Exact triangles in Seiberg-Witten Floer theory. Part I: the geometric triangle

A.L. Carey, M. Marcolli, B.L. Wang

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1 Introduction

This paper is the first part of a program aimed at a better understanding of how the recently defined Seiberg-Witten-Floer homology [5], [10], [15], [16], [29] behaves under surgery.

Throughout the paper we are considering an oriented, closed homology 3-sphere Y and a knot K smoothly embedded in Y. We consider two other manifolds obtained by Dehn surgery on K: a homology 3-sphere Y_1, obtained by $+1$-surgery on K, and a 3-manifold Y_0 which has the homology of $S^1 \times S^2$, obtained by 0-surgery on K. Our main goal is to establish the existence of an exact triangle relating the Seiberg-Witten-Floer homology groups of these manifolds.

A similar setup for instanton homology in Donaldson theory was considered in [2], where Floer's ideas on the corresponding construction of the exact triangle for instanton homology are presented.

Because of various technical difficulties intrinsic in this program, we need to subdivide the problem into several steps. In this first paper we deal with the "geometric triangle", namely we introduce a suitable "surgery perturbation" μ for the Seiberg-Witten equations on Y that simulates the effect of surgery. Let $\mathcal{M}_{Y,\mu}$ be the moduli space of gauge classes of solutions of the perturbed equations. Our main result in this paper is to prove a decomposition

$$\mathcal{M}_{Y,\mu} \cong \mathcal{M}_{Y_1} \cup \bigcup_{\mathfrak{s}_k} \mathcal{M}_{Y_0}(\mathfrak{s}_k), \tag{1}$$

with $\mathcal{M}_{Y_1}$ the moduli space on Y_1 and $\mathcal{M}_{Y_0}(\mathfrak{s}_k)$ the moduli space on Y_0 with respect to the Spin^c-structure $\mathfrak{s}_k$, and to show that the decomposition (1) is compatible with the choice of relative gradings.

Let $\mathfrak{s}_0$ denote the trivial Spin^c structure on Y_0. In the decomposition (1) the moduli space $\mathcal{M}_{Y_0}(\mathfrak{s}_0)$ has to be considered as the moduli space of solutions of the monopole equations perturbed with a cohomologically non-trivial co-closed 1-form ρ, such that $[*\rho] \in H^2(Y_0, \mathbb{R})$ satisfies

$$[*\rho] = \frac{\epsilon}{2} PD_{Y_0}(m),$$

for some small $\epsilon > 0$, depending on the construction of the surgery perturbation μ, and with m the generator of $H_1(Y_0, \mathbb{Z})$. In all other cases, the perturbation considered is cohomologically trivial.

In order to establish the result (1), we prove a gluing theorem for the moduli spaces of critical points of the Chern-Simons-Dirac functional under cutting and gluing the 3-manifold along a torus. Using this gluing result, together with the construction of the perturbation μ that "simulates the effect of surgery", we will be able to derive a corresponding deformation of the moduli spaces, and the expected relation between the generators of the Floer groups defined by (1).

For the picture at the level of generators to be complete, we then compare the relative gradings of the Floer complex on Y, Y_1, and Y_0. This analysis is based essentially on the results of Capell, Lee, and Miller [4] on the spectral flow on manifolds with long cylinders.

We now introduce some notation.

Let $\nu(K)$ be a tubular neighbourhood of K in Y,

$$\nu(K) \cong S^1 \times D^2.$$

On the boundary T^2, we fix a basis m, l of $H_1(T^2, \mathbb{Z})$ such that l is the longitude (parallel to K) and m is the right-handed meridian (intersecting l once). The orientation determined by $m \wedge l$ coincides with the orientation induced from Y. The corresponding longitude and meridian in the knot complement

$$V = Y \backslash \nu(K)$$

are denoted by m', l' respectively. Note that both V and $\nu(K)$ are 3-manifolds with boundary a torus T^2. The homology $H_1(V, \mathbb{Z})$ is generated by the meridian m', and $H_1(\nu(K), \mathbb{Z})$ is generated by the longitude l. In V there is a Seifert surface σ_K bounded by the longitude l'. Similarly, let m'' and l'' be the meridian and longitude in the tubular neighbourhood of the knot $\nu(K)$. The meridian m'' bounds a disk D^2 in $\nu(K)$, and l'' generates $H_1(\nu(K), \mathbb{Z})$.

Dehn surgery on K consists of gluing $\nu(K)$ back to V using an orientation reversing diffeomorphism of T^2. In particular, $(+1)$-surgery on K gives rise to a different homology 3-sphere Y_1, obtained via an orientation reversing diffeomorphism f_1 of T^2 that satisfies

$$(f_1)_*(m'' = [pt \times \partial D^2]) = m' + l' \qquad (f_1)_*(l'') = l'.$$

On the other hand 0-surgery on K gives rise to a three-manifold Y_0 with the homology of $S^1 \times S^2$, obtained via an orientation reversing diffeomorphism f_0 of T^2 that satisfies

$$(f_0)_*(m'') = l' \qquad (f_0)_*(l'') = m'.$$

It is easy to see that $H_1(Y_0, \mathbb{Z}) \cong \mathbb{Z}$ is generated by $m' = l''$ and $H_2(Y_0, \mathbb{Z}) \cong \mathbb{Z}$ is generated by the class of

$$[\sigma_K \cup_{l'=m''} (pt \times D^2)] = \sigma.$$

Since Y and Y_1 are homology 3-spheres, they admit a unique Spin^c structure (the trivial Spin^c structure). We can choose a metric g on Y such that the induced metric on $\nu(K)$ has sufficiently large scalar curvature in the interior (this will be made precise later) and is the standard flat metric on a collar neighbourhood of T^2.

We assume the reader is familiar with the construction of the moduli spaces $\mathcal{M}_Y$, $\mathcal{M}_{Y_1}$ of gauge equivalence classes of solutions to the (suitably perturbed) Seiberg-Witten equations on Y and Y_1 respectively, see [14].

We can now describe more carefully our main result in this paper. We show that there exists a "surgery perturbation" μ for the monopole equations on Y, such that we have an embedding

$$j: \qquad (\mathcal{M}_{Y_1}, \deg_{Y_1}) \hookrightarrow (\mathcal{M}_{Y,\mu}, \deg_Y)$$

with compatible gradings, determined by a choice of the grading $\deg_Y$ for the points of $\mathcal{M}_{Y,\mu}$.

The proof of the decomposition (1) is based on the use of this particular "surgery perturbation" μ and on a gluing result. The gluing theorem represents solutions on Y as a fibred product of reducibles on $\nu(K)$ with solutions on V, fibered over a covering $\chi_0(T^2, Y)$ of the character variety of T^2,

$$\mathcal{M}^*_{Y,\mu} = \mathcal{M}^*_V \backslash \partial_\infty^{-1}(U_\theta) \times_{\chi_0(T^2,Y)} \mathcal{M}^{red}_{\nu(K),\mu}. \qquad (2)$$

Here $*$ denotes the irreducible part of the moduli space, U_θ is a small neighbourhood of the unique singular point θ in $\chi(T^2)$, and ∂_∞ is the asymptotic value map. We shall prove, via a detailed analysis of finite energy monopoles on a three-manifold with an infinite cylindrical end $T^2 \times [0, \infty)$, that this map is well defined and continuous.

It is important to notice that, in general, we do not have a cobordism between the moduli space $\mathcal{M}_Y$, that is, gauge classes of solutions of the Seiberg–Witten equations on Y without the surgery perturbation, and the moduli space $\mathcal{M}_{Y,\mu}$, with the surgery perturbation μ. This is due to the presence of a reducible solution on the integral homology sphere Y. In particular, as we are going to prove in this work, the asymptotic values in $\chi_0(T^2, Y)$ of solutions in $\mathcal{M}_V$ form a one-dimensional manifold with boundary. Along the deformation produced by μ some of the points in $\chi_0(T^2, Y)$ over which the gluing (2) is performed may cross this boundary, hence the surgery formula (1) does not necessarily hold for the moduli space $\mathcal{M}_Y$. This is a phenomenon related to the wall crossing for the Seiberg-Witten invariant of homology 3-spheres, which is by now well understood [12], [16], [25].

The problem of surgery formulae for solutions of the monopole equations has been considered in [6], in the context of proving a conjecture of Kronheimer and Mrowka on the relation between the Seiberg-Witten invariant of homology 3-spheres and the Casson invariant, see also [10], [11], [24]. For this part of the work, our approach is similar to [6], notably in the use of the positive scalar curvature metric on the tubular neighbourhhod of the knot, and in the essential role played by the results of Capell-Lee-Miller [3], [4] on the estimates on the spectrum of first order differential operators on manifolds with a long cylinder and on the splitting of the spectral flow when stretching the cylinder. In particular a proof of (2) can also be found in [6]. Our proof of (2) is based on Taubes' obstruction bundle technique, which shows very clearly how the slow decay of the low modes of the linearization at the approximate solutions, together with the exponential decay of the finite energy solutions on V with non-degenerate asymptotic value, gives the expected gluing theorem. It also shows how there is a non-vanishing obstruction to the gluing when considering solutions on V that decay along the cylinder $T^2 \times [r_0, \infty)$, at a rate $1/s$, for $s \in [r_0, \infty)$, to the degenerate asymptotic value θ in the character variety of T^2.

We are ultimately interested in studying the whole Floer complex and not just its Euler characteristic. Thus, we include in this paper a discussion of a suitable class of perturbations of the Chern-Simons-Dirac functional, similar to the one suggested in [10], which will be useful in studying the moduli spaces of flow lines. Our formulation of the gluing theorem (2) is adapted to the needs of Floer theory: the moduli spaces in (2) are considered with respect to these perturbed equations, and with an additional "surgery perturbation" on the tubular neighbourhood of the knot, which makes the decomposition (1) possible.

The remaining steps of our program may be summarized as follows. The elements of the moduli spaces in (1), which arise from critical points of a perturbed Chern-Simons-Dirac functional, can be graded by the relative index given by the spectral flow of the Hessian operator, fixing as choice of absolute grading the relative index with respect to the unique reducible solution, in the case of the homology spheres. For a dense set of metrics g and g_1 on Y and Y_1, respectively, the Seiberg-Witten-Floer homology groups $HF_*^{SW}(Y, g)$ and $HF_*^{SW}(Y_1, g_1)$ may be defined using this choice of grading [5], [16]. The manifold Y_0 has the homology of $S^1 \times S^2$. Thus, Y_0 has a $\mathbb{Z}$-family of Spin^c structures $\{\mathfrak{s}_k\}_{k\in\mathbb{Z}}$ determined by the first Chern classes:

$$\langle c_1(det\ \mathfrak{s}_k), \sigma\rangle = 2k.$$

Given the choice of perturbations on Y and Y_1, and a further perturbation on Y_0, we observe that there are only finitely many Spin^c structures

$$\{\mathfrak{s}_{i_1}, \cdots, \mathfrak{s}_{i_n}\}$$

with non-trivial Seiberg-Witten monopoles on Y_0, see Remark 4.5.

The construction of Seiberg-Witten-Floer homology groups $HF_*^{SW}(Y_0, \mathfrak{s}_k)$ for each non-trivial Spin^c structure $\mathfrak{s}_k$ is presented in [5]. These Seiberg-Witten-Floer homology groups have a $\mathbb{Z}_{2k}$-grading:

$$HF_0^{SW}(Y_0, \mathfrak{s}_k), \cdots, HF_m^{SW}(Y_0, \mathfrak{s}_k), \cdots, HF_{2k-1}^{SW}(Y_0, \mathfrak{s}_k).$$

However, a satisfactory definition of Seiberg-Witten-Floer homology groups for 3-manifolds with $b_1 > 0$ has not been achieved in the case when the Spin^c-structure is trivial. Since in the exact triangle we need to consider the contribution of the trivial Spin^c structure $\mathfrak{s}_0$ on Y_0, we will provide a suitable definition of Seiberg-Witten-Floer homology groups $HF_*^{SW}(Y_0, \mathfrak{s}_0)$, obtained by introducing a Novikov-type homology $HF_*^{SW}(Y_0, \mathfrak{s}_0; \mathbb{Z}[[t]])$. The details of this construction will be discussed in a subsequent part of this work [19].

The Floer complex $(C_*(Y_0,\mathfrak{s}_k),\partial_{Y_0})$ that computes $HF_*^{SW}(Y_0,\mathfrak{s}_k)$ is generated by the elements in the moduli space $\mathcal{M}_{Y_0}(\mathfrak{s}_k)$ of Seiberg-Witten monopoles on $(Y_0,\mathfrak{s}_k)$. In this paper we extablish a one-to-one correspondence between the critical point sets

$$\mathcal{M}_Y\backslash j(\mathcal{M}_{Y_1}) \xrightarrow{l} \bigcup_{k=1}^{n} \mathcal{M}_{Y_0}(\mathfrak{s}_k).$$

The grading induced by the indices i_Y determines a $\mathbb{Z}$-lifting of the $\mathbb{Z}_{2k}$-valued indices on $\mathcal{M}_{Y_0}(\mathfrak{s}_k)$. Let $\mathcal{M}_{Y_0}^{(m)}(\mathfrak{s}_k)$ be the set of critical points in $\mathcal{M}_{Y_0}(\mathfrak{s}_k)$ with index $m\in\mathbb{Z}$. Then the restriction of ∂_{Y_0} to the $\mathbb{Z}$-graded complex

$$C_{(m)}(Y_0,\mathfrak{s}_k)=\oplus_{\alpha\in\mathcal{M}_{Y_0}^{(m)}(\mathfrak{s}_k)}\mathbb{Z}.<\alpha>$$

defines a homology group, denoted by

$$HF_{(m)}^{SW}(Y_0,\mathfrak{s}_k), \qquad (m\in\mathbb{Z}),$$

as will be proved in [19].

