbb{C}^2$-parameter family of 2-dimensional surfaces on M, being transversal to the complex tangent space to M in any point.*
(ii.) Let ψ_N be a normalization of M and Γ_0 be ψ_N-invariant, i.e. $\psi_N(\Gamma_0 \subset \Gamma_0$. Then ψ_N is an element of $I_0(Q_e)$ having the form (4) with $a^1 = a^2 = 0$.

Corollary 1. *There is a family $\{\eta_\Gamma\}_\Gamma$ of normal parametrizations of the chain Γ. Any normal parameter can be obtained from another normal parameter by a projective transformation.*

$$
(\eta^*) = \frac{\eta}{1 - r\eta}. \tag{6}
$$

3. Scheme of the normalization

Since any transformation of the type Φ_0 preserves the normal form (3), we shall construct a normalization having identical differential at the origin.

We consider a chain Γ passing through the origin and being given by the equation

$$\begin{aligned} z^1 &= p^1(t^1,t^2) \\ z^2 &= p^2(t^1,t^2) \\ \omega^1 &= q^1(t^1,t^2) \\ \omega^2 &= q^2(t^1,t^2), \end{aligned}$$

with $\left.\frac{\partial q^j}{\partial t^s}\right|_0 = \delta^j_s$.

We represent a normalization ψ straightening Γ as a composition of 4 consecutive transformations $\psi_1, \dots, \psi_4$. Each of these ψ_j preserves the form of the equation of M being obtained at the previous step ψ_{j-1} and gives its own contribution to the normalization of the equation.

Step 1. (Straightening the chain, determination of the initial parametrization on Γ and elimination of the harmonic terms)

We look for ψ_1 having the form

$$\begin{aligned} z^1 &= (z^*)^1 + p^1((\omega^*)^1,(\omega^*)^2) + 2i\sum_{k=2}^{\infty} T^1_k(z^*,\omega^*) \\ z^2 &= (z^*)^2 + p^2((\omega^*)^2,(\omega^*)^1) + 2i\sum_{k=2}^{\infty} T^2_k(z^*,\omega^*) \\ \omega^1 &= q^1((\omega^*)^1,(\omega^*)^2) + 2i\sum_{k=1}^{\infty} g^1_k(z^*,\omega^*) \\ \omega^2 &= q^2((\omega^*)^2,(\omega^*)^1) + 2i\sum_{k=1}^{\infty} g^2_k(z^*,\omega^*), \end{aligned} \tag{7}$$

where T^j_k and g^j_k are polynomials of degree k in z with coefficients being analytic functions of ω.

ψ_1 has the following properties:

(*i.*) ψ_1 maps the germ of Γ at the origin onto the germ of the plane $z = 0, V = 0$.

(*ii.*) g^j_k and T^j_k are chosen to eliminate the harmonic terms (F_{k0} and F_{0k}) in the new equation of M.

Step 2. (Normalization of the terms F_{k1} and F_{1k} for $k \geq 1$ and of F_{23} and F_{32})

ψ_2 is a so called (C,ρ)-transformation

$$\begin{aligned} z &= C(\omega^*)z^* \\ \omega &= \rho(\omega^*)\omega^*, \end{aligned} \tag{8}$$

having the following properties:

(*i.*) After applying ψ_2 the term F_{11} takes the form $z^1\bar{z}^2$ and *(ii.)* the terms $F_{k1}, F_{1k}, F_{32}, F_{23}$ belong to the space $\mathcal{N}$.

The normalizing of $F_{21}, F_{12}, F_{32}, F_{23}$ in the new equation of M completely determines the family of chains Γ passing through the origin in the chosen parametrization.

Step 3. (Normalization of the term F_{22})

ψ_3 is a (C, id) transformation having the form

$$
\begin{aligned}
z &= e^{i\phi(\omega^1,\omega^2)} \begin{pmatrix} e^{\mu(\omega^1,\omega^2)} & 0 \\ 0 & e^{-\mu(\omega^1,\omega^2)} \end{pmatrix} z^* \\
\omega &= \omega^*,
\end{aligned} \tag{9}
$$

where $\operatorname{Im}\mu(\eta,\bar\eta) = \operatorname{Im}\phi(\eta,\bar\eta) = 0$.

After applying ψ_3 we get $F_{22} \in \mathcal{N}$.

Step 4. (Normalization of the term F_{33}. Normal parametrization on Γ. The family of normal parameters.)

ψ_4 is a transformation of the form

$$
\begin{aligned}
z^* &= e^{\lambda(\omega)} \begin{pmatrix} e^{i\theta(\omega)} & 0 \\ 0 & e^{-i\theta(\omega)} \end{pmatrix} z \\
\omega^* &= h(\omega) = \begin{pmatrix} h^1(\omega) \\ h^2(\omega) \end{pmatrix},
\end{aligned} \tag{10}
$$

where

$$
\frac{\partial h}{\partial(\omega^1,\omega^2)} = \begin{pmatrix} e^{2(\lambda(\omega)+i\theta(\omega))} & 0 \\ 0 & e^{2(\lambda(\omega)-i\theta(\omega))} \end{pmatrix},
$$

and, $\operatorname{Im}\lambda(\eta,\bar\eta) = \operatorname{Im}\theta(\eta,\bar\eta) = 0$.

This is the general form of a transformation with identical differential at the origin and preserving the partial normalization obtained at the steps 1-3.

We will find λ and θ such that in the new equation the term $F_{33} \in \mathcal{N}$.

One can consider ψ_4 as a reparametrization of Γ. If we suppose that $F_{33} \in \mathcal{N}$, from the very beginning, then ψ_4 is a fractional linear transformation:

$$
\begin{aligned}
(z^*)^1 &= \frac{z^1}{1 - r\omega^1} \\
(z^*)^2 &= \frac{z^2}{1 - \bar r\omega^2} \\
(\omega^*)^1 &= \frac{\omega^1}{1 - r\omega^1} \\
(\omega^*)^2 &= \frac{\omega^2}{1 - \bar r\omega^2},
\end{aligned} \tag{11}
$$

corresponding to some normal parameter on Γ.

4. Initial Parametrization on Γ

In this section we give a construction of ψ_1. The condition $\Gamma \subset M$ implies that

$$\begin{aligned}\frac{1}{2i}[q^1(\eta,\bar\eta) - \bar q^2(\eta,\bar\eta)] &= p^1(\eta,\bar\eta)\bar p^2(\eta,\bar\eta) + \\ &+G(p^1,p^2,\bar p^1,\bar p^2,\frac{1}{2}[q^1+\bar q^2],\frac{1}{2}[q^2+\bar q^1]),\end{aligned}$$

where

$$V = z^1\bar z^2 + G(z,\bar z,\eta,\bar\eta) \tag{12}$$

is the initial equation of M.

After we insert the expression for ψ_1 into equation (12) and solve it with respect to V and $\bar V$ (we remove the star indices) we fix the parameter on Γ taking into consideration the coefficients at V and $\bar V$ only.

By the expression $\alpha \longrightarrow_{(\gamma)} \beta$ we mean that the contribution of the term α of the old equation into the term of type γ in the new equation of M equals to β or to the contribution of β into γ.

The contribution of the left hand side in (12) into the terms being linear with respect to V and $\bar V$ is:

$$\begin{aligned}&\frac{1}{2i}[q^1(\eta+iV,\bar\eta+i\bar V) - q^2(\eta-iV,\bar\eta-i\bar V)] \longrightarrow_{lin(V,\bar V)}\\ &\longrightarrow \frac{1}{2}V[q^1_\eta + \bar q^2_{\bar\eta}] + \frac{1}{2}\bar V[q^1_{\bar\eta} + \bar q^2_\eta] =\\ &= Q^1 + \bar Q^2,\end{aligned}$$

where Q^1 and Q^2 are the components of the vector

$$\begin{pmatrix} Q^1 \\ Q^2 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} q^1_\eta & q^1_{\bar\eta} \\ q^2_\eta & q^2_{\bar\eta}\end{pmatrix}\begin{pmatrix} V \\ \bar V\end{pmatrix}.$$

The contribution of the right hand side is:

$$\begin{aligned}&p^1(\eta+iV,\bar\eta+i\bar V)\bar p^2(\eta-iV,\bar\eta-i\bar V) \longrightarrow_{lin(V,\bar V)}\\ &\longrightarrow iV(p^1_\eta\bar p^2 - \bar p^2_{\bar\eta}p^1) + i\bar V(p^1_{\bar\eta}\bar p^2 - \bar p^2_\eta p^1) =\\ &= 2i(P^1 - \bar P^2),\end{aligned}$$

where

$$\begin{pmatrix} P^1 \\ P^2 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} p^1_\eta \bar{p}^2 & p^1_{\bar\eta}\bar{p}^2 \\ p^2_\eta \bar{p}^1 & p^2_{\bar\eta}\bar{p}^1 \end{pmatrix}\begin{pmatrix} V \\ \bar{V} \end{pmatrix} = \\ = \frac{1}{2}\begin{pmatrix} \bar{p}^2 & 0 \\ 0 & \bar{p}^1 \end{pmatrix}\begin{pmatrix} p^1_\eta & p^1_{\bar\eta} \\ p^2_\eta & p^2_{\bar\eta} \end{pmatrix}\begin{pmatrix} V \\ \bar{V} \end{pmatrix}.$$

The contribution of the function G on the right hand side is

$$\begin{aligned}
&G(z^1 + p^1(\eta + iV, \bar\eta + i\bar{V}), z^2 + p^2(\bar\eta + i\bar{V}, \eta + iV), \\
&\bar{z}^1 + \bar{p}^1(\bar\eta - i\bar{V}, \eta - iV), \bar{z}^2 + \bar{p}^2(\eta - iV, \bar\eta - i\bar{V}), \\
&\frac{1}{2}[q^1(\eta + iV, \bar\eta + i\bar{V}) + \bar{q}^2(\eta - iV, \bar\eta - i\bar{V})], \\
&\frac{1}{2}[q^2(\bar\eta + i\bar{V}, \eta + iV) + \bar{q}^1(\bar\eta - i\bar{V}, \eta - iV)]) \longrightarrow_{lin(V,\bar{V})} \\
&\frac{i}{2}V[G_\eta q^1_\eta + G_{\bar\eta} q^2_\eta - G_\eta \bar{q}^2_{\bar\eta} - G_{\bar\eta}\bar{q}^1_{\bar\eta}] + \\
&+\frac{i}{2}\bar{V}[G_\eta q^1_{\bar\eta} + G_{\bar\eta} q^2_{\bar\eta} - G_\eta \bar{q}^2_\eta - G_{\bar\eta}\bar{q}^1_\eta] + \\
&+iV[G_1 p^1_\eta + G_2 p^2_\eta - G_{\bar1}\bar{p}^1_{\bar\eta} - G_{\bar2}\bar{p}^2_{\bar\eta}] + \\
&+i\bar{V}[G_1 p^1_{\bar\eta} + G_2 p^2_{\bar\eta} - G_{\bar1}\bar{p}^1_\eta - G_{\bar2}\bar{p}^2_\eta] = \\
&= \frac{i}{2}(\tilde{Q}^1 - \bar{\tilde{Q}}^2) + i(\tilde{P}^1 - \bar{\tilde{P}}^2),
\end{aligned}$$

where

$$\begin{pmatrix} \tilde{Q}^1 \\ \tilde{Q}^2 \end{pmatrix} = \begin{pmatrix} G_\eta & G_{\bar\eta} \\ \bar{G}_{\bar\eta} & \bar{G}_\eta \end{pmatrix}\begin{pmatrix} q^1_\eta & q^1_{\bar\eta} \\ q^2_\eta & q^2_{\bar\eta} \end{pmatrix}\begin{pmatrix} V \\ \bar{V} \end{pmatrix},$$

$$\begin{pmatrix} \tilde{P}^1 \\ \tilde{P}^2 \end{pmatrix} = \begin{pmatrix} G_1 & G_2 \\ \bar{G}_{\bar1} & \bar{G}_{\bar2} \end{pmatrix}\begin{pmatrix} p^1_\eta & p^1_{\bar\eta} \\ p^2_\eta & p^2_{\bar\eta} \end{pmatrix}\begin{pmatrix} V \\ \bar{V} \end{pmatrix}.$$

