HIGHER ORDER TRACK CATEGORIES AND THE ALGEBRA OF HIGHER ORDER COHOMOLOGY OPERATIONS

HANS-JOACHIM BAUES

ABSTRACT. We describe a conjecture on the algebra of higher cohomology operations which leads to the computations of the differentials in the Adams spectral sequence. For this we introduce the notion of an n-th order track category which is suitable to study higher order Toda brackets and the differentials in spectral sequences. We describe various examples of higher order track categories which are topological, in particular the track category of higher cohomology operations. Also differential algebras give rise to higher order track categories.

1. DEFORMATION CATEGORIES

We first recall properties of cylinders of spaces. Let $I = [0,1]$ be the unit interval. For a subspace $A \subset B$ we define the *relative cylinder* $I_A B$ by the pushout diagram in **Top**

$$
\begin{array}{ccc}
I \times B & \xrightarrow{\ \bar{p}\ } & I_A B \\
{\scriptstyle I \times i}\Big\uparrow & & \Big\uparrow{\scriptstyle \bar{i}} \\
I \times A & \xrightarrow[\ p\ ]{} & A
\end{array}
$$

where $i : A \subset B$ is the inclusion and p is the projection of the product space $I \times A$. We have *inclusions*

$$i^+, i^- : B \to I_A B$$

by $i^+(x) = \bar{p}(1,x)$, $i^-(x) = \bar{p}(0,x)$, $x \in B$. We call the union

$$\partial(I_A B) = B^- \cup B^+ \text{ with } B^- = i^- B,\ B^+ = i^+ B$$

the *boundary* of $I_A B$ with $B^- \cap B^+ = \bar{i} A$. We also have a *projection* $q : I_A B \to B$ with $q\bar{p} = p$ and

$$qi^+ = 1 \text{ and } qi^- = 1$$

A map $H : I_A B \to C$ in **Top** is termed a *homotopy* $H : f \simeq g$ rel A where $f = Hi^-$ and $g = Hi^+$. The map

$$\varepsilon(f) = fq : I_A B \to B \to C$$

is the *constant homotopy* $\varepsilon(f) : f \simeq f$ rel A. Moreover the map $I \to I$ which carries t to $1 - t$ induces for $H : f \simeq g$ rel A the *opposite homotopy*

$$H^{\mathrm{op}} : g \simeq f \text{ rel } A$$

By *pasting homotopies* $H : f \simeq g$ rel A and $G : g \simeq h$ rel A one gets

$$G \,\square\, H : f \simeq h \text{ rel } A$$

Let **Top** be a convenient category of topological spaces and let $\mathbf{Top}^*$ be the category of pointed spaces $(X, *)$. For pointed spaces X, Y we have the smash product $X \wedge Y = X \times Y / X \vee Y$ where $X \vee Y = X \times * \cup * \times Y$. If A is a non-pointed space then $A^+ = A \,\dot\cup\, *$ is the disjoint union with a base point and we have

$$A^+ \wedge X = A \times X / A \times *$$

We call a map $f : A^+ \wedge X \to Y$ in $\mathbf{Top}^*$ an A-map $X \to Y$. We can identify the A-map f with a map $f : A \to (Y, *)^{(X,*)}$ where $(Y, *)^{(X,*)}$ is the function space of pointed maps $X \to Y$.

Definition 1.1. Let $\mathbf{C}(A)$ be the category of A-maps. Objects are pointed spaces X, Y and morphisms are A-maps. Composition gf of A-maps is determined by the composition

$$A \times X \xrightarrow{\Delta \times X} A \times A \times X \xrightarrow{A \times f} A \times Y \xrightarrow{g} Z$$

where Δ is the diagonal of A. The identity of X in $\mathbf{C}(A)$ is given by the projection $A \times X \to X$.

A map $i : A \to B$ in **Top** induces a functor

$$i^* : \mathbf{C}(B) \to \mathbf{C}(A)$$

which is the identity on objects and carries a B-map f to the A-map $i^* f$ given by the composite $f(i \times X) : A \times X \to B \times X \to Y$. The category $\mathbf{C}(A)$ has a *zero object* $*$ (that is, an initial and final object) given by the basepoint $*$.

We now consider the category of $I_A B$-maps for a relative cylinder $I_A B$.

Proposition 1.2. *Let* $\mathbf{C}_1 = \mathbf{C}(I_A B)$ *and* $\mathbf{C}_0 = \mathbf{C}(B)$. *Then* $\mathbf{C}_0, \mathbf{C}_1$ *are categories with a zero object together with functors*

$$\mathbf{C}_0 \xrightarrow{\varepsilon} \mathbf{C}_1 \underset{\partial^-}{\overset{\partial^+}{\rightrightarrows}} \mathbf{C}_0,$$
$$\mathrm{op} : \mathbf{C}_1 \to \mathbf{C}_1,$$
$$\square : \mathbf{C}_1 \times_\partial \mathbf{C}_1 \to \mathbf{C}_1$$

which are the identity on objects. Here $\mathbf{C}_1 \times_\partial \mathbf{C}_1$ *is the category of pairs* (G, H) *of maps* $X \to Y$ *in* $\mathbf{C}_1$ *satisfying* $\partial^+ H = \partial^- G$. *The functors satisfy the equations (where* id *denotes the identity functor)*

$$\partial^+ \varepsilon = \partial^- \varepsilon = \mathrm{id},$$
$$\partial^+ \mathrm{op} = \partial^-, \quad \mathrm{op} \circ \varepsilon = \varepsilon,$$
$$\mathrm{op} \circ \mathrm{op} = \mathrm{id},$$
$$\mathrm{op}(G \,\square\, H) = H^{\mathrm{op}} \,\square\, G^{\mathrm{op}}$$

We call such a pair $(\mathbf{C}_0, \mathbf{C}_1)$ *a* deformation category.

Proof. With the structure maps of the relative cylinder we define $\varepsilon = q^*$ for $q : I_A B \to B$, $\partial^+ = (i^+)^*$, $\partial^- = (i^-)^*$. Moreover op and $\square$ are given by the opposite homotopy and by pasting respectively. $\square$

In a deformation category $(\mathbf{C}_0, \mathbf{C}_1)$ we write $H : f \simeq g$ if H is a morphism in $\mathbf{C}_1$ with $\partial^- H = f$ and $\partial^+ H = g$.

Deformation categories form a category. Morphisms are pairs of functors $F_0 : \mathbf{C}_0 \to \mathbf{C}_0'$, $F_1 : \mathbf{C}_1 \to \mathbf{C}_1'$ which are the identity on objects and which are compatible with $\varepsilon, \mathrm{op}, \partial^+, \partial^-$ and $\square$.

2. Track categories

A track category by definition is a groupoid enriched category with a strict zero object. In this section we compare track categories and deformation categories. We consider examples of abelian track categories and triple Toda brackets.

Lemma 2.1. *A track category is a deformation category.*

Proof. Let $\mathbf{C}$ be a track category. Then we define the category $\mathbf{C}_0$ by the category of 1-cells f in $\mathbf{C}$. The category $\mathbf{C}_1$ has the morphisms $X \to Y$ which are triples (f, g, H) where $H : f \Rightarrow g$ is a 2-cell (also termed *track*). Composition $(f, g, H)(f', g', H')$ is defined by $(ff', gg', H * H')$ where $H * H'$ is the horizontal composition of tracks defined by

$$H * H' = (g')^* H \,\square\, f_* H' = g_* H' \,\square\, (f')^* H$$

Here $\square$ is the composition of 2-cells (termed *pasting* of tracks). We define $\varepsilon(f) = (f, f, 0_f)$ where 0_f is the identity 2-cell of f. Moreover $\partial^-(f, g, H) = f$, $\partial^+(f, g, H) = g$ and $\mathrm{op}(f, g, H) = (g, f, H^{-1})$ where H^{-1} is the inverse of the 2-cell H in the Hom-groupoid $\mathrm{Hom}(X, Y)$. $\square$

Lemma 2.2. *A deformation category is a track category if and only if the following equations hold:*

$$H \,\square\, \varepsilon(f) = H \qquad \qquad \textit{where } f = \partial^- H,$$
$$\mathrm{op}(H) \,\square\, H = \varepsilon(f)$$
$$(H \,\square\, G) \,\square\, F = H \,\square\, (G \,\square\, F),$$

(*) $$H H' = H\varepsilon(g') \,\square\, \varepsilon(f) H' \qquad \textit{where } g' = \partial^+ H'.$$

Here the last equation is required by the horizontal composition of tracks. The category of track categories is a full subcategory of the category of deformation categories.

Lemma 2.3. *A deformation category $(\mathbf{C}_0, \mathbf{C}_1)$ yields a natural equivalence relation $\simeq$ on the category $\mathbf{C}_0$ so that the homotopy category $\mathbf{C}_0/\simeq$ is defined.*

Proof. For morphisms $f, g : X \to Y$ in $\mathbf{C}_0$ we write $f \simeq g$ if and only if there exists $H : X \to Y$ in $\mathbf{C}_1$ with $\partial^- H = f$ and $\partial^+ H = g$. $\square$

Example 2.4. Let B be a point and let A be the empty set. Then $I_A B$ is the interval I. The category $\mathbf{C}_0 = \mathbf{C}(\text{point})$ coincides with $\mathbf{Top}^*$ and the morphisms in $\mathbf{C}_1 = \mathbf{C}(I)$ are homotopies of pointed maps. The category $\mathbf{C}_0/\simeq$ coincided with the homotopy category $\mathbf{Top}^*/\simeq$. Homotopies between homotopies yield a natural equivalence relation $\simeq$ on $\mathbf{C}_1 = \mathbf{C}(I)$ such that $(\mathbf{C}_0, \mathbf{C}_1/\simeq)$ is the track category associated to $\mathbf{Top}^*/\simeq$. This is a quotient of the deformation category $(\mathbf{C}_0, \mathbf{C}_1)$.

A track category $\mathbf{C} = (\mathbf{C}_0, \mathbf{C}_1)$ is *abelian* if all automorphism groups in homgroupoids are abelian groups. We denote such automorphism groups by $\mathrm{Aut}_\square(f)$ where f is a morphism in $\mathbf{C}_0$ and $H \in \mathrm{Aut}_\square(f)$ is of the form $H : f \simeq f$.

Example 2.5. Let $\mathfrak{X}$ be a class of co-H-groups in $\mathbf{C} = \mathbf{Top}^*$ or let $\mathfrak{X}$ be a class of H-groups in $\mathbf{Top}^*$. Let $\mathbf{Top}^*[[\mathfrak{X}]]$ be the track subcategory of $(\mathbf{C}(\text{point}), \mathbf{C}(I)/\simeq)$ consisting of objects in $\mathfrak{X}$. Then $\mathbf{Top}^*[[\mathfrak{X}]]$ is an abelian track category.

The example has a generalization concerning "under" and "over" categories respectively which play the role of "left" resp. "right" modules.

