INTERSECTION COHOMOLOGY ON NONRATIONAL POLYTOPES

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1. Introduction

For an n-dimensional convex polytope Q R. Stanley ([S]) defined a set of integers $h(Q) = (h_0(Q), h_1(Q), \dots, h_n(Q))$ - the "generalized h-vector" - which are supposed to be the intersection cohomology Betti numbers of the toric variety X_Q corresponding to Q. In case $Q \subset \mathbb{R}^n$ is a rational polytope the variety X_Q indeed exists, and it is known ([S]) that $h_i(Q) = \dim IH^{2i}(X_Q)$. Thus, for a *rational polytope* Q, the integers $h_i(Q)$ satisfy

1. $h_i(Q) \geq 0$,
2. $h_i(Q) = h_{n-i}(Q)$ (Poincaré duality),
3. $h_0(Q) \leq h_1(Q) \leq \dots \leq h_{[n/2]}(Q)$ (follows from the Hard Lefschetz theorem for projective algebraic varieties).

For an arbitrary convex polytope (more generally for an Eulerian poset) Stanley proved ([S], Theorem 2.4) the property 2 above. He conjectured that 1 and 3 also hold without the rationality hypothesis. This is still not known in general. Recently, V. Timorin ([T]), using some of the results of the present paper, established the conjecture for a certain class of polytopes.

In this paper we propose an approach which we expect to lead to a proof of 1 and 3 for general convex polytopes. Our approach is modeled on the "equivariant geometry" of the (non-existent) toric variety X_Q as developed in [BL].

Namely, given a convex polytope $Q \subset \mathbb{R}^n$ we consider the corresponding complete fan $\Phi = \Phi_Q$ in $\mathbb{R}^n$ and work with Φ instead of Q. Let A denote the graded ring of polynomial functions on $\mathbb{R}^n$. Viewing Φ as a partially ordered set (of cones) we consider a category of sheaves of A-modules on Φ. In this category we define a *minimal* sheaf $\mathcal{L}_\Phi$ which corresponds to the T-equivariant intersection cohomology complex on X_Q if the latter exists. Our first main result is the "elementary" decomposition theorem for the direct image of the minimal sheaf under subdivision of fans (Theorem 5.6). (Recall, that a subdivision of a fan

This research was supported in part by the NSF.

corresponds to a proper morphism of toric varieties.) We also develop the Borel-Moore-Verdier duality in the derived category of sheaves of A-modules on Φ. We show that $\mathcal{L}_\Phi$ is isomorphic to its Verdier dual (Corollary 6.26).

Remark 1.1. In fact the usual (equivariant) decomposition theorem for a proper morphism of toric varieties can be deduced from this "elementary" one by the equivalence of categories proved in [L] (Theorem 2.6). However the proof of this last result by itself uses the fundamental properties of the intersection cohomology.

For a complete fan Φ the minimal sheaf $\mathcal{L}_\Phi$ gives rise in a natural way to the graded vector space $IH(\Phi)$ which we declare to be the *intersection cohomology* of Φ. (For rational Q it is proved in [BL] that there is an isomorphism $IH(\Phi_Q) \stackrel{\sim}{=} IH(X_Q)$.) Let $ih_i(\Phi) = \dim IH^i(\Phi)$. We establish the following properties of $IH(\Phi)$:

1. $\dim IH(\Phi) < \infty$;
2. $ih_i(\Phi) = 0$, unless i is even and $0 \le i \le 2n$;
3. $ih_0(\Phi) = ih_{2n}(\Phi) = 1$;
4. $ih_{n-i}(\Phi) = ih_{n+i}(\Phi)$.

The last property follows from Poincaré duality in $IH(\Phi)$ induced by the Verdier duality in sheaves. Similar relations are satisfied by the multiplicities in the decomposition of the direct image of the minimal sheaf under subdivision.

Moreover, there is a natural operator l of degree 2 on the space $IH(\Phi)$, which we expect to have the Lefschetz property as conjectured below:

Conjecture 1.2. *For each $i \geq 1$ the map*

$$l^i : IH^{n-i}(\Phi) \to IH^{n+i}(\Phi)$$

is an isomorphism.

So far we were unable to prove this conjecture, but it seems to be within reach. In case Q is rational the conjecture follows from the results in [BL]. This conjecture has the following standard corollary:

Corollary 1.3 (of the conjecture)**.** *$ih_i(Q) \le ih_{i+2}(Q)$ for $0 \le i < n$.*

In fact the above conjecture implies "everything":

Corollary 1.4. *Assume the above conjecture is true. Then*

1. *$IH(\Phi_Q)$ is a combinatorial invariant of Q, i.e. it depends only on the face lattice of Q.*
2. *Moreover, $ih_{2j}(\Phi_Q) = h_j(Q)$ hence the h-vector $h(Q)$ has the properties conjectured by R. Stanley.*

After the first version of this paper appeared in alg-geom we were informed by G. Barthel, J.-P. Brasselet, K.-H. Fieseler and L. Kaup that their paper [BBFK] (in preparation at the time) contains some similar results; we would like to thank the authors for informing us of their work.

The paper is organized as follows:

- Section 2 gives a brief account of our methods and main results.
- Section 3 discusses the elementary properties of the category of abelian sheaves on a fan and their cohomology.
- In Section 4 we endow a fan with the structure of a ringed space, single out a category of sheaves of modules over the structure sheaf and obtain the first "geometric" result (Theorem 4.7).
- In Section 5 we prove that our category of sheaves is semi-simple and identify the simple objects (Theorem 5.3). We show that our categories of sheaves are stable by direct image under morphisms induced by subdivision of fans (Theorem 5.6).
- Section 6 contains an account of duality on our category of sheaves. As a consequence the existence of the duality involution (Corollary 6.25) and it's effect in cohomology (Theorem 6.21) we obtain the Poincaré duality in intersection cohomology of a complete fan (Corollary 6.29). We show that duality commutes with the direct image under morphisms induced by subdivision of fans (Corollary 6.23) and obtain a relative version of Poincaré duality (Corollary 6.30).
- In Section 7 we make precise Lefschetz type conjectures and discuss their consequences.
- In Section 8 we apply the machinery to a conjecture of G. Kalai (proven recently in the rational case by T. Braden and R.D. MacPherson) and give our version of the proof.

2. Summary of methods and results

1. **Fans as ringed spaces.** Our point of departure is the observation that a fan Φ in a (real) vector space V gives rise to a topological space, which we will denote by Φ as well, and a sheaf of graded rings $\mathcal{A}_\Phi$ on it. Namely, the points of Φ are cones, open subsets are subfans, and the stalk $\mathcal{A}_{\Phi,\sigma}$ of $\mathcal{A}_\Phi$ at the cone $\sigma \in \Phi$ is the graded algebra of polynomial functions on σ (equivalently on the linear span of σ) and the structure maps are given by restriction of functions. All of these rings are quotients of the graded algebra $A = A_V$ of polynomial functions

on V. The grading is assigned so that the linear functions have degree two.

In case V is the Lie algebra of a torus T the graded ring A is canonically isomorphic to $H^*(BT)$ – the cohomology ring of the classifying space of T.

In the case of a rational fan Φ one has the (unique) normal T-toric variety X_Φ such that the T-orbits in X_Φ are in bijective correspondence with the cones of Φ and $\mathcal{A}_{\Phi,\sigma}$ is canonically isomorphic to the cohomology ring of the classifying space of the stabilizer of the corresponding orbit.

All $\mathcal{A}_\Phi$-modules will be regarded by default as A-modules. Let A^+ denote the ideal of functions which vanish at the origin. For a graded A-module M we will denote by $\overline{M}$ the graded vector space M/A^+M.

2. **A category of $\mathcal{A}_\Phi$-modules.** To each fan Φ viewed as the ringed space $(\Phi, \mathcal{A}_\Phi)$ we associate the additive category $\mathfrak{M}(\mathcal{A}_\Phi)$ of (sheaves of finitely generated, graded) $\mathcal{A}_\Phi$-modules which are flabby and locally free over $\mathcal{A}_\Phi$. This latter condition means that, for an object $\mathcal{M}$ of $\mathfrak{M}(\mathcal{A}_\Phi)$, the stalk $\mathcal{M}_\sigma$ is a free graded module of finite rank over $\mathcal{A}_{\Phi,\sigma}$. The flabbyness condition may be restated as follows: for every cone σ the restriction map $\mathcal{M}_\sigma \to \mathcal{M}(\partial\sigma)$ is surjective (where $\mathcal{M}(\partial\sigma)$ is the space of section of $\mathcal{M}$ over the subfan $\partial\sigma$ consisting of cones properly contained in σ). It is easy to see that the sheaf $\mathcal{A}_\Phi$ is flabby if and only if the fan Φ is simplicial.

In the rational case the category $\mathfrak{M}(\mathcal{A}_\Phi)$ is equivalent to the category of equivariant perverse (maybe shifted) sheaves on X_Φ. The following theorems verify that the category $\mathfrak{M}(\mathcal{A}_\Phi)$ and the cohomology of an object $\mathcal{M}$ of $\mathfrak{M}(\mathcal{A}_\Phi)$ have the expected properties.

Since, by definition, the objects of $\mathfrak{M}(\mathcal{A}_\Phi)$ are flabby sheaves, it follows that, for $\mathcal{M}$ in $\mathfrak{M}(\mathcal{A}_\Phi)$, $H^i(\Phi;\mathcal{M}) = 0$ for $i \neq 0$.

Theorem 2.1. *Suppose that Φ is complete (i.e. the union of the cones of Φ is all of V), and $\mathcal{M}$ is in $\mathfrak{M}(\mathcal{A}_\Phi)$. Then, $H^0(\Phi;\mathcal{M})$ is a free A-module.*

In the rational case, $H^0(\Phi;\mathcal{M})$ is the equivariant cohomology of the corresponding perverse sheaf on X_Φ.

The proof of Theorem 2.1 rests on the observation that the cohomology of a sheaf $\mathcal{F}$ on a complete fan may be calculated by a "cellular" complex $C^\bullet(\mathcal{F})$ whose component in degree i is the direct sum of the stalks of $\mathcal{F}$ at cones of codimension i and the differential is given by the sum (with suitable signs) of the restriction maps. In particular, if the sheaf $\mathcal{F}$ is flabby, then the complex $C^\bullet(\mathcal{F})$ is acyclic except in degree zero. This proves the conjecture of J. Bernstein and the second author (Conjecture 15.9 of [BL]) on the acyclicty properties of the "minimal complex", which happens to be the "cellular complex" of the simple object $\mathcal{L}_\Phi$ (see below) of $\mathfrak{M}(\mathcal{A}_\Phi)$. In the simplicial case (when $\mathcal{L}_\Phi \stackrel{\sim}{=} \mathcal{A}_\Phi$) an "elementary" proof of this fact was given by M. Brion in [B].

Concerning the structure of the category $\mathfrak{M}(\mathcal{A}_\Phi)$ we have the following result.

Theorem 2.2. *Every object in $\mathfrak{M}(\mathcal{A}_\Phi)$ is a finite direct sum of indecomposable ones. The indecomposable objects are, up to a shift of the grading, in bijective correspondence with the set of cones. (see Theorem 5.3 below).*

3. IH **and** IP**.** The indecomposable object of $\mathfrak{M}(\mathcal{A}_\Phi)$ which corresponds to the cone σ is a sheaf supported on the star of σ (which constitutes the closure of the set $\{\sigma\}$ in our topology). Let $\mathcal{L}_\Phi$ denote the indecomposable object of $\mathfrak{M}(\mathcal{A}_\Phi)$ which is supported on all of Φ (the star of the origin of V) and whose stalk at the origin is the one dimensional vector space in degree zero. The fan Φ is simplicial if and only if $\mathcal{L}_\Phi \stackrel{\sim}{=} \mathcal{A}_\Phi$.

In the rational case, when Φ is complete (and so is X_Φ), the A-module $H^0(\Phi;\mathcal{L}_\Phi)$ is isomorphic to the T-equivariant intersection cohomology $IH_T(X_\Phi)$ of X_Φ and $\overline{IH_T(X_\Phi)}$ is the usual (non-equivariant) intersection cohomology of X_Φ. This motivates the following notation:

Definition 2.3. Let Φ be a complete fan in V. We set

$$IH(\Phi) \stackrel{def}{=} \overline{H^0(\Phi;\mathcal{L}_\Phi)}$$

and denote by $ih(\Phi)$ the corresponding Poincaré polynomial.

For each cone $\sigma \in \Phi$ we may consider the corresponding local Poincaré polynomial. Namely, in the rational case the graded vector space $\overline{\mathcal{L}_{\Phi,\sigma}}$ is the (cohomology of the) stalk on the corresponding T-orbit O_σ of the intersection cohomology complex of X_Φ. A normal slice to O_σ is an affine cone over some projective variety Y_σ. Then $\overline{\mathcal{L}_{\Phi,\sigma}}$ is the primitive part of the intersection cohomology of Y_σ. This motivates the following notation.

Definition 2.4. For $\sigma \in \Phi$ we set

$$IP(\sigma) \stackrel{def}{=} \overline{\mathcal{L}_{\Phi,\sigma}}$$

and denote by $ip(\sigma)$ the corresponding Poincaré polynomial.

As is well known, the projectivity of a toric variety translates into the following picture. Suppose that Φ is a complete fan in V and $l \in \mathcal{A}_\Phi(\Phi)$ is a (continuous) cone-wise linear (with respect to Φ) strictly convex function on V. Multiplication by l is an endomorphism (of degree 2) of $\mathcal{L}_\Phi$, $H^0(\Phi; \mathcal{L}_\Phi)$ and $IH(\Phi)$. In the rational case it is the Lefschetz operator on $IH(\Phi) = IH(X_\Phi)$ for the corresponding projective embedding of X_Φ. Thus, we make the following conjecture.

Conjecture 2.5 (Hard Lefschetz)**.** *Let Φ be a complete fan. Multiplication by l is a Lefshetz operator on $IH(\Phi)$ i.e. for each $i \geq 1$ the map*

$$l^i : IH^{n-i}(\Phi) \to IH^{n+i}(\Phi)$$

is an isomorphism.

4. **Subdivision and the decomposition theorem.** A fan Ψ is a subdivision of a fan Φ if every cone of the latter is a union of cones of the former. In this case there is a morphism of ringed spaces $\pi : (\Psi, \mathcal{A}_\Psi) \to (\Phi, \mathcal{A}_\Phi)$. In the rational case subdivision corresponds to a proper birational morphism of T-toric varieties.

Theorem 2.6 (Decomposition Theorem)**.** *The functor of direct image under subdivision restricts to the functor $\pi_* : \mathfrak{M}(\mathcal{A}_\Psi) \to \mathfrak{M}(\mathcal{A}_\Phi)$.*

It should be pointed out that the only non-trivial part of Theorem 2.6 is the fact that the direct image of a locally free flabby sheaf is locally free which is proven by essentially the same argument as the one used in the proof of Theorem 2.1.