Collecting all the linear terms in $V, \bar{V}$ on the left hand side of the new equation of M we get

$$\frac{1}{2}(S^1 + \bar{S}^2),$$

where

$$\begin{aligned}\begin{pmatrix} S^1 \\ S^2 \end{pmatrix} &= \left\{ \left(\mathrm{id} - i \begin{pmatrix} G_\eta & G_{\bar\eta} \\ \bar G_{\bar\eta} & \bar G_\eta \end{pmatrix} \right) \begin{pmatrix} q^1_\eta & q^1_{\bar\eta} \\ q^2_\eta & q^2_{\bar\eta} \end{pmatrix} - \right. \\ & \left. -2i \left[\begin{pmatrix} \bar p^2 & 0 \\ 0 & \bar p^1 \end{pmatrix} + \begin{pmatrix} G_1 & G_2 \\ \bar G_{\bar 1} & \bar G_{\bar 2} \end{pmatrix} \right] \begin{pmatrix} p^1_\eta & p^1_{\bar\eta} \\ p^2_\eta & p^2_{\bar\eta} \end{pmatrix} \right\} \begin{pmatrix} V \\ \bar V \end{pmatrix} = \\ &= \left((\mathrm{id} - iB) \left(\frac{\partial q}{\partial(\eta,\bar\eta)} \right) - 2i\chi \left(\frac{\partial p}{\partial(\eta,\bar\eta)} \right) \right) \begin{pmatrix} V \\ \bar V \end{pmatrix}. \end{aligned}$$

Here,

$$\begin{aligned} B &= \begin{pmatrix} G_\eta & G_{\bar\eta} \\ \bar G_{\bar\eta} & \bar G_\eta \end{pmatrix} \\ \chi &= \begin{pmatrix} \bar p^2 & 0 \\ 0 & \bar p^1 \end{pmatrix} + \begin{pmatrix} G_1 & G_2 \\ \bar G_{\bar 1} & \bar G_{\bar 2} \end{pmatrix}. \end{aligned}$$

Consider the matrix

$$H = \begin{pmatrix} h_{11} & h_{12} \\ h_{21} & h_{22} \end{pmatrix} = 2i(\mathrm{id} - iB)^{-1} \chi \left(\frac{\partial p}{\partial(\eta,\bar\eta)} \right).$$

In order to choose the parametrization of the chain we set

$$\begin{aligned} \frac{\partial q^1}{\partial \bar\eta} &= h_{12}, \\ \frac{\partial q^2}{\partial \eta} &= h_{21}, \\ \left.\frac{\partial q^1}{\partial \eta}\right|_0 &= \left.\frac{\partial q^2}{\partial \bar\eta}\right|_0 = 1, \text{ and} \\ \left.\frac{\partial}{\partial \eta}\right|_{\bar\eta=0} (q^1_\eta - h_{11}) &= \left.\frac{\partial}{\partial \bar\eta}\right|_{\eta=0} (q^2_{\bar\eta} - h_{22}) = 0 \end{aligned} \tag{13}$$

These equations uniquely determine q when p is given. Thus, they define the initial parametrization on Γ.

Now we express the functions g^j_k in terms of T, using the condition that the new equation of M does not contain harmonic terms.

We start with $g^1_1(z,\eta,\bar\eta)$ and $g^2_1(z,\eta,\bar\eta)$. Consider the terms F_{10} and F_{01}.

The left hand side of the equation (12) contributes

$$g^1_1(z,\eta,\bar\eta) + \bar g^2_1(\bar z,\eta,\bar\eta).$$

The right hand side contributes

$$z^1\bar{p}^2+\overline{z^2\bar{p}^1}+z^jG_j+\overline{z^j\bar{G}_{\bar{j}}}+$$
$$+iG_\eta g_1^1+iG_{\bar\eta}g_1^2-i\overline{\bar{G}_\eta g_1^2}-i\overline{\bar{G}_{\bar\eta}g_1^1}.$$

Collecting all the terms of the type $(0,1)$ and $(1,0)$ at the left hand side, we get $G_1^1+\bar{G}_1^2$, where

$$\begin{aligned}\begin{pmatrix}G_1^1\\G_1^2\end{pmatrix} &= \left(\mathrm{id}-i\begin{pmatrix}G_\eta & G_{\bar\eta}\bar{G}_{\bar\eta} & \bar{G}_\eta\end{pmatrix}\right)\begin{pmatrix}g_1^1\\g_1^2\end{pmatrix}-\\ &\quad-\left[\begin{pmatrix}\bar{p}^2 & 0\\0 & \bar{p}^1\end{pmatrix}+\begin{pmatrix}G_1 & G_2\\ \bar{G}_{\bar1} & \bar{G}_{\bar2}\end{pmatrix}\right]\begin{pmatrix}p_\eta^1 & p_{\bar\eta}^1\\ p_\eta^2 & p_{\bar\eta}^2\end{pmatrix}\begin{pmatrix}z^1\\z^2\end{pmatrix}=\\ &= (\mathrm{id}-iB)\begin{pmatrix}g_1^1\\g_1^2\end{pmatrix}-\chi\begin{pmatrix}z^1\\z^2\end{pmatrix}.\end{aligned}$$

Finally, we set

$$\begin{pmatrix}g_1^1\\g_1^2\end{pmatrix}=(\mathrm{id}-iB)^{-1}\chi\begin{pmatrix}z^1\\z^2\end{pmatrix} \tag{14}$$

We continue by evaluating g_2^1 and g_2^2. The left hand side contributes to the terms of type $(2,0)$ and $(0,2)$

$$g_2^1+\bar{g}_2^2,$$

the right hand side:

$$\begin{aligned}&2iT_2^1\bar{p}^2-2i\overline{T_2^2\bar{p}^1}+2iT_2^jG_j-2i\overline{T_2^j\bar{G}_{\bar j}}+iG_\eta g_2^1+iG_{\bar\eta}g_2^2-i\overline{\bar{G}_\eta g_2^2}-i\overline{\bar{G}_{\bar\eta}g_2^1}+\\ &+\frac{1}{2}(G_{kj}z^kz^j+2iG_{j\eta}z^jg_1^1+2iG_{j\bar\eta}z^jg_1^2-G_{\eta\eta}(g_1^1)^2-G_{\bar\eta\bar\eta}(g_1^2)^2-2G_{\eta\bar\eta}g_1^1g_1^2)+\\ &+\frac{1}{2}\overline{(\bar{G}_{\bar k\bar j}z^kz^j+2i\bar{G}_{\bar j\bar\eta}z^jg_1^1+2i\bar{G}_{\bar j\eta}z^jg_1^2-\bar{G}_{\bar\eta\bar\eta}(g_1^1)^2-\bar{G}_{\eta\eta}(g_1^2)^2-2\bar{G}_{\eta\bar\eta}g_1^1g_1^2)}.\end{aligned}$$

Collecting these terms on the left hand side we get

$$G_2^1+\bar{G}_2^2,$$

where

$$\begin{pmatrix}G_2^1\\G_2^2\end{pmatrix}=(\mathrm{id}-iB)\begin{pmatrix}g_2^1\\g_2^2\end{pmatrix}-2i\chi\begin{pmatrix}T_2^1\\T_2^2\end{pmatrix}-\frac{1}{2}z'Q_2z.$$

Thus, in order to eliminate the $(2,0)$-term, we set

(15) $$\begin{pmatrix} g_2^1 \\ g_2^2 \end{pmatrix} = 2i(\mathrm{id} - iB)^{-1}\chi \begin{pmatrix} T_2^1 \\ T_2^2 \end{pmatrix} - \frac{1}{2}(\mathrm{id} - iB)^{-1} z' Q_2 z.$$

Analogously, we evaluate F_{k0} for $k \geq 3$. The left hand side contributes to the terms of type $(0,k)$ and $(k,0)$

$$g_k^1(z,\eta,\bar\eta) + \bar g_k^2(z,\eta,\bar\eta),$$

the right hand side:

$$2iT_k^1 \bar p^2 - 2i\overline{T_k^2 \bar p^1} + 2iT_k^j G_j - 2i\overline{T_k^j G_j} +$$
$$+ iG_\eta g_k^1 + iG_{\bar\eta} g_k^2 - i\overline{\bar G_\eta g_k^2} - i\overline{\bar G_{\bar\eta} g_k^1} + Q_k(z),$$

where $Q_k(z)$ is a k-form which does not depend on T_k^j org_k^j.
Collecting the $(k,0)$ and $(0,k)$-terms at the left hand side we obtain

$$G_{k0}^1 + \bar G_{k0}^2,$$

where

$$\begin{pmatrix} G_{k0}^1 \\ G_{k0}^2 \end{pmatrix} = (\mathrm{id} - iB) \begin{pmatrix} g_k^1 \\ g_k^2 \end{pmatrix} - 2i\chi \begin{pmatrix} T_k^1 \\ T_k^2 \end{pmatrix} - Q_k(z).$$

Hence, we can eliminate the harmonic terms in the equation of M, setting

(16) $$\begin{pmatrix} g_k^1 \\ g_k^2 \end{pmatrix} = 2i(\mathrm{id} - iB)^{-1}\chi \begin{pmatrix} T_k^1 \\ T_k^2 \end{pmatrix} - (\mathrm{id} - iB)^{-1} Q_k(z)$$

.