Our final goal is to prove the existence of an exact triangle relating the Seiberg-Witten-Floer homology groups $HF_*^{SW}(Y,g)$, $HF_*^{SW}(Y_1,g_1)$, and $HF_{(*)}^{SW}(Y_0,\mathfrak{s}_k)$ $(1\leq k\leq n)$.

$$\begin{tikzcd}
HF_*^{SW}(Y_1,g_1) \arrow[rr, "I_*"] & & HF_*^{SW}(Y,g) \arrow[dl, "L_*"] \\
& \bigoplus_{k=1}^{n} HF_{(*)}^{SW}(Y_0,\mathfrak{s}_k) \arrow[ul, "\Delta_{(*)}"] &
\end{tikzcd} \tag{3}$$

where $\mathfrak{s}_k$ runs over those Spin^c structures with non-empty Seiberg-Witten moduli space for $(Y_0,\mathfrak{s}_k)$.

The remaining crucial steps, in order to establish our main result on the exact triangle, include the analysis of the behaviour of flow lines in the splitting and regluing of the three-manifold Y under Dehn surgery [17]. This is where some of the most serious technical problems arise, having to deal with the four-dimensional Seiberg-Witten equations on non-compact manifolds with ends of the form $T^2\times[0,\infty)\times\mathbb{R}$. In particular, in this paper we prove that there are slow decaying eigenvalues of the linearization of the 3-dimensional monopole equations on the manifold $Y(r)$. These eigenvalues decay like $\lambda_r\sim 1/r$, as $r\to\infty$. The presence of these slow decaying eigenvalues is a subtle technical issue when dealing with the gradient flow equations, that is with the four dimensional monopole equations on $Y(r)\times\mathbb{R}$. In fact, as we will discuss in a subsequent part of this work, the slow decaying eigenvalues are responsible for a phenomenon of non-uniform convergence of the finite energy monopoles on $Y(r)\times\mathbb{R}$, which gives rise to some interesting geometric limits when $r\to\infty$.

Another step will be the construction of the chain maps I_* and L_* form $C_*(Y_1)$ to $C_*(Y,\mu)$ and from $C_*(Y,\mu)$ to the various $C_*(Y_0,\mathfrak{s}_k)$, respectively, and the proof of exactness [18]. In fact, the results of this first paper prove that the inclusion

$$j: \qquad (\mathcal{M}_{Y_1},\deg_{Y_1})\hookrightarrow(\mathcal{M}_{Y,\mu},\deg_Y)$$

induces an exact sequence of *abelian groups*

$$0 \to C_*(Y_1) \stackrel{j_*}{\to} C_*(Y,\mu) \stackrel{\pi_*}{\to} \oplus_k C_{(*)}(Y_0,\mathfrak{s}_k) \to 0.$$

However, the non-uniform convergence of the flow lines implies, as we shall explain in a forthcoming paper, that this maps are not *chain homomorphisms*. Thus, the following step in the project consists of defining chain homomorphisms via the surgery cobordisms W_1 and W_0 between the manifolds Y_1 and Y, and Y and Y_0, respectively, and compare them with the maps j_* and π_* in order to obtain an exact sequence [18].

Finally several topological consequences can be derived from the exact triangles [20].

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2 Perturbation of the Chern-Simons-Dirac functional

In order to achieve transversality for the moduli space of trajectories between critical points, we introduced in [16] a class of perturbations of the 4-dimensional Seiberg-Witten equations (cf. [9]). As suggested by Kronheimer [10], it is possible to achieve the same result by introducing a non-local perturbation of the Chern-Simons-Dirac functional. We introduce here a class of perturbations similar in spirit to [10], but for which it is easier to derive the pointwise estimates needed in the proof of the compactness of the moduli space of critical points and flow lines. It is important to have such pointwise estimates, since we are interested in the behaviour of solutions under the process of stretching the cylinder in the three-manifold and when performing a Dehn surgery. The perturbation parameter is given by a certain class of functions in the space $\bigcup_{N\geq b_1, K>0} C^\infty(\mathbb{R}^N,\mathbb{R}) \times C^\infty(\mathbb{R}^K,\mathbb{R})$, where b_1 is the first Betti number of the 3-manifold Y.

For any 3-manifold Y with a Riemannian metric g and a fixed Spin^c structure $\mathfrak{s}$, we choose a complete L^2-basis $\{\nu_j\}_{j=1}^\infty$ for the imaginary-valued 1-forms on Y. We also need to choose a complete basis $\{\mu_j\}_{j=1}^\infty$ for the co-closed L^2 (imaginary-valued) 1-forms on Y. Under the Hodge decomposition,

$$\Lambda^1_{L^2}(Y,i\mathbb{R}) = H^1(Y,i\mathbb{R}) \oplus Im(d^*) \oplus Im(d),$$

the $\{\mu_j\}_{j=1}^\infty$ span the space $H^1(Y,i\mathbb{R}) \oplus Im(d^*)$.

Fix a $U(1)$-connection A_0 on the determinant bundle $det(\mathfrak{s})$. To each co-closed 1-form μ_j we associate a function on the configuration space $\mathcal{A}$, defined as

$$\tau_j(A,\psi) = \int_Y (A - A_0) \wedge *\mu_j.$$

For simplicity, we assume that the $\{\mu_j\}_{j=1}^{b_1}$ form a basis of $H^1(Y, i\mathbb{R})$, and $[*\mu_j] = 0$ for $j > b_1$. It is easy to see that the following properties are satisfied:

(1) τ_j is invariant under gauge transformations for $j > b_1$;

(2) the map

$$(\tau_1, \cdots, \tau_{b_1}) : \mathcal{A} \to \mathbb{R}^{b_1},$$

is equivariant with respect to the action of gauge transformations $\lambda : Y \to U(1)$ on $\mathcal{A}$ and the action of the corresponding $[\lambda] \in H^1(Y,\mathbb{Z}) \cong \mathbb{Z}^{b_1}$ on $\mathbb{R}^{b_1}$ as a translation by

$$([g^{-1}dg] \cup [*\mu_1], [g^{-1}dg] \cup [*\mu_2], \cdots, [g^{-1}dg] \cup [*\mu_{b_1}]) \cap [Y].$$

To each imaginary-valued 1-form ν_j, we associate a function ζ_j on $\mathcal{A}$, defined as the quadratic form in the spinor ψ,

$$\zeta_j(\psi,\psi) = \int_Y \langle \nu_j.\psi, \psi\rangle dvol_Y,$$

where $\langle , \rangle$ is the Hermition metric on the space of spinors. It is easy to see that ζ_j is gauge invariant and real-valued. We also define

$$\zeta_j(\psi,\phi) = 2\int_Y Re\langle \nu_j.\psi, \phi\rangle dvol_Y.$$

Now we choose any function $U \in C^\infty(\mathbb{R}^N, \mathbb{R})$ and $V \in C^\infty(\mathbb{R}^K, \mathbb{R})$ (for $N \geq b_1$, $K > 0$) where U is invariant under the action of $H^1(Y,\mathbb{Z})$ on $\mathbb{R}^{b_1} \subset \mathbb{R}^N$. We can define a function on $\mathcal{A}/\mathcal{G}$ as

$$U(\tau_1, \cdots, \tau_N) + V(\zeta_1, \cdots, \zeta_K).$$

Let CSD be the Chern-Simons-Dirac functional

$$CSD(A,\psi) = -\frac{1}{2}\int_Y (A - A_0) \wedge (F_A + F_{A_0}) + \int_Y \langle \psi, \partial\!\!\!/_A \psi\rangle dvol_Y. \tag{4}$$

We perturb this functional by adding a term $U(\tau_1, \cdots, \tau_N) + V(\zeta_1, \cdots, \zeta_K)$. We first prove the compactness and transversality property for the critical points, then we also prove transversality for the moduli space of trajectories between any two critical points.

The perturbation parameter space $\mathcal{P}$ is the set of functions (U, V) in

$$\bigcup_{N \geq b_1, K > 0} C^\infty(\mathbb{R}^N, \mathbb{R}) \times C^\infty(\mathbb{R}^K, \mathbb{R})$$

with U invariant under the actions of $H^1(Y,Z)$ on $\mathbb{R}^{b_1}$. For each $(U,V)\in\mathcal{P}$, the gradient of

$$CSD(A,\psi)+U(\tau_1(A,\psi),\cdots,\tau_N(A,\psi))+V(\zeta_1(A,\psi),\cdots,\zeta_K(A,\psi)) \tag{5}$$

with respect to the natural L^2 metric on $\mathcal{A}/\mathcal{G}$ is given by the following lemma.

Lemma 2.1. *For any pair (U,V) in $\mathcal{P}\cap C^\infty(\mathbb{R}^N,\mathbb{R})\times C^\infty(\mathbb{R}^K,\mathbb{R})$, the gradient of $CSD+U+V$ at $[A,\psi]$ is given by*

$$\Big(*F_A-\sigma(\psi,\psi)-\sum_{j=1}^N\frac{\partial U}{\partial\tau_j}\mu_j,\not\partial_A(\psi)+\sum_{j=1}^K\frac{\partial V}{\partial\zeta_j}\nu_j.\psi\Big),$$

with respect to the natural L^2-metric on $\mathcal{A}/\mathcal{G}$.

Therefore, in this paper we shall always consider, as perturbed Seiberg-Witten equations on a 3-manifold, the equations for the critical points of the perturbed Chern-Simons-Dirac functional (5). The equations for the critical points of $CSD+U+V$ are of the form

$$\begin{cases} *F_A=\sigma(\psi,\psi)+\sum_{j=1}^N\frac{\partial U}{\partial\tau_j}\mu_j \\ \not\partial_A(\psi)+\sum_{j=1}^K\frac{\partial V}{\partial\zeta_j}\nu_j.\psi=0. \end{cases} \tag{6}$$

Let $P=(U,V)$ be the perturbation as above. The moduli space of gauge classes of solutions of the equations (6) on the 3-manifold Y is denoted by $\mathcal{M}_{Y,P}$. Its irreducible part (solutions with non-trivial spinor) is denoted by $\mathcal{M}^*_{Y,P}$. Whenever the manifold Y or the perturbation P is understood, we shall use the notation $\mathcal{M}_P$, $\mathcal{M}^*_P$, or $\mathcal{M}_Y$, $\mathcal{M}^*_Y$. We also introduce the universal moduli space

$$\mathcal{M}_{\mathcal{P}}=\bigcup_{P=(U,V)}\mathcal{M}_{Y,P},\qquad \mathcal{M}^*_{\mathcal{P}}=\bigcup_{P=(U,V)}\mathcal{M}^*_{Y,P}.$$

At any solution (A,ψ,U,V), the linearization of the perturbed equations (6) and of the gauge action defines a corresponding extended Hessian operator $H_{(A,\psi,U,V)}$ mapping

$$\Lambda^0_{L^2_2}(Y,i\mathbb{R})\oplus\Lambda^1_{L^2_1}(Y,i\mathbb{R})\oplus L^2_1(W)\oplus\mathcal{P}\to\Lambda^1_{L^2}(Y,i\mathbb{R})\oplus L^2(W)\oplus\Lambda^0_{L^2}(Y,i\mathbb{R}),$$

which sends $(f,a,\phi,\tilde U,\tilde V)$ to

$$\begin{gathered} H_{(A,\psi,U,V)}(f,a,\phi,\tilde U,\tilde V)= \\ \Big(*da-\sigma(\psi,\phi)-df-\textstyle\sum_{j=1}^N\frac{\partial\tilde U}{\partial\tau_j}\mu_j-\sum_{i,j}\frac{\partial^2U}{\partial\tau_j\partial\tau_i}\tau_i(a)\mu_j, \\ \not\partial_A\phi+\frac{1}{2}a.\psi+f\psi+\sum_{j=1}^K\frac{\partial\tilde V}{\partial\zeta_j}\nu_j.\psi+\sum_{j=1}^K\frac{\partial V}{\partial\zeta_j}\nu_j.\phi+\sum_{i,j}\frac{\partial^2V}{\partial\zeta_i\partial\zeta_j}\zeta_i(\psi,\phi)\nu_j.\psi, \\ -d^*a-iIm\langle\psi,\phi\rangle\Big). \end{gathered} \tag{7}$$

Lemma 2.2. *$H_{(A,\psi,U,V)}$ is surjective. Moreover, there exists a Baire set $\mathcal{P}_0 \subset \mathcal{P}$, such that, for any $P = (U,V) \in \mathcal{P}_0$, the moduli space $\mathcal{M}_P^*$ is a smooth manifold cut out transversely by the equations.*

Proof. Let (a_1, ϕ_1, f_1) be L^2-orthogonal to the image of $H_{(A,\psi,U,V)}$ under the L^2-inner product. By varying $\tilde{V}$ alone, with only $\frac{\partial \tilde{V}}{\partial \tau_j} \neq 0$ for some j, we obtain

$$\int_Y Re\langle \nu_j.\psi, \phi_1\rangle dvol_Y = -\int_Y \langle \nu_j, \sigma(\psi,\phi_1)\rangle dvol_Y = 0.$$

Since ν_j is a complete basis of the imaginary 1-form on Y, we then get

$$\sigma(\psi,\phi_1) = 0.$$

This imples

$$\phi_1 = i\xi\psi,$$

where ξ is a real-valued function on Y.

Now let us vary $\tilde{U}$ alone, with only $\frac{\partial \tilde{U}}{\partial \tau_j} \neq 0$ for some j, and use the fact that μ_j is a complete basis for the co-closed imaginary-valued 1-forms on Y. We see that a_1 must be an exact 1-form, so that we can write a_1 as df_0.