Combining Theorem 2.6 with Theorem 2.2 we obtain the statement which in the rational case amounts to the Decompostion Theorem of A. Beilinson J. Bernstein, P. Deligne, and O. Gabber ([BBD]) and its equivariant analog ([BL]) for proper birational morphisms of toric varieties: "the direct image of a pure object is a direct sum of (suitably shifted) pure objects". Continuing with notations introduced above we have the following "estimate":

Corollary 2.7. *Suppose in addition that Φ is complete (therefore so is Ψ). $\pi_*\mathcal{L}_\Psi$ contains $\mathcal{L}_\Phi$ as a direct summand, therefore $IH(\Psi)$ contains $IH(\Phi)$ as a direct summand. Hence, there is an inequality*

$$ih(\Psi) \geq ih(\Phi)$$

(coefficient by coefficient) of polynomials with non-negative coefficients.

5. **Duality.** As is well known, the (middle perversity) intersection cohomology of a compact space admits an intersection pairing (and the same is the case in the equivariant setting). To this end we have the

following version of Borel-Moore-Verdier duality which we develop for the derived category of sheaves of A-modules on Φ. One of the results is the following

Theorem 2.8. *Let Φ be a fan in V. Then,*

1. *There is a contravariant involution $\mathbb{D}_\Phi$ on $\mathfrak{M}(\mathcal{A}_\Phi)$ (i.e. a functor $\mathbb{D}_\Phi : \mathfrak{M}(\mathcal{A}_\Phi)^{op} \to \mathfrak{M}(\mathcal{A}_\Phi)$ and an isomorphism of functors $\mathbb{D}_\Phi \circ \mathbb{D}_\Phi \cong \mathrm{Id}$).*
2. *If Φ is complete, then there is a natural A-linear non-degenerate pairing (of free A-modules)*
$$H^0(\Phi;\mathcal{M}) \otimes_A H^0(\Phi;\mathbb{D}_\Phi(\mathcal{M})) \to \omega_{A/\mathbb{R}}$$
for every object $\mathcal{M}$ of $\mathfrak{M}(\mathcal{A}_\Phi)$.
3. *If $\pi : \Psi \to \Phi$ is a morphism induced by a subdivision, then, for every object $\mathcal{M}$ of $\mathfrak{M}(\mathcal{A}_\Psi)$, there is a natural isomorphism $\pi_*\mathbb{D}_\Psi(\mathcal{M}) \cong \mathbb{D}_\Phi(\pi_*\mathcal{M})$.*

Here $\omega_{A/\mathbb{R}} = A \otimes \det V^*$ is the dualizing A-module, free of rank one, generated in degree $2\dim_{\mathbb{R}} V$ in accordance with our grading convention.

If follows from Theorem 2.8 that $\mathbb{D}$ is an anti-equivalence of categories, so, the dual of an indecomposable object is an indecomposable one. One checks immediately that the dual $\mathbb{D}(\mathcal{L}_\Phi)$ of $\mathcal{L}_\Phi$ has the properties which characterize the latter. Therefore there is a (non-canonical) isomorphism $\mathbb{D}(\mathcal{L}_\Phi) \cong \mathcal{L}_\Phi$. The numerical consequence of the auto-duality of $\mathcal{L}_\Phi$ is given below.

Corollary 2.9. *For a complete fan Φ in a vector space of dimension n the polynomial $ih(\Phi)$ satisfies $ih_{n-k}(\Phi) = ih_{n+k}(\Phi)$.*

This corollary has a "relative" version. Recall that the indecomposable objects in $\mathfrak{M}(\mathcal{A}_\Phi)$ are, up to shift of the grading, are in one-to-one correspondence with cones. For a cone $\sigma \in \Phi$ we denote the corresponding, suitably normalized indecomposable by $\mathcal{L}_\Phi^\sigma$. Thus, for every object $\mathcal{M} \in \mathfrak{M}(\mathcal{A}_\Phi)$ there is a (non-canonical) isomorphism

$$\mathcal{M} \cong \bigoplus_{\sigma\in\Phi}\bigoplus_{k\in\mathbb{Z}} V_{\sigma,k} \otimes \mathcal{L}_\Phi^\sigma(k) \tag{2.1}$$

where $V_{\sigma,k}$ are finite dimensional vector spaces (trivial for almost all k). Here $(\bullet)(k)$ denotes the shift of grading by k. It is not difficult to show that $\mathbb{D}_\Phi(\mathcal{L}_\Phi^\sigma(k)) \cong \mathcal{L}_\Phi^\sigma(-k - 2\dim\sigma)$.

Corollary 2.10. *Suppose that $\pi : \Psi \to \Phi$ is a morphism induced by a subdivision. Then, the vector spaces $V_{\sigma,k}$ in the decomposition* (2.1) *of*

$\mathcal{M} = \pi_* \mathcal{L}_\Psi$ *satisfy*

$$\dim V_{\sigma,k} = \dim V_{\sigma,-k-2\dim\sigma}$$

6. **Kalai type inequalities.** As an application of our technology we give our restatement of the inequality conjectured by G. Kalai, proven in the rational case by T. Braden and R.D. MacPherson in [BM]. Namely, suppose that Φ is a fan in V generated by a single cone σ, i.e. $\Phi = [\sigma]$ and $\tau \subset \sigma$. Let $\mathrm{Star}(\tau)$ be the collection of cones in Φ which contain τ. Consider the indecomposable $\mathcal{L}^\tau_{[\sigma]}$ on $[\sigma]$ which corresponds to τ (see Definition 5.1 below) and put

$$IP(\mathrm{Star}(\tau)) = \overline{\mathcal{L}^\tau_{[\sigma],\sigma}} \ .$$

Theorem 2.11. *There is an inequality, coefficient by coefficient, of polynomials with non-negative coefficients*

$$ip(\sigma) \geq ip(\tau) ip(\mathrm{Star}(\tau)) \ .$$

3. Abelian sheaves on fans

1. **Fans.** A *fan* Φ in a real vector space V of dimension $\dim V = n$ is a collection of closed convex polyhedral cones with vertex at the origin $\underline{o}$ satisfying

- any two cones in Φ intersect along a common face;
- if a cone is in Φ, then so are all of it's faces.

A fan has a structure of a partially ordered set: given cones σ and τ in Φ we write $\tau \leq \sigma$ if τ is a face of σ.

The origin is the unique minimal cone in every fan and will be denoted $\underline{o}$.

Let $d(\sigma)$ denote the dimension of the cone σ. Thus $d(\sigma) = 0$ iff $\sigma = \underline{o}$.

The fan Φ is *complete* if and only if the union of cones of Φ is all of V. The fan Φ is *simplicial* if every cone of Φ is simplicial. A cone of dimension k is *simplicial* if it has k one-dimensional faces (rays).

2. **Topology on a fan.** The (partially ordered) Φ will be considered as a topological space with the open sets the subfans of Φ.

Let $\Phi_{\leq k}$ denote the subset of cones of dimension at most k; this is an open subset of Φ.

A subset, say S, of Φ generates a subfan, denoted $[S]$. We will frequently abuse notation and write $[\sigma]$ for the irreducible open set $[\{\sigma\}]$.

We denote by $\partial\sigma$ the complement of $\{\sigma\}$ in $[\sigma]$. That is, $\partial\sigma$ is the subfan generated by proper faces of σ.

An open subset is *irreducible* if it is not a union of two open subsets properly contained in it. The irreducible open sets are the subfans generated by single cones.

Remark 3.1. Note that the topological space Φ has the following property: the intersection of irreducible open sets is irreducible.

Let σ be a cone in Φ. Denote by $\mathrm{Star}(\sigma) \subset \Phi$ the subset of all the cones τ such that $\sigma \leq \tau$. This is a closed subset of Φ. Its image under the projection $V \to V/\operatorname{Span}(\sigma)$ is a fan that will be denoted by $\overline{\mathrm{Star}(\sigma)}$.

3. **Sheaves on a fan.** Regarding Φ as a topological space with open sets the subfans of Φ, we consider sheaves on Φ.

Let $\mathcal{I}(\Phi)$ denote the partially ordered set of irreducible open sets of Φ and inclusions thereof. This partially ordered set is isomorphic to Φ.

A sheaf on Φ restricts to a presheaf (a contravariant functor) on $\mathcal{I}(\Phi)$ and this correspondence is an equivalence of categories. Since, for a sheaf F and a cone σ, the stalk F_σ is equal to the sections $\Gamma([\sigma]; F)$ of F over the corresponding irreducible open set, the sheaf F is uniquely determined by the assignment $\sigma \mapsto F_\sigma$ and the restriction maps $F_\sigma \to F_\tau$ whenever $\tau \leq \sigma$.

As usual the support $\mathrm{Supp}(F)$ of a sheaf F is the closure of the set of σ's, such that $F_\sigma \neq 0$.

For a sheaf F and a subset S we will denote by F_S the extension by zero of the restriction of F to S. By abuse of notation we will write F_σ for $F_{\{\sigma\}}$.

4. **Flabby sheaves.** Let F be a sheaf and consider the following condition:

for every cone σ the canonical map

$$F_\sigma \to \Gamma(\partial\sigma; F) \text{ is surjective.} \tag{3.1}$$

Lemma 3.2. *A sheaf F satisfies the condition* (3.1) *if and only if it is flabby.*

Proof. The condition is obviously necessary. To see that it is sufficient we need to show that a section of F defined over a subfan Ψ extends to a global section. Clearly, it is sufficient to show that a section defined over $\Psi \bigcup \Phi_{\leq k}$ extends to a section defined over $\Psi \bigcup \Phi_{\leq k+1}$, but this is immediate from (3.1). □

For the rest of this section 3 we restrict our attention to the category of sheaves of $\mathbb{R}$-vector spaces on Φ which we will denote by $\mathrm{Sh}(\Phi)$.

5. **Some elementary properties of the category** $\mathrm{Sh}(\Phi)$**.** For a cone σ let $i_\sigma : \{\sigma\} \hookrightarrow \Phi$ denote the inclusion. The embedding i_σ is locally closed, and closed (respectively open) if and only if σ is maximal (respectively minimal, i.e. the origin).

Suppose that W is a vector space considered as a sheaf on $\{\sigma\}$.

Since every sheaf on a point is flabby the functor $(i_\sigma)_*$ is exact (i.e. $\mathbf{R}^p\, i_{\sigma *}W = 0$ for $p \neq 0$). Since every object in $\mathbb{R} - mod$ is injective the sheaf $i_{\sigma *}W$ is an injective object in $\mathrm{Sh}(\Phi)$ which is equal to the constant sheaf W supported on $\mathrm{Star}(\sigma)$. Every sheaf F is embedded into a direct sum of injective sheaves by the canonical map

$$F \to \bigoplus_{\sigma\in\Phi} i_{\sigma *} i_\sigma^{-1} F\ .$$

Let $i_{[\sigma]} : [\sigma] \hookrightarrow \Phi$ denote the open embedding of the irreducible open set $[\sigma]$. Since

$$F \mapsto \mathrm{Hom}_{\mathrm{Sh}(\Phi)}(i_{[\sigma]!}W, F) = \mathrm{Hom}_{\mathbb{R}}(W, F_\sigma)$$

(where $i_{[\sigma]!}W$ is the extension by zero of the constant sheaf W on $[\sigma]$) is an exact functor, it follows that $i_{[\sigma]!}W$ is a projective object in $\mathrm{Sh}(\Phi)$. Every sheaf F admits an epimorphism from a projective object, for example the canonical map

$$\bigoplus_{\sigma\in\Phi} i_{[\sigma]!} i_\sigma^{-1} F \to F\ .$$

Thus, the Abelian category $\mathrm{Sh}(\Phi)$ has enough projectives.

6. **The cellular complex of a sheaf.** Let Φ be a fan in a (real) vector space V of dimension n. We fix an orientation of each cone on Φ.

Definition 3.3. For a sheaf F on Φ the *cellular complex* $C^\bullet(F)$ of

$$0 \to C^0(F) \to C^1(F) \to \ldots \to C^n(F) \to 0$$

is defined by

$$C^i(F) = \bigoplus_{d(\sigma)=n-i} F_\sigma$$

with the differential $d^i : C^i(F) \to C^{i+1}(F)$ equal to the sum of the restriction maps $F_\sigma \to F_\tau$ with the sign ± 1 depending on whether the orientations of σ and τ agree or disagree.

Remark 3.4. $C^\bullet(\bullet)$ is an exact functor from $\mathrm{Sh}(\Phi)$ to complexes of vector spaces, therefore extends trivially to a functor $C^\bullet(\bullet) : D^b(\mathrm{Sh}(\Phi)) \to D^b(\mathbb{R} - mod)$.

Proposition 3.5. *Assume that the fan Φ is complete. Then, $C^\bullet(\bullet)$ and $\mathbf{R}\,\Gamma(\Phi;\bullet)$ are isomorphic as functors $D^b(\mathrm{Sh}(\Phi)) \to D^b(\mathbb{R}-mod)$.*

Proof. If $n = \dim V = 0$ the statement is trivially true, so we assume that $n \neq 0$ First, notice that, since Φ is complete, there is a natural isomorphism

$$\Gamma(\Phi; F) = H^0(C^\bullet(F)) \ .$$

Indeed, a global section $s \in \Gamma(\Phi; F)$ is equivalent to a collection of local sections $s_\sigma \in F([\sigma])$, for each cone σ of dimension $d(\sigma) = n$ such that $s_\sigma = s_\tau$ in $F([\sigma \cap \tau])$ if $d(\sigma \cap \tau) = n-1$. This shows that $\Gamma(\Phi; F) = H^0(C^\bullet(F))$.

It remains to show that $H^i(C^\bullet(I)) = 0$ for $i \neq 0$ for any injective sheaf I. In fact, it is sufficient to take $I = W_{\mathrm{Star}(\sigma)}$ (i.e. the extension by zero of the constant sheaf with value W on $\mathrm{Star}(\sigma)$).

By induction on dimension, the only case to consider is $\sigma = \underline{o}$ (so that $\mathrm{Star}(\sigma) = \Phi$). In this case $C^\bullet(I)$ is isomorphic (up to reindexing and shift) to the augmented cellular chain complex (with coefficients in W) of a sphere of dimension $n-1$. □

Later on we will need the following generalization of the previous proposition.

Proposition 3.6. *Let Φ be any fan in V. Let $F \in \mathrm{Sh}(\Phi)$ be such that its support $Z = Supp(F)$ satisfies the following condition:*

for each $\sigma \in Z$ the fan $\overline{\mathrm{Star}(\sigma)}$ in $V/\mathrm{Span}(\sigma)$ is complete.

Then, the cellular complex $C^\bullet(F)$ is naturally isomorphic to $\mathbf{R}\,\Gamma(\Phi; F)$.

Proof. Same as that of Proposition 3.5. □

The next two lemmas will be used later on.