After we have fixed the chain-parameter and have eliminated the harmonic terms, the vector equation of M takes the form:

(17) $$\rho_0 \begin{pmatrix} V \\ \bar V \end{pmatrix} = z'\Lambda \bar z + \sum_{\substack{k,l \geq 1 \\ k+l \geq 3}} \tilde F_{kl},$$

where

$$\Lambda = \begin{pmatrix} \Lambda^1 \\ \Lambda^{1*} \end{pmatrix}$$

is a vector-valued Hermitian form, and

$$\rho_0 = \left(\mathrm{id} - \frac{i}{2} \begin{pmatrix} G_\eta(\phi_{11} - \bar\phi_{22}) & G_{\bar\eta}(\phi_{22} - \bar\phi_{11}) \\ \bar G_{\bar\eta}(\phi_{22} - \bar\phi_{11}) & \bar G_\eta(\phi_{11} - \bar\phi_{22}) \end{pmatrix} \right),$$

with

$$\begin{pmatrix} \phi_{11}(\eta,\bar\eta) & 0 \\ 0 & \phi_{22}(\bar\eta,\eta) \end{pmatrix} = \frac{\partial q}{\partial(\eta,\bar\eta)} - H.$$

In terms of g_1^j and p^j we have:

$$(18)\ \frac{\partial q}{\partial(\eta,\bar\eta)} \begin{pmatrix} V \\ \bar V \end{pmatrix} = \begin{pmatrix} g_1^1\left(\frac{\partial p}{\partial(\eta,\bar\eta)}\begin{pmatrix} V \\ \bar V \end{pmatrix}, \eta, \bar\eta\right) \\ g_1^2\left(\frac{\partial p}{\partial(\eta,\bar\eta)}\begin{pmatrix} V \\ \bar V \end{pmatrix}, \bar\eta, \eta\right) \end{pmatrix} + \left((1+\phi_{11}(\eta,\bar\eta)V(1+\phi_{22}(\bar\eta,\eta)\bar V\right)$$

5. Normalization of the Levi form (Construction of ψ_2)

After performing the transformation ψ_1 the equation of M has the form (17). We write down the explicit expression of Λ:

$$\begin{aligned} z'\Lambda^1\bar z &= z^1\bar z^2 + z^j G_{j\bar k}\bar z^k - iz^j G_{j\eta}\bar g_1^2 - iz^j G_{j\bar\eta}\bar g_1^1 + iz^j G_{\bar j\eta}\bar g_1^2 + \\ &+ g_1^1 G_{\eta\eta}\bar g_1^2 + g_1^2 G_{\bar\eta\bar\eta}\bar g_1^1 + G_{\eta\bar\eta}(g_1^1\bar g_1^1 + g_1^2\bar g_1^2), \end{aligned}$$

respectively,

$$\begin{aligned} z'(\bar\Lambda^1)'\bar z &= z^2\bar z^1 + z^k\bar G_{j\bar k}\bar z^j + ig_1^2\bar G_{\eta j}\bar z^j + ig_1^1\bar G_{j\bar\eta}\bar z^j - iz^j\bar G_{\bar j\eta}\bar G_1^1 + \\ &+ g_1^2\bar G_{\eta\eta}\bar g_1^1 + g_1^1\bar G_{\bar\eta\bar\eta}\bar g_1^2 + \bar G_{\eta\bar\eta}(g_1^1\bar g_1^1 + g_1^2\bar g_1^2). \end{aligned}$$

We look for ψ_2 being a (C,ρ)-transformation of the form (reftwo):

$$\begin{aligned} z &= C(\omega^*)z^* \\ \omega &= \rho(\omega^*)\omega^*, \end{aligned}$$

where

$$\rho = \begin{pmatrix} \xi & \zeta \\ \bar\zeta & \bar\xi \end{pmatrix}.$$

It follows then that

$$\begin{aligned} \begin{pmatrix} V \\ \bar V \end{pmatrix} &= \rho\begin{pmatrix} V^* \\ \bar V^* \end{pmatrix} \text{ and} \\ \begin{pmatrix} \eta \\ \bar\eta \end{pmatrix} &= \rho\begin{pmatrix} \eta^* \\ \bar\eta^* \end{pmatrix}. \end{aligned}$$

Let

$$\rho_0\rho = \begin{pmatrix} a & a \\ \bar{b} & \bar{a} \end{pmatrix}.$$

By normalization of the Levi form we mean that after applying ψ_2 the equation of M takes the form

$$V = z^1\bar{z}^2 + \sum_{\substack{k,l\geq 1 \\ k+l\geq 3}} F_{kl},$$

The corresponding equation in C and ρ is

$$C'(\eta,\bar{\eta})\Lambda^1\bar{C}(\bar{\eta},\eta) = aJ^1 + bJ^{1*}, \tag{19}$$

where

$$J^1 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

Equation (19) does not determine C and ρ uniquely. As an additional condition we set

$$(C^{-1})'J^1 = (\alpha J^1 + \beta J^{1*})\bar{C}^{-1} \tag{20}$$

It follows that

$$J^{1*}\bar{C}^{-1} = (C^{-1})'(\bar{\alpha}J^{1*} + \bar{\beta}J^1),$$

and therefore,

$$(C^{-1})'J^{1*} = \frac{1}{|\alpha|^2}(\alpha(1-|\beta|^2)J^{1*} - |\alpha|^2\bar{\beta}J^1)\bar{C}^{-1} \tag{21}$$

Combining (19), (20), and (21) we obtain

$$\begin{aligned}\Lambda^1 &= a(C^{-1})'J^1\bar{C}^{-1} + b(C^{-1})'J^{1*}\bar{C}^{-1} = \\ &= \left[\left(a\alpha - \frac{b\alpha^2\bar{\beta}}{|\alpha|^2}\right)J^1 + \left(a\beta + \frac{b\alpha(1-|\beta|^2)}{|\alpha|^2}\right)J^{1*}\right](\bar{C}^{-1})^2\end{aligned}$$

Respectively,

$$\begin{aligned}\Lambda^{1*} &= \bar{b}(C^{-1})'J^1\bar{C}^{-1} + \bar{a}(C^{-1})'J^{1*}\bar{C}^{-1} = \\ &= \left[\left(\bar{b}\alpha - \frac{\bar{a}\alpha^2\bar{\beta}}{|\alpha|^2}\right)J^1 + \left(\bar{b}\beta + \frac{\bar{a}\alpha(1-|\beta|^2)}{|\alpha|^2}\right)J^{1*}\right](\bar{C}^{-1})^2\end{aligned}$$

Thus,

$$\begin{aligned}\Lambda^1 + \Lambda^{1*} &= L_+(\bar{C}^{-1})^2,\\ \Lambda^1 - \Lambda^{1*} &= L_-(\bar{C}^{-1})^2,\end{aligned}$$

where

$$\begin{aligned}L_+ &= \left[\left(a\alpha - \frac{b\alpha^2\bar\beta}{|\alpha|^2}\right) J^1 + \left(a\beta + \frac{b\alpha(1-|\beta|^2)}{|\alpha|^2}\right) J^{1*}\right]\\ L_- &= \left[\left(\bar b\alpha - \frac{\bar a\alpha^2\bar\beta}{|\alpha|^2}\right) J^1 + \left(\bar b\beta + \frac{\bar a\alpha(1-|\beta|^2)}{|\alpha|^2}\right) J^{1*}\right].\end{aligned}$$

Hence,

$$\begin{aligned}C^2 &= (\bar\Lambda^1 + (\Lambda^1)')L_+^{-1}\\ L_+L_-^{-1} &= (\Lambda^1 + \Lambda^{1*})(\Lambda^1 - \Lambda^{1*})^{-1}\end{aligned}$$

This system uniquely determines (C, a, b, α, β).

6. Normalization of the terms F_{k1}, $k \geq 2$

At first we compute F_{12} and F_{21} after performing ψ_1.
The terms containing T_2^j arise from the right hand side and have the form

$$F_{12}^T + \overline{(F_{12}^T)^*},$$

where

$$\begin{pmatrix} F_{12}^T \\ (F_{12}^T)^* \end{pmatrix} = 2i \begin{pmatrix} z'\Lambda^1\bar T \\ z'\Lambda^{1*}\bar T \end{pmatrix} = 2iz'\Lambda\bar T.$$

We collect the terms that do not contain T_2^j but contain derivatives of p and q at the left hand side:

$$\begin{aligned}&g_1^1(z,\omega) + \bar g_1^2(\bar z,\bar\omega) - (z^1 + p^1(\omega^1,\omega^2))(\bar z^2 + \bar p^2(\bar{\omega^2},\bar\omega^1)) -\\ &-G(z+p, \bar z + \bar p, \frac{1}{2}[q^1 + \bar q^2] + ig_1^1 - i\bar g_1^2, \frac{1}{2}[q^2 + \bar q^1])\end{aligned}$$

Using (14) for g_1^1 and g_1^2 we have:

$$\begin{pmatrix} g_1^1(z,\omega) \\ g_1^2(z,\omega) \end{pmatrix} \longrightarrow_{(2,1)(1,2)} (\mathrm{id} - iB)^{-1}\chi \begin{pmatrix} z^1 \\ z^2 \end{pmatrix} =$$
$$\chi \begin{pmatrix} z^1 \\ z^2 \end{pmatrix} + iB(\mathrm{id} - iB)^{-1}\chi \begin{pmatrix} z^1 \\ z^2 \end{pmatrix}.$$

The contribution of $(z^1 + p^1)(\bar{z}^2 + \bar{p}^2) + G$ we represent as

$$\tilde{\chi} \begin{pmatrix} z^1 \\ z^2 \end{pmatrix} + i\tilde{B}(\mathrm{id} - iB)^{-1}\chi \begin{pmatrix} z^1 \\ z^2 \end{pmatrix}.$$

Hence, we obtain at the left hand side

$$(\chi - \tilde{\chi}) \begin{pmatrix} z^1 \\ z^2 \end{pmatrix} + i(B - \tilde{B})(\mathrm{id} - iB)^{-1}\chi \begin{pmatrix} z^1 \\ z^2 \end{pmatrix},$$

where χ and $\tilde{\chi}$ are the matrices with the following entries:

$$\begin{aligned} \chi_{11} = {} & \bar{p}^2(\eta + iV, \bar{\eta} + i\bar{V}) + G_1(p^1(\eta + iV, \bar{\eta} + i\bar{V}), p^2(\bar{\eta} + i\bar{V}, \eta + iV), \\ & \bar{p}^1(\bar{\eta} + i\bar{V}, \eta + iV), \bar{p}^2(\eta + iV, \bar{\eta} + i\bar{V})), \\ & \frac{1}{2}[q^1(\eta + iV, \bar{\eta} + i\bar{V}) + \bar{q}^2(\eta + iV, \bar{\eta} + i\bar{V})], \\ & \frac{1}{2}[q^2(\bar{\eta} + i\bar{V}, \eta + iV) + \bar{q}^1(\bar{\eta} + i\bar{V}, \eta + iV)]). \end{aligned}$$

We will write this expression as

$$\chi_{11} = \bar{p}^2(hol) + G_1(hol_{1\bar{2}}),$$

where hol and $hol_{1\bar{2}}$ mean the indicated distribution of signs and bars of the variables in brackets.