Definition 2.6. Let $\mathbf{C}$ be a category and let $\mathfrak{X}$ be a class of objects in $\mathbf{C}$ yielding the full subcategory $\mathbf{C}\{\mathfrak{X}\}$ of $\mathbf{C}$. Let W be an object in $\mathbf{C}$. Then the *under category* $\mathbf{C}\{W \to \mathfrak{X}\}$ consists of the objects in $\mathfrak{X}$ and the object W. Morphisms

are all morphisms in $\mathbf{C}\{\mathfrak{X}\}$ and all morphisms $W \to Y$ in $\mathbf{C}$ with $Y \in \mathfrak{X}$. Also the identity of W is a morphism of the under category. The *over category* $\mathbf{C}\{\mathfrak{X} \to W\}$ is defined in a dual way as a subcategory of $\mathbf{C}$.

Example 2.7. Let $\mathfrak{X}$ be a class of co-H-groups in $\mathbf{Top}^*$ and let $\mathbf{Top}^*[[\mathfrak{X} \to W]]$ be the track category given by the over category $\mathbf{Top}^*\{\mathfrak{X} \to W\}$. Then $\mathbf{Top}^*[[\mathfrak{X} \to W]]$ is an abelian track category. Dually for a class $\mathfrak{X}$ of H-groups the under track category $\mathbf{Top}^*[[W \to \mathfrak{X}]]$ is abelian. These are again full track subcategories of $(\mathbf{C}(\text{point}), \mathbf{C}(I)/\simeq)$.

Given a category $\mathbf{K}$ we define the *category $F\mathbf{K}$ of factorizations* in $\mathbf{K}$. Objects in $F\mathbf{K}$ are morphisms f in $\mathbf{K}$ and a morphism $f \to g$ in $F\mathbf{K}$ is a pair (a,b) of morphisms with $bfa = g$. We have morphisms $(b,1) : f \to bf$ and $(1,a) : f \to fa$. A functor $D : F\mathbf{K} \to \mathbf{Ab}$ is termed a *natural system* of abelian groups. We write $D_f = D(f)$ and $(b,1)_* = b_* : D_f \to D_{bf}$ and $(1,a)_* = a^* : D_f \to D_{fa}$. In [BW] the *cohomology* $H^n(\mathbf{K}, D)$ is defined.

Let $(\mathbf{C}_0, \mathbf{C}_1)$ be an abelian track category. Then there is an *associated* natural system D on $\mathbf{C}_0/\simeq$ together with a natural isomorphism of abelian groups

$$\sigma : D_{\{f\}} \cong \mathrm{Aut}_\square(f)$$

for f in $\mathbf{C}_0$, see [BJ,Cl]. Here $\{f\}$ denotes the homotopy class of f in $\mathbf{C}_0/\simeq$. We call $(\mathbf{C}_0, \mathbf{C}_1)$ a *linear track extension* of $\mathbf{C}_0/\simeq$ by D. It is proved in [BD], [P] that equivalence classes of such linear track extensions are in 1-1 correspondence with elements in the cohomology $H^3(\mathbf{C}_0/\simeq, D)$, [BW].

We now describe the natural system associated to the abelian track category in the above examples of 2.5 and 2.7. Here $\mathfrak{X}$ is a class of co-H-groups or of H-groups. For pointed spaces X, Y let $X \vee Y$ be the coproduct in $\mathbf{Top}^*$ with inclusions i_1, i_2 and let $X \times Y$ be the product of spaces with projections p_1, p_2. For a map $f : X \to Y$ in $\mathbf{Top}^*/\simeq$ with $X, Y \in \mathfrak{X}$ we define in the co-H-case

$$\nabla f : X \to Y \vee Y \qquad \text{in } \mathbf{Top}^*/\simeq,$$
$$\nabla f = -i_2 f + f(i_2 + i_1).$$

Since $p_2(\nabla f) = 0$ the *partial suspension* $(n \geq 1)$

$$E^n \nabla f : \Sigma^n X \to (\Sigma^n Y) \vee Y$$

is defined, see [B,Ob]. In the H-case we get

$$\nabla f : X \times X \to Y \qquad \text{in } \mathbf{Top}^*/\simeq,$$
$$\nabla f = -f p_2 + (p_2 + p_1)f.$$

Since $(\nabla f)i_1 = 0$ the *partial loop operation* $(n \geq 1)$

$$L^n \nabla f : (\Omega^n X) \times X \to Y$$

is defined, see [B,Ob]. In [B,Ob] we describe rules to compute $E^n \nabla f$ and $L^n \nabla f$ explicitly, see also [B,Al].

Definition 2.8. Let $\mathfrak{X}$ be a class of co-H-groups. Then we define a natural system D^n_Σ on the over category $\mathbf{K} = \mathbf{Top}^*\{\mathfrak{X} \to W\}/\simeq$. If $\mathfrak{X}$ is a class of H-groups we define a natural system D^n_Ω on the under category $\mathbf{K} = \mathbf{Top}^*\{W \to \mathfrak{X}\}/\simeq$. Consider maps

$$X' \xrightarrow{a} X \xrightarrow{f} Y \xrightarrow{b} Y'$$

in $\mathbf{K}$. In the co-H-case we define

$$D^n_\Sigma(f) = [\Sigma^n X, Y]$$

and for $\alpha \in D_\Sigma^n(f)$ let $b_*(\alpha) = b\alpha$ and $a^*(\alpha) = (\alpha, f)(E^n \nabla a)$. Moreover in the H-case we define

$$D_\Omega^n(f) = [X, \Omega^n Y]$$

and for $\beta \in D_\Omega^n(f)$, let $b_*(\beta) = (L^n \nabla b)(\beta, f)$ and $\alpha^*(\beta) = \beta a$.

We point out that in general D_Σ^n and D_Ω^n are not given by the bifunctors $[\Sigma^n X, Y]$ and $[X, \Omega^n Y]$ respectively. In [BJ] we prove for the examples above:

Proposition 2.9. *The natural systems associated to the abelian track categories* $\mathbf{Top}^*[[\mathfrak{X} \to W]]$ *and* $\mathbf{Top}^*[[W \to \mathfrak{X}]]$ *are* D_Σ^1 *and* D_Ω^1 *respectively.*

Next we describe *triple Toda brackets* in abelian track categories $(\mathbf{C}_0, \mathbf{C}_1)$ with natural system D on $\mathbf{C}_0/\simeq$. Let

$$W \xleftarrow{\alpha} X \xleftarrow{\beta} Y \xleftarrow{\gamma} Z$$

be morphisms in $\mathbf{C}_0/\simeq$ with $\alpha\beta = 0$ and $\beta\gamma = 0$. Here $0 = 0(Y, W) : Y \to * \to W$ is the *trivial map* given by the zero object $*$. Then we can choose representatives f, g, h of α, β, γ respectively and $H : Y \to W, G : Z \to X$ in $\mathbf{C}_1$ with

$$\partial^+ H = 0, \ \partial^- H = fg, \ \partial^+ G = 0, \ \partial^- G = gh$$

Hence the element

$$c(H, G) = (H\varepsilon(g)) \,\square\, (\varepsilon(f)G^{\mathrm{op}}) \in \mathrm{Aut}_\square(0(Z, W)) = D_{0(Z,W)}$$

is defined. The collection of all such elements yields the Toda bracket in the quotient group

$$\langle \alpha, \beta, \gamma \rangle \in D_{0(Z,W)} \big/ \gamma^* D_{0(Y,W)} + \alpha_* D_{0(Z,X)}$$

3. The indexing set of balls

We introduce the indexing set of balls which generalizes the set of cubes, but has more properties concerning relative cylinders. The category $\mathbb{B}$ of such balls yields the $\mathbb{B}$-sets generalizing cubical sets. We consider $\mathbb{B}$-categories.

A *ball* B of dimension n is a finite regular CW-complex together with a subcomplex ∂B such that $(B, \partial B)$ is homeomorphic to the Euclidean ball (E^n, S^{n-1}) with $E^n = \{x \in \mathbb{R}^n, \|x\| \le 1\}$ and $S^{n-1} = \{x \in \mathbb{R}^n, \|x\| = 1\}$. We say that the ball B is *elementary* if the CW-complex B has exactly one open n-cell. If B is a ball then the relative cylinder (as a quotient complex of $I \times B$)

$$J(B) = I_{\partial B} B$$

is a ball of dimension $\dim(B) + 1$. Moreover if B and B' are balls then the *product* CW-complex $B \times B'$ is a ball of dimension $\dim(B) + \dim(B')$. A *ball pair* (B, A) is a ball B together with a subcomplex $A \subset \partial B$ which is a ball such that there exists a homeomorphism of pairs $(J(A), A^-) \approx (B, A)$ extending the identity of A. If A_{op} is the closure of $\partial B - A$ then also (B, A_{op}) is a ball pair which we call the *opposite* of (B, A). If (B, A) and (B', A) are ball pairs then the *gluing*

$$B \cup B' = B \cup_A B'$$

is again a ball of dimension $\dim(B) = \dim(B')$.

Definition 3.1. The *indexing set* $\mathbb{B}$ *of balls* is the smallest set of balls with the following properties:

(1) point $\in \mathbb{B}$,
(2) If $B \in \mathbb{B}$ then $J(B) \in \mathbb{B}$,
(3) If $B, B' \in \mathbb{B}$ then $B \times B' \in \mathbb{B}$,
(4) If (B, A) is a ball pair with $B \in \mathbb{B}$ then $A \in \mathbb{B}$,

(5) If (B, A) and (B', A) are ball pairs with $B, B' \in \mathbb{B}$ then $B \cup_A B' \in \mathbb{B}$. Conversely if $B \cup_A B' \in \mathbb{B}$ then also $B, B' \in \mathbb{B}$.

We now describe some examples of balls in $\mathbb{B}$. Since $\{\text{point}\} \in \mathbb{B}$ also the *interval* from 0 to 1 is a ball

$$I = [0, 1] = J(\text{point}) \in \mathbb{B}.$$

Moreover inductively $J^n, I^n \in \mathbb{B}$, $n \geq 0$, where $J^0 = I^0 = \text{point}$ and $J^1 = I^1 = I$ and for $n \geq 1$

$$J^{n+1} = J(J^n)$$
$$I^{n+1} = I \times I^n$$

Here I^n is the *n-cube* and J^n is the *n-globe* with hemispheres $i^-(J^{n-1}), i^+(J^{n-1})$.

Let $\mathbb{B}^n$ be the set of n-dimensional balls in $\mathbb{B}$. Then $\mathbb{B}^0$ contains only the point. Moreover $\mathbb{B}^1$ is given by all *long intervals* $[0, n] \subset \mathbb{R}$ with 0-cells given by $0, 1, \ldots, n \in \mathbb{R}$. Next $\mathbb{B}^2$ consists of balls J^2, I^2, Δ^2 (where Δ^2 is the 2-simplex) and the balls obtained by a regular sequence of gluings of these balls, for example
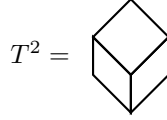
is such a ball, see section 5 below.