Lemma 3.7. *Let $\sigma \subset V$ be a cone of positive dimension (i.e. $\sigma \neq \underline{o}$). For any constant sheaf F on $[\sigma]$ the cellular complex $C^\bullet(F)$ is acyclic.*

Proof. Then the cellular complex $C^\bullet(F)$ is isomorphic (up to reindexing and shift) to an augmented cellular chain complex of a ball of dimension $d(\sigma) - 1$. □

Remark 3.8. In the notation of the previous lemma notice that if $F \neq 0$, then $H^0([\sigma]; F) \neq 0$. Thus, for a fan which is not complete, the cellular complex does not necessarily compute the cohomology of the sheaf.

Lemma 3.9. *Let σ be a cone in V with $\dim V = n$. Let F be a flabby sheaf on the fan $[\sigma]$. Then*

1. *the cellular complex* $C^\bullet(F)$ *is acyclic except in the lowest degree, i.e.* $H^i(C^\bullet(F)) = 0$ *for* $i \neq n - d(\sigma)$*, and*
2. $H^{n-d(\sigma)}(C^\bullet(F)) = \Gamma_{\{\sigma\}}F$.

Proof. The case $d(\sigma) = 0$ being trivial we assume that $d(\sigma) > 0$.

Since $C^\bullet(\bullet)$ is an exact functor, the short exact sequence of sheaves

$$0 \to F_{\partial\sigma} \to F \to F_\sigma \to 0$$

gives rise to the short exact sequence of complexes

$$0 \to C^\bullet(F_{\partial\sigma}) \to C^\bullet(F) \to C^\bullet(F_\sigma) \to 0 \ .$$

Since $\partial\sigma$ is isomorphic to a complete fan, it follows from Proposition 3.5 and the flabbyness of F that

1. $H^i(C^\bullet(F_{\partial\sigma})) = 0$ for $i \neq n - d(\sigma) + 1$;
2. $H^{n-d(\sigma)+1}(C^\bullet(F_{\partial\sigma})) = \Gamma(\partial\sigma; F)$;
3. the (connecting) map

   $$H^{n-d(\sigma)}(C^\bullet(F_\sigma)) = F_\sigma \to \Gamma(\partial\sigma; F) = H^{n-d(\sigma)+1}(C^\bullet(F_{\partial\sigma}))$$

   is surjective. Therefore, the long exact sequence in cohomology reduces to

   $$0 \to H^{n-d(\sigma)}(C^\bullet(F)) \to F_\sigma \to \Gamma(\partial\sigma; F) \to 0 \ .$$

□

7. **Cohomology of some simple sheaves.**

Lemma 3.10. *Let* σ *be a cone in* V*. The functor* $\Gamma([\sigma]; \bullet)$ *is exact. Equivalently, for any sheaf* F*,* $H^i([\sigma]; F) = 0$ *for* $i \neq 0$*.*

Proof. The functor $\Gamma([\sigma]; \bullet)$ is naturally equivalent to the functor "stalk at σ" and the latter is exact. □

Lemma 3.11. *Let* Φ *be a fan in* V *and* $F \in Sh(\Phi)$ *be the constant sheaf on* Φ *with stalk* W*. Then,*

$$H^i(\Phi, F) = \begin{cases} W & \textit{if } i = 0 \\ 0 & \textit{otherwise.} \end{cases}$$

Proof. Since the space Φ is connected $H^0(\Phi, F) = W$. The rest follows from the injectivity of the sheaf F. □

Lemma 3.12. *Let* Φ *be a complete fan in* V *and* W *be a vector space. Consider the sheaf* $W_{\underline{o}}$ *on* Φ *which is the extension by zero of the sheaf* W *on the open point* $\underline{o}$*. Then,*

$$H^i(\Phi, W_{\underline{o}}) = \begin{cases} W & \textit{if } i = n \\ 0 & \textit{otherwise.} \end{cases}$$

Proof. Let $Z = \Phi \setminus \{\underline{o}\}$. Let W_Φ (respectively W_Z) denote the constant sheaf with stalk W on Φ (respectively on Z, extended by zero to Φ). We have the exact sequence

$$0 \to W_{\underline{o}} \to W_\Phi \to W_Z \to 0 \ .$$

By Lemma 3.11 $H^0(\Phi, W_\Phi) = W$ and $H^i(\Phi, W_\Phi) = 0$ if $i > 0$. On the other hand, the cellular complex $C^\bullet(W_Z)$ is isomorphic to a chain complex of a sphere of dimension $n-1$. Hence, by Proposition 3.6,

$$H^j(\Phi, W_Z) = \begin{cases} W & \text{if } j = 0, n-1 \\ 0 & \text{otherwise.} \end{cases}$$

if $n > 1$, and

$$H^j(\Phi, W_Z) = \begin{cases} W \oplus W & \text{if } j = 0 \\ 0 & \text{otherwise.} \end{cases}$$

if $n = 1$. The map $H^0(\Phi, W_\Phi) \to H^0(\Phi, i_* W_Z)$ is injective, so the lemma follows. ☐

4. Fans as ringed spaces

1. **The structure sheaf of a fan.** Let Φ be a fan in V. Let V^* denote the constant sheaf $\sigma \mapsto V^*$ on Φ. Let Ω^1_Φ denote the subsheaf of V^* given by

$$\Phi \ni \sigma \mapsto \Omega^1_{\Phi,\sigma} \stackrel{def}{=} \sigma^\perp \subseteq V^* \ ,$$

where $\sigma^\perp$ denotes the subspace of linear functions which vanish identically on σ.

Let $\mathcal{G}$ denote the sheaf determined by the assignment

$$\Phi \ni \sigma \mapsto \mathcal{G}_\sigma = \mathrm{Span}(\sigma)^* \ .$$

Thus, there is a short exact sequence of sheaves

$$0 \to \Omega^1_\Phi \to V^* \xrightarrow{\pi} \mathcal{G} \to 0 \ . \tag{4.1}$$

We will denote by A the symmetric algebra of V^* with grading determined by assigning degree 2 to V^*. We will use the notation A_Φ for the corresponding constant sheaf on Φ.

Definition 4.1. The structure sheaf $\mathcal{A}_\Phi$ is the symmetric algebra of $\mathcal{G}$, i.e. the sheaf of cone-wise polynomial functions, graded so that the linear functions have degree 2.

Remark 4.2. Clearly, there is an epimorphism of sheaves of graded algebras $A_\Phi \to \mathcal{A}_\Phi$.

Remark 4.3. With these definitions $(\Phi, \mathcal{A}_\Phi)$ is a ringed space over the one point ringed space $(\emptyset, A)$ which we imagine as "the empty fan" in V.

In what follows "an $\mathcal{A}_\Phi$-module" will mean "a (locally) finitely generated graded $\mathcal{A}_\Phi$-module" and similarly for A_Φ-modules. An $\mathcal{A}_\Phi$-module $\mathcal{M}$ is *locally free* if, for every cone σ, $\mathcal{M}_\sigma$ is a free (graded) $\mathcal{A}_{\Phi,\sigma}$-module.

Let A^+ denote the ideal of elements of positive degree. For an A-module M we will denote by $\overline{M}$ the graded vector space M/MA^+.

For a graded A-module (or sheaf) $M = \oplus M_k$ denote by $M(t)$ the corresponding shifted object

$$M(t)_k = M_{k+t}.$$

The flabbiness criterion (3.1) applied to an $\mathcal{A}_\Phi$-module together with Nakayama's Lemma amounts to the following.

Lemma 4.4. *An $\mathcal{A}_\Phi$-module $\mathcal{M}$ is flabby if and only if for every cone σ the canonical map*

$$\overline{\mathcal{M}_\sigma} \to \overline{\Gamma(\partial\sigma; \mathcal{M})}$$

is surjective.

Definition 4.5. Let $\mathfrak{M}(\mathcal{A}_\Phi)$ denote the additive category of flabby locally free $\mathcal{A}_\Phi$-modules considered as a full subcategory of sheaves of $\mathcal{A}_\Phi$-modules and morphisms of dergee zero.

Lemma 4.6. *The fan Φ is* simplicial *if and only if the structure sheaf $\mathcal{A}_\Phi$ is flabby.*

Proof. The statement is easily seen to be equivalent to the following one: *Suppose that $P_1, \ldots, P_n$ are polynomials in variables $x_1, \ldots, x_n$ which satisfy $\dfrac{\partial P_i}{\partial x_i} = 0$ and $P_i|_{x_j=0} = P_j|_{x_i=0}$ for all i and j. Then there is a polynomial Q such that $Q|_{x_i=0} = P_i$.* Verification of the latter fact is left to the reader. □

2. Cohomology of objects in $\mathfrak{M}(\mathcal{A}_\Phi)$ on complete fans.

Theorem 4.7. *Suppose that Φ is a complete fan in V and $\mathcal{M}$ is in $\mathfrak{M}(\mathcal{A}_\Phi)$. Then, $H^i(\Phi; \mathcal{M}) = 0$ for $i \neq 0$ and $H^0(\Phi; \mathcal{M})$ is a free A-module.*

Proof. The vanishing of higher cohomology is a direct consequence of the flabbiness of the objects of $\mathfrak{M}(\mathcal{A}_\Phi)$.

A graded A-module M is free if and only if $\mathrm{Ext}^i_A(M, A) = 0$ for $i \neq 0$. By Proposition 3.5, $H^0(\Phi; \mathcal{M}) = H^0 C^\bullet(\mathcal{M})$. Note that, for every j, $\mathrm{Ext}^i_A(C^j(\mathcal{M}), A) = 0$ for $i \neq j$. Since, in addition, $H^i C^\bullet(\mathcal{M}) = 0$ for

$i \neq 0$ it follows by the standard argument that $\mathrm{Ext}^i_A(H^0C^\bullet(\mathcal{M}), A) = 0$ for $i \neq 0$. □

Corollary 4.8. *Suppose that σ is a cone in V, $\mathcal{M} \in \mathfrak{M}(\mathcal{A}_{[\sigma]})$. Then, $\Gamma_{\{\sigma\}}\mathcal{M}$ is a free module over $\mathcal{A}_{[\sigma],\sigma}$.*

Proof. We may assume that σ spans V, i.e. $A = \mathcal{A}_{[\sigma],\sigma}$. The statement is obvious if $d(\sigma) = 0$, so we assume that $d(\sigma) \geq 1$

Choose a vector v in the "interior" of σ. The projection $V \to V/\mathbb{R}v$ gives rise to an isomorphism (of ringed spaces) between $(\partial\sigma, \mathcal{A}_{\partial\sigma})$ and a complete fan in $V/\mathbb{R}v$. By Theorem 4.7 $\Gamma(\partial\sigma; \mathcal{M})$ is a free module over $A_{V/\mathbb{R}}$, therefore it satisfies

$$\mathrm{Ext}^i_A(\Gamma(\partial\sigma; \mathcal{M}), A) = 0 \text{ for } i \neq 1.$$

The exact sequence

$$0 \to \Gamma_{\{\sigma\}}\mathcal{M} \to \mathcal{M}_\sigma \to \Gamma(\partial\sigma; \mathcal{M}) \to 0$$

shows that $\Gamma_{\{\sigma\}}\mathcal{M}$ satisfies

$$\mathrm{Ext}^i_A(\Gamma_{\{\sigma\}}\mathcal{M}; \mathcal{M}), A) = 0 \text{ for } i \neq 0.$$

The desired statement follows. □

5. Minimal sheaves and intersection cohomology

1. **Minimal sheaves.** Recall that, for a cone σ, the set $\mathrm{Star}(\sigma)$ is defined as the collection of those cones τ which satisfy $\tau \geq \sigma$. Namely, $\mathrm{Star}(\sigma)$ is the closure of the set $\{\sigma\}$.

Fix a cone σ and consider the following conditions on an object $\mathcal{M}$ of $\mathfrak{M}(\mathcal{A}_\Phi)$:

1. $\mathcal{M}_\sigma \neq 0$ and $\mathcal{M}_\tau \neq 0$ only if $\tau \in \mathrm{Star}(\sigma)$;
2. for every cone $\tau \in \mathrm{Star}(\sigma)$, $\tau \neq \sigma$, the canonical map $\overline{\mathcal{M}_\tau} \to \overline{\Gamma(\partial\tau; \mathcal{M})}$ is an isomorphism.

Definition 5.1. In what follows we will refer to an object as above as *a minimal sheaf based at σ*. An instance of a minimal sheaf $\mathcal{M}$ based at σ with $\mathcal{M}_\sigma \cong \mathcal{A}_{\Phi,\sigma}$ will be denoted $\mathcal{L}^\sigma_\Phi$. We will also denote $\mathcal{L}_\Phi = \mathcal{L}^o_\Phi$

Proposition 5.2. *Let Φ be any fan.*

1. *For every $\sigma \in \Phi$ and every finitely generated graded $\mathcal{A}_{\Phi,\sigma}$-module M there exists a unique (up to an isomorphism) minimal sheaf $\mathcal{M}$ based at σ such that $\mathcal{M}_\sigma = M$. In particular, the minimal sheaf $\mathcal{L}^\sigma_\Phi$ exists for each $\sigma \in \Phi$.*
2. *Moreover, if $\mathcal{M}$ is a minimal sheaf based at σ then*

$$\mathcal{M} \simeq \mathcal{L}^\sigma_\Phi \otimes_\mathbb{R} \overline{\mathcal{M}_\sigma}.$$

In particular $\mathcal{M}$ is a direct sum of sheaves $\mathcal{L}^\sigma_\Phi(t)$, $t \in \mathbb{Z}$.

3. *The minimal sheaves $\mathcal{L}^{\sigma}_{\Phi}(t)$, $t \in \mathbb{Z}$ are indecomposable objects in the category of $\mathcal{A}_\Phi$-modules.*
4. *Let $U \subset \Phi$ be an open subset, i.e. U is a subfan. Then*

$$\mathcal{L}_\Phi|_U = \mathcal{L}_U.$$

Proof. Easy exercise. □

Theorem 5.3. *Let Φ be a fan in V. Every object $\mathcal{M}$ of $\mathfrak{M}(\mathcal{A}_\Phi)$ is isomorphic to a direct sum of minimal sheaves. In particular, $\mathcal{M}$ is a direct sum of indecomposable objects $\mathcal{L}^{\sigma}_{\Phi}(t)$, $t \in \mathbb{Z}$.*

Proof. The last part of the theorem follows from the first one using parts 2 and 3 of Proposition 5.2.

Consider an object $\mathcal{M}$ of $\mathfrak{M}(\mathcal{A}_\Phi)$. We will show that it is isomorphic to a direct sum of minimal sheaves by induction on

$$|\mathcal{M}| = \sum_\sigma rank_{\mathcal{A}_{\Phi,\sigma}} \mathcal{M}_\sigma.$$

Consider a cone σ such that $\mathcal{M}_\sigma \neq 0$ and, for every $\tau \in \partial\sigma$, $\mathcal{M}_\tau = 0$. We will show that, for each such σ, $\mathcal{M}$ contains as a direct summand a minimal sheaf $\mathcal{K}$ based at σ with $\mathcal{K}_\sigma = \mathcal{M}_\sigma$. That is, we will construct a direct sum decompostion

$$\mathcal{M} = \mathcal{K} \oplus \mathcal{N} \tag{5.1}$$

with $\mathcal{K}$ as above and $\mathcal{N}$ in $\mathfrak{M}(\mathcal{A}_\Phi)$. This is sufficient, since, clearly, $|\mathcal{N}| < |\mathcal{M}|$.