Analogously,

$$\chi_{12} = G_2(hol_{1\bar{2}}),$$

$$\begin{aligned}
\chi_{21} &= \bar{G}_{\bar{1}}(\bar{p}^1(\bar{\eta}+i\bar{V},\eta+iV),\bar{p}^2(\eta+iV,\bar{\eta}+i\bar{V}),\\
&\quad p^1(\eta+iV,\bar{\eta}+i\bar{V}),p^2(\bar{\eta}+i\bar{V},\eta+iV)),\\
&\quad \frac{1}{2}[\bar{q}^1(\bar{\eta}+i\bar{V},\eta+iV)+q^2(\bar{\eta}+i\bar{V},\eta+iV)],\\
&\quad \frac{1}{2}[\bar{q}^2(\eta+iV,\bar{\eta}+i\bar{V})+q^1(\eta+iV,\bar{\eta}+i\bar{V})]) =\\
&= \bar{G}_{\bar{1}}(hol_{\bar{1}2}),\\
\chi_{22} &= \bar{p}^1(hol)+\bar{G}_{\bar{2}}(hol_{\bar{1}2}).
\end{aligned}$$

The matrix $\tilde{\chi}$ has entries with a different distribution of the variables which we call regular

$$\begin{aligned}
\tilde{\chi}_{11} &= \bar{p}^2(\eta-iV,\bar{\eta}-i\bar{V})+G_1(p^1(\eta+iV,\bar{\eta}+i\bar{V}),p^2(\bar{\eta}+i\bar{V},\eta+iV),\\
&\quad \bar{p}^1(\bar{\eta}-i\bar{V},\eta-iV),\bar{p}^2(\eta-iV,\bar{\eta}-i\bar{V})),\\
&\quad \frac{1}{2}[q^1(\eta+iV,\bar{\eta}+i\bar{V})+\bar{q}^2(\eta-iV,\bar{\eta}-i\bar{V})],\\
&\quad \frac{1}{2}[q^2(\bar{\eta}+i\bar{V},\eta+iV)+\bar{q}^1(\bar{\eta}-i\bar{V},\eta-iV)]) =\\
&= \bar{p}^2(reg)+G_1(reg_{1\bar{2}})\\
\tilde{\chi}_{12} &= G_2(reg_{1\bar{2}})\\
\tilde{\chi}_{21} &= \bar{G}_{\bar{1}}(\bar{p}^1(\bar{\eta}-i\bar{V},\eta-iV),\bar{p}^2(\eta-iV,\bar{\eta}-i\bar{V}),\\
&\quad p^1(\eta+iV,\bar{\eta}+i\bar{V}),p^2(\bar{\eta}+i\bar{V},\eta+iV)),\\
&\quad \frac{1}{2}[\bar{q}^1(\bar{\eta}-i\bar{V},\eta-iV)+q^2(\bar{\eta}+i\bar{V},\eta+iV)],\\
&\quad \frac{1}{2}[\bar{q}^2(\eta-iV,\bar{\eta}-i\bar{V})+q^1(\eta-iV,\bar{\eta}+i\bar{V})]) =\\
&= \bar{G}_{\bar{1}}(reg_{\bar{1}2}),\\
\tilde{\chi}_{22} &= \bar{p}^1(reg)+\bar{G}_{\bar{2}}(reg_{\bar{1}2}).
\end{aligned}$$

For the matrices B and $\tilde{B}$ we obtain:

$$\begin{aligned}
B_{11} &= G_\eta(hol_{1\bar{2}}); & \tilde{B}_{11} &= G_\eta(reg_{1\bar{2}})\\
B_{12} &= G_{\bar{\eta}}(hol_{1\bar{2}}); & \tilde{B}_{12} &= G_{\bar{\eta}}(reg_{1\bar{2}})\\
B_{21} &= \bar{G}_{\bar{\eta}}(hol_{\bar{1}2}); & \tilde{B}_{21} &= \bar{G}_{\bar{\eta}}(reg_{\bar{1}2})\\
B_{22} &= \bar{G}_\eta(hol_{\bar{1}2}); & \tilde{B}_{22} &= \bar{G}_\eta(reg_{\bar{1}2})
\end{aligned}$$

We denote

$$
\begin{aligned}
Dp &:= \left(\frac{\partial p}{\partial(\eta,\bar\eta)}\right)\begin{pmatrix}V\\ \bar V\end{pmatrix};\\
Dq &:= \left(\frac{\partial q}{\partial(\eta,\bar\eta)}\right)\begin{pmatrix}V\\ \bar V\end{pmatrix}.
\end{aligned}
$$

Then we have for $\chi - \tilde\chi$:

$$
\begin{aligned}
(\chi-\tilde\chi)_{11} &\longrightarrow_{(1,1)} 2i[G_{1\bar 1}(\overline{Dp})^1 + G_{1\bar 2}(\overline{Dp})^2 + \\
&\quad +\frac{1}{2}G_{1\eta}(\overline{Dq})^2 + \frac{1}{2}G_{1\bar\eta}(\overline{Dq})^1 + (\overline{Dp})^2]\\
(\chi-\tilde\chi)_{12} &\longrightarrow_{(1,1)} 2i[G_{2\bar 1}(\overline{Dp})^1 + G_{2\bar 2}(\overline{Dp})^2 + \\
&\quad +\frac{1}{2}G_{2\eta}(\overline{Dq})^2 + \frac{1}{2}G_{2\bar\eta}(\overline{Dq})^1\\
(\chi-\tilde\chi)_{21} &\longrightarrow_{(1,1)} 2i[\bar G_{\bar 1 1}(\overline{Dp})^1 + \bar G_{\bar 1 2}(\overline{Dp})^2 + \\
&\quad +\frac{1}{2}\bar G_{\bar 1\eta}(\overline{Dq})^1 + \frac{1}{2}\bar G_{\bar 1\bar\eta}(\overline{Dq})^2\\
(\chi-\tilde\chi)_{22} &\longrightarrow_{(1,1)} 2i[\bar G_{\bar 1\bar 2}(\overline{Dp})^1 + \bar G_{\bar 2\bar 2}(\overline{Dp})^2 + \\
&\quad +\frac{1}{2}\bar G_{\bar 2\eta}(\overline{Dq})^1 + \frac{1}{2}\bar G_{\bar 2\bar\eta}(\overline{Dq})^2 + (\overline{Dp})^1]
\end{aligned}
$$

The entries of $B - \tilde B$ in terms of Dp and Dq are:

$$
\begin{aligned}
(B-\tilde B)_{11} &= 2i[G_{\eta\bar 1}(\overline{Dp})^1 + G_{\eta\bar 2}(\overline{Dp})^2 + \\
&\quad +\frac{1}{2}G_{\eta\eta}(\overline{Dq})^2 + \frac{1}{2}G_{\eta\bar\eta}(\overline{Dq})^1\\
(B-\tilde B)_{12} &= 2i[G_{\bar\eta\bar 1}(\overline{Dp})^1 + G_{\bar\eta\bar 2}(\overline{Dp})^2 + \\
&\quad +\frac{1}{2}G_{\bar\eta\eta}(\overline{Dq})^2 + \frac{1}{2}G_{\bar\eta\bar\eta}(\overline{Dq})^1\\
(B-\tilde B)_{21} &= 2i[\bar G_{\bar\eta 1}(\overline{Dp})^1 + \bar G_{\bar\eta 2}(\overline{Dp})^2 + \\
&\quad +\frac{1}{2}\bar G_{\bar\eta\eta}(\overline{Dq})^1 + \frac{1}{2}\bar G_{\bar\eta\bar\eta}(\overline{Dq})^2\\
(B-\tilde B)_{22} &= 2i[\bar G_{\eta 1}(\overline{Dp})^1 + \bar G_{\eta 2}(\overline{Dp})^2 + \\
&\quad +\frac{1}{2}\bar G_{\eta\eta}(\overline{Dq})^1 + \frac{1}{2}\bar G_{\eta\bar\eta}(\overline{Dq})^2].
\end{aligned}
$$

Hence,

$$
\begin{aligned}
(\chi-\tilde{\chi})^{z^1}_{z^2} &= 2i(z^j G_{j\bar{k}}(\overline{Dp})^k + \frac{1}{2}z^j G_{j\eta}(\overline{Dq})^2 + \\
&+\frac{1}{2}z^j G_{j\bar{\eta}}(\overline{Dq})^1 + z^1(\overline{Dp})^2) \times \\
&\times 2i(z^2(\overline{Dp})^1) + z^j \bar{G}_{k\bar{j}}(\overline{Dp})^k + \frac{1}{2}z^j \bar{G}_{\bar{j}\eta}(\overline{Dq})^1 + \\
&+\frac{1}{2}z^j \bar{G}_{\bar{j}\bar{\eta}}(\overline{Dq})^2,
\end{aligned}
$$

and,

$$
i(B-\tilde{B})(\mathrm{id}-iB)^{-1}\chi\begin{pmatrix} z^1 \\ z^2 \end{pmatrix} = (B-\tilde{B})\begin{pmatrix} g_1^1 \\ g_1^2 \end{pmatrix}
$$

is the vector with the first component

$$
\begin{aligned}
&2i[i(\overline{Dp})^k G_{\eta\bar{k}} g_1^1 + \frac{i}{2}(\overline{Dq})^2 G_{\eta\eta} g_1^1 + \\
&+\frac{i}{2}(\overline{Dq})^1 G_{\eta\bar{\eta}} g_1^1 + i(\overline{Dp})^k G_{\bar{\eta}\bar{k}} g_1^2 + \frac{i}{2}(\overline{Dq})^2 G_{\bar{\eta}\eta} g_1^2 + \\
&+\frac{i}{2}(\overline{Dq})^1 G_{\bar{\eta}\bar{\eta}} g_1^2]
\end{aligned}
$$

and the second component

$$
\begin{aligned}
&2i[i(\overline{Dp})^k \bar{G}_{\bar{\eta}k} g_1^1 + \frac{i}{2}(\overline{Dq})^1 \bar{G}_{\bar{\eta}\eta} g_1^1 + \\
&+\frac{i}{2}(\overline{Dq})^2 \bar{G}_{\bar{\eta}\bar{\eta}} g_1^1 + i(\overline{Dp})^k \bar{G}_{\eta k} g_1^2 + \frac{i}{2}(\overline{Dq})^1 \bar{G}_{\eta\eta} g_1^2 + \\
&+\frac{i}{2}(\overline{Dq})^2 \bar{G}_{\eta\bar{\eta}} g_1^2].
\end{aligned}
$$

Inserting into the latter expression

$$
\begin{aligned}
(\overline{Dq})^1 &= -2i\bar{g}_1^1(\overline{Dp}) + \bar{\phi}_{11}(\bar{\eta},\eta)z'\Lambda^1\bar{z} \\
(\overline{Dq})^2 &= -2i\bar{g}_1^2(\overline{Dp}) + \bar{\phi}_{22}(\eta,\bar{\eta})z'\Lambda^{1*}\bar{z},
\end{aligned}
$$

and collecting the terms with derivatives of p we obtain:

$$\begin{aligned}(\chi & - \tilde{\chi})z + i(B - \tilde{B})g_1 = \\ & = 2i \begin{bmatrix} z'\Lambda^1 \overline{Dp} \\ z'\Lambda^{1*}\overline{Dp} \end{bmatrix} + \dots\end{aligned}$$

where the omitted terms do not depend on the derivatives of p.
Finally,

$$\begin{aligned} F_{21} &= 2i(T_2{}'\Lambda^1 \bar{z} - z'\Lambda^1 \overline{Dp}) + \tilde{F}_{21} \\ \bar{F}_{12} &= 2i(T_2{}'\Lambda^{1*}\bar{z} - z'\Lambda^{1*}\overline{Dp}) + \bar{\tilde{F}}_{12} \end{aligned} \tag{22}$$

where $\tilde{F}_{21}$ and $\tilde{F}_{12}$ do not depend on T and the derivatives of p.
Thus, before applying the transformation ψ_2 the equation of M has the form

$$\rho_0 \begin{pmatrix} V \\ \bar{V} \end{pmatrix} = z'\Lambda\bar{z} + 2i(T_2{}'\Lambda\bar{z}) - 2i\left(z'\Lambda\overline{\frac{\partial p}{\partial(\eta,\bar{\eta})}\begin{pmatrix} V \\ \bar{V}\end{pmatrix}}\right) + \dots$$

After applying ψ_2 it takes the form

$$\begin{aligned}\rho_0\rho \begin{pmatrix} V \\ \bar{V} \end{pmatrix} &= \rho_0\rho\langle z, z\rangle + 2i((C^{-1}T_2)'C'\Lambda\bar{C}\bar{z} - \\ & \quad 2i\left(z'C'\Lambda\bar{C}\overline{\bar{C}^{-1})\frac{\partial p}{\partial(\eta,\bar{\eta})}\rho\begin{pmatrix} V \\ \bar{V}\end{pmatrix}}\right) + \dots\end{aligned}$$