Let $\mathbb{B}$ be the full subcategory of **Top** with objects in the indexing set of balls. Let $\mathbb{B}_\partial$ be the subcategory of pair maps $f : (B, \partial B) \to (A, \partial A)$ with $A, B \in \mathbb{B}$ and let $B_\partial / \simeq$ be the homotopy category relative the boundary with $f \simeq g \Leftrightarrow f|\partial B = g|\partial B$ and $f \simeq g$ rel ∂B.

Remark 3.2. Given a small category $\mathbf{D}$ a $\mathbf{D}$-*object* X in a category $\mathbf{U}$ is a contravariant functor $X : \mathbf{D} \to \mathbf{U}$. This is a $\mathbf{D}$-set if $\mathbf{U}$ is the category of sets. Moreover let $\mathcal{O}$ be a fixed class of objects with $* \in \mathcal{O}$ and let $\mathbf{cat}(\mathcal{O})$ be the category of categories for which the class of objects is $\mathcal{O}$ with zero object $*$. Morphisms are functors which are the identity on $\mathcal{O}$. Then a $\mathbf{D}$-*category* is a $\mathbf{D}$-object in $\mathbf{cat}(\mathcal{O})$. For the category $\mathbb{B}$ of balls we shall considers $\mathbb{B}$-sets and $\mathbb{B}$-categories. They generalize cubical sets and cubical categories respectively since the category of cubes is a subcategory of $\mathbb{B}$.

For $B \in \mathbb{B}$ we have the *category* $\mathbf{C}(B)$ *of B-maps* defined in 1.1. Moreover $f : B \to A$ in $\mathbb{B}$ induces a functor $f^* : \mathbf{C}(A) \to \mathbf{C}(B)$ in $\mathbf{cat}(\mathcal{O})$ where $\mathcal{O}$ is the class of pointed spaces. Hence

$$\mathbf{C} = \{\mathbf{C}(B), B \in \mathbb{B}\}$$

is a $\mathbb{B}$-category which we call the *topological $\mathbb{B}$-category*. The morphism sets $\mathbf{C}(X, Y)$ of B-maps $X \to Y$ for $B \in \mathbb{B}$ form a $\mathbb{B}$-set.

For any $\mathbb{B}$-category $\mathbf{C}$ and a ball pair (B, A) let

$$\partial_A = (i_A)^* : \mathbf{C}(B) \to \mathbf{C}(A)$$

be induced by the inclusion $i_A : A \subset B$. We also write $\partial_A f = f|A$. For example for $(JB, B^\pm)$ we get

$$\partial^\pm = \partial_{B^\pm} : \mathbf{C}(JB) \to \mathbf{C}(B)$$

and for H in $\mathbf{C}(JB)$ we write $H : f \simeq g$ if $\partial^- H = f$ and $\partial^+ H = g$.

4. $\mathbb{B}$-CATEGORIES WITH UNIONS

Since the union of balls is defined in $\mathbb{B}$ we can consider unions and gluings in a $\mathbb{B}$-category.

Let $\mathbf{C}$ be the $\mathbb{B}$-category of B-maps defined in section 3. Given ball pairs (B, A) and (B', A) with $B, B' \in \mathbb{B}$ one obtains the *gluing functor*

$$\cup : \mathbf{C}(B) \times_{\mathbf{C}(A)} \mathbf{C}(B') \to \mathbf{C}(B \cup_A B').$$

Here the pullback category is given by pairs $(F : X \to Y) \in \mathbf{C}(B), (G : X \to Y) \in \mathbf{C}(B')$ with $\partial_A F = \partial_A G$ and $F \cup G$ is defined by

$$(B \cup_A B') \times X = (B \times X)_{A \times X}(B' \times X) \xrightarrow{F \cup G} Y.$$

The gluing functor is *natural* with respect to pair maps $f : (B, A) \to (B_1, A_1), g : (B', A) \to (B_1', A_1)$ in $\mathbb{B}$ which coincide on A, $f|A = g|A$. Then

$$f^*F \cup g^*G = (f \cup g)^*(F \cup G)$$

We now describe the gluing rule in the $\mathbb{B}$-category $\mathbf{C}$.

Let $\overline{B}$ be a ball and let M be a set of balls B which are subcomplexes of $\overline{B}$ and $\dim B = \dim \overline{B}$. Moreover assume that for $B, B' \in M$ with $B \neq B'$ we have $(B - \partial B) \cap (B' - \partial B') = \emptyset$ and

$$\bigcup_{B \in M} B = \overline{B}$$

Then we say that $\overline{B}$ is a *union of balls* in M. A *regular sequence* of balls in $\overline{B}$ is a bijection $\{B_1, \ldots, B_k\} \approx M$ (where k is the number of elements in M) such that

$$B_{\leq i} = \bigcup_{j \leq i} B_j, \quad 1 \leq i < k,$$

is a ball and $(B_{\leq i}, A_i), (B_{i+1}, A_i)$ are ball pairs with $A_i = B_{\leq i} \cap B_{i+1}$. If B-maps $f_B : X \to Y$ in $\mathbf{C}(B)$ are given for $B \in M$ then each regular sequence in $\overline{B}$ yields the iterated gluing operation

$$(\ldots((f_{B_1} \cup f_{B_2}) \cup f_{B_3}) \ldots \cup f_{B_k})$$

which is a $\overline{B}$-map $X \to Y$. The *gluing rule* is the fact that this $\overline{B}$-map is independent of the choice of the regular sequence in $\overline{B}$.

Lemma 4.1. *Let $\mathbf{C}$ be a $\mathbb{B}$-category with unions. Then for $B \in \mathbb{B}$ the pair $(\mathbf{C}(B), \mathbf{C}(JB))$ is a deformation category with structure $\varepsilon, \mathrm{op}, \square$. This yields the homotopy category $\mathbf{C}[B] = \mathbf{C}(B)/\simeq$ and $\mathbf{C}[\_] = \{\mathbf{C}[B], B \in \mathbb{B}\}$ is a $(\mathbb{B}_\partial/\simeq)$- category.*

Proof. The maps $\varepsilon' : JB \to B, i^\pm : B \to JB, \mathrm{op}' : JB \to JB, \square' : JB \to JB \cup JB$ are maps in $\mathbb{B}$. They induce $\varepsilon = (\varepsilon')^*$, $\mathrm{op} = (\mathrm{op}')^*$, $\partial^\pm = (i^\pm)^*$. Moreover $\square'$ yields $\square$ by the union property, that is

$$(\square')^*(F \cup G) = F \square G.$$

$\square$

Since $i^+ i_A = i^- i_A$ we have $\partial_A \partial^+ = \partial_A \partial^-$ and therefore there is a commutative diagram

$$\begin{array}{ccc} \mathbf{C}(B) & \xrightarrow{\quad q \quad} & \mathbf{C}[B] \\ & \searrow{\scriptstyle \partial_A} \quad \swarrow{\scriptstyle \partial_A} & \\ & \mathbf{C}(A) & \end{array}$$

for each ball pair (B, A). Here q is the quotient map.

The topological $\mathbb{B}$-category $\mathbf{C}$ has the property termed *quotient property* that there is also a well defined gluing functor $\cup$ on homotopy categories $\mathbf{C}[B] = \mathbf{B}/\simeq$, that is, the following diagram commutes where q is the quotient functor.

$$\begin{array}{ccc} \mathbf{C}(B) \times_{\mathbf{C}(A)} \mathbf{C}(B') & \xrightarrow{\quad \cup \quad} & \mathbf{C}(B \cup_A B') \\ \downarrow{\scriptstyle q \times q} & & \downarrow{\scriptstyle q} \\ \mathbf{C}[B] \times_{\mathbf{C}(A)} \mathbf{C}[B'] & \xrightarrow{\quad \cup \quad} & \mathbf{C}[B \cup_A B'] \end{array}$$

Definition 4.2. A $\mathbb{B}$-*category with unions* is a $\mathbb{B}$-category $\mathbf{C}$ with a gluing functor, which is natural and satisfies the union rule and the quotient property.

The topological $\mathbb{B}$-category is an example of a $\mathbb{B}$-category with unions.

Remark 4.3. The quotient property can not be deduced from the properties of an abstract $\mathbb{B}$-category since $JJB \cup_B JJB$ is not a ball.

Lemma 4.4. *Let $\mathbf{C}$ be a $\mathbb{B}$-category with unions. Then for $B \in \mathbb{B}$ the pair $(\mathbf{C}(B), \mathbf{C}[JB])$ is a deformation category which is a quotient of the deformation category $(\mathbf{C}(B), \mathbf{C}(JB))$. Moreover for objects X, Y the morphism sets*

$$(\mathbf{C}(B)(X, Y), \mathbf{C}[JB](X, Y))$$

form a groupoid.

For the proof below we use thin fillers obtained as follows. If $s, t : B \to B'$ are maps in $\mathbb{B}$ with $s|\partial B = t|\partial B$ then there exists $T : JB \to B'$ with $\partial^+ T = t$, $\partial^- T = s$. We can find T since B' is contractible. Hence $T^* : \mathbf{C}(B') \to \mathbf{C}(JB)$ satisfies $\partial^- T^* = s^*$, $\partial^+ T^* = t^*$. We call $T^*(x) : s^*(x) \simeq t^*(x)$ a *thin filler* for the pair $(s^*(x), t^*(x))$.

Proof. We have to show that the equations in Lemma 2.2 are satisfied, that is, there exist elements $\bar{\varepsilon}, \overline{\mathrm{op}}, \overline{\square}$ in $\mathbf{C}(JJ(B))$ which are homotopies

$$\bar{\varepsilon} : H \,\square\, \varepsilon(f) \simeq H \quad \text{for } \partial^- H = f \text{ in } \mathbf{C}(B)$$

$$\overline{\mathrm{op}} : \mathrm{op}(H) \,\square\, H \simeq \varepsilon(f)$$

$$\overline{\square} : (H \,\square\, G) \,\square\, F \simeq H \,\square\, (G \,\square\, F)$$

We obtain these homotopies by use of thin fillers. We consider the diagram

$$\begin{array}{ccc} JB & & \\ \downarrow{\scriptstyle \square'} & \searrow{\scriptstyle 1} & \\ JB \cup JB & \xrightarrow[i^- \varepsilon' \cup 1]{} & JB \end{array}$$

Here $\square'$ induces $\square$ and $\varepsilon' : JB \to B$ induces ε. Using a thin filler for the induced operators one gets $\bar{\varepsilon}$. In a similar way one gets $\overline{\mathrm{op}}, \overline{\square}$. $\qquad\square$

5. $\mathbb{B}$-CATEGORIES WITH $\otimes$-PRODUCTS

Let $\mathbf{C}$ be the topological $\mathbb{B}$-category of B-maps in section 3. Since products of balls are defined in $\mathbb{B}$ we can define $\otimes$-products in $\mathbb{B}$-categories as follows.