Note, that we need to specify the direct sum decompostion (5.1) only on Star(σ). Therefore it is sufficient to treat the case when σ is the origin $\underline{o}$.

We proceed to construct the decomposition (5.1) by induction on the dimension of the cone and the number of cones of the given dimension.

Let $\mathcal{K}_{\underline{o}} = \mathcal{M}_{\underline{o}}$ and $\mathcal{N}_{\underline{o}} = 0$. Assume that a direct sum decomposition

$$\mathcal{M}|_{\Phi_{\leq k}} = \mathcal{K}_{\leq k} \oplus \mathcal{N}_{\leq k}$$

in $\mathfrak{M}(\mathcal{A}_\Phi|_{\Phi_{\leq k}})$ has been defined and consider a cone σ of dimension $k+1$. Since $\partial\sigma$ consists of cones of dimension at most k the induction hypothesis says that there is a direct sum decompostion of $\Gamma(\partial\sigma; \mathcal{A}_\Phi)$-modules

$$\Gamma(\partial\sigma; \mathcal{M}) = \Gamma(\partial\sigma; \mathcal{K}_{\leq k}) \oplus \Gamma(\partial\sigma; \mathcal{N}_{\leq k}) \ .$$

Claim 5.4. *There is a decomposition*

$$\mathcal{M}_\sigma = K_\sigma \oplus N_\sigma$$

into a direct sum of free $\mathcal{A}_{\Phi,\sigma}$-modules, such that the restriction homomorphism

$$\mathcal{M}_\sigma \twoheadrightarrow \Gamma(\partial\sigma;\mathcal{M})$$

maps K_σ to $\Gamma(\partial\sigma;\mathcal{K}_{\leq k})$ and N_σ to $\Gamma(\partial\sigma;\mathcal{N}_{\leq k})$ and induces an isomorphism

$$\overline{K_\sigma} \xrightarrow{\cong} \overline{\Gamma(\partial\sigma;\mathcal{K}_{\leq k})}$$

and an epimorphism

$$\overline{N_\sigma} \twoheadrightarrow \overline{\Gamma(\partial\sigma;\mathcal{N}_{\leq k})}$$

Assume the claim for the moment. The desired extension $\mathcal{K}_{\leq k+1}$ (respectively $\mathcal{N}_{\leq k+1}$) of $\mathcal{K}_{\leq k}$ (respectively $\mathcal{N}_{\leq k}$) is given, for every cone σ of dimension $k+1$, by $\mathcal{K}_{\leq k+1,\sigma} = K_\sigma$ (respectively $\mathcal{N}_{\leq k+1,\sigma} = N_\sigma$) and has all the required properties.

Proof of claim. Choose a subspace $Z \subset \Gamma(\partial\sigma;\mathcal{K}_{\leq k})$ which maps isomorphically onto $\overline{\Gamma(\partial\sigma;\mathcal{K}_{\leq k})}$ under the residue map. Choose a subspace $S \subset \mathcal{M}_\sigma$ so that the map $\mathcal{M}_\sigma \to \Gamma(\partial\sigma;\mathcal{M})$ restricts to an isomorphisms $S \xrightarrow{\sim} Z$. Since $S \bigcap A^+\mathcal{M}_\sigma = 0$ there is a subspace $T \subset \mathcal{M}_\sigma$ such that $S \bigcap T = 0$ and $S \oplus T$ generates $\mathcal{M}_\sigma$ freely. Subtracting, if necessary, elements of $A \cdot S$ from elements of T we may assume that the image of T under the map $\mathcal{M}_\sigma \to \Gamma(\partial\sigma;\mathcal{M})$ is contained in $\Gamma(\partial\sigma;\mathcal{N}_{\leq k})$. Thus we may take $K_\sigma = \mathcal{A}_{\Phi,\sigma}S$ and $N_\sigma = \mathcal{A}_{\Phi,\sigma}T$. This concludes the proof of the claim and of the theorem. □

Remark 5.5. In the course of the proof we have constructed an isomorphism

$$\mathcal{M} \cong \bigoplus_{\sigma\in\Phi} V_\sigma \otimes \mathcal{L}^\sigma_\Phi$$

where V_σ is a finite dimensional graded vector space such that

$$V_\sigma \cong \ker(\overline{\mathcal{M}}_\sigma \to \overline{\Gamma(\partial\sigma;\mathcal{M})}) \ .$$

In terms of homogeneous components we may write

$$V_\sigma = \bigoplus_{k\in\mathbb{Z}} V_{\sigma,k}(k)$$

for suitable vector spaces $V_{\sigma,k}$ (trivial for almost all k), and

$$\mathcal{M} \cong \bigoplus_{\sigma\in\Phi}\bigoplus_{k\in\mathbb{Z}} V_{\sigma,k} \otimes \mathcal{L}^\sigma_\Phi(k) \ .$$

Note that the multiplicities $\dim V_{\sigma,k}$ in the decompostion of $\mathcal{M}$ into a direct sum of indecomposable objects are uniquely determined.

2. **Subdivision of fans and the decomposition theorem.** Suppose that Φ and Ψ are two fans in V and Ψ is a subdivision of Φ, which is to say, every cone of Φ is a union of cones of Ψ. (In the rational case this induces a proper morphism of toric varieties). This corresponds to a morphism of ringed spaces $\pi : (\Psi, \mathcal{A}_\Psi) \to (\Phi, \mathcal{A}_\Phi)$. The next theorem combined with the structure Theorem 5.3 is a combinatorial analog of the decomposition theorem ([BBD], [BL]).

Theorem 5.6. *In the notations introduced above, for $\mathcal{M}$ in $\mathfrak{M}(\mathcal{A}_\Psi)$,*

1. $\mathbf{R}^i\pi_*\mathcal{M} = 0$ *for* $i \neq 0$ *and* $\pi_*\mathcal{M}$ *is flabby;*
2. $\pi_*\mathcal{M}$ *is locally free.*

In other words, the direct image under subdivision π restricts to an exact functor $\pi_* : \mathfrak{M}(\mathcal{A}_\Psi) \to \mathfrak{M}(\mathcal{A}_\Phi)$.

Proof. The first claim follows from the flabbiness of $\mathcal{M}$.

Since the issue is local on Φ we may assume that the latter is generated by a single cone σ of top dimension n, i.e. $\Phi = [\sigma]$. By induction on dimension it is sufficient to show that the stalk $(\pi_*\mathcal{M})_\sigma$ is a free A-module.

Let $Z = \pi^{-1}(\sigma)$. This is a closed subset of Ψ which consists of the cones which subdivide the interior of σ.

Claim 5.7. *For any sheaf F on Ψ the restriction map*

$$H^0(\Psi; F) \to H^0(Z; F)$$

is an isomorphism.

Proof. Indeed, a global section $\alpha \in \Gamma(\Psi; F)$ is the same as a collection of local sections $\alpha_\tau \in F_\tau = \Gamma([\tau]; F)$, $d(\tau) = n$ such that $\alpha_\tau = \alpha_\xi$ in $F_{\tau\cap\xi}$ in case $d(\tau \cap \xi) = n-1$. The same local data specifies an element in $\Gamma(Z; F)$. This proves the claim. □

By Claim 5.7 pull-back and restriction to Z establish

$$(\pi_*\mathcal{M})_\sigma \stackrel{\sim}{=} H^0(\Phi; \pi_*\mathcal{M}) \stackrel{\sim}{=} H^0(\Psi; \mathcal{M}) \stackrel{\sim}{=} H^0(\Psi; \mathcal{M}_Z)\ .$$

By Proposition 3.6 the cellular complex $C^\bullet(\mathcal{M}_Z)$ is quasi-isomorphic to $\mathbf{R}\Gamma(\Psi; \mathcal{M}_Z)$. Note that the sheaf $\mathcal{M}_Z$ is flabby. Now the same argument as in the proof of Theorem 4.7 shows that the A-module $H^0(\Psi; \mathcal{M}_Z)$ is free. □

3. **Intersection cohomology of fans.**

Definition 5.8. For a complete fan Φ we define *the intersection cohomology of* Φ as the graded vector space

$$IH(\Phi) \stackrel{def}{=} \overline{H^0(\Phi; \mathcal{L}_\Phi)}$$

and denote by $ih(\Phi)$ the corresponding Poincaré polynomial.

Lemma 5.9. *The intersection cohomology of a complete fan Φ enjoys the following properties:*

1. $\dim IH(\Phi) < \infty$*;*
2. $ih_j(\Phi) = 0$ *for* $j < 0$ *or* j *odd;*
3. $ih_0(\Phi) = 1$.

Proof. The A-module $H^0(\Phi; \mathcal{L}_\Phi)$ is finitely generated, hence the first claim follows.

Using induction on the dimension of the cone it follows from the definition of $\mathcal{L}_\Phi$ that, for each $\sigma \in \Phi$ the (graded) A-module $\mathcal{L}_{\Phi,\sigma}$ has no non-trivial components of negative of odd degree. Hence the same is true for $H^0(\Phi; \mathcal{L}_\Phi)$ and the second claim follows.

Using induction on the dimension of the cone and Lemma 3.11 one checks easily that the degree sero component of $\mathcal{L}_{\Phi,\sigma}$ is one-dimesional. In other words, the degree zero component of the sheaf $\mathcal{L}_\Phi$ is the constant sheaf $\mathbb{R}_\Phi$. Thus, by Lemma 3.11, $ih_0(\Phi) = 1$. □

Lemma 5.10. *Suppose tht Φ is a complete fan. Let Ψ be a subdivision of Φ. Then $IH(\Phi)$ is a direct summand of $IH(\Psi)$. In particular, one has the inequality*

$$ih(\Psi) \geq ih(\Phi)$$

(coefficient by coefficient) of polynomials with nonnegative coefficients.

Proof. Let $\pi : (\Psi, \mathcal{A}_\Psi) \to (\Phi, \mathcal{A}_\Phi)$ denote the corresponding morphism of ringed spaces. Since $(\pi_*\mathcal{L}_\Psi)_{\underline{o}} = \mathbb{R}$ it follows that the sheaf $(\pi_*\mathcal{L}_\Psi)$ contains $\mathcal{L}_\Phi$ as a direct summand (see Theorem 5.3). The lemma follows. □

Definition 5.11. For a cone σ in V we define *the local intersection cohomology (space)* by

$$IP(\sigma) = \overline{\mathcal{L}_{[\sigma],\sigma}}$$

and denote by $ip(\sigma)$ the corresponding Poincaré polynomial.

Remark 5.12. Note that $ip_j(\sigma) = 0$ if j is odd or negative.

6. Borel-Moore-Verdier duality

Let Φ be a fan in $V = \mathbb{R}^n$. Let A_Φ as usual be the constant sheaf on Φ with stalk A. Denote by $D^b_c(A_\Phi - mod)$ the bounded derived category of (locally finitely generated) A_Φ-modules. In particular the additive category of sheaves $\mathfrak{M}(\mathcal{A}_\Phi)$ is a full subcategory of $D^b_c(A_\Phi - mod)$.

In this section we define the duality functor, i.e. a contravariant involution $\mathbb{D}$ on the category $D^b_c(A_\Phi - mod)$. We show that duality preserves the subcategory $\mathfrak{M}(\mathcal{A}_\Phi)$ and $\mathbb{D}(\mathcal{L}_\Phi) \simeq \mathcal{L}_\Phi$. Among other things, it gives rise to Poicaré duality in $IH(\Phi)$.

1. **The dualizing object.** Let $j^\Phi_{\underline{o}} : \{\underline{o}\} \hookrightarrow \Phi$ denote the (open) inclusion.

Definition 6.1. The object

$$D_\Phi \overset{def}{=} (j^\Phi_{\underline{o}})_! \omega[n]$$

of $D^b_c(A_\Phi - mod)$ is called *the dualizing object.*

We now introduce a flabby resolution of the dualizing object. For a cone $\sigma \in \Phi$ let $i_\sigma : \{\sigma\} \hookrightarrow \Phi$ denote the inclusion. Consider the dualizing module A-module ω as a sheaf on the point σ. Then, the A_Φ-module $i_{\sigma *}\omega$ is a constant sheaf on $\mathrm{Star}(\sigma)$ with stalk ω. If $\tau \leq \sigma$ then there is a natural surjection of sheaves $r_{\tau\sigma} : i_{\tau *}\omega \to i_{\sigma *}\omega$. Let

$$K^{-n+j} \overset{def}{=} \bigoplus_{d(\sigma)=j} i_{\sigma *}\omega \ .$$

As usual, the maps $r_{\tau\sigma}$ with the sign ± 1 define the differential in the complex $K^\bullet_A$

$$0 \to K^{-n} \to K^{-n+1} \to \cdots \to K^0 \to 0 \ .$$

Lemma 6.2. *The complex $K^\bullet$ is a flabby resolution of D_Φ*

Proof. The sheaves K^i are direct sums of flabby sheaves, therefore flabby. It remains to show that, for $\underline{o} \neq \sigma \in \Phi$ the complex of stalks

$$K^{-n}_\sigma \to K^{-n+1}_\sigma \to \cdots \to K^0_\sigma$$

is acyclic. Now, the above complex is isomorphic to the complex $\mathrm{Hom}_\mathbb{R}(C^\bullet(\mathbb{R}_{[\sigma]}, \omega)$, and $C^\bullet(\mathbb{R}_{[\sigma]})$ is acyclic by Lemma 3.7. □

Remark 6.3. Note that $\mathrm{Hom}_{A_\Phi}(F, i_{\sigma *}\omega) = \mathrm{Hom}_A(F_\sigma, \omega)$ for any A_Φ-module F. Hence, for any A_Φ-module F, there is a natural isomorhism of complexes

$$\mathrm{Hom}_{A_\Phi}(F, K^\bullet_A) \overset{\sim}{=} \mathrm{Hom}_A(C^\bullet(F), \omega) \ , \tag{6.1}$$

Since the functor $C^\bullet(\bullet)$ is exact, it extends naturally to a functor $C^\bullet(\bullet) : D^b_c(A_\Phi - mod) \to D^b_c(A - mod)$.