Plugging into the latter formula

$$\frac{\partial p}{\partial(\eta,\bar{\eta})} = \frac{\partial p}{\partial(\eta^*,\bar{\eta}^*)} = \rho^{-1},$$

we get

$$\begin{pmatrix} V \\ \bar{V} \end{pmatrix} = \langle z, z\rangle + 2i(\langle\tilde{T}_2\rangle z\rangle - \langle z, \Omega\langle z, z\rangle\rangle) + \dots$$

Here, $\tilde{T} = C^{-1}T_2$ and $\Omega = C^{-1}\frac{\partial p}{\partial(\eta^*,\bar{\eta}^*)}$, $\langle z, z\rangle = z'J\bar{z}$, or, in terms of V:

$$\begin{aligned} V &= z^1\bar{z}^2 + 2i(\tilde{T}_2^1\bar{z}^2 - \bar{\omega}_{22}(z^1)^2\bar{z}^2 - \bar{\omega}_{21}z^1z^2\bar{z}^1 - \\ & \quad -2i(z^1\bar{\tilde{T}}_2^2 - \omega_{11}(z^1)(\bar{z}^2)^2 - \omega_{12}z^2\bar{z}^1\bar{z}^2 \end{aligned}$$

To normalize F_{21} we choose T_2^1 eliminating any terms which contain $\bar{z}^2$ in F_{21}. T_2^2 will be chosen to eliminate all terms containing z^1 in F_{12} in (22):

$$
\begin{aligned}
\tilde{T}^1 &= \bar{\omega}_{22}(z^1)^2 + \ldots \\
\tilde{T}^2 &= \bar{\omega}_{11}(z^1)^2 + \ldots
\end{aligned} \tag{23}
$$

We denote $(C^{-1})_{11} = 1 + \delta_{11}$, $(C^{-1})_{21} = \delta_{21}$, $(C^{-1})_{12} = \delta_{12}$, $(C^{-1})_{22} = 1 + \delta_{22}$. Then

$$
\begin{aligned}
\omega^{11} &= (1 + \delta_{11})\frac{\partial p^1}{\partial \eta} + \delta_{12}\frac{\partial p^2}{\partial \eta}, \\
\omega^{22} &= \delta_{21}\frac{\partial p^1}{\partial \bar{\eta}} + (1 + \delta_{12})\frac{\partial p^2}{\partial \bar{\eta}},
\end{aligned}
$$

and,

$$
\begin{aligned}
\tilde{T}^1 &= \left((1 + \delta_{11})\frac{\partial p^1}{\partial \eta} + \delta_{12}\frac{\partial p^2}{\partial \eta}\right)(z^1)^2 + \ldots, \\
\tilde{T}^2 &= \left(\delta_{21}\frac{\partial p^1}{\partial \bar{\eta}} + (1 + \delta_{12})\frac{\partial p^2}{\partial \bar{\eta}}\right)(z^1)^2 + \ldots
\end{aligned}
$$

The coefficient at $z^1 z^2 \bar{z}^1$ in F_{21} equals to

$$
-2i\bar{\omega}_{21} + \cdots = -2i(\bar{\delta}_{21}\bar{p}^1_\eta + (1 + \bar{\delta}_{22})\bar{p}^2_\eta) + \ldots
$$

We choose p^2_η to eliminate this term

$$
p^2_\eta = -\frac{\delta_{21}}{1 + \delta_{22}} p^1_\eta + \ldots \tag{24}
$$

Then we choose $p^1_{\bar{\eta}}$ to eliminate the term at $z^2 \bar{z}^1 \bar{z}^2$ in F_{12} which equals to

$$
(2i\bar{\omega}_{12} + \ldots)z^2\bar{z}^1\bar{z}^2 = 2i((1 + \bar{\delta}_{11})\bar{p}^1_{\bar{\eta}} + \bar{\delta}_{12})\bar{p}^2_{\bar{\eta}} + \ldots)z^2\bar{z}^1\bar{z}^2.
$$

This implies

$$
p^1_{\bar{\eta}} = -\frac{\delta_{12}}{1 + \delta_{11}} p^2_{\bar{\eta}} + \ldots \tag{25}
$$

We have omitted the terms which do not depend on the derivatives of p.

The convergence of the solutions of the system (24), (25) follows from some argument used by Loboda in [Lob88], section 4.

Thus, after performing ψ_1 and ψ_2 the terms F_{12} and F_{21} take the form

$$F_{12} = \phi_{11\bar{1}}(z^1)^2\bar{z}^1 + \phi_{22\bar{1}}(z^2)^2\bar{z}^1$$
$$F_{21} = \phi_{2\bar{1}\bar{1}}z^2(\bar{z}^1)^2 + \phi_{2\bar{2}\bar{2}}(z^2)(\bar{z}^2)^2.$$

These polynomials belong to the space $\mathcal{N}$.

Now, we shall normalize the terms F_{k1}, F_{1k} for $k \geq 3$.

Since g_k^j are expressed in terms of T_k^j, we have only to determine the T_k^j.

After applying ψ_1 we observe that the terms in F_{k1} and F_{1k} containing T_k are equal to $F_{1k}^T + \overline{(F_{1k}^T)^*}$, where

$$\begin{pmatrix} F_{1k}^T \\ (F_{1k}^T)^* \end{pmatrix} = 2iz'\Lambda\bar{T}_k.$$

Analogously,

$$\begin{pmatrix} F_{k1}^T \\ (F_{k1}^T)^* \end{pmatrix} = 2iT'_k\Lambda\bar{z}.$$

After performing $z = Cz^*$ and dropping the stars

$$\begin{pmatrix} F_{k1}^T \\ (F_{k1}^T)^* \end{pmatrix} = 2i\begin{pmatrix} (C^{-1}T_k)'C'\Lambda^1\bar{C}\bar{z} \\ (C^{-1}T_k)'C'\Lambda^{1*}\bar{C}\bar{z} \end{pmatrix}.$$

Multiplying the left hand side by $(\rho_0\rho)^{-1}$ we get:

$$2i\tilde{T}_k^1\bar{z}^2 - 2i\overline{\tilde{T}_k^2\bar{z}^1},$$

where $\tilde{T}_k = C^{-1}T_k$.

Since the only freedom we have to normalize F_{k1} is the choice of T_k^1, one can eliminate at most the coefficients at $\bar{z}^2$, i.e. the normalized form of F_{k1} is

$$F_{k1} = A_k(z)\bar{z}^1,$$

and the normalized form of F_{1k} is, analogously,

$$F_{1k} = z^2\overline{B_k(z)}.$$

Since among T_k, $k \geq 2$ only T_2 and T_3 contribute to the terms being related to the special conditions of the normal form, for our purposes we need a more or less explicit expression only for T_3.

The contribution of the right hand side of (12) into the term of F_{31} (we consider the highest order terms with respect to the derivatives of p) equals

$$-2i(2iT'\Lambda^1\overline{Dp}) \longrightarrow -2i\left(2i\tilde{T}'C'\Lambda^1\bar{C}\overline{\Omega\begin{pmatrix} z^1\bar{z}^2 \\ z^2\bar{z}^1 \end{pmatrix}}\right).$$

After cancelling $\rho_0\rho$ we obtain

$$-2i(2i\tilde{T}^1(\omega_{21}z^2\bar{z}^1 + \omega_{22}z^1\bar{z}^2)).$$

Thus, the pure term in η in $\tilde{T}^1_3$ equals to

$$\tilde{T}^1_3(\eta) = 2iz^1\tilde{T}^1_2(\eta)\bar{\omega}_{22}(\eta),$$

where $\tilde{T}^1_3(\eta), \tilde{T}^1_2(\eta), \bar{\omega}_{22}(\eta)$ denote the the pure terms in η in $\tilde{T}^1_3, \tilde{T}^1_2$, and $\bar{\omega}_{22}$, respectively.

The corresponding contribution into F_{13} equals to

$$\overline{-2i(2iT_2{}'\Lambda^{1*}\overline{Dp})} = \overline{-2i(2i\tilde{T}_2'C'\Lambda^{1*}\bar{C}\Omega\begin{pmatrix} z^1\bar{z}^2 \\ z^2\bar{z}^1 \end{pmatrix}}$$
$$\longrightarrow \overline{-2i(2i\tilde{T}^2_2(\bar{\omega}_{11}z^2\bar{z}^1 + \bar{\omega}_{12}z^1\bar{z}^2))}.$$

Thus, the pure component in $\bar{\eta}$ in T^2_3 equals

$$\tilde{T}^2_3(\bar{\eta}) = 2iz^2\tilde{T}^2_2(\eta)\bar{\omega}_{11}(\bar{\eta}) + \dots,$$

and, since

$$\tilde{T}^1_2(\eta) = (z^1)^2\bar{\omega}_{22}(\eta) + \dots,$$

then

$$T^1_3(\eta) = 2i(z^1)^3\bar{\omega}^2_{22}(\eta) + \dots, \tag{26}$$

and, since

$$\tilde{T}^2_2(\bar{\eta}) = (z^2)^2\bar{\omega}_{11}(\bar{\eta}) + \dots,$$

then

$$T^2_3(\bar{\eta}) = 2i(z^2)^3\bar{\omega}^2_{11}(\bar{\eta}) + \dots, \tag{27}$$

This completes the normalization of the terms F_{k1} and F_{1k} for $k \geq 2$.

7. Normalization of the term F_{22}

In this section we give a construction of the transformation ψ_3 having the form (9):

$$\begin{aligned} z &= e^{i\phi(\omega^1,\omega^2)}\begin{pmatrix} e^{\mu(\omega^1,\omega^2)} & 0 \\ 0 & e^{-\mu(\omega^1,\omega^2)}\end{pmatrix} z^* \\ \omega &= \omega^*, \end{aligned}$$

where $\mu(\eta,\bar\eta)$ and $\phi(\eta,\bar\eta)$ are both real functions.

At first, we study what changes in the term F_{22} in the equation of M can be reached by applying ψ_3.

We set $\mu(\omega^1,\omega^2)+i\phi(\omega^1,\omega^2)=s(\omega^1,\omega^2)$, thus

$$\begin{aligned} \mu(\eta,\bar\eta) &= \frac{1}{2}(s(\eta,\bar\eta)+\bar s(\bar\eta,\eta)) \\ \phi(\eta,\bar\eta) &= \frac{1}{2i}(s(\eta,\bar\eta)-\bar s(\bar\eta,\eta)) \\ \overline{\phi(\eta,\bar\eta)} &= \frac{1}{2i}(\bar s(\bar\eta,\eta)-s(\eta,\bar\eta)) \end{aligned}$$

Therefore,

$$\begin{aligned} \mu(\omega^1,\omega^2) &= \frac{1}{2}(s(\eta+iV,\bar\eta+i\bar V)+\bar s(\bar\eta+i\bar V,\eta+iV)) \\ \overline{\mu(\omega^1,\omega^2)} &= \frac{1}{2}(s(\eta-iV,\bar\eta-i\bar V)+\bar s(\bar\eta-i\bar V,\eta-iV)) \\ \phi(\omega^1,\omega^2) &= \frac{1}{2i}(s(\eta+iV,\bar\eta+i\bar V)-\bar s(\bar\eta+i\bar V,\eta+iV)) \\ \overline{\phi(\omega^1,\omega^2)} &= \frac{1}{2i}(\bar s(\bar\eta-i\bar V,\eta-iV)-s(\eta-iV,\bar\eta-i\bar V)) \end{aligned}$$

The terms of the type $(2,2)$ may arise from F_{22} itself and from the Levi form as well.