We then vary a alone, and note that a_1 is exact and $\sigma(\psi,\phi_1) = 0$. Thus, we obtain

$$df_1 = -\sigma(\psi,\phi_1) = 0$$

hence f_1 is constant. By varying ϕ alone, we get

$$\int_Y \big(Re\langle \partial\!\!\!/_A\phi + \sum_j \frac{\partial V}{\partial \zeta_j}\nu_j.\phi + \frac{\partial^2 V}{\partial \zeta_j \zeta_i}\zeta_i(\psi,\phi)\nu_j.\psi, \phi_1\rangle\big) dvol_Y$$
$$-\int_Y \langle\big(\sigma(\psi,\phi), a_1\rangle + \langle iIm(\langle\psi,\phi\rangle), f_1\rangle\big) dvol_Y = 0.$$

We apply then the equation $\sigma(\psi,\phi_1) = 0$. This gives

$$\int_Y Re\langle \frac{\partial^2 V}{\partial \zeta_j \zeta_i}\zeta_i(\psi,\phi)\nu_j.\psi, \phi_1\rangle dvol_Y = 0,$$

hence we obtain an equation for ϕ_1

$$\partial\!\!\!/_A\phi_1 + \frac{\partial V}{\partial \zeta_j}\nu_j.\phi_1 + \frac{1}{2}a_1.\psi + f\psi = 0. \tag{8}$$

We apply d^* to $\sigma(\psi,\phi) = 0$ and use the Dirac equation for ψ and the above equation (8). We get

$$\begin{aligned} 0 &= d^*\sigma(\psi,\phi_1) \\ &= -\tfrac{i}{2}Im\langle \partial\!\!\!/_A\psi, \phi_1\rangle + \tfrac{i}{2}Im\langle\psi, \partial\!\!\!/_A\phi_1\rangle \\ &= -\tfrac{i}{2}Im\langle \partial\!\!\!/_A\psi + \textstyle\sum_j \frac{\partial V}{\partial \zeta_j}\nu_j.\psi, \phi_1\rangle + \tfrac{i}{2}Im\langle\psi, \partial\!\!\!/_A\phi_1 + \textstyle\sum_j \frac{\partial V}{\partial \zeta_j}\nu_j.\phi_1\rangle \\ &= \tfrac{i}{2}Im\langle\psi, -\tfrac{1}{2}a_1.\psi + f_1.\psi\rangle \\ &= \tfrac{1}{4}f|\psi|^2. \end{aligned}$$

This implies $f_1 = 0$. Using the identity $\phi_1 = i\xi\psi$ in the equation (8), and using the Dirac equation for ψ, we obtain

$$id\xi.\psi + \frac{1}{2}a_1.\psi = 0.$$

so $a_1 = -2id\xi$.

Finally, we vary f alone. We get $d^*a_1 + iIm\langle\psi, \phi_1\rangle = 0$, which implies

$$2d^*d\xi + \xi|\psi|^2 = 0,$$

and so $\xi = 0$ by the maximal principle. Thus, we prove the surjectivity of $H_{(A,\psi,U,V)}$. This implies the universal moduli space $\mathcal{M}^*_{\mathcal{P}}$ is a smooth Banach manifold. The second claim follows from the infinite dimensional Sard-Smale theorem applied to the projection map

$$\mathcal{M}_{\mathcal{P}} \to \mathcal{P}.$$

□

The perturbation we introduced does not destroy the compactness of the moduli space of solutions of the unperturbed Seiberg-Witten equations: the proof is standard once we have a priori pointwise estimates for the solutions.

Lemma 2.3. *Suppose given a solution (A, ψ) of the perturbed Seiberg-Witten equations (6) for a choice of $P = (U, V) \in \mathcal{P}$. Then we have an estimate*

$$|\psi|^2 \leq max_{y \in Y}\{0, -s(y) + C(P)\}$$

where $s(y)$ is the scalar curvature on the compact manifold (Y, g), and $C(P) > 0$ is a constant depending only on the perturbation $P = (U, V)$.

Proof. Apply the Weitzenböck formula for the Dirac operator at the maxima of $|\psi|^2$. We obtain:

$$\begin{aligned}
0 &\geq \Delta\psi|^2 = 2|\nabla_A\psi|^2 - 2\langle\nabla_A^*\nabla_A\psi, \psi\rangle \\
&\geq -2\langle\nabla_A^*\nabla_A\psi, \psi\rangle \\
&= \frac{s}{2}|\psi|^2 - \langle *F_A.\psi, \psi\rangle + \langle\frac{\partial U}{\partial\zeta_i}\nu_i.\psi, \frac{\partial U}{\partial\zeta_j}\nu_i.\psi\rangle - \langle\frac{\partial U}{\partial\zeta_i}d\nu_i.\psi, \psi\rangle \\
&\geq \frac{s}{2}|\psi|^2 + \frac{1}{2}|\psi|^4 - C|\psi|^2.
\end{aligned}$$

for some positive constant $C > 0$ which depends only on the perturbation $P = (U, V)$. This completes the proof of Lemma. □

Corollary 2.4. *For $b_1 > 0$, for generic $(U,V) \in \mathcal{P}$, there are only finitely many irreducible critical points of $CSD + U + V$ on $\mathcal{A}/\mathcal{G}$ for each* Spin^c *structure and they are all non-degenerate. In the case $b_1 = 0$, Y is a rational homology sphere and for each* Spin^c *structure the moduli space of critical points consists of finitely many non-degenerate irreducible points and a unique reducible point given by the gauge class of solutions of the equation*

$$*F_A = \sum_j \frac{\partial U}{\partial \tau_j}\mu_j.$$

The downward gradient flow equation of $CSD + U + V$ is given by

$$\begin{cases} \dfrac{\partial A}{\partial t} &= -*F_A + \sigma(\psi,\psi) + \sum_{j=1}^N \dfrac{\partial U}{\partial \tau_j}\mu_j \\ \dfrac{\partial \psi}{\partial t} &= -\partial\!\!\!/_A \psi - \sum_{j=1}^K \dfrac{\partial V}{\partial \zeta_j}\nu_j.\psi \end{cases} \tag{9}$$

for a family $(A(t),\psi(t))$ of connections and spinors. The equations (9) are the 4-dimensional Seiberg-Witten equations on $Y \times \mathbb{R}$, for a pair $(A(t),\psi(t))$ in a temporal gauge.

Proposition 2.5. *For a generic perturbation of CSD by an element $P = (U,V) \in \mathcal{P}_0$, the moduli space of trajectories is smooth at any irreducible solution of (9).*

Proof. Assume that $(A(t),\psi(t),U,V)$ is an irreducible solution to (9), then $(A(t),\psi(t),U,V)$ is a 4-dimensional Seiberg-Witten monopole on $Y\times\mathbb{R}$ in a temporal gauge. Let $\pi : Y\times\mathbb{R} \to Y$ be the projection. Then, as bundles over $Y\times\mathbb{R}$, we have the identifications

$$\Lambda^1(Y\times\mathbb{R}, i\mathbb{R}) \cong \pi^*\Lambda^1(Y,i\mathbb{R}) \oplus \pi^*(\Lambda^0(Y,i\mathbb{R})$$

$$\Lambda^{2,+}(Y\times\mathbb{R}, i\mathbb{R}) \cong \pi^*\Lambda^1(Y,i\mathbb{R}).$$

Also, using the Clifford multiplication by dt, we can identify the spinor bundles

$$W^+ \cong W^- \cong \pi^* W.$$

The linearization of (9) at (A,ψ,U,V) gives the following map:

$$\mathcal{D}_{(A,\psi,U,V)} : L^2_k(\mathbb{R}, \Lambda^1(Y,i\mathbb{R}) \oplus \Lambda^0(Y,i\mathbb{R}) \oplus \Gamma(W)) \oplus \mathcal{P} \to L^2_{k-1}(\mathbb{R}, \Lambda^1(Y,i\mathbb{R}) \oplus \Gamma(W)).$$

where for any $(b,f,\phi,\tilde U,\tilde V)$,

$$\begin{aligned} &\mathcal{D}_{(A,\psi,U,V)}(b,f,\phi,\tilde U,\tilde V) \\ = \;&\Big(-df + (\frac{\partial}{\partial t} + *d)b - \sigma(\phi,\psi) + \sum_j \frac{\partial \tilde U}{\partial \tau_j}\mu_j - \sum_{i,j}\frac{\partial^2 U}{\partial\tau_j\partial\tau_i}\tau_i(b)\mu_j, \\ &(\frac{\partial}{\partial t} + \partial\!\!\!/_A)\phi + b.\psi + f.\psi + \sum_j \frac{\partial V}{\partial \zeta_j}\nu_j.\phi + \sum_j \frac{\partial \tilde V}{\partial \zeta_j}\nu_j.\psi + \sum_{i,j}\frac{\partial^2 V}{\partial\zeta_i\partial\zeta_j}\zeta_i(\psi,\phi)\nu_j.\psi\Big). \end{aligned} \tag{10}$$

Suppose that $(\rho(t), s(t)) \in L^2_{k-1}(\mathbb{R}, \Lambda^1(Y, i\mathbb{R}) \oplus \Gamma(W))$ is L^2-orthogonal to the range of $\mathcal{D}_{(A,\psi,U,V)}$. Under the Hodge decomposition, write $\rho(t)$ as $h(t) + \rho_1(t) + d\rho_2(t)$ where $h(t) + \rho_1(t)$ is co-closed and $\rho_2(t)$ is an imaginary valued function. Now we vary $\tilde{U}$ alone in $\mathcal{P}$, with only $\dfrac{\partial \tilde{U}}{\partial \tau_j} \neq 0$ for some j. We have,

$$\int_R \frac{\partial \tilde{U}}{\partial \tau_j} \langle \mu_j, h(t) + \rho_1(t) \rangle_Y = 0,$$

for any $\tilde{U} \in \mathcal{P}$. This leads to

$$\langle \mu_j, h(t) + \rho_1(t) \rangle_Y = 0$$

for any $t \in \mathbb{R}$, and any j. Notice that the $\{\mu_j\}_{j=1}^{\infty}$ form a complete basis for the co-closed 1-forms, hence we know that $h(t) + \rho_1(t) = 0$ identically in $t \in \mathbb{R}$.

Next we vary $\tilde{V}$ alone in $\mathcal{P}$, with only $\dfrac{\partial \tilde{V}}{\partial \zeta_j} \neq 0$ for some j. We have,

$$\int_{\mathbb{R}} Re \langle \frac{\partial \tilde{V}}{\partial \zeta_j} \nu_j . \psi, s(t) \rangle_Y dt = 0. \tag{11}$$

We can see that $Re\langle \nu_j . \psi, s(t) \rangle_Y = -\langle \nu_j, \sigma(\psi, s(t)) \rangle_Y = 0$ for any t and any j. Notice that the $\{\nu_j\}_{j=1}^{\infty}$ form a complete basis for the 1-forms. we obtain that

$$\sigma(\psi, s(t)) \equiv 0.$$

This implies $s(t) = i\xi(t).\psi$, where $\xi(t)$ is a real valued function on $Y \times \mathbb{R}$.

We vary $b = db'(t)$ alone. Notice that $\rho(t)$ is exact. We obtain

$$\int_{\mathbb{R}} \big(\langle \partial_t db'(t), d\rho_2(t) \rangle_Y + Re\langle db'.\psi, s(t) \rangle_Y \big) dt = 0.$$

Since $\sigma(\psi, s(t)) = 0$, and $\partial_t b'(t)$ can vary arbitrarily, we obtain $d^* d\rho_2(t) = 0$ for any t, that is, $\rho(t) = d\rho_2(t) = 0$.

Then, we can vary ϕ alone and use $\sigma(\psi, s(t)) = 0$. We have the following identity:

$$(-\partial_t + \partial\!\!\!/_A) s(t) + \frac{\partial V}{\partial \zeta_j} \nu_j . s(t) = 0.$$

We apply $s(t) = i\xi(t)\psi$ and the Dirac equation for ψ. We get

$$-\partial_t \xi(t) \psi + d\xi(t).\psi = 0,$$

hence $\xi(t) = \xi$ is a constant. Finally, by varying f alone, we get

$$\xi |\psi|^2 = 0.$$

We see then that $\xi = 0$, hence $\mathcal{D}_{(A,\psi,U,V)}$ is surjective. The second claim follows from the infinite dimensional Sard-Smale theorem. $\square$

Using the pointwise estimate derived in Lemma 2.3, the results of [16] on the convergence of flow lines on compact sets can be adapted to the perturbed equations. Thus, the argument of [16] on the compactification of the moduli spaces of flow lines between critical points continues to hold for the perturbed equations (9). The properties of the moduli spaces of flow lines, and their behaviour under surgery will be analysed in a subsequent part of this work [17].

3 Monopoles on a 3-manifold with a cylindrical end

In this section we use techniques developed in [22], in order to construct the moduli space of Seiberg-Witten monopoles on the knot complement V with an infinite cylindrical end $T^2 \times [0, \infty)$.

Consider the three-manifolds V and $\nu(K)$, respectively the knot complement and the tubular neighbourhood of the knot K in the homology sphere Y. Both are 3-manifolds with boundary a torus T^2. In this subsection we will study the Seiberg-Witten monopoles on such a manifold.