Proposition 6.4. *There is a canonical isomorphism of functors $D^b_c(A_\Phi - mod)^{op} \to D^b_c(A - mod)$*

$$\mathbf{R}\,\Gamma(\Phi; \mathbb{D}_\Phi(\bullet)) \overset{\sim}{=} \mathbf{R}\,\mathrm{Hom}_A(C^\bullet(\bullet), \omega) \ .$$

Proof. Since the category $A_\Phi - mod$ has enough projectives, it is sufficient to give, for each projective P an isomorphism

$$\mathbf{R}\,\Gamma(\Phi;\mathbb{D}(P)) \stackrel{\sim}{=} \mathbf{R}\,\mathrm{Hom}_A(C^\bullet(P),\omega)$$

natual in P. Such an isomorphism is given by

$$\begin{aligned}\mathbf{R}\,\Gamma(\Phi;\mathbb{D}(P)) &\stackrel{\sim}{=} \mathbf{R}\,\mathrm{Hom}_{A_\Phi}(P,D_\Phi)\\ &\stackrel{\sim}{=} \mathbf{R}\,\mathrm{Hom}_{A_\Phi}(P,K_A^\bullet)\\ &\stackrel{\sim}{=} \mathrm{Hom}_{A_\Phi}(P,K_A^\bullet)\\ &\stackrel{\sim}{=} \mathrm{Hom}_A(C^\bullet(P),\omega)\\ &\stackrel{\sim}{=} \mathbf{R}\,\mathrm{Hom}_A(C^\bullet(P),\omega)\end{aligned}$$

using Remark 6.3 and the fact that $C^\bullet(P)$ is a complex of free A-modules. □

Corollary 6.5. *The canonical natural transformation*

$$\mathbf{R}\,\Gamma(\Phi;\mathbb{D}_\Phi(\bullet)) \to \mathbf{R}\,\mathrm{Hom}_A(\mathbf{R}\,\Gamma(\Phi;\bullet),\mathbf{R}\,\Gamma(\Phi;D_\Phi))$$

is an isomorphism.

2. **Trace maps.**

Lemma 6.6. *Suppose that Φ is complete. Then there is a canonical isomorphism $\mathbf{R}\,\Gamma(\Phi;D_\Phi) \to \omega$.*

Proof. Proposition 6.4 applied to $D_\Phi = \mathbb{D}_\Phi(A_\Phi)$ gives the isomorphism

$$\mathbf{R}\,\Gamma(\Phi;D_\Phi) \stackrel{\sim}{=} \mathbf{R}\,\mathrm{Hom}_A(C^\bullet(A_\Phi),\omega)$$

By Proposition 3.5 and the flabbyness of the sheaf A_Φ the inclusion $A \to C^\bullet(A_\Phi)$ is a quasiisomorphism, hence so is

$$\mathbf{R}\,\mathrm{Hom}_A(C^\bullet(A_\Phi),\omega) \to \mathbf{R}\,\mathrm{Hom}_A(A,\omega) \stackrel{\sim}{=} \omega\ .$$

The composition of the two provides the desired map. □

Definition 6.7. The isomorphism of Corollary 6.6 will be denoted by $\int_\Phi$ and called the *integration map* or *the (absolute) trace map.*

Proposition 6.8. *Suppose that $\pi : \Psi \to \Phi$ is a subsdivision.*

1. *There is a natural isomorphism $\pi^{-1}D_\Phi \stackrel{\sim}{=} D_\Psi$ in $D_c^b(A_\Psi - mod)$.*
2. *The canonical morphism $D_\Phi \to \mathbf{R}\,\pi_*\pi^{-1}D_\Phi \stackrel{\sim}{=} \mathbf{R}\,\pi_* D_\Psi$ is an isomorphism (in $D_c^b(A_\Phi - mod)$).*

Proof. The first claim follows directly from the definitions.

The second claim is proved by inspecting the map induced on stalks. Since the issue is local on Φ we may assume that $\Phi = [\sigma]$. The claim is clearly true at the origin. Since D_Φ is supported at the origin it is sufficient to show that so is $\mathbf{R}\pi_*\pi^{-1}D_\Phi \overset{\sim}{=} \mathbf{R}\pi_*D_\Psi$.

By induction on the dimension we may assume that the statement holds at every cone of $\partial\sigma$.

The stalk of $\mathbf{R}\pi_*\pi^{-1}D_\Phi$ at σ is given by

$$\begin{aligned}\left(\mathbf{R}\pi_*\pi^{-1}D_\Phi\right)_\sigma &\overset{\sim}{=} \mathbf{R}\Gamma(\Phi;\mathbf{R}\pi_*\pi^{-1}D_\Phi)\\ &\overset{\sim}{=} \mathbf{R}\Gamma(\Psi;\pi^{-1}D_\Phi)\\ &\overset{\sim}{=} \mathbf{R}\Gamma(\Psi;D_\Psi)\\ &\overset{\sim}{=} \mathrm{Hom}^\bullet_A(C^\bullet(A_\Psi),\omega)\end{aligned}$$

using Proposition 6.4. The complex $C^\bullet(A_\Psi)$ is acyclic since it is isomorphic to the augmented cellular chain complex (with coefficients in A) of a contractible space. □

Definition 6.9. In what follows we will denote by $\int_\pi : \mathbf{R}\pi_*D_\Psi \to D_\Phi$ the (inverse) isomorphism of Proposition 6.8.

3. **Duality à la Verdier.** In what follows we use $\underline{\mathrm{Hom}}$ to denote the "sheaf Hom". Mote that, for $\sigma\in\Phi$ and any two sheaves F and G

$$\underline{\mathrm{Hom}}(F,G)_\sigma = \mathrm{Hom}(F_{[\sigma]},G)\ .$$

Definition 6.10. The functor

$$\mathbb{D}_\Phi(\bullet) \overset{def}{=} \mathbf{R}\,\underline{\mathrm{Hom}}_{A_\Phi}(\bullet,D_\Phi) : D^b_c(A_\Phi - mod)^{op} \to D^b_c(A_\Phi - mod)$$

is called *the duality functor.*

Remark 6.11. Note the isomorphism $D_\Phi \overset{\sim}{=} \mathbb{D}_\Phi(A_\Phi)$.

Our next goal is to prove Theorem 6.20 below. Let us begin with some preparations.

Definition 6.12. A nonempty open subset $U\subset\Phi$ is *saturated* if whenever the boundary of a cone is in U, then the cone itself is in U.

Example 6.13. Φ and $\{\underline{o}\}$ are saturated.

Definition 6.14. For a nonempty open subset $U\subset\Phi$ define its *opposite* U' as follows:

$$U' := \{\sigma\in\Phi|\forall\tau\in U,\quad \tau\cap\sigma=\underline{o}\}.$$

Remark 6.15. 1. $\Phi'=\{\underline{o}\}$, $\{\underline{o}\}'=\Phi$.
2. For any U its opposite U' is saturated.
3. We have $U\subset U''$.

Lemma 6.16. *If U is saturated then $U=U''$.*

Proof. Let $\underline{o} \neq \sigma \in U''$. Let $\tau \leq \sigma$ be a face of dimension 1. Then $\tau \notin U'$. Hence $\tau \in U$. Thus by induction on dimension all faces of σ are in U and so σ is in U. □

Corollary 6.17. *The map $U \mapsto U'$ is an involution of the collection of saturated open subsets of Φ.*

For an open subset $U \subset \Phi$ and an A-module M we denote as usual by M_U the extension by zero to Φ of the constant sheaf M on U. In case $U = \{\underline{o}\}$ we will also denote this sheaf by $M_{\underline{o}}$.

Remark 6.18. Let $U \subset \Phi$ be open. Note the equality of sheaves

$$\underline{\mathrm{Hom}}(A_U, \omega_{\underline{o}}) = \omega_{U'} \ .$$

Hence, if U is saturated, then by Lemma 6.16 the obvious map

$$\begin{array}{rcl} A_U & \to & \underline{\mathrm{Hom}}(\underline{\mathrm{Hom}}(A_U, \omega_{\underline{o}}), \omega_{\underline{o}}) \\ a & \mapsto & (f \mapsto f(a)) \end{array}$$

is an isomorphism.

Lemma 6.19. *Let $\sigma \in \Phi$ and put $U := [\sigma]'$. Then,*

1. $H^{-n}\,\mathbf{R}\,\underline{\mathrm{Hom}}(A_U, D_\Phi) \tilde{=} \underline{\mathrm{Hom}}(A_U, \omega_{\underline{o}})$ *and*
2. $H^{i}\,\mathbf{R}\,\underline{\mathrm{Hom}}(A_U, D_\Phi) = 0$ *for $i > -n$.*

Equivalently, $\mathbf{R}\,\underline{\mathrm{Hom}}(A_U, D_\Phi) \tilde{=} \underline{\mathrm{Hom}}(A_U, \omega_{\underline{o}})[n]$.

Proof. The first claim is clear. To prove the second claim we need to show that it holds on stalks.

For $\tau \in \Phi$ we put $W = U \cap [\tau]$ and calculate the stalk of $\mathbf{R}\,\underline{\mathrm{Hom}}(A_U, D_\Phi)$ at τ using Proposition 6.4:

$$\begin{array}{rcl} \mathbf{R}\,\underline{\mathrm{Hom}}(A_U, D_\Phi)_\tau & \tilde{=} & \mathbf{R}\,\mathrm{Hom}(A_W, D_\Phi) \\ & \tilde{=} & \mathrm{Hom}(C^\bullet(A_W), \omega) \ . \end{array}$$

If $\tau = \underline{o}$, then $W = \{\underline{o}\}$ and the complex $C^\bullet(A_W)$ is concentrated in degree n, so $\mathrm{Hom}(C^\bullet(A_W), \omega)$ is concentrated in degree $-n$.

If $\tau \neq \underline{o}$, then $C^\bullet(A_W)$ is isomorphic to the augmented chain complex of a contractible space, hence acyclic. □

We are ready to prove the main result of this section.

Theorem 6.20. *The canonical natural transformation* $\mathrm{Id} \to \mathbb{D}_\Phi \circ \mathbb{D}_\Phi$ *given on an object $F^\bullet$ of $D^b_c(A_\Phi - mod)$ by*

$$\begin{array}{rcl} F^\bullet & \to & \mathbf{R}\,\underline{\mathrm{Hom}}_{A_\Phi}(\mathbf{R}\,\underline{\mathrm{Hom}}_{A_\Phi}(F^\bullet, D_\Phi), D_\Phi) \\ a & \mapsto & (f \mapsto f(a)) \end{array}$$

is an isomorphism of functors.

Proof. Let $\sigma \in \Phi$. Then $A_{[\sigma]}$ is an indecomposable projective A_Φ-module and every indecomposable projective is of this form. Since every object of $D^b_c(A_\Phi - mod)$ is isomorphic to a bounded complex of modules which are finite direct sums of indecomposable projectives we may assume that $F^\bullet = A_{[\sigma]}$.

By Remark 6.18 and Lemma 6.19 we have

$$\begin{aligned}
\mathbf{R}\,\underline{\mathrm{Hom}}_{A_\Phi}(\mathbf{R}\,\underline{\mathrm{Hom}}_{A_\Phi}(A_{[\sigma]}, D_\Phi), D_\Phi) &\cong \mathbf{R}\,\underline{\mathrm{Hom}}_{A_\Phi}(\underline{\mathrm{Hom}}_{A_\Phi}(A_{[\sigma]}, \omega_{\underline{o}})[n], \omega_{\underline{o}}[n]) \\
&\cong \mathbf{R}\,\underline{\mathrm{Hom}}_{A_\Phi}(\omega_{[\sigma]'}, \omega_{\underline{o}}) \\
&\cong \underline{\mathrm{Hom}}_{A_\Phi}(\omega_{[\sigma]'}, \omega_{\underline{o}}) \\
&\cong A_{[\sigma]}\ .
\end{aligned}$$

□

4. **Global and relative duality.** Here we show that the duality functor commutes in the appropriate sence with direct image under subdivision and global sections over complete fans.

Theorem 6.21. *Suppose that Φ is complete. Then, the natural transformation of functors $D^b_c(A_\Phi - mod)^{op} \to D^b_c(A - mod)$*

$$\mathbf{R}\,\Gamma(\Phi; \mathbb{D}_\Phi(\bullet)) \to \mathbf{R}\,\mathrm{Hom}_A(\mathbf{R}\,\Gamma(\Phi; \bullet), \omega)$$

given on an object $F^\bullet$ of $D^b_c(A_\Phi)$ by the composition

$$\begin{aligned}
\mathbf{R}\,\Gamma(\Phi; \mathbb{D}_\Phi(F^\bullet)) = \mathbf{R}\,\mathrm{Hom}_{A_\Psi}(F^\bullet, D_\Psi)& \\
\to \mathbf{R}\,\mathrm{Hom}_A(\mathbf{R}\,\Gamma(\Phi; F^\bullet), \mathbf{R}\,\Gamma(\Phi; D_\Psi))& \\
\xrightarrow{\int_\Phi} \mathbf{R}\,\mathrm{Hom}_A(\mathbf{R}\,\Gamma(\Phi; F^\bullet), \omega)&
\end{aligned}$$

is an isomorphism.

Proof. Follows from Corollary 6.5 and Lemma 6.6. □

Theorem 6.22. *Suppose that $\pi : \Psi \to \Phi$ is a subdivision. The canonical natural transformation of functors $D^b(A_\Psi)^{op} \to D^b(A_\Phi)$*

$$\mathbf{R}\,\pi_*\,\mathbf{R}\,\underline{\mathrm{Hom}}_{A_\Psi}(\bullet, D_\Psi) \to \mathbf{R}\,\underline{\mathrm{Hom}}_{A_\Phi}(\mathbf{R}\,\pi_*(\bullet), \mathbf{R}\,\pi_* D_\Psi)$$

is an isomorphism.

Proof. Since the issue is local (on Φ) we may assume that $\Phi = [\sigma]$. By induction on dimension we may assume that the statement holds on every cone of $\partial\sigma$. It remains to show that the statement holds on stalks at σ. We begin with a calculation of respective stalks.

If $F^\bullet$ is in $D^b(A_\Psi)$, then

$$\begin{aligned}\left(\mathbf{R}\,\pi_*\,\mathbf{R}\,\underline{\mathrm{Hom}}_{A_\Psi}(F^\bullet, D_\Psi)\right)_\sigma &\cong \mathbf{R}\,\Gamma([\sigma]; \mathbf{R}\,\pi_*\,\mathbf{R}\,\underline{\mathrm{Hom}}_{A_\Psi}(F^\bullet, D_\Psi))\\ &\cong \mathbf{R}\,\Gamma(\Psi; \mathbf{R}\,\underline{\mathrm{Hom}}_{A_\Psi}(F^\bullet, D_\Psi))\\ &\cong \mathbf{R}\,\mathrm{Hom}_{A_\Psi}(F^\bullet, D_\Psi)\\ &\cong \mathbf{R}\,\mathrm{Hom}_A(C^\bullet(F^\bullet), \omega)\end{aligned}$$

and

$$\begin{aligned}\Big(\mathbf{R}\,\underline{\mathrm{Hom}}_{A_{[\sigma]}}(\mathbf{R}\,\pi_*(F^\bullet), \mathbf{R}\,\pi_* D_\Psi)\Big)_\sigma &\cong \mathbf{R}\,\Gamma([\sigma]; \mathbf{R}\,\underline{\mathrm{Hom}}_{A_{[\sigma]}}(\mathbf{R}\,\pi_* F^\bullet, \mathbf{R}\,\pi_* D_\Psi))\\ &\cong \mathbf{R}\,\mathrm{Hom}_{A_{[\sigma]}}(\mathbf{R}\,\pi_* F^\bullet, \mathbf{R}\,\pi_* D_\Psi)\\ &\cong \mathbf{R}\,\mathrm{Hom}_{A_\Psi}(\pi^{-1}\,\mathbf{R}\,\pi_* F^\bullet, D_\Psi)\\ &\cong \mathbf{R}\,\mathrm{Hom}_A(C^\bullet(\pi^{-1}\,\mathbf{R}\,\pi_* F^\bullet), \omega)\end{aligned}$$

Under these identifications the map in question corresponds to the map

$$\mathbf{R}\,\mathrm{Hom}_A(C^\bullet(F^\bullet), \omega) \to \mathbf{R}\,\mathrm{Hom}_A(C^\bullet(\pi^{-1}\,\mathbf{R}\,\pi_* F^\bullet), \omega)$$

induced by the adjunction map $\pi^{-1}\,\mathbf{R}\,\pi_* F^\bullet \to F^\bullet$. Thus, it is sufficient to show that the map

$$C^\bullet(\pi^{-1}\,\mathbf{R}\,\pi_* F^\bullet) \to C^\bullet(F^\bullet)$$

(induced by the adjunction map) is an isomorphism.