For $B, B' \in \mathbb{B}$ and for a B'-map $g : Y \to Z$ and a B-map $f : X \to Y$ we have the $B' \times B$-map $g \otimes f : X \to Z$ induced by the composite

$$B' \times B \times X \xrightarrow{B' \times f} B' \times Y \xrightarrow{g} Z.$$

This yields the $\otimes$-product in $\mathbf{C}$ which has properties as described in the next Lemmas.

Lemma 5.1. *The operation $\otimes$ has the following properties:*

(1) *$\otimes$ is associative.*

(2) *For B-maps $f : X \to Y, g : Y \to Z$ the composite gf in $\mathbf{C}(B)$ satisfies*

$$\Delta^*(g \otimes f) = gf$$

where $\Delta^* : \mathbf{C}(B \times B) \to \mathbf{C}(B)$ is induced by the diagonal $B \to B \times B$.

(3) *Let $0_{X,Y}^B$ be the trivial map $X \to * \to Y$ in $\mathbf{C}(B)$. Then we have for $(g : Y \to Z) \in \mathbf{C}(B'), (f : X \to Y) \in \mathbf{C}(B)$,*

$$g \otimes 0_{X,Y}^B = 0_{X,Z}^{B \times B'},$$
$$0_{Y,Z}^{B'} \otimes f = 0_{X,Z}^{B \times B}.$$

The point is the final object in $\mathbb{B}$ and the unique map $\varepsilon_0' : B \to$ point induces the functor

$$\varepsilon_0 : \mathbf{C}(\text{point}) \to \mathbf{C}(B)$$

with $f^* \varepsilon_0 = \varepsilon_0$ for all $f : B \to B'$ in $\mathbb{B}$.

Lemma 5.2. *For $g : Y \to Z$ in $\mathbf{C}(B')$ and $f : X \to Y$ in $\mathbf{C}(B)$ we have $g \otimes f : X \to Z$ in $\mathbf{C}(B' \times B)$. If B' is the point then $f \otimes g = f \varepsilon_0(g)$ in $\mathbf{C}(B)$ and if B is the point then $f \otimes g = \varepsilon_0(f)g$ in $\mathbf{C}(B')$.*

Moreover we have the following *naturality of $\otimes$-products.*

Lemma 5.3. *For maps $g_1 : B_1' \to B', f_1 : B_1 \to B$ in $\mathbb{B}$ one has*

$$(g_1^* g) \otimes (f_1^* f) = (f_1 \times g_1)^*(g \otimes f)$$

In particular, for all ball pairs (B, A) the product $(B' \times B, B' \times A)$ is again a ball pair and we have

$$g \otimes \partial_A f = \partial_{B' \times A}(g \otimes f)$$

Definition 5.4. An abstract $\mathbb{B}$-*category* $\mathbf{C}$ *with $\otimes$-products* is defined by $\otimes$-products which satisfy the properties in the Lemmata above.

6. $\mathbb{B}$-DEFORMATION CATEGORIES

Let $\mathbf{C}$ be the topological $\mathbb{B}$-category. Then we have seen that $\mathbf{C}$ has unions and $\otimes$-products. Moreover the following formulas are satisfied.

Let $g : Y \to Z$ in $\mathbf{C}(B')$ and $f : X \to Y$ in $\mathbf{C}(B)$. Then $g \otimes f : X \to Z$ in $\mathbf{C}(B \times B')$ is defined. Now let $B = B_1 \cup B_2$ or $B' = B_1' \cup B_2'$ be unions of ball pairs. Then for $f = f_1 \cup f_2$ one has the *compatibility of $\otimes$ and $\cup$*

$$g \otimes (f_1 \cup f_2) = (g \otimes f_1) \cup (g \otimes f_2)$$

and for $g = g_1 \cup g_2$ one has

$$(g_1 \cup g_2) \otimes f = (g_1 \otimes f) \cup (g_2 \otimes f)$$

Definition 6.1. An abstract $\mathbb{B}$-*deformation category* $\mathbf{C}$ is a $\mathbb{B}$-category $\mathbf{C}$ with unions $\cup$ and $\otimes$-products such that these formulas on the compatibility of $\otimes$ and $\cup$ hold.

The next Lemma extends the corresponding Lemma 4.4.

Lemma 6.2. *Let $\mathbf{C}$ be an abstract $\mathbb{B}$-deformation category. Then for $B \in \mathbb{B}$ the deformation category $(\mathbf{C}(B), \mathbf{C}[JB])$ is a quotient of $(\mathbf{C}(B), \mathbf{C}(JB))$ and this quotient is a track category.*

Proof. We have to find R in $\mathbf{C}(JJB)$ such that

$$R : HE \simeq (H\varepsilon(e')) \,\square\, (\varepsilon(f)E)$$

where $H : f \simeq f'$, $E : e \simeq e'$. Here HE is the composite in the category $\mathbf{C}(JB)$. Now R is obtained by the diagram

$$
\begin{array}{ccc}
JB & & \\
{\scriptstyle \square'}\big\downarrow & \searrow {\scriptstyle \Delta} & \\
JB \cup JB & \xrightarrow[H' \cup E']{} & JB \times JB
\end{array}
$$

Here Δ is the diagonal and we define

$$E' = (i^- \varepsilon', 1) : JB \to JB \times JB,$$
$$H' = (1, i^+ \varepsilon') : JB \to JB \times JB.$$

Then $\Delta^*(H \otimes E) = HE$ and $((H' \cup E')\square')^*(H \otimes E) = (H\varepsilon(e')) \,\square\, (\varepsilon(f)E)$. Hence a thin filler yields R. $\qquad\square$

Let $\mathbf{C}$ be an abstract $\mathbb{B}$-deformation category and let (B, A) be a ball pair in $\mathbb{B}$. Then we can choose a map

$$\square'_A : B \to B \cup_A JA$$

which is the identity on the boundary. The union is the pushout of i_A and i^+. We call $\square'_A$ the *action map* which induces the *action functor*

$$\square_A : \mathbf{C}[B] \times_{\mathbf{C}(A)} \mathbf{C}[JA] \to \mathbf{C}[B]$$
$$\square_A(F, G) = F \,\square_A\, G = (\square'_A)^*(F \cup G)$$

Here $\square_A$ does not depend on the choice of $\square'_A$. For $B = JA$ the functor $\square_A = \square$ is given by the track category $(\mathbf{C}(A), \mathbf{C}[JA])$ above.

Lemma 6.3. *Let $\mathbf{C}$ be an abstract $\mathbb{B}$-deformation category. Then $\square_A$ satisfies the equations*

$$F \,\square_A\, \varepsilon(f) = F \quad for\ \partial_A F = f,$$
$$F \,\square_A\, (G \,\square\, H) = (F \,\square_A\, G) \,\square_A\, H,$$
$$\partial_{A_{\mathrm{op}}}(F \,\square_A\, G) = \partial_{A_{\mathrm{op}}} F,$$
$$\partial_A(F \,\square_A\, G) = \partial^- G.$$

Moreover given $F \in \mathbf{C}[B]$ with $\partial_A F = f$, $\partial_{A_{\mathrm{op}}} F = g$, the group $\mathrm{Aut}_\square(f)$ in the track category $(\mathbf{C}(A), \mathbf{C}[JA])$ acts transitively and effectively on the set

$$\{F' \in \mathbf{C}[B], \partial_A F' = f, \partial_{A_{\mathrm{op}}} F' = g\}$$

by $F' \,\square_A\, \alpha$ for $\alpha \in \mathrm{Aut}_\square(f)$

The Lemma describes the *action property* of $\square_A$.

7. Abelian $\mathbb{B}$-deformation categories

For all $B \in \mathbb{B}$ one has the *projection functor*

$$\partial_0 : \mathbf{C}(B) \to \mathbf{C}(\text{point})/\simeq$$

which is induced by any map $f : \text{point} \to B$ and which by thin fillers is independent of this choice.

Definition 7.1. An abstract $\mathbb{B}$-deformation category $\mathbf{C}$ is *pre-abelian* if the track categories

$$(\mathbf{C}(B), \mathbf{C}[JB])$$

are abelian for all $B \in \mathbb{B}$ and if the associated natural systems D_B are composites of the form

$$D_B : F(\mathbf{C}(B)/\simeq) \xrightarrow{\partial_0} F(\mathbf{C}(\text{point})/\simeq) \xrightarrow{D^n} \mathbf{Ab}$$

Here D^n with $n = \dim(B)$ depends only on n and not on B.

Now let $\mathbf{C}$ be pre-abelian with natural systems $D^n, n \geq 1$. Then we have for a ball pair (B, A) and $f : X \to Y$ in $\mathbf{C}(A)$ the group $\text{Aut}_\square(f)$ in the track category $(\mathbf{C}(A), \mathbf{C}[JA])$ and we have the isomorphism, $n = \dim(A) + 1$, (see section 2)

$$\sigma_A = \sigma : D^n_{\partial_0 f} \cong \text{Aut}_\square(f)$$

which is natural in f. Hence the $\square_A$-action of $\text{Aut}_\square(f)$ on the set $\{F \in \mathbf{C}[B], \partial_A F = f\}$ yields an action of $\alpha \in D^n_{\partial_0 f}$ on this set by $F \square_A \alpha = F \square_A \sigma(\alpha)$. Here we have $\partial_0 f = \partial_0 \partial_A F = \partial_0 F$.

Definition 7.2. Let $\text{or}(B)$ be a manifold orientation of the topological manifold $B \in \mathbb{B}$. there are two such orientations, $\text{or}(B)$ and $\overline{\text{or}}(B)$. Now $\text{or}(B)$ yields a manifold orientation $\text{or}(\partial B)$ on the boundary ∂B. If (B, A) is a ball pair $\text{or}(\partial B)$ yields by restriction a manifold orientation $\text{or}(A)$. Now the product $I \times A$, where I is the oriented interval from 0 to 1, yields a manifold orientation $\text{or}(JA)$. If $\text{or}(B) \cup \text{or}(JA)$ define a manifold orientation of $B \cup_A JA$ we write $\varepsilon(B, A) = +1$ and $\varepsilon(B, A) = -1$ otherwise.

Definition 7.3. Let $\mathbf{C}$ be pre-abelian and consider for a ball pair (B, A) and $F \in \mathbf{C}[B]$, $\partial_A F = f$ the element

$$F \square_{\text{or}(A)} \alpha = F \square_A (\varepsilon(B, A)\alpha) \in \mathbf{C}[B]$$

where $\alpha \in D^n_{\partial_0 F}$. We call $\mathbf{C}$ an *abelian* $\mathbb{B}$-deformation category if this element does not depend on the ball pair (B, A) but only on $F, \text{or}(B)$ and α. For the opposite orientation $\overline{\text{or}}(B)$ we have the formula

$$F \square_{\overline{\text{or}}(B)} \alpha = F \square_{\text{or}(B)} (-\alpha)$$

Moreover the *abelian union property* is satisfied, that is, for a union $\overline{B} = B_1 \cup \ldots \cup B_k$ of balls and $\text{or}(B_i)$ induced by $\text{or}(\overline{B})$, $i = 1, \ldots, k$, we have for $F_i \in \mathbf{C}[B_i]$ the formula

$$F_1 \cup \ldots \cup (F_i \square_{\text{or}(B_i)} \alpha) \cup \ldots \cup F_k = (F_1 \cup \ldots \cup F_k) \square_{\text{or}(\overline{B})} \alpha$$

where $\partial_0(F_i) = \partial_0(F_1 \cup \ldots \cup F_k)$.