Let Z be a 3-manifold with a boundary T^2. Equip Z with a Riemannian metric such that Z has a collar near the boundary $T^2 \times (-1, 0]$, endowed with the product metric. On T^2 we use the standard flat metric induced from $\mathbb{R}^2$. Let $\mathfrak{s}$ be a Spin^c structure on Z with trivial determinant on the boundary. On the collar $\mathfrak{s}$ is the pull-back of the canonical Spin^c structure induced by the complex structure on T^2:

$$\bar\partial + \bar\partial^* : \qquad \Lambda^{0,0} \oplus \Lambda^{1,1} \to \Lambda^{0,1} \oplus \Lambda^{1,0}.$$

It is useful to glue the half-cylinder $T^2 \times [0, \infty)$ to Z in order to obtain a complete 3-manifold, which, as no confusion will arise, we still denote by Z. Thus Z is a 3-manifold with a cylindrical end $T^2 \times [0, \infty)$ with the product metric and the trivial Spin^c-structure along the half cylinder.

In this section we analyse the asymptotic behaviour of the solutions of the Seiberg-Witten equations. As will be clear in the following, we are interested in analysing both the case of the unperturbed and perturbed equations, with a perturbation in the class $\mathcal{P}$ previously introduced. Since we are now dealing with a non-compact 3-manifold, we need to add some condition on the perturbations in $\mathcal{P}$ that prescribes the behaviour at infinity.

The unperturbed Seiberg-Witten equations on $(Z, \mathfrak{s})$ are the equations

$$\begin{cases} \partial\!\!\!/_A \psi = 0, \\ *F_A = \sigma(\psi, \psi) \end{cases} \tag{12}$$

for a pair (A, ψ) consisting of a $U(1)$ connection and a spinor section. The perturbed equations are of the form (6), where the perturbation $P = (U, V)$ is defined as in the previous section. On the non-compact manifold Z, we assume that the elements $\{\nu_j\}_{j=1}^{\infty}$ and $\{\mu_j\}_{j=1}^{\infty}$ are a complete basis of δ-decaying forms, for a suitable weight δ that will be specified in the following. We denote the corresponding class of perturbations by $\mathcal{P}_\delta$.

We need to impose a "finite energy" condition for the Seiberg-Witten monopoles. Specifically, we assume that the curvature is L^2,

$$\int_Z |F_A|^2 dvol_Z < \infty. \tag{13}$$

As we are going to discuss later, this finite energy condition can be restated as the finite variation condition for a functional of connections and spinors on T^2 with the canonical Spin^c-structure.

Let $\mathcal{M}_Z$ denote the Seiberg-Witten moduli space of solutions of the equations (12) with finite energy condition (13), modulo the gauge transformations $\mathcal{G}_Z = Maps(Z, U(1))$.

We now summarise the properties of Seiberg-Witten monopoles on Z that we will establish in this section. Suppose we are given a smooth solution (A, ψ) of the Seiberg-Witten equations, satisfying the finite energy condition (13). Then we will see that there is a choice of a connection $\tilde{A}$ in the gauge class of A that approaches a flat connection on T^2, while the spinor ψ vanishes in the limit $s \to \infty$ on the cylindrical end. That is, we will have $\lim_{s\to\infty}(\tilde{A}, \psi) = (a_\infty, 0)$ in the appropriate topology, for each finite energy solution (A, ψ) to the Seiberg-Witten equations (12). The same result holds for solutions of the perturbed equations (6).

The flat connections on T^2, modulo the gauge group $\mathcal{G}_{T^2} = Map(T^2, U(1))$, form a torus $\chi(T^2) \cong \mathbb{R}^2/\mathbb{Z}^2$. Let $\chi_0(T^2, Z)$ be the moduli space of flat connections on T^2 modulo the subgroup of the gauge transformations on T^2 which can be extended to the whole of Z. These are restrictions of the elements of $\mathcal{G}_Z$ to the boundary T^2. Let π denote the quotient map $\pi : \chi_0(T^2, Z) \to \chi(T^2)$. The asymptotic limit of the Seiberg-Witten monopoles on the manifold Z with a cylindrical end defines a boundary value map

$$\mathcal{M}_Z \overset{\partial_\infty}{\to} \chi_0(T^2, Z) \overset{\pi}{\to} \chi(T^2). \tag{14}$$

We will show that, in a suitable topology, this boundary value map is well-defined and continuous.

The fiber of the map π is described in the following Lemma.

Lemma 3.1. *We have $\mathcal{G}_{T^2}/\mathcal{G}_Z \cong H^1(T^2, \mathbb{Z})/(Im(H^1(Z, \mathbb{Z}) \overset{i^*}{\to} H^1(T^2, \mathbb{Z}))$, where i is the inclusion map T^2 to $T^2 \times \{0\} \subset Z$. Thus, if $Z = V$ is the knot complement of a knot in an integral homology 3-sphere, then $\chi_0(T^2, Z)$ is a $\mathbb{Z}$-covering of the torus $\chi(T^2)$.*

Proof. The restriction map $\mathcal{G}_Z \to \mathcal{G}_{T^2}$ maps onto those connected components of $\mathcal{G}_{T^2}$ which are represented by

$$Im(i^* : H^1(Z, \mathbb{Z}) \to H^1(T^2, \mathbb{Z})).$$

Therefore, we have $\mathcal{G}_{T^2}/\mathcal{G}_Z \cong H^1(T^2,\mathbb{Z})/(Im(H^1(Z,\mathbb{Z}) \to^{i^*} H^1(T^2,\mathbb{Z}))$. Notice that we have the following commutative diagram

$$\begin{tikzcd}
H^1(Z,\mathbb{Z}) \arrow[r, "i^*"] \arrow[d, "\cong"] & H^1(T^2,\mathbb{Z}) \arrow[d, "\cong"] \\
H_2(Z,T^2;\mathbb{Z}) \arrow[r, "\partial"] & H_1(T^2,\mathbb{Z})
\end{tikzcd}$$

where the vertical isomorphisms are given by Poincare duality. If $Z = V$ is a knot complement in a homology 3-sphere, then $H_2(Z,T^2;\mathbb{Z}) \cong \mathbb{Z}$ is generated by the Seifert surface of the knot, hence we have $Im(\partial) \cong \mathbb{Z}$, which proves the remaining claim in the Lemma. $\square$

In the next subsection, we will investigate the behaviour of the solutions of the Seiberg-Witten equations (12) on the cylindrical end $T^2 \times [0,\infty)$ and the continuity of the boundary map ∂_∞ defined in (14). Then, we will describe the structure of the moduli space $\mathcal{M}_Z$.

3.1 Monopoles on $T^2 \times [0,\infty)$

Lemma 3.2. *Choose the coordinate $s \in [0,\infty)$ on the cylindrical end $T^2 \times [0,\infty)$. Choose the* Spin^c *structure over $T^2 \times [0,\infty)$ to be the pull-back of the canonical* Spin^c *structure on T^2 induced by the complex structure. We can write (A,ψ) as*

$$\begin{cases} A = a(s) + h(s)ds, \\ \psi = (\alpha(s),\beta(s)) \in \Lambda^{0,0} \oplus \Lambda^{0,1}. \end{cases}$$

where $a(s) = a^{1,0}(s) + a^{0,1}(s) \in \Lambda^1(T^2, i\mathbb{R})$, $h(s) \in \Lambda^0(T^2, i\mathbb{R})$, then the Seiberg-Witten equations can be written in the form

$$\begin{cases} F_a = \dfrac{i}{2}(|\alpha|^2 - |\beta|^2)\omega, \\ \dfrac{\partial a^{0,1}(s)}{\partial s} = i\bar{\alpha}\beta + \bar{\partial}h, \\ \begin{pmatrix} i(\partial_s + h) & \bar{\partial}^*_{a(s)} \\ \bar{\partial}_{a(s)} & -i(\partial_s + h) \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = 0. \end{cases} \tag{15}$$

where ω is the area 2-form on T^2 with $\int_{T^2} \omega = 1$.

Proof. We may choose a trivialisation of the cotangent bundle to $T^2 \times [0,\infty)$ so that, using a fullstop to denote Clifford multiplication by a one form, we can make the identifications:

$$ds. = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \quad dx. = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad dy. = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}, \tag{16}$$

Letting the Hodge $*$ on forms on $T^2 \times [0,\infty)$ be denoted by $*_3$, then under the preceding identifications we have

$$*_3(\sigma(\psi,\psi)) = \frac{i}{2}(|\alpha|^2 - |\beta|^2)\omega - i(\alpha\bar{\beta} + \bar{\alpha}\beta) \wedge ds$$

and

$$F_A = F_a + (d_{T^2}h - \partial_s a) \wedge ds,$$

hence we get

$$\begin{cases} F_a = \dfrac{i}{2}(|\alpha|^2 - |\beta|^2)\omega, \\ \dfrac{\partial a^{0,1}(s)}{\partial s} = i\bar{\alpha}\beta + \bar{\partial}h. \end{cases}$$

The form $a(s) \in \Lambda^1(T^2, i\mathbb{R})$, is uniquely determined, as an $i\mathbb{R}$-valued 1-form, by its $(0,1)$-part $a^{0,1} \in \Lambda^{0,1}(T^2)$. Similarly, the Dirac operator on $T^2 \times [0,\infty)$ can be expressed as

$$\partial\!\!\!/_{a(s)+h(s)ds} = \begin{pmatrix} i(\partial_s + h) & \bar{\partial}^*_{a(s)} \\ \bar{\partial}_{a(s)} & -i(\partial_s + h) \end{pmatrix}$$

which gives rise to the Dirac equation as in the Lemma. $\square$

Note that $F_a = \frac{i}{2}(|\alpha|^2 - |\beta|^2)\omega$ is a constant constraint with respect to s. Thus, we can regard the Seiberg-Witten equations (15) as equations on the submanifold

$$\left\{(a,\alpha,\beta) | F_a = \frac{i}{2}(|\alpha|^2 - |\beta|^2)\omega\right\}.$$

If (A,ψ) is an irreducible solution of the Seiberg-Witten equations on the manifold Z, then they can be written in standard form along the cylindrical end $T^2 \times [0,\infty)$ as the equations

$$\begin{cases} \partial_s a^{0,1} = i\bar{\alpha}\beta + \bar{\partial}h, \\ \partial_s \alpha = i\bar{\partial}^*_{a(s)}\beta - h\alpha, \\ \partial_s \beta = -i\bar{\partial}_{a(s)}\alpha - h\beta. \end{cases}$$

These are gauge-equivalent, through a gauge transformation in $\mathcal{G}_Z$, to the following gradient-flow equations, in a temporal gauge along the infinite cylinder $T^2 \times [0,\infty)$:

$$\begin{cases} \partial_s a^{0,1} = i\bar{\alpha}\beta, \\ \partial_s \alpha = i\bar{\partial}^*_{a(s)}\beta, \\ \partial_s \beta = -i\bar{\partial}_{a(s)}\alpha. \end{cases} \tag{17}$$

On the configuration space of triples (a,α,β) we consider the action of the gauge group of all gauge transformations on T^2 that extend to gauge transformations on Z.

Lemma 3.3. *The equations (17) are the downward gradient flow equations of the $\mathcal{G}_{T^2} = Map(T^2, U(1))$-invariant functional*

$$f(a,\alpha,\beta) = -\int_{T^2} \langle \alpha, i\bar{\partial}_a^* \beta \rangle \omega \tag{18}$$

on the space $\Lambda^1(T^2, i\mathbb{R}) \oplus \Lambda^{0,0}(T^2) \oplus \Lambda^{0,1}(T^2, i\mathbb{R})$, where the product $\langle , \rangle$ denotes the natural inner product using the Hodge star operator.

Proof. Direct calculation shows that we have

$$\nabla\ f(a,\alpha,\beta) = (-i\bar{\alpha}\beta, -i\bar{\partial}_{a(s)}^* \beta, i\bar{\partial}_{a(s)}\alpha).$$

□

Critical points of the functional (18) are all the elements $[a_\infty, 0, 0]$, with a_∞ a flat connection on T^2. (square brackets denote the effect of quotienting by $\mathcal{G}_{T^2}$.) The functional (18) was first considered in [11] and [29].