By induction we may assume that the statement holds on $\partial\Psi = \pi^{-1}(\partial\sigma)$, i.e. that the map

$$C^\bullet((\pi^{-1}\,\mathbf{R}\,\pi_* F^\bullet)_{\partial\Psi}) \to C^\bullet((F^\bullet)_{\partial\Psi})$$

is an isomorphism. Therefore it is sufficient to show that the map

$$C^\bullet((\pi^{-1}\,\mathbf{R}\,\pi_* F^\bullet)_Z) \to C^\bullet(F^\bullet_Z)\ ,$$

where $Z = \pi^{-1}(\{\sigma\})$, is an isomorphism as well.

The stalks of the sheaf $\pi^{-1}\,\mathbf{R}\,\pi_* F^\bullet$ are given by

$$\begin{aligned}\left(\pi^{-1}\,\mathbf{R}\,\pi_* F^\bullet\right)_\tau &\cong (\mathbf{R}\,\pi_* F^\bullet)_{\pi(\tau)}\\ &\cong \mathbf{R}\,\Gamma([\pi(\tau)]; \mathbf{R}\,\pi_* F^\bullet)\\ &\cong \mathbf{R}\,\Gamma(\pi^{-1}([\pi(\tau)]); F^\bullet)\ .\end{aligned}$$

In particular, the sheaf $(\pi^{-1}\,\mathbf{R}\,\pi_* F^\bullet)_Z$ is the extension by zero of the constant sheaf with stalk $\mathbf{R}\,\Gamma(\Psi; F^\bullet)$ on Z.

For any constant sheaf W the cellular complex $C^\bullet(W_Z)$ is isomorphic (up to reindexing and shift) to the reduced cellular chain complex with coefficients in W of a sphere (of dimension $\dim\Psi - 1 = \dim\sigma - 1$)

so that the natural map $W[\dim\sigma - n - 1] \to C^\bullet(W_Z)$ is a quasi-isomorphism. Therefore it is sufficient to show that the composition

$$\mathbf{R}\,\Gamma(\Psi; F^\bullet)[n - \dim\sigma + 1] \to C^\bullet((\pi^{-1}\mathbf{R}\,\pi_* F^\bullet)_Z) \to C^\bullet(F^\bullet_Z)$$

is an isomorphism.

It follows from Proposition 3.6 that there is a natural isomorphism $C^\bullet(F^\bullet_Z) \overset{\sim}{=} \mathbf{R}\,\Gamma(\Psi; F^\bullet_Z)[\dim\sigma - n - 1]$. The map

$$\mathbf{R}\,\Gamma(\Psi; F^\bullet) \to \mathbf{R}\,\Gamma(\Psi; F^\bullet_Z)$$

induced after the above identification is the natural map induced by the restriction map $F^\bullet \to F^\bullet_Z$. It is a natural isomorphism because

1. the restriction map induces an isomorphism $\Gamma(\Psi; F^\bullet) \to \Gamma(\Psi; F^\bullet_Z)$, and
2. if $F^\bullet$ is a complex of flabby sheaves then so is $F^\bullet_Z$.

□

Corollary 6.23. *Suppose that $\pi : \Psi \to \Phi$ is a subdivision. The natural transformation of functors $D^b_c(A_\Psi)^{op} \to D^b_c(A_\Phi)$*

$$\mathbf{R}\,\pi_* \circ \mathbb{D}_\Psi \to \mathbb{D}_\Phi \circ \mathbf{R}\,\pi_*$$

given on an object $F^\bullet$ of $D^b(A_\Psi)$ by the composition

$$\begin{aligned} \mathbf{R}\,\pi_* \mathbb{D}_\Psi(F^\bullet)) = \mathbf{R}\,\pi_*\,\mathbf{R}\,\underline{\mathrm{Hom}}_{A_\Psi}(F^\bullet, D_\Psi) \\ \to \mathbf{R}\,\underline{\mathrm{Hom}}_{A_\Phi}(\mathbf{R}\,\pi_* F^\bullet, \mathbf{R}\,\pi_* D_\Psi) \\ \xrightarrow{\int_\pi} \mathbf{R}\,\underline{\mathrm{Hom}}_{A_\Phi}(\mathbf{R}\,\pi_* F^\bullet, D_\Phi) = \mathbb{D}_\Phi(\mathbf{R}\,\pi_* F^\bullet) \end{aligned}$$

is an isomorphism.

5. **Poincare duality.**

Theorem 6.24. *Suppose that $\mathcal{M} \in \mathfrak{M}(\mathcal{A}_\Phi)$. Then,*

1. *$H^i\mathbb{D}_\Phi(\mathcal{M}) = 0$ for $i \neq 0$; we put*
$$\mathbb{D}_\Phi(\mathcal{M}) \overset{def}{=} H^0\mathbb{D}_\Phi(\mathcal{M})\ ;$$
2. *the sheaf $\mathbb{D}_\Phi(\mathcal{M})$ is flabby.*
3. *the A_Φ-module $\mathbb{D}_\Phi(\mathcal{M})$ is, in fact, a $\mathcal{A}_\Phi$-module;*
4. *the $\mathcal{A}_\Phi$-module $\mathbb{D}_\Phi(\mathcal{M})$ is locally free;*

Proof. Since the issue is local we may assume that $\Phi = [\sigma]$. By induction on dimension we may assume that the statement holds on $\partial\sigma$.

Applying $\mathbb{D}_\Phi(\bullet)$ to the exact sequence

$$0 \to \mathcal{M}_{\partial\sigma} \to \mathcal{M} \to \mathcal{M}_\sigma \to 0$$

we obtain the exact triangle

$$\mathbb{D}_\Phi(\mathcal{M}_\sigma) \to \mathbb{D}_\Phi(\mathcal{M}) \to \mathbb{D}_\Phi(\mathcal{M}_{\partial\sigma}) \to \mathbb{D}_\Phi(\mathcal{M}_\sigma)[1]\ . \tag{6.2}$$

The isomorphisms

$$\mathbb{D}_\Phi(\mathcal{M}_\sigma) = \mathbf{R}\,\underline{\mathrm{Hom}}_{A_\Phi}(\mathcal{M}_\sigma, D_\Phi) \overset{\cong}{\longleftarrow}$$
$$\overset{\cong}{\longleftarrow} \mathbf{R}\,\underline{\mathrm{Hom}}_{A_\Phi}(A_\sigma, \mathbf{R}\,\underline{\mathrm{Hom}}_{A_\Phi}(\mathcal{M}, D_\Phi)) = \mathbf{R}\,\Gamma_{\{\sigma\}}\mathbb{D}_\Phi(\mathcal{M})$$

lead to the identification of the exact triangle (6.2) with the canonical exact triangle

$$\mathbf{R}\,\Gamma_{\{\sigma\}}\mathbb{D}_\Phi(\mathcal{M}) \to \mathbb{D}_\Phi(\mathcal{M}) \to \mathbf{R}\,j_*j^{-1}\mathbb{D}_\Phi(\mathcal{M}) \to \mathbf{R}\,\Gamma_{\{\sigma\}}\mathbb{D}_\Phi(\mathcal{M})[1]\ ,$$

where $j : \partial\sigma \hookrightarrow [\sigma]$ is the inclusion (of an open subset).

By inductive assumption $j^{-1}\mathbb{D}_\Phi(\mathcal{M}) \cong \mathbb{D}_{\partial\sigma}(j^{-1}\mathcal{M})$ satisfies the conclusions of the thorem. In particular, it is a flabby sheaf. Therefore $\mathbf{R}^i\,j_*j^{-1}\mathbb{D}_\Phi(\mathcal{M}) = 0$ for $i \neq 0$ and $j_*j^{-1}\mathbb{D}_\Phi(\mathcal{M})$ is flabby. Consequently, $\mathbb{D}_\Phi(\mathcal{M}_{\partial\sigma})$ is (isomorphic to) a flabby sheaf.

Next we examine $\mathbb{D}_\Phi(\mathcal{M}_\sigma)$, which is supported on $\{\sigma\}$. Thus, it is concentrated in degree zero if and only if $\mathbf{R}\,\Gamma(\Phi; \mathbb{D}_\Phi(\mathcal{M}_\sigma))$ is. By Proposition 6.4

$$\begin{aligned} \mathbf{R}\,\Gamma(\Phi; \mathbb{D}_\Phi(\mathcal{M}_\sigma)) &\cong \mathbf{R}\,\mathrm{Hom}_A(C^\bullet(\mathcal{M}_\sigma), \omega) \\ &\cong \mathbf{R}\,\mathrm{Hom}(\mathcal{M}_\sigma[d(\sigma)-n], \omega) \\ &\cong \mathrm{Ext}_A^{n-d(\sigma)}(\mathcal{M}_\sigma, \omega) \end{aligned}$$

(because $\mathcal{M}_\sigma$ is a free module over $\mathcal{A}_{\Phi,\sigma}$). This calculation shows that $\mathbb{D}_\Phi(\mathcal{M}_\sigma)$ is (isomorphic to) the skyscraper sheaf $\mathrm{Ext}_A^{n-d(\sigma)}(\mathcal{M}_\sigma, \omega)_\sigma$, in particular it is flabby and locally free over $\mathcal{A}_\Phi$.

So far we have shown that both $\mathbb{D}_\Phi(\mathcal{M}_\sigma)$ and $\mathbb{D}_\Phi(\mathcal{M}_{\partial\sigma})$ are complexes concentrated in degree zero. Hence so is $\mathbb{D}_\Phi(\mathcal{M})$. This proves the first claim.

It follows that the exact triangle (6.2) is equivalent to the short exact sequence of sheaves

$$0 \to \mathbb{D}_\Phi(\mathcal{M}_\sigma) \to \mathbb{D}_\Phi(\mathcal{M}) \to \mathbb{D}_\Phi(\mathcal{M}_{\partial\sigma}) \to 0\ .$$

(where we have written $\mathbb{D}$ for $H^0\mathbb{D}$). Since $\mathbb{D}_\Phi(\mathcal{M}_\sigma)$ and $\mathbb{D}_\Phi(\mathcal{M}_{\partial\sigma})$ are flabby, so is $\mathbb{D}_\Phi(\mathcal{M})$. This proves the second claim.

By inductive assumption, for $\tau \in \partial\sigma$, the stalk $\mathbb{D}_\Phi(\mathcal{M})_\tau$ is a free $\mathcal{A}_{\Phi,\tau}$-module. It remains to show that the same hold with $\tau = \sigma$. Since $\Phi = [\sigma]$, the stalk at σ is given by

$$\begin{aligned} \mathbb{D}_\Phi(\mathcal{M})_\sigma &\cong \mathbf{R}\,\Gamma(\Phi; \mathbb{D}_\Phi(\mathcal{M})) \\ &\cong \mathbf{R}\,\mathrm{Hom}_A(C^\bullet(\mathcal{M}), \omega) \\ &\cong \mathbf{R}\,\mathrm{Hom}_A(\Gamma_{\{\sigma\}}\mathcal{M}[d(\sigma)-n], \omega) \\ &\cong \mathrm{Ext}_A^{n-d(\sigma)}(\Gamma_{\{\sigma\}}\mathcal{M}, \omega)\ , \end{aligned}$$

(using Proposition 6.4, Lemma 3.9, and Corollary 4.8) and the latter is a free $\mathcal{A}_{\Phi,\sigma}$-module. $\square$

Corollary 6.25. *The duality functor induces an anti-involution of the category $\mathfrak{M}(\mathcal{A}_\Phi)$.*

Proof. Since $\mathbb{D}$ is an anti-involution of $D^b_c(A_\Phi - mod)$ and it preserves $\mathfrak{M}(\mathcal{A}_\Phi)$ the corollary follows. $\square$

Corollary 6.26. *Suppose that σ is a cone in Φ. The minimal sheaf $\mathcal{L}^\sigma_\Phi(k)$ on Φ based at σ (see Definition 5.1) satsisfies*

$$\mathbb{D}_\Phi(\mathcal{L}^\sigma_\Phi(k)) \cong \mathcal{L}^\sigma_\Phi(-k-2d(\sigma))$$

(non-canonically). In particular, $\mathbb{D}_\Phi(\mathcal{L}_\Phi) \cong \mathcal{L}_\Phi$.

Proof. By Corollary 6.25 the dual of an indecomposable object is indecomposable. In addition, $\operatorname{Supp}(\mathbb{D}(\mathcal{L}^\sigma_\Phi(k))) \subseteq \operatorname{Supp}(\mathcal{L}^\sigma_\Phi(k)) = \operatorname{Star}(\sigma)$. It follows from the proof of Theorem 6.24 that $\mathbb{D}_\Phi(\mathcal{L}^\sigma_\Phi(k))_\sigma = \operatorname{Ext}^{n-d(\sigma)}_A(\mathcal{A}_{\Phi,\sigma}(k),\omega) \cong \mathcal{A}_{\Phi,\sigma}(-k-2d(\sigma))$. Thus, the corollary follows from Proposition 5.2 and Theorem 5.3. $\square$

Corollary 6.27. *Let Φ be a complete fan. Then there exists a non-canonical isomorphism of A-modules*

$$\Gamma(\Phi;\mathcal{L}_\Phi) \cong \operatorname{Hom}_A(\Gamma(\Phi;\mathcal{L}_\Phi),\omega) \ ,$$

i.e. the free A-module $\Gamma(\Phi;\mathcal{L}_\Phi)$ is self-dual.