Let $\mathbf{C}$ be the topological $\mathbb{B}$-category and let $\mathfrak{X}$ be a class of objects in $\mathbf{Top}^*$ and let W be a space in $\mathbf{Top}^*$. For each $B \in \mathbb{B}$ we obtain the over category

$$\mathbf{C}(B)\{\mathfrak{X} \to W\} \subset \mathbf{C}(B)$$

and dually the under category $\mathbf{C}(B)\{W \to \mathfrak{X}\}$. Then the collection of over categories

$$\mathbf{C}\{\mathfrak{X} \to W\} = \{\mathbf{C}(B)\{\mathfrak{X} \to W\}, B \in \mathbb{B}\}$$

is again a $\mathbb{B}$-deformation category. The same holds for the under category $\mathbf{C}\{W \to \mathfrak{X}\}$.

Proposition 7.4. *Let $\mathfrak{X}$ be a class of co-H-groups. Then $\mathbf{C}\{\mathfrak{X} \to W\}$ is an abelian $\mathbb{B}$-deformation category with natural systems $D^n = D^n_\Sigma, n \geq 1$. Dually, if $\mathfrak{X}$ is a class of H-groups then $\mathbf{C}\{W \to \mathfrak{X}\}$ is an abelian $\mathbb{B}$-deformation category with natural system $D^n = D^n_\Omega, n \geq 1$.*

Compare section 2 where D^n_Σ and D^n_Ω are defined.

8. Higher order track categories

Let $\mathbf{C}$ be a $\mathbb{B}$-deformation category. Then we define for $n \geq 1$ the collection of categories $\mathbf{C}^n$, termed the *n-truncation* of $\mathbf{C}$, by

$$\mathbf{C}^n(B) = \begin{cases} \mathbf{C}(B)/\simeq\, = \mathbf{C}[B] & \text{if } \dim(B) = n, \\ \mathbf{C}(B) & \text{if } \dim(B) < n \end{cases}$$

with $B \in \mathbb{B}$. Hence only in dimension n we take the homotopy category.

The following definition of an "n-th order track category" is motivated by those properties of a $\mathbb{B}$-deformation category which remain visible in the truncation $\mathbf{C}^n$. Moreover these properties should be minimal to allow the definition of higher order Toda brackets and certain spectral sequences below (see section 10), and they should be fulfilled by algebraic examples given by chain complexes (see section 12).

Let $\mathbb{B}(n), n \geq 0$ be the subcategory of $\mathbb{B}$ consisting of balls B with $\dim(B) \leq n$ and which is generated by the maps:

- ball pair inclusions $i_A : A \subset B$,
- projections $\epsilon' : JA \to A$,
- trivial maps $\varepsilon'_0 : B \to \text{point}$.

Let $\mathbb{B}_\partial(n), n \geq 0$, be the subcategory of $\mathbb{B}$ consisting of balls B with $\dim(B) \leq n$ and which is generated by the following maps:

- The *opposite map* $\text{op}' : JA \to JA$,
- For a ball pair (B, A) we have the ball $B \cup_A JA$ by pushout of i_A, i^+. The boundary satisfies $\partial(B \cup_A JA) = A_{\text{op}} \cup A^- = A_{\text{op}} \cup A = \partial B$. Then there is a map
$$\square'_A \colon B \to B \cup_A JA \quad \text{in } \mathbb{B}_\partial(n)$$
  which is the identity on the boundary. We call $\square'_A$ an *action map*. Also $1 \cup \varepsilon : B \cup_A JA \to B$ is a map in $\mathbb{B}_\partial(n)$.
- There is a map $h_A : A \to A_{\text{op}}$ which extends the identity on $\partial A_{\text{op}} = \partial A$. This map is called a *comparison map*.

All the maps of $\mathbb{B}_\partial(n)$ are isomorphisms in the homotopy category $\mathbb{B}_\partial(n)/\simeq\, \subset \mathbb{B}_\partial/\simeq$.

As above we use only functors in $\mathbf{cat}^*(\mathcal{O})$ which are the identity on objects.

Definition 8.1. An *n-th order track category* $\mathbf{K}$ is a $\mathbb{B}(n)$-category given by categories $\mathbf{K}(B)$ with zero object and functors $f^* : \mathbf{K}(B) \to \mathbf{K}(A)$ for $f : A \to B$ in $\mathbb{B}(n)$ such that the following properties hold:

(1) The relation $\simeq$ on $\mathbf{K}(B)$, defined by

$$f \simeq g \Leftrightarrow \begin{cases} f = g & \text{if } \dim(B) = n, \\ \exists F \in \mathbf{K}(JB) \text{ with } \partial^- F = f,\ \partial^+ F = g & \text{if } \dim(B) < n, \end{cases}$$

is a *natural equivalence relation*. Here the functor $\partial^\pm = (i^\pm)^*$ is induced by the ball pair inclusion $i^\pm : B \subset JB$.

Let $\mathbf{K}[B] = \mathbf{K}(B)/\simeq$ be the homotopy category with quotient functor $q : \mathbf{K}(B) \to \mathbf{K}[B]$. Then $\partial_A = i_A^*$ induces $\partial_A : \mathbf{K}[B] \to \mathbf{K}(A)$ with $\partial_A q = \partial_A$ since $\partial_A \partial^- = \partial_A \partial^+$.

(2) The categories $\mathbf{K}[B]$ have the structure of a $\mathbb{B}_\partial(n)$-*category* given by functors $f^* : \mathbf{K}[B] \to \mathbf{K}[A]$ for $f : A \to B$ in $\mathbb{B}_\partial(n)$. Moreover $f, g : A \to B$ in $\mathbb{B}_\partial(n)$ with $f|\partial A = g|\partial A$ and $f \simeq g$ rel ∂A satisfy $f^* = g^*$ so that $\mathbf{K}[\_]$ is actually a $\langle \mathbb{B}_\partial(n)/\simeq \rangle$-category. The induced functors f^* are compatible with the restriction functors ∂_A (as in a $\mathbb{B}$-deformation category).

(3) For ball pairs $(B, A), (B', A)$ in $\mathbb{B}(n)$ a *gluing functor* $\cup$ is given for which the following diagram commutes

$$\begin{array}{ccc} \mathbf{K}(B) \times_{\mathbf{K}(A)} \mathbf{K}(B') & \xrightarrow{\cup} & \mathbf{K}(B \cup_A B') \\ \downarrow{\scriptstyle q \times q} & & \downarrow{\scriptstyle q} \\ \mathbf{K}[B] \times_{\mathbf{K}(A)} \mathbf{K}[B'] & \xrightarrow{\cup} & \mathbf{K}[B \cup_A B'] \end{array}$$

The gluing functor is natural with respect to maps in $\mathbb{B}(n)$, resp. $\mathbb{B}_\partial(n)$, and the gluing rule is satisfied, see section 4

(4) For a ball pair (B, A) we have the *action functor*

$$\square_A : \mathbf{K}[B] \times_{\mathbf{K}(A)} \mathbf{K}[JA] \to \mathbf{K}[B]$$

defined by $\square_A(G, F) = (\square_A')^*(G \cup F)$. For $B = JA$ the tuple

$$\left(\mathbf{K}(A), \mathbf{K}[JA], \varepsilon = q^*(\varepsilon')^*, \mathrm{op} = (\mathrm{op}')^*, \square = \square_A, \partial^\pm \right)$$

is a *track category* with associated homotopy category $\mathbf{K}[A]$. The action functor has the action property in section 6.

(5) Given a ball pair (B, A) and f in $\mathbf{K}(A)$. Then $f \simeq 0$ if and only if there exists F in $\mathbf{K}(B)$ with $\partial_A F = f$ and $\partial_{A_{\mathrm{op}}} F = 0$. This is the *extension property* of $\mathbf{K}$.

(6) For $B' \times B$ in $\mathbb{B}(n)$ and $g : Y \to Z$ in $\mathbf{K}(B')$ and $f : X \to Y$ in $\mathbf{K}(B)$ the $\otimes$-*product*

$$g \otimes f : X \to Y \text{ in } \mathbf{K}(B' \times B)$$

is given. We also write $q(g \otimes f) = g \otimes f$ in $\mathbf{K}[B \times B']$. The tensor product is associative and natural and $g \otimes 0 = 0$, $0 \otimes f = 0$. Moreover for $\varepsilon_0 = (\varepsilon_0')^*$ we have $f \otimes g = f\varepsilon_0(g)$ if $B' = $ point and $f \otimes g = \varepsilon_0(f)g$ if $B = $ point. Also $g \otimes \partial_A f = \partial_{B' \times A}(g \otimes f)$, $(\partial_{A'} g) \otimes f = \partial_{A' \times B}(g \otimes f)$ and $\otimes$ and $\cup$ are compatible.

(7) Let $\dim(B \times B_1) \leq n - 1$ for balls B, B_1 in $\mathbb{B}$. We consider the boundary of the product $JB \times JB_1$ which is the following union of balls

$$\partial(JB \times JB_1) = U_{\mathrm{op}} \cup U$$
$$U = B^+ \times JB_1 \cup JB \times B_1^+$$
$$U_{\mathrm{op}} = B^- \times JB_1 \cup JB \times B_1^-$$

and we use a comparison map $h_U : U \to U_{\mathrm{op}}$ in $\mathbb{B}_\partial(n)$. Let $G \in \mathbf{K}(JB)$, $F \in \mathbf{K}(JB_1)$, then we have in $\mathbf{K}[U]$ the following *boundary formula*,

$$(*) \qquad h_U^* \left((\partial^- G) \otimes F \cup G \otimes (\partial^- F) \right) = (\partial^+ G) \otimes F^{\mathrm{op}} \cup G^{\mathrm{op}} \otimes (\partial^+ F)$$

For $\dim(B \times B_1) = 0$ this corresponds to formula $(*)$ in a track category, compare the horizontal composition in section 2.

Lemma 8.2. *Given a $\mathbb{B}$-deformation category $\mathbf{C}$ one has for $n \geq 1$ an n-th order track category $\mathbf{C}^n$ by truncation of $\mathbf{C}$, see the definition of $\mathbf{C}^n(B)$ above. Similarly the truncation $\mathbf{K}^{n-1}$ of an n-th order track category $\mathbf{K}$ is an $(n-1)$-st order track category.*

Definition 8.3. A *map* $\alpha : \mathbf{K} \to \mathbf{K}'$ *between n-th order track categories* is given by functors $\alpha : \mathbf{K}(B) \to \mathbf{K}'(B)$, $\alpha : \mathbf{K}[B] \to \mathbf{K}'[B]$ with $q\alpha = \alpha q$. They are natural with respect to $\mathbb{B}(n)$ and $\mathbb{B}_\partial(n)$ respectively. Moreover α is compatible with gluing and $\otimes$-products. A map α is a *weak equivalence* if $\alpha : \mathbf{K}[\text{point}] \to \mathbf{K}'[\text{point}]$ is an isomorphism and α induces isomorphisms of groups

$$\text{Aut}_\square(f) \cong \text{Aut}_\square(\alpha f)$$

for all f in $\mathbf{K}(B)$, $\dim(B) \leq n$.