Now consider the perturbed equations (6). On the cylindrical end $T^2 \times [0,\infty)$, we can write the forms μ_j as

$$*\mu_j = p_j + q_j \wedge ds.$$

Similarly, we write the imaginary valued 1-forms ν_j as $\nu_j = \nu_j^1 dx + \nu_j^2 dy + \nu_j^0 ds$. The Clifford multiplication by the 1-forms ν_j is determined by (16). Thus, we obtain the following non-autonomous system of perturbed equations

$$\begin{cases} \partial_s a^{0,1} = i\bar{\alpha}\beta + \sum_{j=1}^N \frac{\partial U}{\partial \tau_j} q_j^{0,1}, \\ \partial_s \alpha = i\bar{\partial}_{a(s)}^* \beta + \sum_{j=1}^N \frac{\partial V}{\partial \zeta_j}((-\nu_j^1 + i\nu_j^2)\beta + i\nu_j^0\alpha), \\ \partial_s \beta = -i\bar{\partial}_{a(s)}\alpha + \sum_{j=1}^N \frac{\partial V}{\partial \zeta_j}((\nu_j^1 + i\nu_j^2)\alpha - i\nu_j^0\beta), \end{cases} \tag{19}$$

on the perturbed submanifold

$$\mathcal{C}_{T^2}(U,V) = \Big\{(s,(a,\alpha,\beta)) | F_a = \frac{i}{2}(|\alpha|^2 - |\beta|^2)\omega + \sum_{j=1}^N \frac{\partial U}{\partial \tau_j} p_j(s)\Big\}.$$

For simplicity of notation, let us write $x(s) = (a(s), \alpha(s), \beta(s))$. In the following we shall write the perturbation term in (19) as

$$P(x(s),s) = \begin{cases} \sum_{j=1}^N \frac{\partial U}{\partial \tau_j} q_j^{0,1}(s), \\ \sum_{j=1}^N \frac{\partial V}{\partial \zeta_j}((-\nu_j^1(s) + i\nu_j^2(s))\beta(s) + i\nu_j^0(s)\alpha(s)), \\ \sum_{j=1}^N \frac{\partial V}{\partial \zeta_j}((\nu_j^1(s) + i\nu_j^2(s))\alpha(s) - i\nu_j^0(s)\beta(s)) \end{cases}$$

Equivalently, we shall use the notation

$$
\begin{aligned}
&P_0(s) = \textstyle\sum_{j=1}^{N} \frac{\partial U}{\partial \tau_j} p_j(s) \\
&P_1^{0,1}(s) = \textstyle\sum_{j=1}^{N} \frac{\partial U}{\partial \tau_j} q_j^{0,1}(s) \\
&P(s) = \textstyle\sum_{j=1}^{N} \frac{\partial V}{\partial \zeta_j} \begin{pmatrix} i\nu_j^0 & -\nu_j^1 + i\nu_j^2 \\ \nu_j^1 + i\nu_j^2 & -i\nu_j^0 \end{pmatrix}
\end{aligned}
\tag{20}
$$

and we shall write the equations (19) as

$$
\partial_s a^{0,1}(s) = i\bar{\alpha}(s)\beta(s) + P_1^{0,1}(s)
$$

$$
\partial_s \begin{pmatrix} \alpha(s) \\ \beta(s) \end{pmatrix} = i \begin{pmatrix} 0 & \bar{\partial}^*_{a(s)} \\ -\bar{\partial}_{a(s)} & 0 \end{pmatrix} \begin{pmatrix} \alpha(s) \\ \beta(s) \end{pmatrix} + P(s) \begin{pmatrix} \alpha(s) \\ \beta(s) \end{pmatrix}.
$$

3.2 Decay estimate

Now we will prove that there is a unique bad point θ in $\chi(T^2)$, such that the following holds. Any gradient flow line of f on the submanifold $\mathcal{C}_{T^2}$ of the configuration space $\Lambda^1(T^2, i\mathbb{R}) \oplus \Lambda^{0,0}(T^2) \oplus \Lambda^{0,1}(T^2, i\mathbb{R})$, with asymptotic limit away from a unique bad point θ in $\chi(T^2)$, has an exponential decay property.

The Hessian operator of f at the critical point $(a_\infty, 0, 0)$ is given by

$$
Q_{(a_\infty,0,0)}(a_1, \alpha_1, \beta_1) = \Big(0, -i\bar{\partial}^*_{a_\infty}\beta_1, i\bar{\partial}_{a_\infty}\alpha_1\Big). \tag{21}
$$

Lemma 3.4. *(a) Restricted to the submanifold*

$$
\mathcal{C}_{T^2} = \Big\{(a, \alpha, \beta) | F_a = \frac{i}{2}(|\alpha|^2 - |\beta^2|)\omega\Big\}/\mathcal{G}_{T^2},
$$

the kernel of the Hessian operator at the critical point $[a_\infty, 0, 0]$, for $a_\infty \in \chi(T^2)$, is

$$
H^1(T^2, i\mathbb{R}) \oplus ker\bar{\partial}_{a_\infty} \oplus ker\bar{\partial}^*_{a_\infty}
$$

(b) There exists a unique point $\theta \in \chi(T^2)$ with $ker(\bar{\partial}_\theta + \bar{\partial}^*_\theta)$ non-trivial. This point θ is determined by the condition that its holonomy is trivial. We have

$$
ker\bar{\partial}_\theta \cong ker\bar{\partial}^*_\theta \cong \mathbb{C}.
$$

We call θ the singular critical point of f, or simply the "bad point" in $\chi(T^2)$.

(c) For $a_\infty \neq \theta$, f is non-degenerate at a_∞ in the sense of Morse-Bott on $\mathcal{C}_{T^2}$, that is the Hessian operator Q at $[a_\infty, 0, 0]$ is non-degenerate in the normal direction to the critical manifold in the tangent space of $\mathcal{C}_{T^2}$ at $[a_\infty, 0, 0]$. We call any such $a_\infty \neq \theta$ a smooth critical point of f, or a "good point" in $\chi(T^2)$.

Proof. The claims in this Lemma follow, upon noticing that tangent space of $\mathcal{C}_{T^2}$ at $[a_\infty, 0, 0]$ is given by

$$H^1(Z, i\mathbb{R}) \oplus \Lambda^{0,0} \oplus \Lambda^{0,1}.$$

$\square$

Let U_θ be a small open neighbourhood of θ in $\chi(T^2)$. For any point $a_\infty \in \chi(T^2)\backslash U_\theta$, let $\lambda_{a_\infty} > 0$ be the smallest absolute value of the non-zero eigenvalues of the Hessian operator Q at $[a_\infty, 0, 0]$ defined in (21). Since $Q_{[a_\infty,0,0]}$ is invertible in the directions normal to the critical manifold in $\mathcal{C}_{T^2}$, the operator $Q^{-1}_{[a_\infty,0,0]}$ restricted to the normal directions in the tangent space has norm bounded by $\lambda^{-1}_{a_\infty}$. Hence there exists a small neighbourhood of $[a_\infty, 0, 0]$ in $\mathcal{C}_{T^2}$, denoted by $K_{[a_\infty,0,0]}$, such that, for $x = (a, \alpha, \beta)$, we obtain

$$\begin{aligned} dist_{L^2}(x, [a_\infty, 0, 0]) &\leq \frac{3}{2}\|Q^{-1}_{[a_\infty,0,0]}(\nabla f(x))\|_{L^2(T^2)} \\ &\leq \frac{3}{2\lambda_{a_\infty}}\|\nabla f(x)\|_{L^2(T^2)} \end{aligned} \tag{22}$$

for any $x \in K_{[a_\infty,0,0]}$. From this estimate we obtain the Lojasiewicz–type inequality with best exponent for f

$$|f(x)| < \frac{3}{2\lambda_{a_\infty}}\|\nabla f(x)\|^2_{L^2(T^2)} \tag{23}$$

for $x \in K_{[a_\infty,0,0]}$. Using this Lojasiewicz–type inequality, we can establish the exponential decay result for the Seiberg-Witten monopoles along the cylindrical end of Z.

Proposition 3.5. *Suppose that $x(s) = [a(s), \psi(s)]$ is an irreducible flow line of f, with asymptotic limit $[a_\infty, 0, 0]$ where $[a_\infty] \neq \theta \in \chi(T^2)$. Then, there exist gauge representatives $(a(s), \psi(s))$ for $x(s)$ and a_∞ for $[a_\infty, 0, 0]$ such that $(a(s) - a_\infty, \psi(s))$ decays exponentially along with its first derivative as $s \to \infty$.*

Choose $R >> 1$. Then, for all $s > R$, we define

$$\mathcal{E}(s) = \int_s^\infty \|\nabla f(x(\tau))\|^2_{L^2(T^2)} d\tau.$$

We prove a preliminary estimate for the energy. This is the analogue of Lemma 6.14 of [23].

Lemma 3.6. *The estimate*

$$dist_{L^2_1}(x(s), [a_\infty, 0, 0]) \leq C_1\mathcal{E}(s-1)$$

holds, with some constant $C_1 > 0$.

Proof. Using the flow equations (17), we can estimate the energy as

$$\mathcal{E} \geq \int_s^{s'} \|i\bar{\alpha}\beta\|^2 + \|i\alpha\bar{\beta}\|^2 d\tau + \int_s^{s'} \|i\bar{\partial}^*_{a(s)}\beta - i\bar{\partial}_{a(s)}\alpha\|^2 d\tau,$$

with $s' > s \geq R_0$. The Weitzenböck formula then gives

$$|i\bar{\partial}^*_{a(s)}\beta - i\bar{\partial}_{a(s)}\alpha|^2 = 2(|\nabla_a\alpha|^2 + |\nabla_a\beta|^2) + \frac{s}{2}(|\alpha|^2 + |\beta|^2) + \langle F_a \cdot \psi, \psi\rangle.$$

Using the equation

$$F_a = \frac{i}{2}(|\alpha|^2 - |\beta|^2)\omega$$

and the Clifford action

$$\frac{1}{2}\langle \omega \cdot \psi, \psi\rangle = \langle *_3\omega, \sigma(\psi,\psi)\rangle = -\frac{i}{2}(|\alpha|^2 - |\beta|^2)$$

we can rewrite the last term as

$$\langle F_a \cdot \psi, \psi\rangle = |(|\alpha|^2 - |\beta|^2)|^2.$$

Thus, we obtain

$$\mathcal{E} \geq \int_s^{s'} (\|\sigma(\psi,\psi)\|^2 + 2\|\nabla_a\psi\|^2 - \frac{s_0}{2}\|\psi\|^2) d\tau$$

$$\geq \int_s^{s'} (\frac{1}{4}\|\psi\|^4_{L^4} - \frac{s_0}{2}\|\psi\|^2) d\tau,$$

where $s_0 = \max_{x \in Y(r_0)}\{-s(x), 0\}$, with $s(x)$ the scalar curvature. This implies the estimate

$$\|\psi\|^4_{L^4(T^2\times[s,s'])} \leq 4\mathcal{E} + 2s_0\|\psi\|^2_{L^2(T^2\times[s,s'])}.$$

We proceed as in Lemma 6.14 of [23] and differentiate the Seiberg-Witten equations to get

$$*F_{\partial_s A} = \sigma(\partial_s\psi, \psi)$$

$$\partial\!\!\!/_A \partial_s\psi + (\partial_s A)\cdot\psi = 0.$$

We introduce a cutoff function ρ identically equal to 1 on the smaller interval $[s+1, s'-1]$, with a bound $|d\rho| \leq \frac{M}{s'-s}$. We can then estimate the quantity

$$SW(\rho\partial_s A, \rho\partial_s\psi) =$$

$$(F_{\rho\partial_s A} - \sigma(\rho\partial_s\psi,\psi), \partial\!\!\!/_A(\rho\partial_s\psi) + (\rho\partial_s A)\cdot\psi, d^*(\rho\partial_s A) + iIm\langle\rho\partial_s\psi,\psi\rangle)$$

by

$$\|SW(\rho\partial_s A, \rho\partial_s\psi)\|^2_{L^2(T^2\times[s,s'])}$$

$$\leq \frac{K}{(s'-s)^2}\left(\|\partial_s A\|^2_{L^2(T^2\times[s,s'])} + \|\partial_s\psi\|^2_{L^2(T^2\times[s,s'])}\right).$$

On the other hand, we can estimate

$$\|SW(\rho\partial_s A, \rho\partial_s\psi)\|^2 \geq \frac{1}{2}\left(\|F_{\rho\partial_s A}\|^2 + \|\partial\!\!\!/_A(\rho\partial_s\psi)\|^2 + \|d^*(\rho\partial_s A)\|^2\right)$$

$$-2\left(\|\sigma(\rho\partial_s\psi,\psi)\|^2 + \|(\rho\partial_s A)\cdot\psi\|^2 + \|iIm\langle\rho\partial_s\psi,\psi\rangle)\|^2\right),$$

and obtain further estimates

$$\|\sigma(\rho\partial_s\psi,\psi)\|^2_{L^2(T^2\times[s,s'])} \leq (C\sqrt{s'-s} + D\sqrt{\mathcal{E}})\|\rho\partial_s\psi\|^2_{L^2_1(T^2\times[s,s'])},$$

$$\|(\rho\partial_s A)\cdot\psi\|^2_{L^2(T^2\times[s,s'])} \leq (C\sqrt{s'-s} + D\sqrt{\mathcal{E}})\|\rho\partial_s A\|^2_{L^2_1(T^2\times[s,s'])},$$

and

$$\|iIm\langle\rho\partial_s\psi,\psi\rangle)\|^2_{L^2(T^2\times[s,s'])} \leq (C\sqrt{s'-s} + D\sqrt{\mathcal{E}})\|\rho\partial_s\psi\|^2_{L^2_1(T^2\times[s,s'])}$$

derived from the initial estimate

$$\|\psi\|^4_{L^4(T^2\times[s,s'])} \leq 4\mathcal{E} + 2s_0\|\psi\|^2_{L^2(T^2\times[s,s'])}.$$

Thus we obtain

$$\left(\|\rho\partial_s A\|^2_{L^2_1(T^2\times[s,s'])} + \|\rho\partial_s\psi\|^2_{L^2_1(T^2\times[s,s'])}\right)$$

$$\leq \frac{K'}{(s'-s)^2}\left(\|\partial_s A\|^2_{L^2(T^2\times[s,s'])} + \|\partial_s\psi\|^2_{L^2(T^2\times[s,s'])}\right).$$

This implies the estimate

$$\left(\|\partial_s A\|^2_{L^2_1(T^2\times[s+1,s'-1])} + \|\partial_s\psi\|^2_{L^2_1(T^2\times[s+1,s'-1])}\right)$$

$$\leq \frac{K'}{(s'-s)^2}\left(\|\partial_s A\|^2_{L^2(T^2\times[s,s'])} + \|\partial_s\psi\|^2_{L^2(T^2\times[s,s'])}\right).$$

Finally this implies the estimate of the Lemma, since we have

$$dist_{L^2_1}(x(s),[a_\infty,0,0]) = \int_s^\infty \|\nabla f(\tau)\|^2_{L^2_1} d\tau$$

$$= \|\partial_s A\|^2_{L^2_1(T^2\times[s,\infty))} + \|\partial_s\psi\|^2_{L^2_1(T^2\times[s,\infty))} \leq C_1\int_{s-1}^\infty \|\nabla f(\tau)\|^2_{L^2} d\tau.$$

This completes the proof of the Lemma. □

Now we can prove the Proposition.