Proof. By Theorem 4.7 and Corollary 6.25 the natural isomorphism of Theorem 6.21

$$\mathbf{R}\,\Gamma(\Phi;\mathbb{D}_\Phi(\mathcal{L}_\Phi)) \cong \mathbf{R}\operatorname{Hom}(\mathbf{R}\,\Gamma(\Phi;\mathcal{L}_\Phi),\omega)$$

reduces to

$$\Gamma(\Phi;\mathbb{D}(\mathcal{L}_\Phi)) \cong \operatorname{Hom}_A(\Gamma(\Phi;\mathcal{L}_\Phi),\omega) \ .$$

A choice of an isomorphism of $\mathbb{D}_\Phi(\mathcal{L}_\Phi)$ and $\mathcal{L}_\Phi$ provides an isomorphism of A-modules

$$\Gamma(\Phi;\mathcal{L}_\Phi) \cong \operatorname{Hom}_A(\Gamma(\Phi;\mathcal{L}_\Phi),\omega) \ .$$

$\square$

Consider the 1-dimensional (graded) vector space $\overline{\omega}$. It has degree $2n$.

Corollary 6.28. *Let Φ be a complete fan. Then, there exists an isomorphism of graded vector spaces*

$$IH(\Phi) \cong \operatorname{Hom}_{\mathbb{R}}(IH(\Phi),\overline{\omega}) \ .$$

Proof. Immediate from the previous corollary. $\square$

Corollary 6.29. *Let Φ be a complete fan. Then,*

1. $ih_{n-j}(\Phi) = ih_{n+j}(\Phi)$ *for all* j;
2. $ih_j(\Phi) = 0$ *unless* j *is even and* $j \in [0, 2n]$;
3. $ih_0(\Phi) = ih_{2n}(\Phi) = 1$.

Proof. Immediate from the previous corollary and Lemma 5.9. □

Corollary 6.30. *Suppose Φ is a fan and $\pi : \Psi \to \Phi$ is a morphism induced by subdivision. Let*

$$\pi_* \mathcal{L}_\Psi \cong \bigoplus_{\sigma \in \Phi} \bigoplus_{k \in \mathbb{Z}} V_{\sigma,k} \otimes \mathcal{L}^\sigma_\Phi(k) \tag{6.3}$$

be a decomposition of $\pi_* \mathcal{L}_\Psi$ into a direct sum of indecomposable objects of $\mathfrak{M}(\mathcal{A}_\Phi)$ (see Theorem 5.3 and Remark 5.5) with $V_{\sigma,k}$ finite dimensional vector spaces. Then,

$$\dim V_{\sigma,k} = \dim V_{\sigma,-k-2d(\sigma)} \ .$$

Proof. Applying $\mathbb{D}_\Phi$ to both sides of (6.3) and using Corollary 6.26 we obtain

$$\mathbb{D}_\Phi(\pi_* \mathcal{L}_\Psi) \cong \bigoplus_{\sigma \in \Phi} \bigoplus_{k \in \mathbb{Z}} V_{\sigma,k} \otimes \mathcal{L}^\sigma_\Phi(-k - 2d(\sigma)) \tag{6.4}$$

By Corollary 6.23 and Corollary 6.26 there is an isomorphism

$$\mathbb{D}_\Phi(\pi_* \mathcal{L}_\Psi) \cong \pi_* \mathcal{L}_\Psi \ .$$

The claim is now established by matching the multiplicities in (6.3) and (6.4). □

6. **Duality à la Borel-Moore.** We conclude with a brief account of duality patterned after Borel-Moore duality for sheaves on locally compact spaces. Informally speaking, the Borel-Moore dual of a sheaf F is defined as the "pointwise" linear dual of the co-sheaf $\Gamma_c(F)$ of "compactly supported" sections of F.

6.1. *Co-sheaves.* Suppose that $\mathcal{C}$ is a category. While a $\mathcal{C}$-valued sheaf on a fan Φ is a functor $\Phi^0 \to \mathcal{C}$, a ($\mathcal{C}$-valued) *co-sheaf* is a functor $\Phi \to \mathcal{C}$.

Assume that $\mathcal{C}$ is Abelian. Then, so is the category of co-sheaves.

Suppose that W is in $\mathcal{C}$ and σ is a cone. Let $W_{\mathrm{Star}(\sigma)}$ denote the co-sheaf, obtained by extending by zero the constant co-sheaf with value W on $\mathrm{Star}(\sigma)$. Every co-sheaf is a quotient of a direct sum of co-sheaves of this form.

If W is projective then so is $W_{\mathrm{Star}(\sigma)}$. If $\mathcal{C}$ has enough projectives, then so does the category of $\mathcal{C}$-valued co-sheaves.

6.2. *Homology of co-sheaves.* Recall that, for a sheaf F on Φ its space of global sections, defined as $\Gamma(\Phi; F) = \varinjlim_{\Phi^{op}} F$, is a left exact functor of F.

Definition 6.31. For a co-sheaf $\mathcal{V} : \Phi \to \mathcal{C}$ we define its space of global co-sections as the direct limit $\varinjlim_{\Phi} F$.

Lemma 6.32. *The functor of global co-sections is right exact.*

Assume from now on that the category $\mathcal{C}$ has enough projectives.

Definition 6.33. For a co-sheaf $\mathcal{V}$ on Φ we define *the i-th homology of (Φ with coefficients in) $\mathcal{V}$* as the i-th left derived functor of global co-sections:

$$H_i(\Phi; \mathcal{V}) \stackrel{def}{=} H^{-i}\, \mathbf{L} \varinjlim_{\Phi} \mathcal{V} \ .$$

6.3. *Co-sheaf of sections with compact support.* Suppose that Φ is a fan in a vector space of dimension n. Fix an orientation of each cone σ in Φ.

For a *sheaf* F on Φ and $\sigma \in \Phi$ let

$$\Gamma_c(F)_\sigma \stackrel{def}{=} C^\bullet(F_{[\sigma]}) \ . \tag{6.5}$$

This is a complex which is concentrated in degrees $[n - d(\sigma), n]$. For $\tau \le \sigma$ we have the obvious inclusion of complexes $\Gamma_c(F)_\tau \hookrightarrow \Gamma_c(F)_\sigma$. This makes the assignment

$$\Gamma_c(F) : \sigma \mapsto \Gamma_c(F)_\sigma \tag{6.6}$$

a complex of co-sheaves.

Definition 6.34. The complex of co-sheaves $\Gamma_c(F)$ defined by (6.6) and (6.5) is called *the co-sheaf of compactly supported sections (of F).*

Clearly, $\Gamma_c(F)$ is functorial in F. The functor $\Gamma_c(\bullet)$ is exact, thus it extends trivially to the derived category of sheaves on Φ.

Lemma 6.35. *The functors (with values in the category of complexes) $\varinjlim_{\Phi} \Gamma_c(\bullet)$ and $C^\bullet(\bullet)$ are isomorphic.*

Proof. Straightforward consequence of the definitions. $\square$

Corollary 6.36. *The natural map $\mathbf{L} \varinjlim_{\Phi} \Gamma_c(\bullet) \to \varinjlim_{\Phi} \Gamma_c(\bullet)$ is a quasi-isomorphism.*

Proof. Follows from Lemma 6.35 and the exactness of $C^\bullet(\bullet)[n]$. $\square$

Corollary 6.37. *Assume that the fan* Φ *is complete. Then, the functors* $\mathbf{L}\varinjlim_{\Phi}\Gamma_c(\bullet)$ *and* $\mathbf{R}\,\Gamma(\Phi;\bullet)$ *are naturally isomorphic.*

Proof. Follows from Proposition 3.5, Lemma 6.35 and Corollary 6.36. □

6.4. *Borel-Moore duality.* Note that, if $\mathcal{V}$ is a co-sheaf on Φ with values in $\mathcal{C}$ and $T : \mathcal{C}^{op} \to \mathcal{D}$ is a functor, then $T \circ \mathcal{V}$ is a $\mathcal{D}$-valued sheaf on Φ which we denote $T(\mathcal{V})$.

Note also that, for a A_Φ-module F, the co-sheaf $\Gamma_c(F)$ takes values in the category of complexes of A-modules.

Definition 6.38. The functor $\mathbb{D}_\Phi^{BM} : D_c^b(A_\Phi - mod)^{op} \to D_c^b(A_\Phi - mod)^{op}$ is defined by

$$\mathbb{D}_\Phi^{BM}(F^\bullet) : \sigma \mapsto \operatorname{Hom}_A(\Gamma_c(F^\bullet)_\sigma, I^\bullet)$$

where $I^\bullet$ is an injective resolution of ω, i.e.

$$\mathbb{D}_\Phi^{BM}(F^\bullet) = \operatorname{Hom}_A(\Gamma_c(F^\bullet), I^\bullet) = \mathbf{R}\operatorname{Hom}_A(\Gamma_c(F^\bullet), \omega)\ .$$

It is clear that the functor $\mathbb{D}_\Phi^{BM}$ is essentially independent of the choice of the injective resolution $I^\bullet$.

Proposition 6.39. *The functors* $\mathbb{D}_\Phi$ *and* $\mathbb{D}_\Phi^{BM}$ *are naturally isomorphic.*

Proof. Since the category $A_\Phi - mod$ has enough projectives it is sufficient to give, for each projective P, an isomorphism

$$\mathbb{D}_\Phi^{BM}(P) \cong \mathbb{D}_\Phi(P)$$

natural in P.

Suppose that P is a projective A_Φ-modules. Then, for every $\sigma \in \Phi$, $\Gamma_c(P)_\sigma$ is a complex of projective A-modules, therefore the map

$$\operatorname{Hom}_A(\Gamma_c(P)_\sigma, \omega) \to \operatorname{Hom}_A(\Gamma_c(P)_\sigma, I^\bullet)$$

is a quasiisomorphism compatible with restriction maps. Thus, $\mathbb{D}_\Phi^{BM}(P) \cong \operatorname{Hom}_A(\Gamma_c(P), \omega)$.

By Remark 6.3 one has the isomorphism(s)

$$\begin{aligned} \operatorname{Hom}_A(\Gamma_c(P)_\sigma, \omega) &= \operatorname{Hom}_A(C^\bullet(P_{[\sigma]}), \omega) \\ &\cong \operatorname{Hom}_{A_\Phi}(P_{[\sigma]}, K_A^\bullet) \\ &= \underline{\operatorname{Hom}}_{A_\Phi}(P, K_A^\bullet)_\sigma \end{aligned}$$

compatible with restriction maps. Thus

$$\mathbb{D}_\Phi^{BM}(P) \cong \underline{\operatorname{Hom}}_{A_\Phi}(P, K_A^\bullet) \cong \mathbb{D}_\Phi(P)$$

(since P is projective). □

Remark 6.40. Perhaps the main motivation for the Borel-Moore version of the duality functor is that global duality for a sheaf F on a complete fan Φ (Theorem 6.21) follows very naturally from Corollary 6.37 and general properties of derived functors of limits:

$$\begin{aligned} \mathbf{R}\,\Gamma(\Phi;\mathbb{D}_\Phi^{BM}(F)) &\cong \mathbf{R}\varinjlim_{\Phi^{op}}\mathbf{R}\,\mathrm{Hom}_A(\Gamma_c(F),\omega) \\ &\cong \mathbf{R}\,\mathrm{Hom}_A(\mathbf{L}\varinjlim_{\Phi}\Gamma_c(F),\omega) \\ &\cong \mathbf{R}\,\mathrm{Hom}_A(\mathbf{R}\,\Gamma(\Phi;F),\omega) \end{aligned}$$

7. Toward Hard Lefschetz and the combinatorial invariance

Throughout this section Φ will denote a complete fan in a vector space V of dimension n.

1. **Ampleness in the context of fans.** Consider the short exact sequence of sheaves

$$0 \to \Omega^1_\Phi \to V^* \to \mathcal{G} \to 0\ ,$$

where V^* denotes the constant sheaf and $\Omega^1_{\Phi,\sigma} = \mathrm{Span}(\sigma)^\perp$. Since constant sheaves have trivial higher cohomology and Ω^1_Φ is supported on $\Phi_{\leq n-1}$, the long exact sequence in cohomology reduces, in low degrees, to the short exact sequence of vector spaces

$$0 \to V^* \to \Gamma(\Phi,\mathcal{G}) \to H^1(\Phi;\Omega^1_\Phi) \to 0\ .$$

The space $\Gamma(\Phi;\mathcal{G})$ consists of continuous, cone-wise linear functions on Φ.

For any object $\mathcal{M}$ of $\mathfrak{M}(\mathcal{A}_\Phi)$, the elements of $\Gamma(\Phi;\mathcal{G})$ act naturally on the free graded A-module $\Gamma(\Phi;\mathcal{M})$ by endomorphisms of degree two. Clearly, the induced action on the graded vector space $\overline{\Gamma(\Phi;\mathcal{M})}$ factors through $H^1(\Phi;\Omega^1_\Phi)$.

Definition 7.1. An element $\bar{l}$ of $H^1(\Phi;\Omega^1_\Phi)$ is called *ample* iff it admits a lifting $l \in \Gamma(\Phi;\mathcal{M})$ which is strictly convex.

2. **Hard Lefschetz for complete fans.** The statement of Conjecture 7.2 (below) is the analog of the Hard Lefschetz Theorem in the present context. Recall that $\mathcal{L}_\Phi$ denotes the indecomposable object of $\mathfrak{M}(\mathcal{A}_\Phi)$ which is based at the origin and satisfies $\mathcal{L}_{\Phi,\underline{o}} = \mathbb{R}$.

For a graded vector space W we will denote by $W^{(i)}$ the subspace of homogeneous elements of degree i.

Conjecture 7.2 (Hard Lefschetz Conjecture). *An ample $\bar{l} \in H^1(\Phi; \Omega^1_\Phi)$ induces a Lefschetz operator on the graded vector space $IH(\Phi)$, i.e. for every i the map $\bar{l}^i : IH(\Phi)^{(n-i)} \to IH(\Phi)^{(n+i)}$ is an isomorphism.*

For a rational fan Φ this conjecture follows immediately from results in [BL], ch.15.

The above conjecture has the following standard corollary.

Corollary 7.3. *Assume the Hard Lefschetz Conjecture. Then for an ample $\bar{l}$ the map $\bar{l} : IH(\Phi)^{(i)} \to IH(\Phi)^{(i+2)}$ is injective for $i \leq n-1$ and surjective for $i \geq n-1$. In particular*

$$ih_0(\Phi) \leq ih_2(\Phi) \leq \ldots \leq ih_{2[n/2]}(\Phi).$$

3. **The global-local formula.** Suppose that σ is a cone of dimension $d(\sigma) = d+1 \geq 2$ in V. Let $W = \mathrm{Span}(\sigma) \subseteq V$. Choose a linear isomorphism

$$W \tilde{=} \mathbb{R}^d \times \mathbb{R} \tag{7.1}$$

so that the ray $(0, \mathbb{R}^+)$ lies in the interior of σ. Let $p : W \to \mathbb{R}^d$ denote the projection.

Let $\overline{\partial\sigma}$ denote the image of $\partial\sigma$ under p. Then, $\overline{\partial\sigma}$ is a complete fan in $\mathbb{R}^d$ and $\partial\sigma$ is the graph of a contiuous piecewise linear function $l : \mathbb{R}^d \to \mathbb{R}$ which strictly convex with respect to the fan $\overline{\partial\sigma}$. In particular $\bar{l} \in H^1(\overline{\partial\sigma}; \Omega^1_{\overline{\partial\sigma}})$ is an ample class.