Remark 8.4. There is also a notion of *pseudo functor* and *pseudo equivalence* between higher track categories, see [BM] for $n = 1$.

Composites of inclusions of ball pairs yield for B in $\mathbb{B}(n)$ maps $\{\text{point}\} \to B$ in $\mathbb{B}(n)$ which induce a well defined functor

$$\partial_0 : \mathbf{K}(B) \to \mathbf{K}(\text{point})/\simeq = \mathbf{K}[\text{point}].$$

Definition 8.5. An n-th order track category $\mathbf{K}$ is *abelian* if all track categories $(\mathbf{K}(B), \mathbf{K}[JB])$, $\dim(B) \leq n-1$, are abelian and the associated natural system D^B is a composite of the form

$$F(\mathbf{K}[B]) \xrightarrow{\partial_0} F(\mathbf{K}[\text{point}]) \xrightarrow{D^k} \mathbf{Ab}$$

where D^k depends only on $k = \dim(B)$. Moreover the $\square_A$-action yields a well defined action $\square_{\text{or}(B)}$ which satisfies the abelian union property, see section 7.

The truncation of the abelian $\mathbb{B}$-deformation categories in section 7 yield examples of abelian n-th order track categories.

Example 8.6. A first order track category $\mathbf{K}$ is the same as a track category in section 2. In fact, iterate composites of action maps yield a map $h : I \to [0, n] = I(n)$ inducing the isomorphism

$$h^* : \mathbf{K}[I(n)] \approx \mathbf{K}(I(n)) \xrightarrow{\approx} \mathbf{K}(I) \approx \mathbf{K}[I]$$

which does not depend on h. Now unions are defined by

$$
\begin{array}{ccc}
\mathbf{K}(I(n)) \times_\partial \mathbf{K}(I(m)) & =\!=\!= & \mathbf{K}(I) \times_\partial \mathbf{K}(I) \\
\cup \downarrow & & \downarrow \square \\
\mathbf{K}(I(n+m)) & =\!=\!= & \mathbf{K}(I)
\end{array}
$$

and $\otimes$-products are composites $\varepsilon(f)H, H\varepsilon(g)$ with H in $\mathbf{K}(I)$ and f, g in $\mathbf{K}(\text{point})$. Of course $\mathbf{K}$ is abelian iff the associated track category $(\mathbf{K}(\text{point}), \mathbf{K}[I])$ is abelian.

Remark 8.7. Let $\mathbf{K}$ be an abelian n-th order track category. Then the truncation $\mathbf{K}^{n-1}$ is an abelian $(n-1)$-st order track category. We consider $\mathbf{K}$ as a *linear track extension* of $\mathbf{K}^{n-1}$. The set of equivalence classes of such extensions is denoted by $H^{n+2}(\mathbf{K}^{n-1}, D^n)$. This leads to a cohomology which for $n = 1$ coincides with the cohomology in section 2, [BW].

9. Higher order Toda brackets and higher order chain complexes

The properties of an abelian n-th order track category are chosen in such a way that it is possible to define higher order Toda brackets generalizing the triple Toda brackets in a track category of section 2.

Proposition 9.1. *Let $\mathbf{C}$ be an abelian n-th order track category with natural system $D^i, i = 1, \ldots, n$ and let*

$$Y = X_0 \xleftarrow{\alpha_1} X_1 \xleftarrow{\alpha_2} X_2 \leftarrow \cdots \xleftarrow{\alpha_{n+2}} X_{n+2} = X$$

be a sequence of morphisms α_i in $\mathbf{C}(\text{point})/\simeq$. Then the higher Toda bracket is defined as a subset

$$\langle \alpha_1, \ldots, \alpha_{n+2} \rangle \subset D^n_{0(X,Y)}$$

For $n = 1$ this is the triple Toda bracket in the abelian track category $(\mathbf{C}(\text{point}),$ $\mathbf{C}(I)/\simeq)$. The set $\langle \alpha_1, \ldots, \alpha_{n+2} \rangle$ is possibly empty. If $\mathbf{C} \sim \mathbf{K}$ is a weak equivalence then the brackets coincide, that is $\langle \alpha_1, \ldots, \alpha_{n+2} \rangle_{\mathbf{C}} = \langle \alpha_1, \ldots, \alpha_{n+2} \rangle_{\mathbf{K}}$.

Definition 9.2. Let $\mathbf{C}$ be an abelian n-th order track category. Let B be a ball in $\mathbb{B}$ and let $F \in \mathbf{C}(B)$, $F : X \to Y$, be *trivial on the boundary* (that is, for any ball pair (B, A) we have $\partial_A F = 0$ and $\partial_{A_{\text{op}}} F = 0$) then there is a unique element $\alpha \in D^i_{\partial_0 F}$, $i = \dim(B)$, with

$$0 \, \square_{\text{or}(B)} \, \alpha = \{F\} \text{ in } \mathbf{C}[B]$$

Here $\partial_0 F = 0(X, Y)$ in $\mathbf{C}[\text{point}]$. We call

$$\text{ob}(F) = \alpha \in D^i_{0(X,Y)}$$

the *obstruction* associated to F. If (E, B) is a ball pair the extension property shows that $\text{ob}(F) = 0$ if and only if there is $\overline{F} \in \mathbf{C}(E)$ with $\partial_B(\overline{F}) = F$ and $\partial_{B_{\text{op}}}(\overline{F}) = 0$.

We now consider the ball pair (I^{n+1}, T^n) where T^n is the union of all n-dimensional faces of I^{n+1} which contain the origin $0 = (0, \ldots, 0) \in I^{n+1}$. Let $(I^{n+1}, T^n_{\text{op}})$ be the opposite ball pair. Hence T^n_{op} is the union of all faces of I^{n+1} which contain the point $(1, \ldots, 1) \in I^{n+1}$. We shall construct $F : X \to Y$ in $\mathbf{C}(T^n)$ such that F is trivial on the boundary and such that

$$\text{ob}(F) \in \langle \alpha_1, \ldots, \alpha_{n+2} \rangle$$

In fact, all possible choices of F yield this way the set $\langle \alpha_1, \ldots, \alpha_{n+2} \rangle$. If F is not constructable then this set is empty.

The element $\text{ob}(F)$ is the zero element in the abelian group $D^n_{0(X,Y)}$ if and only if there exists $\overline{F}$ in $\mathbf{C}(I^{n+1})$ with $\overline{F}|T^n = F$ and $\overline{F}|T^n_{\text{op}} = 0(X, Y) \in \mathbf{C}(T^n_{\text{op}})$.

Here we call $\overline{F}$ a *cubical extension* of F. We shall see that F is constructible if and only if inductively certain cubical extensions exist.

We start the induction by choosing representatives f_i of the homotopy class α_i, $i = 1, \ldots, n + 2$. Then we choose

$$f_i^1 \in \mathbf{C}(I), \quad f_i^1 : f_i f_{i+1} \simeq 0.$$

Assume now $f_i^k \in \mathbf{C}(I^k)$, $1 \leq k < n$, $i \leq n - k + 2$, are chosen. Then we define $f_i^{k+1} \in \mathbf{C}(I^{k+1})$ as follows, in fact, f_i^{k+1} is a cubical extension of

$$F_i^k \in \mathbf{C}(T^k)$$

where F_i^k is obtained by the following gluing. The faces of I^{k+1} in T^k are of the form $I^k \times 0$, $I^{k-1} \times 0 \times I$, $I^{k-2} \times 0 \times I^2, \ldots, 0 \times I^k$ and this is a regular sequence of balls in T^k. Then F_i^k is given by the restrictions

$$F_i^k | I^k \times 0 = f_i^k \varepsilon_0(f_{i+k+1})$$
$$F_i^k | 0 \times I^k = \varepsilon_0(f_i) f_{i+1}^k$$
$$F_i^k | I^r \times 0 \times I^k = f_i^r \otimes f_{i+r+1}^k, \quad r + s = k, \ 1 \leq r < k.$$

One can check that by the inductive construction this gluing, defining F_i^k, is well defined. Now let $F = F_0^n \in \mathbf{C}(T^n)$. All possible choices of F yield the set of elements $\text{ob}(F)$ defining $\langle \alpha_1, \ldots, \alpha_{n+2} \rangle$.

We now consider *higher order chain complexes*. Let $\mathbf{C}$ be an n-th order track category and let

$$(X, \alpha) = \left(\cdots \leftarrow X_{i-1} \xleftarrow{\alpha_i} X_i \leftarrow \cdots, \ i \in \mathbb{Z} \right)$$

be a sequence of morphisms in the homotopy category $\mathbf{C}[\text{point}]$. A $\mathbf{C}$-*chain complex* (X, f, F) *associated to* (X, α) is defined in the same way as a representation of a higher Toda bracket. In fact, (X, f, F) consists of the following data. For $i \in \mathbb{Z}$ the element f_i in $\mathbf{C}(\text{point})$ is a representative of α_i. Then

$$f_i^1 \in \mathbf{C}(I), \quad f_i^1 : f_i f_{i+1} \simeq 0$$

Assume now $f_i^k \in \mathbf{C}(I^k)$, $1 \le k < n$, is given. Then $f_i^{k+1} \in \mathbf{C}(I^{k+1})$ is a cubical extension of $F_i^k \in \mathbf{C}(T^k)$ where F_i^k is obtained by gluing as above, that is, F_i^k has the restrictions

$$
\begin{aligned}
F_i^k | I^k \times 0 &= f_i^k \varepsilon_0(f_{i+k+1}) \\
F_i^k | 0 \times I^k &= \varepsilon_0(f_i) f_{i+1}^k \\
F_i^k | I^r \times 0 \times I^k &= f_i^r \otimes f_{i+r+1}^k,
\end{aligned}
$$

where $r + s = k$, $1 \le r \le k - 1$. Now f_i^k and F_i^k, $1 \le k \le n$, describe (X, f, F). This is a $\mathbf{C}$-chain complex if the obstruction $\mathrm{ob}(F_i^n) = 0$ vanishes for $i \in \mathbb{Z}$, see 9.2.

Remark 9.3. For $n = 1$ we have the track category $\mathbf{C}$ and in this case a $\mathbf{C}$-chain complex is the same as a *secondary chain complex* in [BJ,Se]. Secondary chain complexes form a category but $\mathbf{C}$-chain complexes in general do not though morphisms between $\mathbf{C}$-chain complexes can be defined.

10. Higher order cohomology operations and the Adams spectral sequence

Let p be a prime and let $\mathbb{F} = \mathbb{Z}/p$ be the field of p elements. Let $Z^n = K(\mathbb{F}, n)$ be the Eilenberg-MacLane space which is an H-group.