Proof. The proof follows the similar arguments in [5] [16]. We prove that $\mathcal{E}(s)$ (for $s > R$) decays exponentially as $s \to \infty$. Note that

$$\frac{\partial}{\partial s}|f(x(s)| = \langle \nabla f(x(s)), -\partial_s x(s) \rangle_{L^2(T^2)} = \|\nabla f(x(s))\|^2_{L^2(T^2)}.$$

We integrate over $[s, \infty)$ to obtain

$$\begin{aligned}
\mathcal{E}(s) &\leq |f(x(s)| \\
&\leq \frac{3}{2\lambda_{a_\infty}} \|\nabla f(x(s))\|_{L^2(T^2)} \qquad \text{(by the Łojasiewicz inequality (23))} \\
&= -\frac{3}{2\lambda_{a_\infty}} \frac{\partial \mathcal{E}(s)}{\partial s}.
\end{aligned}$$

This implies

$$\frac{\partial \mathcal{E}(s)}{\partial s} \leq -\frac{2\lambda_{a_\infty}}{3} \mathcal{E}(s).$$

Therefore there exists a positive constant C_1 and $R > 0$ such that

$$\mathcal{E}(s) \leq C_1 \exp(-\frac{2\lambda_{a_\infty}}{3} s)$$

for all $s > R$. □

Since $\chi(T^2)\backslash U_\theta$ is compact, we can set

$$\delta = \frac{1}{2} min\{\lambda_{a_\infty} | a_\infty \in \chi(T^2)\backslash U_\theta\}.$$

Then, when restricted to the cylindrical end, any Seiberg-Witten monopole on Z with finite energy and with asymptotic limit in $\chi(T^2)\backslash U_\theta$ has an exponential decay at a rate at least δ.

In order to prove that the boundary value map (14) is well-defined and continuous, we need to consider the decay rate at the singular point θ as well. The following Remark reduces this case to the "finite energy implies finite length" principle.

Remark 3.7. *Given that f is a real analytic function, the work of L. Simon can be employed to prove a more general Łojasiewicz inequality for f at any critical point in $\chi(T^2)$. However, in the case of the "bad point" θ, this will not have the best exponent. Let $x(s)$ be a flow line of f, corresponding to an irreducible finite energy solution of the Seiberg-Witten equations on Z. Then, there exist constants $0 < \beta \leq 1$ and $0 < \alpha \leq \frac{1}{2}$ such that, when $s > R >> 1$, we have*

$$\begin{aligned}
&\inf_{a_\infty \in \chi(T^2)} \|x(s) - a_\infty\|_{L^2} \leq (\|\nabla f(x(s))\|_{L^2})^\beta, \\
&|f(x(s))|^{1-\alpha} \leq \|\nabla f(x(s))\|_{L^2}.
\end{aligned} \tag{24}$$

At the smooth critical points in $\chi(T^2)$, the Lojasiewicz inequalities have the best exponents $\beta = 1$ and $\alpha = \frac{1}{2}$, as in (22) and (23). The direct consequence of these Lojasiewicz inequalities is the following finite length result for flow lines:

$$\int_{s_1}^{s_2} \|\frac{\partial x(s)}{\partial s}\|_{L^2} ds \leq \frac{4}{\alpha} |f(x(s_1)) - f(x(s_2))|.$$

Notice that the Lojasiewicz inequality, for decay at the bad point θ, only guarantees a polynomial decay, since(24) does not have the best exponent. In general it is dififcult to derive a precise estimate of the rate of decay simply from the result (24), however, we'll see later in this section that in our case we can explicitly obtain the decay rate at the bad point θ. That will turn out to be a very slow decay, at a rate $1/s$, as $s \to \infty$, as we prove in Corollary 3.15. This has important consequences in the gluing theorem as we shall explain in the next section, see Proposition 4.12.

In the case of the perturbed equations (19) we have a similar estimate, for a non-degenerate asymptotic value a_∞ in $\chi(T^2)\backslash U_\theta$. We have an estimate

$$dist_{L^2}(x(s), [a_\infty, 0, 0]) \leq \frac{3}{2\lambda_{a_\infty}} \|\nabla f(x(s)) + P(x(s), s)\|_{L^2(T^2 \times \{s\})}, \tag{25}$$

where $P(x(s), s)$ is the perturbation term (20). Here we need to introduce the extra assumption on the class $\mathcal{P}$ for the case of the non-compact manifold Z. Namely, we are assuming that the exponential decay of the perturbation along the cylindrical end has a rate at least of the order of

$$\delta = \frac{1}{2} min\{\lambda_{a_\infty} | a_\infty \in \chi(T^2)\backslash U_\theta\}.$$

An analogue of Remark 3.7 holds for the perturbed equations, under the assumption of real-analyticity of the perturbation term as a function on $\mathcal{C}_{T^2}(U, V)$.

We have now established analogues for our situation of the key results in [22] (pages 60-70). Applying the arguments in [22] to the present context will prove that the boundary value map (14) is well-defined and continuous.

3.3 Asymptotic values

Let $x(s) = (a(s), \alpha(s), \beta(s))$ be an irreducible solution (that is a solution where the pair (α, β) is not identically vanishing) of equations (17) on $[s_1, s_2] \times T^2$. Then we have

$$\begin{aligned} & f(x(s_1)) - f(x(s_2)) \\ = & -\int_{s_1}^{s_2} \langle \nabla f(x(s)), \partial_s x(s) \rangle ds \\ = & \int_{s_1}^{s_2} \|\partial_s x(s)\|^2 ds \\ = & \int_{s_1}^{s_2} (\|\alpha\|^2 \|\beta\|^2 + \|\bar{\partial}_{a(s)} \alpha\|^2 + \|\bar{\partial}^*_{a(s)} \beta\|^2) ds. \end{aligned}$$

We now formulate some consequences of the previous discussion.

Lemma 3.8. *Let $(a(s),\psi(s))$ be the restriction to the cylindrical end $T^2\times[0,\infty)$ of a smooth irreducible finite energy solution of the Seiberg-Witten equations (17) on Z. Then the following properties are satisfied.*

(a) $f(a(s),\psi(s))$ is a non-increasing function of s and $\lim_{s\to\infty} f(a(s),\psi(s)) = 0$.

(b) We have $lim_{s\to\infty}\psi(s) = 0$ (pointwise) and there are connections $\{a'(s)\}, s\in[0,\infty)$, with $a'(s)$ gauge equivalent to $a(s)$, for which there is a strong limit $a_\infty = \lim_{s\to\infty} a'(s)$ in $L^2(T^2)$.

(c) The set of critical points of f modulo the gauge group $\mathcal{G}_{T^2} = Map(T^2, U(1))$ is $\chi(T^2)$.

Proof. Notice that $|F_A|^2 = \dfrac{1}{4}|\psi|^4$. Thus, the finite energy condition implies that the spinor ψ is in L^4. Using smoothness of the solutions, guaranteed by elliptic regularity, we therefore have the pointwise limit $\lim_{s\to\infty}\psi(s) = 0$. This implies that $\lim_{s\to\infty} f(a(s),\psi(s)) = 0$. Notice that we have

$$\frac{\partial f(a,\psi)}{\partial s} = \langle \nabla f(a,\psi), \partial_s(a,\psi)\rangle = -\|\nabla f(a,\psi)\|^2_{L^2},$$

so we see that $f(a,\psi)$ is a non-increasing function of s. Item (b) follows by Uhlenbeck's weak convergence argument together with the decay estimate of the previous section. In fact, the finite energy condition implies that

$$\lim_{R\to\infty}\int_R^{R'} \|F_{a(s)}\|_{L^2(T^2)} ds = 0,$$

for any $R' > R$. Then, by Proposition 8.3 in [8], we have

$$\lim_{s\to\infty} \|F_{a(s)}\|_{L^2_1(T^2)} = 0.$$

This implies that, for any sequence $\{s_n\}$ going to ∞, there is a subsequence $\{s_{n'}\}$, such that, after appropriately gauge transforming (Cf. [8]), the corresponding $a(s_{n'})$ converge weakly in L^2_2 to a flat connection on T^2. Thus, we obtain a subsequence converging strongly in $L^2(T^2)$. We then need to show that if $\{s_n\}$ and $\{s'_n\}$ are sequences going to ∞ and if $a(s_n)\to a_\infty$ and $a(s'_n)\to a'_\infty$, for flat connections a_∞ and a'_∞ in $\chi(T^2)$, then $a_\infty = a'_\infty$. This follows from the Lojasiewicz–type inequality (22). Then items (b) and (c) follow directly. $\square$

Notice that the finite energy condition (13) is equivalent to the finite variation of the functional $f(a,\alpha,\beta)$.

Lemma 3.9. *Let $x = (a, \alpha, \beta)$ be a smooth irreducible solution of equations (17). We assume T^2 is endowed with the canonical* Spin^c*-structure, with $S^+ = \Lambda^{0,0} \oplus \Lambda^{1,1}$. Then (a, α, β) satisfies the L^2 finite energy condition (13) if and only if it satisfies the finite variation condition on the functional (18):*

$$\int_0^\infty \|\partial_s(x(s))\|^2 ds = \int_0^\infty (\|\alpha\|^2\|\beta\|^2 + \|\bar\partial_{a(s)}\alpha\|^2 + \|\bar\partial^*_{a(s)}\beta\|^2) ds < \infty. \tag{26}$$

Proof. A smooth irreducible solution (a, α, β) of equations (17) that satisfies the L^2 finite energy condition (13) has limits

$$\begin{cases} |\alpha|^2 - |\beta|^2 \to 0, & \text{as } s \to \infty \ , \\ \bar\alpha\beta \to 0, & \text{as } s \to \infty \ . \end{cases}$$

Therefore, any irreducible solution of the equations (17) satisfying the finite energy condition (13) also satisfies the finite variation condition (26) on the functional. Conversely, suppose a smooth irreducible (a, α, β) satisfies equations (17) and condition (26). Then, as $s \to \infty$, the element $(a(s), \alpha(s), \beta(s))$ approaches in the L^2_1-topology a critical point of the functional, namely a solution of

$$\begin{cases} \bar\alpha\beta = 0 \\ \bar\partial^*_a\beta = 0 \\ \bar\partial_a\alpha = 0. \end{cases} \tag{27}$$

This follows essentially from the estimate (22). Since we are considering the canonical Spin^c-structure on T^2, equations (27) imply that the spinor is vanishing and the connection is flat. Thus, the spinor ψ is L^2_1 on the three-manifold Z, hence it is L^4, which implies by the equations that the curvature is L^2.

□

3.4 Deformation complex

The exponential decay property implies that we can introduce weighted Sobolev norms in order to study the Fredholm theory of the linearization of the equations, and the structure of the moduli space $\mathcal{M}^*_Z \backslash (\tilde\partial_\infty)^{-1}(U_\theta)$. Here $*$ indicates the irreducible part and $\tilde\partial_\infty$ is the composition of ∂_∞ and π in (14). With δ as in the previous section, we define the space

$$\mathcal{A}_{Z,T^2} = \left\{ (A, \psi) \;\middle|\; \begin{array}{l} (1)\ A \text{ is an extended } L^2_{2,\delta}\text{-connection on } det(W) \\ (2)\ \psi \text{ is an } L^2_{2,\delta}\text{-spinor on } W \end{array} \right\}, \tag{28}$$

where extended $L^2_{2,\delta}$-connection means that there exists an imaginary-valued harmonic 1-form A_∞ in $H^1(T^2, i\mathbb{R})$ such that $A - A_\infty$ is an $L^2_{2,\delta}$-connection on $det(W)$, where $L^2_{2,\delta}$ denotes the Sobolev norm with weight as in [13]. Here we choose the weight $e_\delta(t) = e^{\tilde{\delta}t/2}$, where $\tilde{\delta}$ is a smooth function with bounded derivatives, $\tilde{\delta} : \mathbb{R} \to [-\delta, \delta]$ for some fixed positive number δ, such that $\tilde{\delta}(t) \equiv -\delta$ for $t \leq -1$ and $\tilde{\delta}(t) \equiv \delta$ for $t \geq 1$. The $L^2_{k,\delta}$ norm is defined as $\|f\|_{2,k,\delta} = \|e_\delta f\|_{2,k}$. The weight e_δ imposes an exponential decay as an asymptotic condition along the cylinder. We define the gauge group $\mathcal{G}_{Z,T^2}$ to be the $L^2_{3,loc}$-gauge transformations such that there exists $g_\infty \in U(1)$ with $g_\infty^{-1} g$ an $L^2_{3,\delta}$-gauge transformation.

Assume that $x = (A, \psi) \in \mathcal{A}_{Z,T^2}$ is an irreducible ($\psi \neq 0$) Seiberg-Witten monopole on Z with finite energy. We can assume further that A_∞ represents a flat connection a_∞ in $\chi(T^2)\backslash U_\theta$. Then the irreducible part of the fiber $(\tilde{\partial}_\infty)^{-1}(a_\infty)$ has a deformation complex

$$0 \to \Lambda^0_{L^2_{3,\delta}}(Z, i\mathbb{R}) \xrightarrow{G} \Lambda^1_{L^2_{2,\delta}}(Z, i\mathbb{R}) \oplus L^2_{2,\delta}(W) \xrightarrow{L} \Lambda^1_{L^2_{1,\delta}}(Z, i\mathbb{R}) \oplus L^2_{1,\delta}(W) \tag{29}$$

where G is the map which gives the infinitesimal gauge transformations:

$$G\mid_{(A,\psi)}(f) = (-df, f\psi)$$

and L is the linearization

$$L_{A,\psi}(\alpha, \phi) = \begin{cases} *d\alpha - 2iIm(\sigma(\psi, \phi)) \\ \not{\partial}_A \phi + \alpha.\psi \end{cases} \tag{30}$$

of the Seiberg-Witten equations on Z (12). Taking into account the gauge fixing condition $G^*_{A,\psi}(\alpha, \phi) = 0$, we know that the index of the above deformation complex is 0, [5] [14] [16].