Note that $A_W = A_{\mathbb{R}^d}[l]$. Let $\mathfrak{m}_1 \subset A_W$ and $\mathfrak{m}_2 \subset A_{\mathbb{R}^d}$ denote the maximal ideals, so that $\mathfrak{m}_1 = \mathfrak{m}_2[l]$.

The $\mathcal{A}_{[\partial\sigma]}$-module structure on the minimal sheaf $\mathcal{L}_{[\partial\sigma]}$ is obtained by extension of scalars:

$$\mathcal{L}_{[\partial\sigma]} = \mathcal{A}_{[\partial\sigma]} \otimes_{\mathcal{A}_{\overline{\partial\sigma}}} \mathcal{L}_{\overline{\partial\sigma}} \; .$$

Thus, the intersection cohomology $IH(\overline{\partial\sigma})$ is an $\mathbb{R}[l]$-module. We have

$$IH(\overline{\partial\sigma})/l \cdot IH(\overline{\partial\sigma}) \simeq \Gamma(\partial\sigma; \mathcal{L}_{[\partial\sigma]})/\mathfrak{m}_1\Gamma(\partial\sigma; \mathcal{L}_{[\partial\sigma]}) \simeq \overline{\mathcal{L}_{[\sigma],\sigma}} = IP(\sigma) \; ,$$

where $IP(\sigma)$ is defined in 5.11.

The Hard Lefschetz Conjecture (for $\overline{\partial\sigma}$) implies that $\bar{l}$ is a Lefschetz operator on $IH(\overline{\partial\sigma})$. Thus $IP(\sigma)$ is isomorphic to the l-primitive part of $IH(\overline{\partial\sigma})$. In particular, the Poincaré polynomial $ih(\overline{\partial\sigma})$ depends only on σ and not on a particular choice of the isomorphism (7.1).

Let us summarize our discussion in the following corollary.

Corollary 7.4. *Let $\sigma \subset V$ be a cone of dimension $d+1 \geq 2$. Choose an isomorphism*

$$\mathrm{Span}(\sigma) \simeq \mathbb{R}^d \times \mathbb{R}$$

as in (7.1), *so that the image* $\overline{\partial\sigma}$ *of* $\partial\sigma$ *under the projection to* $\mathbb{R}^d$ *is a complete fan in* $\mathbb{R}^d$. *Then, the Hard Lefschetz Conjecture implies that*

1. *The Poincare polymonial* $ih(\overline{\partial\sigma})$ *is independent of the choices made.*
2. *The polynomial* $ip(\sigma)$ *(see Definition 5.11) is given by*

$$ip_j(\sigma) = \begin{cases} ih_j(\overline{\partial\sigma}) - ih_{j-2}(\overline{\partial\sigma}) & \text{for } 0 \le j \le d \\ 0 & \text{otherwise} \end{cases}$$

Definition 7.5. For a cone σ in V we define the polynomial $ih(\sigma)$ by

$$ih(\sigma) = \begin{cases} 1 & \text{if } d(\sigma) \le 1 \\ ih(\overline{\partial\sigma}) & \text{if } d(\sigma) \ge 2 \end{cases}$$

Remark 7.6. Corollary 7.4 implies that $ih(\sigma)$ is well defined and

$$ip_j(\sigma) = \begin{cases} ih_j(\sigma) - ih_{j-2}(\sigma) & \text{for } 0 \le j \le d \\ 0 & \text{otherwise} \end{cases}$$

We call the last equation the *global-local formula.* Note also that if cones σ and σ' are linearly isomorphic, then $ip(\sigma) = ip(\sigma')$ and $ih(\sigma) = ih(\sigma')$.

4. **The local-global formula.** In this section we express the Poincaré polynomial $ih(\Phi)$ of a complete fan Φ in terms of the local Poincaré polynomials $ip(\sigma)$ for $\sigma \in \Phi$. The argument is standard and is independent of any conjectures.

Proposition 7.7. *For a complete fan* Φ *in* $\mathbb{R}^n$ *we have the following relation between Poincaré polynomials in the variable* q*:*

$$ih(\Phi)(q) = \sum_{\sigma\in\Phi} (q^2-1)^{n-\dim\sigma} ip(\sigma)(q).$$

Proof. The quasi-isomorphism (Proposition 3.5)

$$\Gamma(\Phi;\mathcal{L}_\Phi) \to C^\bullet(\mathcal{L}_\Phi)$$

induces the quasi-isomorphism

$$\Gamma(\Phi;\mathcal{L}_\Phi) \otimes^{\mathbb{L}}_A \mathbb{R} \to C^\bullet(\mathcal{L}_\Phi) \otimes^{\mathbb{L}}_A \mathbb{R}$$

and the equality of the graded Euler characteristics

$$\chi(\Gamma(\Phi;\mathcal{L}_\Phi) \otimes^{\mathbb{L}}_A \mathbb{R}) = \chi(C^\bullet(\mathcal{L}_\Phi) \otimes^{\mathbb{L}}_A \mathbb{R}) \ .$$

Since $\Gamma(\Phi;\mathcal{L}_\Phi)$ is free over A, the canonical map

$$\Gamma(\Phi;\mathcal{L}_\Phi) \otimes^{\mathbb{L}}_A \mathbb{R} \to \Gamma(\Phi;\mathcal{L}_\Phi) \otimes_A \mathbb{R} = IH(\Phi)$$

is a quasi-isomorphism and

$$\chi(\Gamma(\Phi;\mathcal{L}_\Phi) \otimes^{\mathbb{L}}_A \mathbb{R}) = ih(\Phi) \ .$$

Since $C^\bullet(\mathcal{L}_\Phi))$ is a complex of finitely generated A-modules and A has finite Tor-dimension it follows that

$$\chi(C^\bullet(\mathcal{L}_\Phi) \otimes_A^{\mathbb{L}} \mathbb{R}) = \sum_i (-1)^i \chi(C^i(\mathcal{L}_\Phi) \otimes_A^{\mathbb{L}} \mathbb{R})$$

Since $C^i(\mathcal{L}_\Phi)$ is isomorphic to $\bigoplus_{\dim \sigma = n-i} \mathcal{L}_{\Phi,\sigma}$ the above formulas imply the equality

$$ih(\Phi) = \sum_{\sigma \in \Phi} (-1)^{n-\dim \sigma} \chi(\mathcal{L}_{\Phi,\sigma} \otimes_A^{\mathbb{L}} \mathbb{R}) \ .$$

By definition of $\mathcal{L}_\Phi$, the stalk $\mathcal{L}_{\Phi,\sigma}$ is a free module over $\mathcal{A}_{\Phi,\sigma}$ of graded rank $ip(\sigma)$. The standard caclulation with the Koszul complex shows that $\mathcal{A}_{\Phi,\sigma} \otimes_A^{\mathbb{L}} \mathbb{R}$ is represented by the complex (with trivial differential) $\bigoplus_i \bigwedge^i \sigma^\perp$. It follows that

$$\chi(\mathcal{L}_{\Phi,\sigma} \otimes_A^{\mathbb{L}} \mathbb{R}) = (1 - q^2)^{n-\dim \sigma} ip(\sigma)$$

and

$$ih(\Phi) = \sum_{\sigma \in \Phi} (-1)^{n-\dim \sigma} (1 - q^2)^{n-\dim \sigma} ip(\sigma) = \sum_{\sigma \in \Phi} (q^2 - 1)^{n-\dim \sigma} ip(\sigma)$$

□

5. **Summary.** Assuming the Hard Lefschetz Conjecture for complete fans we have associated two polynomials $ip(\sigma)$ and $ih(\sigma)$ to any cone $\sigma \subset V$. The odd coefficients of these polynomials vanish and the following relations hold (here $d + 1 = d(\sigma)$):

1. $ip(\underline{o}) = ih(\underline{o}) = 1$
2. $ip_j(\sigma) = \begin{cases} ih_j(\sigma) - ih_{j-2}(\sigma) & \text{for } 0 \le j \le d \\ 0 & \text{otherwise} \end{cases}$
3. $ih(\sigma)(q) = \sum_{\tau < \sigma} (q^2 - 1)^{d-d(\tau)} ip(\tau)(q).$

Indeed, the first two relations are contained in Definition 7.5 and the third one follows from Proposition 7.7 applied to the complete fan $\overline{\partial\sigma}$ as in Corollary 7.4.

As an immediate consequence of the above relations we obtain (by induction on the dimension $d(\sigma)$) that the polynomials $ip(\sigma)$ and $ih(\sigma)$ are combinatorial invariants of σ, i.e. they depend only on the face lattice of σ.

Recall that in case $d > 0$ the polynomial $ih(\sigma)$ is defined as $ih(\overline{\partial\sigma})$ for a complete fan $\overline{\partial\sigma}$ of dimension d. Hence it follows from Corollary 6.29 and Corollary 7.3 that

1. $ih_0(\sigma) = 1 = ih_{2d}(\sigma)$
2. $ih_j(\sigma) = 0$, unless j is even and $j \in [0, 2d]$

3. for all j $ih_{d-j}(\sigma) = ih_{d+j}(\sigma)$ for all j
4. $ih_0(\sigma) \leq ih_2(\sigma) \leq \ldots \leq ih_{2[d/2]}(\sigma)$.

6. **The h-vector and Stanley's conjectures.** Let $Q \subset \mathbb{R}^n$ be a convex polytope of dimension d. In [S] Stanley defined two polynomials $g(Q)$ and $h(Q)$. These polynomials are defined simultaneously and recursively for faces of Q, including the empty face $\emptyset$, as follows:

1. $g(\emptyset) = h(\emptyset) = 1$
2. $g_j(Q) = \begin{cases} h_j(Q) - h_{j-1}(Q) & \text{for } 0 \leq j \leq [d/2] \\ 0 & \text{otherwise} \end{cases}$
3. $h(Q)(t) = \sum_{P\subset Q}(t-1)^{d-d(P)-1}g(P)(t)$, where the last summation is over all proper faces P of Q including the empty face $\emptyset$. Here $d(P)$ is the dimension of P and $d(\emptyset) = -1$.

Stanley proved (in a more general context of Eulerian posets) the "Poincaré duality" for $h(Q)$:

$$h_j = h_{d-j}$$

and conjectured that

$$0 \leq h_0 \leq h_1 \leq ... \leq h_{[d/2]} \ .$$

Let us show how this conjecture follows from the Hard Lefschetz Conjecture. Namely, consider the space $\mathbb{R}^n$ (which contains Q) as a hyperplane $(\mathbb{R}^n, 1) \subset \mathbb{R}^{n+1}$. Let $\sigma \subset \mathbb{R}^{n+1}$ be the cone with vertex at the origin $\underline{o}$ which is spanned by Q. Then $d(\sigma) = d + 1$. Nonempty faces of σ are in bijective correspondence with faces of Q (with a shift of dimension by 1), where the origin $\underline{o}$ corresponds to the empty face $\emptyset \subset Q$. Assuming the Hard Lefschetz Conjecture the polynomials $ih(\sigma)$ and $ip(\sigma)$ are defined, and, by induction on dimension, one concludes that

$$ih(\sigma)(q) = h(Q)(q^2), \quad ip(\sigma)(q) = g(Q)(q^2) \ .$$

Thus Stanley's conjecture follows from the corresponding statement about the coefficients of $ih(\sigma)$.

8. Calai conjecture (after T. Braden and R. MacPherson)

The statement of following theorem is the ip-analog of the inequalities conjectured by G. Kalai and proven, in the rational case, by T. Braden and R.D. MacPherson in [BM]. Our proof follows the same pattern as the one in [BM]. However, major simpifications result from abscence of rationality hypotheses and, consequently, any ties to geometry whatsoever.

Suppose that σ is a cone in V and let $[\sigma]$ denote as usual the corresponding "affine" fan which consists of σ and all of it's faces. Let $\tau \leq \sigma$ be a face. By Proposition 5.2 $\mathcal{L}_{[\sigma]}|_{[\tau]} = \mathcal{L}_{[\tau]}$. Recall the graded vector spaces (Definition 5.11)

$$IP(\sigma) = \overline{\mathcal{L}_{[\sigma],\sigma}}, \quad IP(\tau) = \overline{\mathcal{L}_{[\tau],\tau}}$$

and the corresponding Poincaré polynomials $ip(\sigma)$, $ip(\tau)$. Consider the minimal sheaf $\mathcal{L}^{\tau}_{[\sigma]} \in \mathfrak{M}(\mathcal{A}_{[\sigma]})$. Its support is $\mathrm{Star}(\tau)$ and we put

$$IP(\mathrm{Star}(\tau)) := \overline{\mathcal{L}^{\tau}_{[\sigma],\sigma}} \ .$$

Let $ip(\mathrm{Star}(\tau))$ denote the corresponding Poincaré polynomial.

Theorem 8.1. *Suppose that σ is a cone (in V) and τ is a face of σ. Then, there is an inequality, coefficient by coefficient, of polynomials with non-negative coefficients*

$$ip(\sigma) \geq ip(\tau) \cdot ip(Star(\tau)) \ .$$

Proof. Let $\iota : \mathrm{Star}(\tau) \to \Phi$ denote the closed embedding. Then $\iota_* \iota^{-1} \mathcal{L}_{[\sigma]} \in \mathfrak{M}(\mathcal{A}_{[\sigma]})$. Indeed, the sheaf $\iota^{-1}\mathcal{L}_{[\sigma]}$ is flabby, hence so is $\iota_*\iota^{-1}\mathcal{L}_{[\sigma]}$. Moreover, ι_* is the extension by zero, so $\iota_*\iota^{-1}\mathcal{L}_{[\sigma]}$ is locally free.

Thus by the structure Theorem 5.3 there is a direct sum decomposition

$$\iota_*\iota^{-1}\mathcal{L}_{[\sigma]} \simeq \bigoplus_{\rho \geq \tau} \mathcal{L}^{\rho}_{[\sigma]} \otimes V_\rho$$

where the multiplicities V_ρ are certain graded vector spaces. Comparing the stalks at τ we find that

$$\mathcal{L}_{[\sigma],\tau} \simeq \mathcal{L}^{\tau}_{[\sigma],\tau} \otimes V_\tau.$$

Hence $V_\tau = IP(\tau)$.

On the other hand, comparing the stalks at σ we find

$$\mathcal{L}_{[\sigma],\sigma} \simeq \mathcal{L}^{\tau}_{[\sigma],\sigma} \otimes V_\tau \oplus \bigoplus_{\rho > \tau} \mathcal{L}^{\rho}_{[\sigma],\sigma} \otimes V_\rho \ .$$

In particular

$$IP(\sigma) \simeq IP(\mathrm{Star}(\tau)) \otimes IP(\tau) \oplus \bigoplus_{\rho > \tau} \overline{\mathcal{L}^{\rho}_{[\sigma],\sigma}} \otimes V_\rho \ .$$

Numerically this amounts to the inequality

$$ip(\sigma) \geq ip(\tau) ip(\mathrm{Star}(\tau)) \ .$$

□

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