Definition 10.1. The *track theory of n-th order cohomology operations* is the abelian n-th order track category

$$\mathbf{C}^n\{\mathcal{Z}\}$$

where $\mathbf{C}$ is the topological $\mathbb{B}$-category and $\mathcal{Z}$ is the set of all products $Z^{n_1} \times \ldots \times Z^{n_r}$ with $n_1, \ldots, n_r \ge 1$ and $r \ge 1$. For a pointed space X let

$$\mathbf{C}^n\{X \to \mathcal{Z}\}$$

be the under category in section 7 which is also an abelian n-th order track category. Here $\mathbf{C}^n\{X \to \mathcal{Z}\}$ is considered as a left module over $\mathbf{C}^n\{\mathcal{Z}\}$.

Remark 10.2. The homotopy category $\mathbf{C}^n\{\mathcal{Z}\}[\text{point}]$ is the theory of Eilenberg-MacLane spaces constructed in [B,Se, 1.1.5]. Models of this theory are connected unstable algebras over the Steenrod algebra $\mathcal{A}$. For example $\mathbf{C}\{X \to \mathcal{Z}\}[\text{point}]$ is such a model which is equivalently given by the cohomology $H^*(X, \mathbb{F})$. Given a sequence

$$X \xrightarrow{\beta} X^0 \xrightarrow{\alpha^0} X^1 \xrightarrow{\alpha^1} \cdots \xrightarrow{\alpha^n} X^{n+1}$$

in $\mathbf{Top}^*/\simeq$ with $X^i \in \mathcal{Z}$ the associated Toda bracket $\langle \alpha^n, \ldots, \alpha^0, \beta \rangle$ is termed a *higher matrix Massey product* in the $\mathcal{A}$-module $H^*(X)$. If $X \in \mathcal{Z}$ this is a higher Massey product in the Steenrod algebra.

Remark 10.3. It is possible to describe the analogue of the topological $\mathbb{B}$-deformation category $\mathbf{C}$ in $\mathbf{Top}^*$ in the stable homotopy category of spectra. For example the stable track theory of Eilenberg-MacLane spaces in [B,Se, 2.2.6] uses Eilenberg-MacLane spectra. The use of spectra, however, leads to technical complications which we want to avoid in this paper. We therefore use the "stable range" in the next definition.

Definition 10.4. For a "large" number N let $\mathcal{Z}_N$ be the subset of $\mathcal{Z}$ above consisting of all products $Z^{n_1} \times \ldots \times Z^{n_r}$ with $N \leq n_i < 2N$, $i = 1, \ldots, r$. This is a *stable range* of $\mathcal{Z}$. Accordingly we get the *stable theories*

$$\mathbf{C}^n\{\mathcal{Z}_N\}, \quad \mathbf{C}^n\{\Sigma^N X \to \mathcal{Z}_N\}$$

where $\Sigma^N X$ is the N-fold suspension of a CW-complex X with $\dim(X) < N$.

Let X and Y be finite CW-complexes. Then the *Adams spectral sequence* $(E_n, n \geq 2)$ with

$$E_2 = \mathcal{E}xt_{\mathcal{A}}(H^*X, H^*Y)$$

converges to the p-local part of the stable homotopy set $\{Y, X\}$. For the computation of $\mathcal{E}xt$ we choose a resolution of the left $\mathcal{A}$-modules H^* by finitely generated free modules M_i

$$H^*X \leftarrow M_0 \leftarrow M_1 \leftarrow \cdots$$

Let B_i be a basis of M_i and let

$$X^i = \underset{b \in B_i}{\bigtimes} \mathcal{Z}^{N+|b|}$$

Then a finite part $H^*X \leftarrow M_0 \leftarrow \cdots \leftarrow M_n$ of the resolution corresponds to a sequence

$$(X, \alpha) = \left(\Sigma^N X \xrightarrow{\alpha^{-1}} X^0 \xrightarrow{\alpha^0} X^1 \xrightarrow{\alpha^1} \cdots \to X^m \right)$$

which for large N lies in the homotopy category $\mathbf{K}[\text{point}]$ with $\mathbf{K} = \mathbf{C}^n\{W \to \mathcal{Z}_N\}$, $W = \Sigma^N X \vee \Sigma^{N+s} Y$. Moreover an element $\overline{\beta} \in \text{Ext}_{\mathcal{A}}^r(H^*X, H^*Y)^s$ gets represented by a cocycle $c \in \text{Hom}(M_r, \Sigma^{-s} H^*Y)$ which corresponds to a map β in the following diagram which for N large lies in $\mathbf{K}[\text{point}]$.

$$\Sigma^N X \longrightarrow X^0 \xrightarrow{\alpha^0} \cdots \longrightarrow X^r \xrightarrow{\alpha^r} \cdots \xrightarrow{\alpha^{r+n}} X^{r+n+1}, \ r+n+1 \leq m,$$

$$\uparrow \beta$$

$$\Sigma^{N+s} Y$$

We use this diagram for the determination of the differential

$$d_{n+1} : E_n^{r,s} \to E_n^{r+n+1,s+n}$$

Proposition 10.5. *There is a $\mathbf{K}$-chain complex (X, f, F) associated to (X, α) above. Moreover since $\overline{\beta}$ represents an element in $E_n^{r,s}$ there is a $\mathbf{K}^{n-1}$-chain complex (Y, g, G) associated to*

$$(Y, \beta) = \left(\Sigma^{n+s} Y \xrightarrow{\beta} X^r \xrightarrow{\alpha^r} \cdots \to X^{r+n+1} \right).$$

Here the restriction of (Y, g, G) to $(X^r \to \cdots \to X^{r+n+1})$ is given by (X, f, F) and $\mathbf{K}^{n-1}$ is the truncation of $\mathbf{K}$. Since (Y, β) is a $\mathbf{K}^{n-1}$-chain map we can choose a cubical extension defining $G_0^n \in \mathbf{K}(T^n)$. The obstruction $\text{ob}(G_0^n) \in [\Sigma^n(\Sigma^{N+s}Y), X^{r+n+1}]$ yields an element in $\mathcal{E}xt_{\mathcal{A}}^{r+n+1}(H^*X, H^*Y)^{n+s}$ which represents $d_{n+1}\{\beta\} \in E_n^{r+n-1,n+s}$.

This result is proved for $n = 1$ in [BJ].

Corollary 10.6. *Let* $\mathbf{K}'$ *be an n-th order track category quasi isomorphic to* $\mathbf{K} = \mathbf{C}^n\{W \to \mathcal{Z}_N\}$, $W = \Sigma^N X \vee \Sigma^{N+s} Y$. *Then the differential d_n can be computed in* $\mathbf{K}'$.

For $n = 1$ this is proved in [BJ,Se, 5.1].

11. Δ-BALLS

The diagonal Δ of a CW-complex X is not a cellular map but is homotopic to a cellular map $\overline{\Delta}$ which is called a *diagonal approximation*. Then $\overline{\Delta} = \overline{\Delta}_X$ induces a chain map

$$\overline{\Delta}_* : C_* X \to C_*(X \times X) = C_* X \otimes C_* X$$

where $C_* X$ is the cellular chain complex with $C_n X = H_n(X^n, X^{n-1})$. We say that $(C_* X, \overline{\Delta}_*)$ is a *coalgebra* if $\overline{\Delta}_*$ is coassociative and the augmentation $\varepsilon_0 = (\varepsilon_0')_* : C_* B \to C_*(\text{point}) = R$ satisfies $(1 \otimes \varepsilon_0)\overline{\Delta}_* = 1 = (\varepsilon_0 \otimes 1)\overline{\Delta}_*$. In general it is not possible to find a diagonal approximation $\overline{\Delta}$ such that $(C_* X, \overline{\Delta}_*)$ is a coalgebra, but we consider CW-complexes with the following nice properties.

Definition 11.1. A Δ-*CW-complex* X is a regular CW-complex together with a diagonal approximation $\overline{\Delta}$ and a homotopy $D : \overline{\Delta} \simeq \Delta$ in **Top** such that the following properties are satisfied.

(a) Each subcomplex Y of X admits a commutative diagram

$$
\begin{array}{ccc}
X & \xrightarrow{\overline{\Delta}} & X \times X \\
\uparrow & & \uparrow \\
Y & \xrightarrow{\overline{\Delta}_Y} & Y \times Y
\end{array}
$$

and the homotopy D induces a homotopy $D_Y : \overline{\Delta}_Y \simeq \Delta_Y$.

(b) The cellular chain complex $(C_* X, \overline{\Delta}_*)$ is a coalgebra.

We have the following properties of Δ-CW-complexes.

(c) The interval I is a Δ-CW-complex using $\overline{\Delta}$ in $I \times 0 \cup 1 \times I$.

(d) The product $X \times Y$ of Δ-CW-complexes is a Δ-CW-complex given by

$$\overline{\Delta} = (1 \times T \times 1)(\overline{\Delta}_X \times \overline{\Delta}_Y), \quad D^t = (1 \times T \times 1)(D_X^t \times D_Y^t), \ t \in I,$$

where $T : X \times Y \to Y \times X$ is the interchange map.

(e) A subcomplex Y of a Δ-CW-complex X is a Δ-CW-complex.

(f) Let X, X' be Δ-CW-complexes and let $Y \subset X$ and $Y \subset X'$ be the inclusions of Δ-CW-subcomplexes. Then the union $X \cup_Y X'$ is a Δ-CW-complex with $\overline{\Delta} = \overline{\Delta}_X \cup \overline{\Delta}_{X'}$.

(g) Let X, X', Y be Δ-CW-complexes and let $Y \subset X$ be the inclusion of a Δ-CW-subcomplex. Then the pushout P as in the diagram

$$
\begin{array}{ccc}
X \times X' & \longrightarrow & P \\
\uparrow & & \uparrow \\
Y \times X' & \xrightarrow{q} & X'
\end{array}
$$

is a Δ-CW-complex. Here q is the projection. For example a relative cylinder is such a pushout.

Definition 11.2. A Δ-*ball* is a Δ-CW-complex for which the underlying CW-complex is a ball, see section 3.

Proposition 11.3. *Each ball B in the indexing set $\mathbb{B}$ of balls in section 3 has the canonical structure of a Δ-ball.*

This follows from the properties of Δ-CW-complexes above, compare the definition of $\mathbb{B}$ in section 3.

12. Track categories associated to truncated chain algebras

Let R be a commutative ring with unit. We use the category of R-modules with tensor product $\otimes = \otimes_R$.