The deformation complex for the moduli space $\mathcal{M}_Z\backslash(\bar{\partial}_\infty)^{-1}(U_\theta)$ is given by

$$0 \to T_{id}\mathcal{G}_{Z,T^2} \xrightarrow{G} T_x\mathcal{A}_{Z,T^2} \xrightarrow{L} \Lambda^1_{L^2_{1,\delta}}(Z, i\mathbb{R}) \oplus L^2_{1,\delta}(W) \xrightarrow{G^*} \Lambda^0_{L^2_\delta}(Z, i\mathbb{R}) \to 0. \tag{31}$$

These two complexes are related by the fact that (29) is a sub-complex of (31) with the quotient complex

$$0 \to Lie(Stab(a_\infty)) \xrightarrow{0} H^1(T^2, i\mathbb{R}) \to 0.$$

Therefore, the virtual dimension of $\mathcal{M}_Z\backslash(\bar{\partial}_\infty)^{-1}(U_\theta)$ at $x = (A, \psi)$ is

$$\dim(\bar{\partial}_\infty)^{-1}(a_\infty) + \dim H^1(T^2, i\mathbb{R}) - \dim Stab(a_\infty) = 1,$$

where $\dim(\bar{\partial}_\infty)^{-1}(a_\infty)$ is the virtual dimension of the fiber.

In the case of the perturbed equations (6), the Hessian

$$H_{(A,\psi)}(f, \alpha, \phi) = \begin{cases} L\mid_{(A,\psi)}(\alpha, \phi) + G\mid_{(A,\psi)}(f) \\ G^*\mid_{(A,\psi)}(\alpha, \phi) \end{cases}, \tag{32}$$

that is, the operator of the assembled complex of (29), should be replaced by the perturbed Hessian (7).

Theorem 3.10. *There exists a Baire set $\mathcal{P}_0$ of perturbations $\pi = (U,V)$ on Z, such that the perturbed Seiberg-Witten moduli space $\mathcal{M}^*_Z \backslash (\bar{\partial}_\infty)^{-1}(U_\theta)$ is a smooth, compact, oriented manifold of dimension* 1.

Proof. The transversality argument is the same as in the closed case, see Lemma 2.2. The computation of the dimension follows from the above deformation complexes (29) and (31). Compactness of the moduli space $\mathcal{M}_Z$ follows the lines of the argument for the closed manifold case, and it relies essentially on the pointwise estimate of Lemma 2.3. In fact, suppose we are given a sequence (A_k, ψ_k) of solutions on Z. The pointwise bound on the spinor, together with the finite energy condition (13) on the curvature, implies that, up to gauge transformations λ_k in $\mathcal{G}_Z$, there is a subsequence that converges smoothly on compact sets to a solution. An orientation of $\mathcal{M}_Z$ is obtained from a trivialization of the determinant line bundle of the assembled operator of the deformation complex (31). The trivialization of the determinant line of the operator (7) in the assembled complex of (29) is obtained from the orientation of $H^0_\delta(Z) \oplus H^1_\delta(Z)$, the cohomology groups of δ-decaying forms. In fact, we can deform the operator $H_{(A,\psi)}$ with a homotopy $\epsilon\psi$, $\epsilon \in [0,1]$. The asymptotic operator $Q_{(a_\infty,0,0)}$ is preserved in the deformation. Thus, if the weight δ is chosen in such a way that $\delta/2$ is not in the spectrum of $Q_{(a_\infty,0,0)}$, then ([13], [22] Lemma 8.3.1) we can ensure that the operator $H_{(A,\epsilon\psi)}$ is Fredholm, for all $\epsilon \in [0,1]$. Since the Dirac operator is complex linear and it preserves the orientation induced by the complex structure on the spinor bundle, a trivialization of the determinant line bundle at $\epsilon = 0$ is obtained by the orientation of $H^0_\delta(Z) \oplus H^1_\delta(Z)$. This in turn determines a trivialization of the determinant line for $\epsilon = 1$, hence an orientation of $\mathcal{M}_Z$.

□

3.5 Center manifold

We have to understand the local structure of $\mathcal{M}^*_Y$ around $(\bar{\partial}_\infty)^{-1}(U_\theta)$. We first analyse the case of the unperturbed equations and then we extend the results to the case of the perturbed equations. The center manifold technique developed in [22] is a useful model to study the structure of $(\bar{\partial}_\infty)^{-1}(U_\theta)$. We already know that the virtual tangent space of $\chi(T^2)$ at the singular point θ is

$$\mathcal{H}^1_\theta = H^1(T^2, i\mathbb{R}) \oplus ker\bar{\partial}_\theta \oplus ker\bar{\partial}^*_\theta.$$

We briefly recall a few facts about center manifolds [22], [28]. We are given the autonomous system (17) of equations on the Hilbert manifold $\mathcal{C}_{T^2}$. In general, suppose given a system of the form

$$\dot{x} = Qx + N(x), \tag{33}$$

with Q a linear operator acting on a Hilbert space $\mathcal{X}$. Assume given a decomposition $\mathcal{X} = \mathcal{X}_h^+ \oplus \mathcal{X}_c \oplus \mathcal{X}_h^-$ determined by the positive, negative, and zero spectrum of the operator Q. Let $\mathcal{X}_h = \mathcal{X}_h^+ \oplus \mathcal{X}_h^-$, and consider the projections $\pi_c : \mathcal{X} \to \mathcal{X}_c$ and $\pi_h : \mathcal{X} \to \mathcal{X}_h$. We denote by $Q_h^\pm$ and Q_c the induced operators on $\mathcal{X}_h^\pm$ and $\mathcal{X}_c$. By construction Q_c is trivial. In our case we are considering the operator Q_θ, the Hessian of the functional f at the degenerate critical point θ. The evolution semigroups $e^{-sQ_\theta^+}$ and $e^{sQ_\theta^-}$, for $s \geq R_0 > 0$, satisfy

$$\sup_{s \geq R_0} \max\{e^{\delta s} \|e^{-sQ_\theta^+}\|, e^{-\delta s}\|e^{sQ_\theta^-}\|\} \leq C, \tag{34}$$

for some constant $C > 0$. This follows from the bound

$$\frac{1}{2} \inf\{|\lambda| \mid \lambda \in spec(Q_\theta), \lambda \neq 0\} = \delta > 0.$$

In this case we have the space $\mathcal{X}_c$ given by

$$\mathcal{X}_c = \mathcal{H}_\theta^1 = H^1(T^2, i\mathbb{R}) \oplus ker\bar{\partial}_\theta \oplus ker\bar{\partial}_\theta^* \cong \mathbb{C}^3.$$

The center manifold theorem [22] [28] states that there exists a unique map $\varphi : \mathcal{X}_c \to \mathcal{X}_h$ that vanishes of second order at the origin, and such that an element $\tilde{x}(s)$ is a solution of (33) if and only if the projection $\pi_c \tilde{x}(s)$ is a solution of the equation

$$\dot{x_c} = \pi_c N(x_c + \varphi(x_c)). \tag{35}$$

The *center manifold* $\mathcal{H}$ is defined as

$$\mathcal{H} = \{x_c + \varphi(x_c) | x_c \in \mathcal{X}_c\}.$$

The degree of regularity $\mathcal{C}^m$ of the center manifold is determined by the Sobolev norm on $\mathcal{X}$, as in [22]. In our case the center manifold $\mathcal{H}_\theta$ for the functional f at the degenerate critical point θ is a C^2-manifold which is invariant under the gradient flow of f, contains a small neighbourhood U_θ of θ, and has tangent space at θ given by $\mathcal{H}_\theta^1$.

The restriction of the gradient flow of f to the center manifold provides a model for the structure of the space of flows with asymptotic values in a small neighbourhood U_θ of the "bad point" θ, in the following sense. Given a point x in $\mathcal{H}_\theta$, the *stable set* $\mathcal{S}_x$ at x is defined as

$$\mathcal{S}_x = \{y \in \mathcal{H}_\theta \text{ such that the flowline starting at } y \text{ converges to } x\}.$$

Once the center manifold is determined, the stable set $\mathcal{S} = \cup_{x \in \mathcal{H}_\theta} \mathcal{S}_x$ defines a refinement of the boundary value map, as described in the following commutative diagram:

$$\begin{tikzcd} (\bar{\partial}_\infty)^{-1}(U_\theta) \arrow[rr, "\tilde{\partial}_\infty"] \arrow[dr, "\Upsilon"'] & & U_\theta \\ & \mathcal{S} \arrow[ur, "\Gamma"'] & \end{tikzcd} \tag{36}$$

Here Υ is a continuous map defined by taking a flow line on the stable set $\mathcal{S}$ that is exponentially close to a monopole in $(\tilde{\partial}_\infty)^{-1}(U_\theta)$. The map Γ is the limit value map under the flow line of f on $\mathcal{S}$.

In fact, we summarize some results of [22], which explain the construction of the map Υ. Lemma 5.3.1 of [22] shows that if the projection $\pi_c\tilde{x}(s)$ of a flow line $\tilde{x}(s)$ satisfies an estimate

$$\|\partial_s\pi_c\tilde{x}(s) - \nabla f(\pi_c\tilde{x}(s))\| < Ce^{-\delta s}, \tag{37}$$

for all $s \geq R_0$, then there exists a unique flow line $x_c(s)$ in the center manifold $\mathcal{H}_\theta$ that is exponentially close to $\pi_c\tilde{x}(s)$ for large $s \geq R_0$, with the same exponent δ determined by the smallest absolute value of the non-trivial eigenvalues of Q_θ. Moreover, Lemma 5.4.1 of [22] shows that, for a flow line $\tilde{x}(s)$ satisfying

$$\|\pi_c\tilde{x}(s) - a_\infty\|_{L^2_2(T^2\times\{s\})} + \|\pi_h\tilde{x}(s)\|_{L^2_2(T^2\times\{s\})} \leq C$$

for all $s \geq R_0$, the projection $\pi_h\tilde{x}(s)$ is exponentially small for large s, with exponent δ. The condition (37), will follow from our explicit construction of the center manifold. This shows that every flow line in $(\tilde{\partial}_\infty)^{-1}(U_\theta)$ is exponentially close to a flow line in the center manifold. Thus, the refinement Υ of the boundary map is well defined and continuous.

We describe explicitly the center manifold and the stable set for the unperturbed equations (17).

Lemma 3.11. *At every point $x = (a, \alpha, \beta) \in \mathcal{H}^1_\theta$, the gradient vector $\nabla f(a, \alpha, \beta)$ is tangent to $\mathcal{H}^1_\theta$, hence $\mathcal{H}^1_\theta$ is a center manifold of f around θ.*

Proof. Using the natural complex structure on T^2, we can identify $\mathcal{H}^1_\theta$ as the space of constant sections of

$$\Lambda^1(T^2, i\mathbb{R}) \oplus \Lambda^{0,0}(T^2, \mathbb{C}) \oplus \Lambda^{0,1}(T^2, \mathbb{C}).$$

For $(a, \alpha, \beta) \in \mathcal{H}^1_\theta$, we have

$$\nabla f(a, \alpha, \beta) = (-i\bar{\alpha}\beta, -i\bar{\partial}^*_a\beta, i\bar{\partial}_a\alpha),$$

which is a constant section. Take (z_1, z_2, z_3) as the coordinates on $\mathcal{H}^1_\theta \cong \mathbb{C}^3$, we have

$$\nabla f(z_1, z_2, z_3) = \big(-\bar{z}_2 z_3, -\frac{\bar{z}_1 z_3}{2}, -\frac{z_1 z_2}{2}\big).$$

□

The downward gradient flow of f on $\mathcal{H}^1_\theta$ is given by

$$\begin{cases} \dfrac{\partial z_1}{\partial s} = \bar{z}_2 z_3, \\ \dfrac{\partial z_2}{\partial s} = \dfrac{\bar{z}_1 z_3}{2}, \\ \dfrac{\partial z_3}{\partial s} = \dfrac{z_1 z_2}{2}. \end{cases} \tag{38}$$

Note that this gradient flow is invariant under the $U(1)$-action (the constant gauge transformation):

$$(z_1, z_2, z_3) \overset{e^{i\alpha} \in U(1)}{\to} (z_1, e^{i\alpha} z_2, e^{i\alpha} z_3).$$

Lemma 3.12. *The quantities $|z_2|^2 - |z_3|^2$, $|z_1|^2 - |z_2|^2 - |z_3|^2$ and $Im(z_1 z_2 \bar{z}_3)$ are preserved under the gradient flow on $\mathcal{H}_\theta$.*

Proof. This is a direct calculation using the gradient flow equations (38). $\square$

The stable set of $(a_\infty, 0) \in \mathcal{H}_\theta$ is defined to be

$$\mathcal{S}_{a_\infty} = \{x \in \mathcal{H}_\theta \text{ such that the flowline of (38) starting at } x \text{ converges to } (a_\infty, 0)\}.$$

The stable set $\mathcal{S}$ of f in $\mathcal{H}_\theta$ is the union of these $\mathcal{S}_{a_\infty}$, for $(a_\infty, 0) \in \mathcal{H}_\theta$.