The latter is

$$\begin{aligned} z^1\bar z^2 \longrightarrow\ & z^1\bar z^2\exp[\mu(\omega^1,\omega^2)-\overline{\mu(\omega^1,\omega^2)}+ \\ & +i(\phi(\omega^1,\omega^2)-\overline{\phi(\omega^1,\omega^2)})]. \end{aligned}$$

In the brackets we have then

$$\begin{aligned} & \mu(\omega^1,\omega^2) - \overline{\mu(\omega^1,\omega^2)} + i(\phi(\omega^1,\omega^2) - \overline{\phi(\omega^1,\omega^2)}) = \\ = \quad & s(\eta + iV, \bar\eta + i\bar V) - s(\eta - iV, \bar\eta - i\bar V) \longrightarrow \\ \longrightarrow \quad & 2i(s_\eta V + s_{\bar\eta}\bar V). \end{aligned}$$

Hence,

$$\begin{aligned} & z^1\bar z^2 \longrightarrow_{(1,1),(2,2)} z^1\bar z^2 + 2iz^1\bar z^2(s_\eta z^1\bar z^2 + s_{\bar\eta} z^2\bar z^1) = \\ & = z^1\bar z^2 + 2is_\eta(z^1)^2(\bar z^2)^2 + 2is_{\bar\eta}|z^1|^2|z^2|^2. \end{aligned}$$

Thus, s will be uniquely determined, if we require that after applying ψ_3 the coefficient at $|z^1|^2|z^2|^2$ in the term F_{22} vanishes as well as the coefficient at the pure term in η at $(z^1)^2(\bar z^2)^2$.

Let $s(\eta,\bar\eta) = s_0(\eta) + \bar\eta\tilde s(\eta,\bar\eta)$.

Before writing down the equations determining s_0 and $\tilde s$, we have to evaluate the leading terms in F_{22} after applying $\psi_2 \circ \psi_1$. These terms equal to the sum:

$$2((Dp)'\Lambda^1\overline{Dp}) + 4(T_2)'\Lambda^1\bar T_2.$$

$$\begin{aligned} & 4(T_2)'\Lambda^1\bar T_2 \longrightarrow 4(\tilde T_2)'C'\Lambda^1\bar C\bar{\tilde T}_2 = \\ & = 4((1+\bar\delta_{22})\overline{p^2_{\bar\eta}}(\eta,\bar\eta) + \bar\delta_{21})\overline{p^1_{\bar\eta}}(\bar\eta,\eta)) \times \\ & ((1+\delta_{11})p^1_\eta(\eta,\bar\eta) + \delta_{12}p^2_\eta(\bar\eta,\eta))(z^1)^2(\bar z^2)^2 + \cdots = \\ & = 4\bar\omega_{22}\omega_{11}(z^1)^2(\bar z^2)^2. \end{aligned}$$

The component of the latter term pure in η equals to

$$\begin{aligned} & = 4((1+\bar\delta_{22}(\eta,0)\overline{\pi^2_{\bar\eta}}(\eta) + \bar\delta_{21}(\eta,0))\overline{p^1_{\bar\eta}}(0,\eta)) \times \\ & ((1+\delta_{11}(\eta,0))\pi^1_\eta(\eta) + \delta_{12}(\eta,0)p^2_\eta(0,\eta))(z^1)^2(\bar z^2)^2, \end{aligned}$$

where $\pi^1 = p^1(\eta,0)$, $\pi^2 = p^2(\bar\eta,0)$ - the pure components of p^1 and p^2, respectively. Analogously,

$$
\begin{aligned}
&2(Dp)'\Lambda^1\overline{Dp}) = \\
&2\left(C^{-1}\frac{\partial p}{\partial(\eta^*,\bar\eta^*)}\rho^{-1}\rho\begin{pmatrix} z^1\bar z^2 \\ z^2\bar z^1\end{pmatrix}\right)' \times \\
&\times C'\Lambda^1\bar C\overline{\left(C^{-1}\frac{\partial p}{\partial(\eta^*,\bar\eta^*)}\rho^{-1}\rho\begin{pmatrix} z^1\bar z^2 \\ z^2\bar z^1\end{pmatrix}\right)} = \\
&2\left(\Omega\begin{pmatrix} z^1\bar z^2 \\ z^2\bar z^1\end{pmatrix}\right)^1\overline{\left(\Omega\begin{pmatrix} z^1\bar z^2 \\ z^2\bar z^1\end{pmatrix}\right)}^2 = \\
&2(\omega_{11}\bar\omega_{22}(z^1)^2(\bar z^2)^2 + (\omega_{11}\bar\omega_{21} + \omega_{12}\bar\omega_{22})|z^1|^2|z^2|^2 + \\
&\omega_{12}\bar\omega_{21}(z^2)^2(\bar z^1)^2).
\end{aligned}
$$

Thus, the leading terms in F_{22} can be written as

$$
6\omega_{11}\bar\omega_{22}(z^1)^2(\bar z^2)^2 + 2(\omega_{11}\bar\omega_{21} + \omega_{12}\bar\omega_{22})|z^1|^2|z^2|^2 + \dots,
$$

Hence, the contribution to F_{22} available for normalization is

$$
\begin{aligned}
&2(\omega_{11}\bar\omega_{21} + \omega_{12}\bar\omega_{22})|z^1|^2|z^2|^2 + \\
&+6((1+\bar\delta_{22})\overline{p^2_{\bar\eta}}(\eta,\bar\eta) + \bar\delta_{21})\overline{p^1_{\bar\eta}}(\bar\eta,\eta)) \times \\
&((1+\delta_{11})p^1_\eta(\eta,\bar\eta) + \delta_{12}p^2_\eta(\bar\eta,\eta))(z^1)^2(\bar z^2)^2 + \dots = \\
&= 4\bar\omega_{22}\omega_{11}(z^1)^2(\bar z^2)^2.
\end{aligned}
$$

After setting

$$
\frac{\partial s}{\partial \eta} = i(\omega_{11}\bar\omega_{21} + \omega_{12}\bar\omega_{22}) + \dots \tag{28}
$$

to eliminate the coefficient at $|z^1|^2|z^2|^2$, and

$$
\begin{aligned}
s_0'(\eta) \;=\; 3i\Big[&(1+\bar\delta_{22})\overline{p^2_{\bar\eta}}(\eta,\bar\eta) + \bar\delta_{21})\overline{p^1_{\bar\eta}}(\bar\eta,\eta))\times \\
&((1+\delta_{11})p^1_\eta(\eta,\bar\eta) + \delta_{12}p^2_\eta(\bar\eta,\eta)\Big]
\end{aligned} \tag{29}
$$

we normalize F_{22}.

The convergence of s follows again from Loboda's argument in [Lob88] applied to (28). (29) is just an ordinary equation.

8. Normalization of F_{23} and F_{32}. The equation of the chain.

We have alredy obtained the equations determining $p^1_{\bar\eta}$ and p^2_η after normalizing F_{21} and F_{12}.

The normalization of F_{23} and F_{32} leads to some ordinary second order differential equations determining the functions $\pi^1(\eta) = p^1(\eta, 0)$ and $\pi^2(\bar\eta) = p^2(\bar\eta, 0)$.

At first we select the terms in F_{23} and F_{32} which might contain second derivatives of π^1 and π^2:

$$-2(z'\Lambda^1\overline{DT_2} + \overline{z'\Lambda^{1*}\overline{DT_2}}),$$

where

$$DT_2 = \frac{\partial T_2}{\partial(\eta, \bar\eta)}\begin{pmatrix} V \\ \bar V \end{pmatrix},$$

$$\longrightarrow -2(z'C'\Lambda\bar C\overline{C^{-1}DT_2} + \overline{z'C'\Lambda\bar C\overline{C^{-1}DT_2}} \longrightarrow$$
$$\longrightarrow -2\left(z^1\overline{(D\tilde T_2)}^2 + z^2\overline{(D\tilde T_2)}^1\right) + \dots,$$

where we omitted the terms of degree ≤ 2 in p-derivatives.

They give the following contribution into F_{23}:

$$\longrightarrow_{(2,3)} -2((\bar{\tilde T}^2_2)_\eta z^1 z^2 \bar z^1 + (\bar{\tilde T}^2_2)_{\bar\eta}(z^1)^2\bar z^2) =$$
$$= -2\left\{\frac{\partial}{\partial\eta}[(1+\bar\delta_{11})\bar p^1_\eta(\bar\eta,\eta) + \bar\delta_{12}\bar p^2_\eta(\eta,\bar\eta)]|z^1|^2|z^2|^2\bar z^2 + \right.$$
$$\left.\frac{\partial}{\partial\eta}[(1+\bar\delta_{11})\bar p^1_\eta(\bar\eta,\eta) + \bar\delta_{12}\bar p^2_\eta(\eta,\bar\eta)](z^1)^2(\bar z^2)^3\right\}.$$

The pure contribution in η equals:

$$-2(1+\delta_{11}(\eta,0))(\pi^1)''(\eta)(z^1)^2(\bar z^2)^3 + \dots, \tag{30}$$

where we omitted the terms of degree ≤ 2 in p-derivatives.

Analogously for F_{32}:

$$\longrightarrow_{(3,2)} -2((\tilde{T}_2^1)_\eta z^1(\bar{z}^2)^2 + (\tilde{T}_2^1)_{\bar{\eta}}(z^2)\bar{z}^1\bar{z}^2) =$$

$$= -2\left\{\frac{\partial}{\partial\bar{\eta}}[(1+\bar{\delta}_{22})\bar{p}_{\bar{\eta}}^2(\eta,\bar{\eta}) + \bar{\delta}_{21}\bar{p}_{\bar{\eta}}^1(\bar{\eta},\eta)]z^1|z^1|^2|z^2|^2 + \right.$$

$$\left.\frac{\partial}{\partial\eta}[(1+\bar{\delta}_{22})\bar{p}_{\bar{\eta}}^2(\eta,\bar{\eta}) + \bar{\delta}_{21}\bar{p}_{\bar{\eta}}^1(\bar{\eta},\eta)](z^1)^3(\bar{z}^2)^2\right\}$$

$$-2(1+\bar{\delta}_{22})(\bar{\pi}^2)''(\eta)(z^1)^3(\bar{z}^2)^2 + \ldots. \tag{31}$$

Thus, we use the remaining freedom in π^1 and π^2 to eliminate the pure terms in η ($\bar{\eta}$) in F_{23} (F_{32}), respectively.

To give an appropriate approximation of π^1 and π^2, we are going now to determine the leading terms in the equations of the π-s. The leading terms will be those of degree 3 in the first order derivatives of the p-s. They arise from $4T_3'\Lambda\bar{T}_2$ and the contribution of F_{22}.