A *chain algebra* Q is a non-negatively bigraded R-module $Q = \{Q_s^r ; r, s \geq 0\}$ with unit $1 \in Q_0^0$, associative multiplication

$$\mu : Q_s^r \otimes Q_{s'}^{r'} \to Q_{s+s'}^{r+r'}, \ \mu(x \otimes y) = x \cdot y,$$

and differential $d : Q_s^r \to Q_{s-1}^r$ satisfying $d \circ d = 0$ and

$$d(x \cdot y) = (dx) \cdot y + (-1)^s x \cdot (dy).$$

(If Q is concentrated in upper degree 0 then Q is a chain algebra in the usual sense.) A *Q-module* is a non negatively bigraded R-module $M = \{M_s^r ; r, s \geq 0\}$ with a differential $d : M_s^r \to M_{s-1}^r$, $d \circ d = 0$, and an action

$$\mu : M_s^r \otimes Q_{s'}^{r'} \to M_{s+s'}^{r+r'}, \ \mu(x \otimes y) = x \cdot y,$$

satisfying the formula for $d(x \cdot y)$ above. (If M is concentrated in upper degree 0 then M is a chain complex in the usual sense.) A *Q-morphism* $f : M \to N$ between Q-modules is a map $f : M_s^r \to N_s^r$ with $df = fd$ and $f(x \cdot y) = f(x) \cdot y$.

Below we shall consider Q-modules of the form

$$C \otimes L \otimes Q$$

where C is a chain complex and L is a finitely generated free graded R-module. Here C and L are concentrated in upper degree 0.

Definition 12.1. A chain algebra Q is *n-truncated* if $Q_s^r = 0$ for $s > n$. The *n-th truncation* $Q(n)$ of a chain algebra Q is given by

$$Q(n)_s^r = \begin{cases} 0 & \text{for } s > n \\ Q_n^r / d Q_{n+1}^r & \text{for } s = n \\ Q_s^r & \text{for } s < n \end{cases}$$

Then $Q(n)$ is an n-truncated chain algebra.

We now define for an n-truncated chain algebra Q the *n-th order track category* $\mathbf{K}^Q$ *associated to Q* by the properties in (1), (2) and (3) below.

(1) The objects of $\mathbf{K}^Q$ are the finitely generated free graded R-modules; they form the set $\mathcal{O}$ of objects. The zero object $0 \in \mathcal{O}$ is the trivial module. For $B \in \mathbb{B}(n)$ let $\mathbf{K}^Q(B)$ be the category with objects in $\mathcal{O}$ and morphisms $f : L \to L'$ with $L, L' \in \mathcal{O}$ given by Q-morphisms

$$C_*(B) \otimes L \otimes Q \xrightarrow{f} L' \otimes Q$$

Composition of such morphisms $gf : L \to L' \to L''$ is defined by the composite

$$C_* B \otimes L \otimes Q \xrightarrow{\overline{\Delta}_*} C_* B \otimes C_* B \otimes L \otimes Q \xrightarrow{C_* B \otimes f} C_* B \otimes L' \otimes Q \xrightarrow{g} L'' \otimes Q$$

Since $(C_* B, \overline{\Delta}_*)$ is a coalgebra this is a well defined category. The identity $1 : L \to L$ is given by $C_*(B) \otimes L \to L$ induced by $(\varepsilon_0')_* : C_*(B) \to C_*(\text{point}) = R$. Let $j : A \to B$ be a morphism in $\mathbb{B}(n)$ then j induces a coalgebra morphism $j_* : C_* A \to C_* B$ and the functor $j^* : \mathbf{K}^Q(B) \to \mathbf{K}^Q(A)$ carries f to $f \circ (j_* \otimes L \otimes Q)$.

Lemma 12.2. *The relation of $\simeq$ defined by $\mathbf{K}^Q$ is a natural equivalence relation.*

Proof. For the ball JB we know that $C_*(JB)$, as a quotient of $C_*I \otimes C_*B$, is a relative cylinder in the category of chain complexes. $\square$

Hence the category $\mathbf{K}^Q[B] = \mathbf{K}^Q(B)/\simeq$ is well defined.

Lemma 12.3. *If* $\dim(A) = n$ *then* $\simeq$ *on* $\mathbf{K}(B)$ *is the trivial relation so that* $\mathbf{K}(B) = \mathbf{K}[B]$.

This follows readily by the assumption that Q is n-truncated.

(2) Let $j : A \to B$ be a map in $\mathbb{B}_\partial(n)$. Then $\partial j : \partial A \to \partial B$ induces a coalgebra map $j^\partial = C_*(\partial j)$ in the commutative diagram

$$
\begin{array}{ccc}
C_*\partial A & \xrightarrow{\;j^\partial\;} & C_*\partial B \\
\cap & & \cap \\
C_*A & \xrightarrow{\;\bar{j}\;} & C_*B
\end{array}
$$

Here we can choose a chain map $\bar{j}$ extending j^∂ since B is contractible. We now define

$$j^* : \mathbf{K}[B] \to K[A]$$

by $j^*\{f\} = \{f(\bar{j} \otimes L \otimes Q)\}$

Lemma 12.4. j^* *is a functor which does not depend on the choice of* $\bar{j}$.

Proof. j^∂ is a coalgebra map but $\bar{j}$ is only a chain map extending j^∂. This yields the diagram

$$
\begin{array}{ccc}
C_*A & \xrightarrow{\;\bar{j}\;} & C_*B \\
\overline{\Delta}_* \downarrow & & \downarrow \overline{\Delta}_* \\
C_*A \otimes C_*A & \xrightarrow{\;\bar{j} \otimes \bar{j}\;} & C_*B \otimes C_*B
\end{array}
$$

which commutes on the boundary $C_*\partial A$. Since B is contractible there is a homotopy $(\bar{j} \otimes \bar{j})\overline{\Delta}_* \simeq \overline{\Delta}_*\bar{j}$ rel $C_*\partial A$. This shows that $j^*(gf) = (j^*g)(j^*f)$. $\square$

(3) We now consider the *gluing functor* in $\mathbf{K}^Q$. Given ball pairs (B, A) and (B, A') in $\mathbb{B}(n)$ we have

$$C_*(B \cup_A B') = C_*B \cup_{C_*A} C_*B'$$

where the right hand side is a pushout of chain complexes. Given F in $\mathbf{K}^Q(B)$ and G in $\mathbf{K}^Q(B')$ with $\delta_A F = \delta_A G$ we get $F \cup G$ by

$$C_*(B \cup_A B') \otimes L \otimes Q = C_*B \otimes L \otimes Q \cup_{C_*A \otimes L \otimes Q} C_*B' \otimes L \otimes Q \xrightarrow{F \cup G} L' \otimes Q$$

(4) Finally we obtain $\otimes$-*products* in $\mathbf{K}^Q$ as follows. Let $g : Y \to Z \in \mathbf{K}^Q(B')$ and $f : X \to Y \in \mathbf{K}^Q(B)$. Then $g \otimes f : X \to Z \in \mathbf{K}^Q(B' \times B)$ is the composite

$$C_*(B' \times B) \otimes X \otimes Q = C_*B' \otimes C_*B \otimes X \otimes Q \xrightarrow{C_*B' \otimes f} C_*B' \otimes Y \otimes Q \xrightarrow{g} Z \otimes Q.$$

Theorem 12.5. *The data in* $(1) \ldots (4)$ *above describe a well-defined* n*-th order track category* $\mathbf{K}^Q$ *with all the properties in section 8. Moreover for the* $(n-1)$*-truncation we get*

$$(\mathbf{K}^Q)^{n-1} = \mathbf{K}^{Q(n-1)}.$$

The homology $H_0 = H_0^*(Q)$ is a graded algebra. Let $\mathbf{mod}(H_0)$ be the category of finitely generated free right H_0-modules $L \otimes H_0$, $L \in \mathbf{Ab}$. Then we get the bifunctor

$$D^k : \mathbf{mod}(H_0)^{\mathrm{op}} \times \mathbf{mod}(H_0) \to \mathbf{Ab}$$

$$D^k(L, L') = \mathrm{Hom}_{H_0} \left(L \otimes H_0, L' \otimes H_0 \otimes_{H_0} H_k \right)$$

where $H_k = H_k^*(Q)$ is an H_0-bimodule.

Theorem 12.6. *The n-th order track category $\mathbf{K}^Q$ is abelian with homotopy category*

$$K^Q[\mathrm{point}] = \mathbf{mod}(H_0^* Q)$$

and natural systems D^k defined above $1 \leq k \leq n$.

Remark 12.7. Higher order Toda brackets in the abelian n-th order track category $\mathbf{K}^Q$ coincide with *higher order matrix Massey products* in the differential algebra Q.

Let $\mathcal{M}^N$ be the set of finitely generated free graded R-modules concentrated in degree $< N$. Then $\mathbf{K}^Q$ defines the n-th order track category $(\mathbf{K}^Q(\mathcal{M}^N))^{\mathrm{op}}$ with objects in $\mathcal{M}^N$ which is formally dual to $\mathbf{K}^Q(\mathcal{M}^N)$.

Conjecture 12.8. *There exists a bigraded differential algebra Q over $R = \mathbb{Z}/p^2$ such that for $n \geq 0$ the truncation $Q(n)$ of Q yields an n-th order track category $\left(\mathbf{K}^{Q(n)}(\mathcal{M}^N) \right)^{\mathrm{op}}$ which is weakly equivalent to the stable track category $\mathbf{C}^n\{\mathcal{Z}_N\}$ of higher cohomology operations. We call Q the "algebra of higher cohomology operations".*

Theorem 12.9. *The conjecture is true for $n = 0$ and $n = 1$. For $n = 0$ we get the Steenrod algebra $Q(0) = \mathcal{A}$. For $n = 1$ we get the pair algebra $Q(1) = \mathcal{B}$ of secondary cohomology operations. The weak equivalence of track categories is established for $n = 1$ in [B,Se, 5.5.6].*

If the algebra Q is computed one has by the conjecture and section 10 a direct way to compute the differentials in the Adams spectral sequence which then allows the computation of stable homotopy groups of spheres.

References

[BD] H.-J. Baues, W. Dreckmann, The cohomology of homotopy categories and the general linear group, K-Theory **3** (1989), 307–338

[P] T. Pirashvili, H^3 and models for the homotopy theory, Trudy Tbiliss **94** (1992), 81–98

[B,Ob] H.-J. Baues, *Obstruction theory*, Lecture Notes in Math. **628**, Springer 1977, 387 pages

[B,Al] H.-J. Baues, *Algebraic homotopy*, Cambridge studies in Advanced Math. **15**, Cambridge 1989, ISBN 0-521-33376-8

[BJ] H.-J. Baues, M. Jibladze, Georgian Math. J. **8**, 683–696 and 697–712.

[BJ,Cl] H.-J. Baues, M. Jibladze, Classification of abelian track categories, K-Theory **25** (2002), 299–311

[BM] H.-J. Baues, F. Muro, The homotopy category of pseudofunctors and translation cohomology, JPAA **211** (2007), 821–850

[BW] H.-J. Baues, G. Wirsching, Cohomology of small categories, JPAA **38** (1985), no. 2-3, 187–211

[BJ,Se] H.-J. Baues, M. Jibladze, Secondary derived functors and the Adams spectral sequence, Topology **45** (2006), 295–324

[B,Se] H.-J. Baues, *The Algebra of secondary cohomology operations*, Progress in Math. **247**, Birkhäuser 2006