We have:

$$\begin{aligned} 4T_3'\Lambda^1\bar{T}_2 &= 4(C^{-1}T_3)C'\Lambda^1\bar{C}\overline{C^{-1}T_2} \longrightarrow \\ &\longrightarrow 4\tilde{T}_3^1\bar{\tilde{T}}_2^2 \\ &\longrightarrow 8i\bar{\omega}_{22}^2\omega_{11}(z^1)^3(\bar{z}^2)^2 \\ &\longrightarrow_{(3,2)} 8i(1+\bar{\delta}_{22})^2(\bar{\pi}_{\bar{\eta}}^2(\eta))^2(1+\delta_{11})\pi_\eta^2(\eta)(z^1)^3(\bar{z}^2)^2, \end{aligned}$$

$$\begin{aligned} 4T_2'\Lambda^{1*}\bar{T}_3 &= 4(C^{-1}T_2)C'\Lambda^{1*}\bar{C}\overline{C^{-1}T_3} \longrightarrow \\ &\longrightarrow 4\tilde{T}_2^1\bar{\tilde{T}}_3^2 \\ &\longrightarrow -8i\bar{\omega}_{22}\omega_{11}^2(z^1)^2(\bar{z}^2)^3 \\ &\longrightarrow_{(2,3)} -8i(1+\delta_{11})^2(\pi_\eta^1(\eta))^2(1+\bar{\delta}_{22})\bar{\pi}_{\bar{\eta}}^2(\eta)(z^1)^2(\bar{z}^2)^3, \end{aligned}$$

The contribution of F_{22} eaquals to

$$\longrightarrow_{(3,2)} -2iz'\Lambda^1\overline{\frac{\partial p}{\partial(\eta,\bar{\eta})}\begin{pmatrix}F_{22}^1\\F_{22}^2\end{pmatrix}} \longrightarrow$$

$$-4iz^1\overline{\left[\Omega\begin{pmatrix}2\tilde{T}_2^1\bar{\tilde{T}}_2^2 + \left(\Omega\begin{pmatrix}z^1 & \bar{z}^2\\ z^2 & \bar{z}^1\end{pmatrix}\right)^1\overline{\left(\Omega\begin{pmatrix}z^1 & \bar{z}^2\\ z^2 & \bar{z}^1\end{pmatrix}\right)^2}\\ 2\tilde{T}_2^2\bar{\tilde{T}}_2^1 + \left(\Omega\begin{pmatrix}z^1 & \bar{z}^2\\ z^2 & \bar{z}^1\end{pmatrix}\right)^2\overline{\left(\Omega\begin{pmatrix}z^1 & \bar{z}^2\\ z^2 & \bar{z}^1\end{pmatrix}\right)^1}\end{pmatrix}\right]}^2.$$

Within the matrix in brackets:

$$2\tilde{T}_2^2\bar{\tilde{T}}_2^1 \longrightarrow 2(1+\bar{\delta}_{11})\bar{\pi}_\eta^1(\bar{\eta})(1+\delta_{22})\pi_{\bar{\eta}}^2(\bar{\eta})(z^2)^2(\bar{z}^1)^2.$$

And, the whole contribution into $F_{3,2}$ eaquals to

$$2\tilde{T}_2^2\bar{\tilde{T}}_2^1 \longrightarrow 2(1+\bar{\delta}_{11})\bar{\pi}_\eta^1(\bar{\eta})(1+\delta_{22})\pi_{\bar{\eta}}^2(\bar{\eta})(z^2)^3(\bar{z}^1)^2.$$

$$\longrightarrow_{(3,2)} -12i(1+\bar{\delta}_{22})^2(1+\delta_{11})(\bar{\pi}_{\bar{\eta}}^2(\eta))^2\pi_\eta^1(\eta)(z^1)^3(\bar{z}^2)^2+\dots$$

Analogously,

$$\longrightarrow_{(2,3)} 12i(1+\delta_{11})^2(1+\bar{\delta}_{22})(\pi_\eta^1(\eta))^2\bar{\pi}_{\bar{\eta}}^2(\eta)(z^1)^2(\bar{z}^2)^3+\dots$$

Collecting all the leading terms, both in F_{23} and F_{32}, we get:

$$\longrightarrow_{(2,3)} 4i(1+\delta_{11}(\eta,0))^2(1+\bar{\delta}_{22}(0,\eta))(\pi_\eta^1(\eta))^2\bar{\pi}_{\bar{\eta}}^2(\eta)(z^1)^2(\bar{z}^2)^3$$
$$\longrightarrow_{(3,2)} -4i(1+\bar{\delta}_{22})^2(1+\delta_{11})(\bar{\pi}_{\bar{\eta}}^2(\eta))^2\pi_\eta^1(\eta)(z^1)^3(\bar{z}^2)^2.$$

Combining these formulas with (30) and (31) and taking into account the condition that the coefficient at $(z^1)^2(\bar{z}^2)^3$ in F_{23} and the coefficient at $(z^1)^3(\bar{z}^2)^2$ in F_{32} vanish, we obtain the second order equations determining π^1 and π^2:

$$(32)\qquad \begin{aligned}(\pi^1)''(\eta) &= 2i(1+\delta_{11}(\eta,0))(1+\bar{\delta}_{22}(0,\eta))\pi_\eta^1(\eta)\bar{\pi}_{\bar{\eta}}^2(\eta)\pi_\eta^1(\eta)+\dots\\ (\pi^2)''(\bar{\eta}) &= 2i(1+\delta_{22}(0,\bar{\eta}))(1+\delta_{11}(\bar{\eta},0))\pi_{\bar{\eta}}^2(\bar{\eta})\bar{\pi}_\eta^1(\bar{\eta})\pi_{\bar{\eta}}^2(\bar{\eta})+\dots\end{aligned}$$

These equations completely define the chain Γ up to the parameters

$$\begin{aligned} a^1=(\pi^1)'(0) &= \left.\frac{\partial p^1}{\partial\eta}\right|_0, \text{ and}\\ a^2=(\pi^2)'(0) &= \left.\frac{\partial p^2}{\partial\bar{\eta}}\right|_0. \end{aligned}$$

The normalized terms F_{32} and F_{23} satisfy the conditions:

$$\begin{aligned} \left.\frac{\partial^4}{\partial\nu^2\partial z^1\partial\eta}\right|_{\bar{\eta}=0} F_{32} &= 0,\\ \left.\frac{\partial^4}{\partial\nu^2\partial\bar{z}^2\partial\eta}\right|_{\bar{\eta}=0} F_{23} &= 0. \end{aligned}$$

9. Normalization of F_{33}

Any transformation ψ_4 preserving the pre-normalized form of the equation obtained in the previous sections has the form (10)

$$\begin{aligned} z^* &= e^{\lambda(\omega)} \begin{pmatrix} e^{i\theta(\omega)} & 0 \\ 0 & e^{-i\theta(\omega)} \end{pmatrix} z \\ \omega^* &= h(\omega) = \begin{pmatrix} h^1(\omega) \\ h^2(\omega) \end{pmatrix}, \end{aligned}$$

where

$$\frac{\partial h}{\partial(\omega^1, \omega^2)} = \begin{pmatrix} e^{2(\lambda(\omega)+i\theta(\omega))} & 0 \\ 0 & e^{2(\lambda(\omega)-i\theta(\omega))} \end{pmatrix},$$

and, $\operatorname{Im} \lambda(\eta, \bar{\eta}) = \operatorname{Im} \theta(\eta, \bar{\eta}) = 0$.
If λ and θ are constants then ψ_4 is an linear (C, ρ)-automorphism of the elliptic quadric. Therefore, we presume that $\lambda(0,0) = \theta(0,0) = 0$.
The Levi-form in $*$-coordinates is

$$\begin{aligned} z^{1*}\bar{z}^{2*} &= e^{2(\lambda+i\theta)} z^1 \bar{z}^2 \\ z^{2*}\bar{z}^{1*} &= e^{2(\lambda-i\theta)} z^2 \bar{z}^1. \end{aligned}$$

The form of the Jacobian $\frac{\partial h}{\partial(\omega^1,\omega^2)}$ implies that

$$\begin{aligned} \frac{\partial h^1}{\partial \omega^2} &= 0 \\ \frac{\partial h^1}{\partial \omega^1} &= e^{2(\lambda+i\theta)(\omega)} \text{ and} \\ \frac{\partial h^2}{\partial \omega^1} &= 0 \\ \frac{\partial h^2}{\partial \omega^2} &= e^{2(\lambda-i\theta)(\omega)} \end{aligned}$$

We denote

$$\begin{aligned} \zeta^1(\omega^1) &= \zeta(\omega^1) = (\lambda + i\theta)(\omega) \\ \zeta^2(\omega^2) &= \bar{\zeta}(\omega^2) = (\lambda - i\theta)(\omega). \end{aligned}$$

Consequently,

$$\begin{aligned}\lambda &= \frac{\zeta^1(\omega^1)+\zeta^2(\omega^2)}{2},\\ \theta &= \frac{\zeta^1(\omega^1)-\zeta^2(\omega^2)}{2i}.\end{aligned}$$

And, since $\zeta(\eta)=\lambda(\eta,\bar\eta)+i\theta(\eta,\bar\eta)$, we get for λ and θ:

$$\begin{aligned}\lambda(\eta,\bar\eta) &= \frac{\zeta(\eta)+\bar\zeta(\bar\eta)}{2},\\ \theta(\eta,\bar\eta) &= \frac{\zeta(\eta)-\bar\zeta(\bar\eta)}{2i},\end{aligned}$$

and, finally in terms of ζ:

$$\begin{aligned}\lambda(\omega^1,\omega^2) &= \frac{\zeta(\eta+iV)+\bar\zeta(\bar\eta+i\bar V)}{2},\\ \theta(\omega^1,\omega^2) &= \frac{\zeta(\eta+iV)-\bar\zeta(\bar\eta+i\bar V)}{2i},\\ (h^1)'(\omega^1) &= e^{2\zeta(\omega^1)},\\ (h^2)(\omega)'(\omega^2) &= e^{2\bar\zeta(\omega^2)}.\end{aligned}$$

Now we are able to compute the contribution to F_{33} after performing ψ_4. The "new" Levi form contributes into the "old" equation of M:

$$z^{1*}\bar z^{2*}\longrightarrow e^{\lambda(\omega)+i\theta(\omega))+\overline{\lambda(\omega)}+i\overline{\theta(\omega)}}z^1\bar z^2.$$

For the expression in the brackets we have:

$$\begin{aligned}&\frac{\zeta(\eta+iV)+\bar\zeta(\bar\eta+i\bar V)}{2}+\frac{\bar\zeta(\bar\eta-i\bar V)+\zeta(\eta-iV)}{2}\longrightarrow\\ &\longrightarrow\frac{2\zeta(\eta)-2\frac{V^2}{2}\zeta''(\eta)+2\bar\zeta(\bar\eta)-2\frac{\bar V^2}{2}\bar\zeta''(\bar\eta)}{2}\\ &\longrightarrow\zeta(\eta)+\bar\zeta(\bar\eta)-\frac{V^2}{2}(\zeta''(\eta)+\bar\zeta''(\bar\eta)).\end{aligned}$$

Similarly,

$$i\left[\frac{\zeta(\eta+iV)-\bar\zeta(\bar\eta+i\bar V)}{2i}-\frac{\bar\zeta(\bar\eta-i\bar V)-\zeta(\eta-iV)}{2i}\right]\longrightarrow$$
$$\longrightarrow\frac{2\zeta(\eta)-2\frac{V^2}{2}\zeta''(\eta)-2\bar\zeta(\bar\eta)-2\frac{\bar V^2}{2}\bar\zeta''(\bar\eta)}{2}$$
$$\longrightarrow\zeta(\eta)-\bar\zeta(\bar\eta)-\frac{V^2}{2}(\zeta''(\eta)-\bar\zeta''(\bar\eta)).$$

Hence, the contribution of the $*$-Levi form equals to

$$z^{1*}\bar z^{2*}\longrightarrow\exp(2\zeta(\eta)-V^2\zeta''(\eta))z^1\bar z^2\longrightarrow e^{2\zeta}(1-V^2\zeta''(\eta))z^1\bar z^2 \tag{33}$$

The contribution of the $*$-left hand side:

$$\begin{aligned}V^* &= \frac{\omega^{1*}-\bar\omega^{2*}}{2i}=\frac{h^1(\eta+iV)-h^1(\eta-iV)}{2i}\longrightarrow\\ &\longrightarrow\frac{2iV(h^1)'(\eta)}{2i}-\frac{2iV^3}{12i}(h^1)'''(\eta)=\\ &(h^1)'(\eta)V-\frac{V^3}{6}(h^1)'''(\eta).\end{aligned}$$

And, since

$$\begin{aligned}(h^1)' &= e^{2\zeta(\eta)}\\ (h^1)'' &= 2\zeta'(\eta)e^{2\zeta(\eta)}\\ (h^1)''' &= (2\zeta''(\eta)+4(\zeta'(\eta))^2)e^{2\zeta(\eta)},\end{aligned}$$

we get for the left hand side:

$$V^*=e^{2\zeta(\eta)}\left(V-\frac{1}{3}\left(\zeta''(\eta)+2(\zeta'(\eta))^2V^3\right)+\dots\right) \tag{34}$$

Combining (33) and (34), and cancelling $e^{2\zeta}$ we get in the right hand side:

$$\longrightarrow\left(-\zeta''(\eta)+\frac{1}{3}\zeta''(\eta)+\frac{2}{3}(\zeta'(\eta))^2\right)(z^1\bar z^2)^3=$$
$$=-\frac{2}{3}(\zeta''(\eta)-(\zeta'(\eta))^2)(z^1\bar z^2)^3.$$

This means, that in order to obtain the normal form of F_{33} we can eliminate the η-pure term at $(z^1\bar{z}^2)^3$. If this η-pure term in the pre-normalized form equals to $\phi(\eta)(z^1\bar{z}^2)^3$ then the equation for ζ takes the form:

$$\zeta''(\eta) - (\zeta'(\eta))^2 + \frac{3}{2}\phi(\eta) = 0, \tag{35}$$

with $\zeta(0) = 0$.

This equation defines ζ up to $\zeta'(0)$, that gives a $\mathbf{C}^1$, or $\mathbf{R}^2$,-freedom in the initial data related to the parameter r (see (5)).

Now we compute the leading terms of degree 4 in the first derivatives of the π-s. Since,

$$\begin{aligned} 4T_3'\Lambda^1\bar{T}_3 &= 4(C^{-1}T_3)'C'\Lambda^1\bar{C}\overline{C^{-1}T_3} \longrightarrow \\ &\longrightarrow 4T_3^1\bar{T}_3^2 \longrightarrow 16\bar{\omega}_{22}^2(\eta,0)\omega_{11}^2(\eta,0)(z^1\bar{z}^2)^3 \longrightarrow \\ &\longrightarrow 16(1+\bar{\delta}_{22}(\eta,0))^2(1+\delta_{11}(\eta,0))^2((\bar{\pi}^2)'(\eta))^2((\pi^1)'(\eta))^2(z^1\bar{z}^2)^3 \end{aligned}$$

and, as was shown in [ES93], the leading terms of F_{33} have certain proportionality with respect to the contribution of different terms, the contribution of $4T_3'\Lambda\bar{T}_3$ is (-24)-times the actual value of the leading term at $\phi(\eta)$. Therefore,

$$\phi(\eta) = -\frac{2}{3}(1+\bar{\delta}_{22}(\eta,0))^2(1+\delta_{11}(\eta,0))^2((\bar{\pi}^2)'(\eta))^2((\pi^1)'(\eta))^2(z^1\bar{z}^2)^3.$$

Hence, the so-called parametrization equation defining $\zeta(\eta)$ takes the form:

(36)
$$\zeta''(\eta) - (\zeta'(\eta))^2 - (1+\bar{\delta}_{22}(\eta,0))^2(1+\delta_{11}(\eta,0))^2((\bar{\pi}^2)'(\eta))^2((\pi^1)'(\eta))^2(z^1\bar{z}^2)^3 + \cdots = 0,$$

where the dots indicate the terms of lower degree in π-derivatives.

The normalized F_{33} satisfies the condition

$$\left.\frac{\partial^4}{\partial\nu^3\partial\eta}\right|_{\bar{\eta}=0} F_{33} = 0.$$

This completes the proof of Theorem 1.

10. Parametrization of the normalizations by $\mathrm{Aut}_0 Q$. The family of normal parameters on Γ.

We prove Theorems 2 and 3 in this section.

Proof of Theorem 2. Following the proof of Theorem 1 step by step we see that the only freedom, we have in the normalization with identical differential at 0 relates to the parameters

$$\begin{aligned} a^1 &= (\pi^1)'(0), \\ a^2 &= (\pi^2)'(0), \\ r &= \zeta'(0). \end{aligned}$$

It is easy to observe that any linear (C,ρ)-automorphism preserving the Levi form (a linear (C,ρ)-automorphism of Q_e) preserves the normal form of M as well. Therefore, removing the condition that the differential of the normalization ψ at 0 is identical, we obtain the freedom

$$\begin{aligned} C &= \left.\frac{\partial z^*}{\partial z}\right|_0, \\ \rho &= \left.\frac{\partial w^*}{\partial w}\right|_0, \end{aligned}$$

where $\langle Cz, Cz\rangle = \rho\langle z, z\rangle$, as well.

Since the entire set of parameters (C,ρ,a,r) arises from the first and one of the second derivatives of ψ, the multiplication law for the set of parameters is the same as in the case of Q_e.

This statement makes sense in the case when the original M is already given in normal form (One can presume this according to Theorem 1).

Thus, we obtained the group $\mathcal{N}(M)$ of normalizations of M at the origin which is actually isomorphic to $\mathrm{Aut}_0 Q_e$.

This completes the proof of Theorem 2.

Proof of Theorem 3. Suppose that M is given in normal form in which Γ is the plane $\Gamma_0:\ z=0,\ v=0$. Since any linear (C,ρ)-transformation from $\mathrm{Aut}_0 Q_e$ preserves Γ_0 and the normal form we can represent ψ as a composition $\psi=\psi_e\circ\psi_0$, where ψ_0 has identical differential at 0, and ψ_e is a linear (C,ρ)-transformation.

Since ψ_0 keeps Γ_0 and the normal form of M it has to be a transformation of type ψ_4. Since the term F_{33} has normal form both in original and $*$-coordinates, the corresponding equation (36) defining the function $\zeta(\eta)$ does not contain $\phi(\eta)$. Thus, it is the homogenious equation:

$$\zeta''-(\zeta')^2=0,$$

with $\zeta(0) = 0$.
Set $\zeta'(\eta) =: \chi(\eta)$, then $\chi' = \chi^2$, and, consequently, $\chi = \zeta' = -\frac{1}{\eta+C}$.
Setting $C := -\frac{1}{r}$ and integrating the equation above we obtain

$$h^1(\eta) = \frac{\eta}{1 - r\eta}.$$

(We used that $h^1(0) = 0$.)
This implies

$$\begin{aligned} h^1(\omega^1) &= \frac{\omega^1}{1 - r\omega^1}, \\ h^2(\omega^2) &= \frac{\omega^2}{1 - \bar{r}\omega^2}, \quad r \in \mathbb{C}. \end{aligned} \tag{37}$$

Theorem 3 has been proved.

11. Final remarks

In this work we do not cover the following questions concerning a nonquadric (i.e. not locally equivalent to Q_e) elliptic CR surface M in $\mathbb{C}^4$.

(*i.*) What is the sharp estimate of the order of contact of the chains passing through the origin of an elliptic surface M given in normal form and the chains on the tangent elliptic quadric being intersections of Q_e with so-called "matrix lines".

(*ii.*) Since $\mathcal{N}(M)$ has a structure of $\mathrm{Aut}_0\, Q_e$, we obtain a faithful representation $\mathrm{Aut}_0(M) \hookrightarrow \mathcal{N}(M) \cong \mathrm{Aut}_0(Q_e)$, as in the hypercase. Now, the question is whether this embedding is uniquely defined by the CR projection of the automorphism? We call this effect "Lobodization", after A.Loboda, who proved the corresponding result for codimension 1 [Lob82].

(*iii.*) In the case of an affirmative answer to the question (ii.) the following questions arise: Has any element of $\mathrm{Aut}_0(M)$ an invarinat chain? Does there exist a chain Γ passing through the origin being invariant with respect to the entire $\mathrm{Aut}_0(M)$?

We hope that the construction of a normal form developped in this paper might help to find the answers to the questions stated above.

References

[Bel89] V.K. Beloshapka. Finite-dimensionality of the group of automorphisms of a real analytic surface. *USSR Izvestiya*, 32(2):437–442, 1989.

[Bel90] V.K. Beloshapka. A uniqueness theorem for automorphisms of a nondegenerate surface in a complex space. *Math. Notes*, 47:239–242, 1990.

[EKV84] V.V. Ežov, N.G. Kružilin, and A.G. Vituškin. Extention of holomorphic mappings along real-analytic hypersurfaces (Russian). *Trudy MIAN*, 167:60–95, 1984.

[ES92] V.V. Ežov and G. Schmalz. Biholomorphic automorphisms of Siegel domains in $\mathbb{C}^4$. preprint, Max-Planck-Institut für Mathematik Bonn, 1992.

[ES93] V.V. Ežov and G. Schmalz. Holomorphic automorphisms of real nondegenerate quadrics of codimension 2 in $\mathbb{C}^4$. preprint 4, Oklahoma State University, 1993.

[Ežo83] V.V. Ežov. Asymptotics of a strictly pseudoconvex hypersurface along its chain (Russian). *Izv. AN SSSR*, 47(4):560–586, 1983.

[Kru83] N.G. Kružilin. Estimate of the variation of the normal parameter on the chain on a stricly pseudoconvex hypersurface (Russian). *Izv. AN SSSR*, 47(5):1091–1113, 1983.

[Lob82] A.V. Loboda. Linearization of the automorphisms of nonspherical hypersurfaces (Russian). *Izv.AN.SSSR*, 46(4):864–880, 1982.

[Lob88] A.V. Loboda. Generic real analytic manifoldsof codimension 2 in $\mathbb{C}^n$ and their biholomorphic mappings (Russian). *Izv.AN.SSSR*, 52:970–990, 1988.

[MS74] J.K. Moser and Chern S.S. Real hypersurfaces in complex manifolds. *Acta Math.*, 133(3-4):219–271, 1974.

[Vit85] A.G. Vituškin. Holomorphic mappings and geometry of hypersurfaces (Russian). In *Itogi nauki i tehniki, Sovr.probl.matematiki, Fund.napr. vol. 7*, pages 167–226. Viniti, Moscow, 1985.

(V.V. Ežov) Oklahoma State University, Department of Mathematics, College of Arts and Sciences, Stillwater, Oklahoma 74078-0613

E-mail address: ezhov@hardy.math.okstate.edu

(G. Schmalz) Mathematisches Institut, der Universität Bonn, Wegelerstrasse 10, D-5300 Bonn-1

E-mail address: schmalz@mpim-bonn.mpg.de