Lemma 3.13. *Let $(a, \phi) = (z_1, z_2, z_3) \in \mathcal{H}_\theta$. Then $(a, \phi) \in \mathcal{S}$ if and only if we have*

$$\begin{cases} |z_2|^2 - |z_3|^2 = 0, \\ |z_1|^2 - |z_2|^2 - |z_3|^2 = |a_\infty|^2 \geq 0, \\ Im(z_1 z_2 \bar{z}_3) = 0. \end{cases} \tag{39}$$

In particular, $(a, \phi) = (z_1, z_2, z_3) \in \mathcal{S}_\theta$ (the stable set of the point θ) if and only if

$$\begin{cases} |z_2|^2 - |z_3|^2 = 0, \\ |z_1|^2 - |z_2|^2 - |z_3|^2 = 0, \\ Im(z_1 z_2 \bar{z}_3) = 0. \end{cases} \tag{40}$$

These equations describe a cone over a torus T^2 with vertex at θ. Furthermore, $\mathcal{S} \backslash \{(\theta, 0)\}$ is a 4-dimensional manifold with boundary $\mathcal{S}_\theta \backslash \{(\theta, 0)\}$.

Proof. It follows from Lemma 3.12 that $(a, \phi) \in \mathcal{S}$ converges to some $(a_\infty, 0)$ as $t \to \infty$. It is easy to see that the equations (40) define a torus over θ and as $|a_\infty|^2 \to 0$, points defined by (39) approach points in $\mathcal{S}_\theta$. $\square$

Now consider the perturbed equations (19) on $T^2 \times [0, \infty)$. We now have a non-autonomous system of the form

$$\dot{x}(s) = Q(s)x(s) + N(s, x(s)). \tag{41}$$

Some results about center manifolds have been extended to the case of non-autonomous systems [7]. The non-autonomous system (19) defines an associated *evolution family* $\mathcal{U}(s, t)$, defined by the condition that $x(s)$ is a solution of (41) if and only if it satisfies

$$x(s) = \mathcal{U}(s, t)x(t) + \int_t^s \mathcal{U}(s, \tau) N(\tau, x(\tau)) d\tau. \tag{42}$$

In other words, $\mathcal{U}(s,t)$ is the evolution family associated to the non-autonomous linear system $\dot{x}(s) = Q(s)x(s)$.

We again want to decompose the space $\mathcal{X}$ as $\mathcal{X}_c(s) \oplus \mathcal{X}_h^+(s) \oplus \mathcal{X}_h^-(s)$, with projections π_c and π_h as before, so that there are well defined restrictions of the evolution family,

$$\mathcal{U}_c(s,t) = \mathcal{U}(s,t)|_{\mathcal{X}_c(t)} : \mathcal{X}_c(t) \to \mathcal{X}_c(s),$$

and

$$\mathcal{U}_h(s,t) = \mathcal{U}(s,t)|_{\mathcal{X}_h(t)} : \mathcal{X}_h(t) \to \mathcal{X}_h(s).$$

The estimate (34) is then replaced by estimates

$$\begin{aligned} \|\mathcal{U}_c(s,t)\| &\leq C \\ \|\mathcal{U}_h^+(s,t)\| &\leq Ce^{-\delta(s-t)} \\ \|(\mathcal{U}_h^-(s,t))^{-1}\| &\leq Ce^{-\delta(s-t)}, \end{aligned} \tag{43}$$

for all $s > t \geq R_0$.

Lemma 3.14. *In our case the decomposition*

$$\mathcal{X}_c(s) \oplus \mathcal{X}_h^+(s) \oplus \mathcal{X}_h^-(s)$$

is defined by considering the positive, negative, and zero spectrum of the operator $Q(s)$, given by

$$Q_x(s)(b(s),\gamma(s),\eta(s)) = \begin{pmatrix} i(\bar{\alpha}\eta + \bar{\gamma}\beta) \\ \left(i\begin{pmatrix} 0 & \bar{\partial}_a^* \\ -\bar{\partial}_a & 0 \end{pmatrix} + P_x(s)\right)\begin{pmatrix} \gamma \\ \eta \end{pmatrix}, \end{pmatrix}$$

with $x(s) = (a(s),\alpha(s),\beta(s))$.

Moreover, the estimate (43) follows from the exponential decay of the perturbation $P(s,x(s))$, $P \in \mathcal{P}_\delta$, and the estimates of the unperturbed case.

Theorem 3.4 of [7] then ensures the existence of a map

$$\varphi(s,\cdot) : \mathcal{X}_c(s) \to \mathcal{X}_h(s),$$

such that

$$\mathcal{H} = \{(s, x_c(s) + \varphi(s,x_c(s))) \mid (s,x_c(s)) \in \mathbb{R} \times \mathcal{X}_c(s)\} \tag{44}$$

is a smooth invariant manifold for the equations (19). An element $x(s) = (a(s),\alpha(s),\beta(s))$ is a solution of (19) if and only if the projection $x_c(s) = \pi_c(x(s))$ satisfies the integral equation

$$x_c(s) = \mathcal{U}_c(s,t)x_c(t) + \int_t^s \mathcal{U}_c(s,\tau)\pi_c N(x_c(\tau) + \varphi(\tau,x_c(\tau)))d\tau,$$

If the perturbation $P(x(s),s)$ is compactly supported on $T^2 \times [0,\infty)$, then the center manifold $\mathcal{H}$ reduces asymptotically to the center manifold for the unperturbed equations, explicitly described above. We can derive the case of an exponentially decaying perturbation by rephrasing, in this non-autonomous setting, the parametric version of the center manifold theorem (see [28]). Consider a family of perturbations $P_\sigma(x(s),s)$, with $-\epsilon \le \sigma \le \epsilon$, such that P_0 is compactly supported on $T^2 \times [0,\infty)$ and $P_\sigma(x(s),s)$ is uniformly bounded with exponential decay along $[0,\infty)$ at a uniform rate δ, for all $-\epsilon \le \sigma \le \epsilon$. Then, by analogy to (24) of [28], we have a smooth local version of the center manifold theorem with (44) replaced by

$$\mathcal{H}(\sigma) = \{(s, x_c(s) + \varphi(s, x_c(s), \sigma), \sigma)\}.$$

One last useful observation that we can derive from the analysis of the center manifold is the following estimate of the rate of decay of solutions approaching the singular point θ.

Corollary 3.15. *Let $x(s) = (a(s), \alpha(s), \beta(s))$ be an irreducible finite energy solution of the Seiberg–Witten equations on Z, with asymptotic value θ, that is, $[x] \in \partial_\infty^{-1}(\theta)$. Then the rate of decay in the $s \to \infty$ direction is polynomial with*

$$\|(a(s), \alpha(s), \beta(s))\|_{L^2(T^2\times\{s\})} \sim \frac{1}{s},$$

for $s \to \infty$.

Proof. The result follows by direct analysis of the equations (38), and the center manifold theorem.

$\square$

This estimate will be essential in the gluing theorem, in order to ensure that we can avoid gluing at the reducible point θ.

3.6 Moduli space

The results of the previous section and the arguments in [22] (Page 82-100) imply the following structure theorem for our moduli space $\mathcal{M}_{Z,P}$.

Theorem 3.16. *Fix a metric g and perturbation $P \in \mathcal{P}$ as in Theorem 3.10. Let $\mathcal{K} \subset \mathcal{M}^*_{Z,P}$ be defined as $\mathcal{K} = \left(\bar\partial_\infty\right)^{-1}(\theta)$, and let $\mathcal{K}'$ denote the subset $\Upsilon^{-1}(\theta,0)$. Then, generically, $\mathcal{K}'$ is empty and $\mathcal{K}$ consists of only finitely many points. Moreover, $\mathcal{M}^*_{Z,P}$ is a smooth manifold of dimension 1, with boundary $\mathcal{K} = \left(\bar\partial_\infty\right)^{-1}(\theta)$. The image $\partial_\infty(\mathcal{M}^*_{Z,P})$ under the boundary value map is a compact set in the covering of $\chi(T^2)$*

$$\chi_0(T^2, Z) = H^1(T^2,\mathbb{Z})/(Im(i^* : H^1(Z,\mathbb{Z}) \to H^1(T^2,\mathbb{Z}))).$$

Moreover, the boundary value map ∂_∞ maps $\mathcal{M}^*_{Z,P}$ into a collection of immersed curves in $\chi_0(T^2, Z)$. It is possible to choose the perturbation P in such a way that these curves are transverse to any fixed given curve in $\chi_0(T^2, Z)$.

Proof. By the center manifold theorem in [22], the restriction of any finite energy monopole

$$[A, \psi] \in \mathcal{M}_{Z,\mu}(U_\theta) = \left(\bar{\partial}_\infty\right)^{-1}(U_\theta)$$

to the tube $T^2 \times [T_0, \infty)$ (for a fixed large T_0) is exponentially close to a flow line in the center manifold starting from the point $\Upsilon([A, \psi])$ given by the refinement boundary map (36). The exponential weight is at least a half of the smallest absolute value of the non-zero eigenvalues of $(\bar{\partial}_\theta + \bar{\partial}^*_\theta)$. Theorem 3.10 shows that, away from $\Upsilon^{-1}(\mathcal{S}_\theta)$, the moduli space $\mathcal{M}^*_{Z,P}(U_\theta)$ is a smooth manifold of dimension 1, for a generic choice of the perturbation. We know from the analysis of the unperturbed case (or the case with compactly supported perturbation) that $\mathcal{S}\backslash\{(\theta, 0)\}$ is a 4-manifold with boundary $\mathcal{S}_\theta\backslash\{(\theta, 0)\}$. Moreover, upon considering the local stable set $\mathcal{S} = \mathcal{S}(\sigma)|_{\sigma=0}$ as above, the theorem follows from the Sard-Smale theorem. □

3.7 The reducible set

In this subsection we return to our setting, where we consider the two manifolds V and $\nu(K)$ with boundary T^2. We can add an infinite cylinder $T^2 \times [0, \infty)$ to the boundary, in order to apply to these two manifolds the results stated in the previous section.

We first analyse the set of reducible solutions of the monopole equations on V and $\nu(K)$.

If V is the knot complement for a knot K in a homology sphere, then the reducibles in $\mathcal{M}_V$ cannot be perturbed away. They consist of the space of flat connections over V, $Hom(H_1(V, \mathbb{Z}), U(1)) = \chi(V)$.

Unlike the case of irreducible solutions, analysed in Proposition 3.5, for reducibles the asymptotic value map ∂_∞ cannot be defined by means of the exponential decay property. However, in this case, elements of $\chi(V)$ define, by restriction, flat connections in $\chi_0(T^2, V)$. Thus, in this case, we can define ∂_∞ to be simply the restriction map. The image of $\chi(V)$ under ∂_∞ is a non-trivial circle in $\chi(T^2)$, parametrized by the holonomy around the meridian m. This lifts to a straight line $\{(0, v), v \in \mathbb{R}\}$ in the covering $\chi_0(T^2, V) = \mathbb{R}^2$.

Next we make an observation that will be useful in the following.

Remark 3.17. *Let V be the knot complement in a homology sphere, with a cylindrical end modelled on $T^2 \times [0, \infty)$. Let $\chi(V)$ be the set of flat connections modulo gauge transformations on Z,*

$$\chi(V) = Hom(H_1(V, \mathbb{Z}), U(1)).$$

Then the boundary value map

$$\partial_\infty : H^1(V, \mathbb{R})/H^1(V, \mathbb{Z}) \to H^1(T^2, \mathbb{R})/Im(H^1(V, \mathbb{Z}) \to H^1(T^2, \mathbb{Z}))$$

is an embedding of the compact set $\chi(V)$ in $\chi_0(T^2,V)$. Similarly, for the tubular neighbourhood of the knot $\nu(K)$ we have an embedding

$$\partial_\infty : H^1(\nu(K),\mathbb{R})/H^1(\nu(K),\mathbb{Z}) \to H^1(T^2,\mathbb{R})/Im(H^1(\nu(K),\mathbb{Z}) \to H^1(T^2,\mathbb{Z})).$$

In order to study the moduli space of solutions on $\nu(K)$, we need some preliminary results.

In the next section, we shall introduce a perturbation of the equations on $\nu(K)$ by a coclosed, cohomologically trivial, 1-form μ, compactly supported inside $\nu(K)$. For finite energy solutions (A,ψ) on $\nu(K)$, the Weitzenböck formula gives an estimate $|\psi| \leq \max_{\nu(K)}\{0, -s+4\sqrt{2}|\mu|\}$. If the metric has sufficiently large positive curvature away from the boundary of $D^2\times S^1$, then the estimate implies that the only finite energy solutions on $\nu(K)$ are reducibles.

Lemma 3.18. *There exists a metric g on the solid torus $D^2\times S^1$ with non-negative scalar curvature, such that it coincides with the flat product metric near the boundary of D^2, and it has positive scalar curvature $s_g>0$ away from a neighbourhood of the boundary. Moreover, given a metric g on $D^2\times S^1$, with the properties stated above, there exists a coclosed, cohomologically trivial 2-form μ, supported inside $D^2\times S^1$, satisfying the condition $\int_{D^2}\mu = 1$, such that we have*

$$-s_g + 4\sqrt{2}|\mu| \leq 0. \tag{45}$$

Proof. Choose a metric g_0 on D^2 with non-negative scalar curvature, such that it is the standard flat metric near the boundary and has scalar curvature given by a function $s(r)$, which is strictly positive away from a neighbourhood of the boundary S^1. This produces the desired metric on the solid torus $D^2\times S^1$. By Gauss-Bonnet, $s(r)$ must satisfy the condition

$$\int_0^1 s(r)rdr = 1.$$

Consider the function $\mu(r) = \frac{s(r)}{2\pi}$. This defines a two form on D^2 satisfying

$$\int_{D^2}\mu = 1$$

and such that we have $4\sqrt{2}|\mu| - s \leq 0$ on all $D^2\times S^1$.

□

We have the following result on the moduli space of solutions on $\nu(K)$.

Lemma 3.19. *Given a metric on $\nu(K)$ as in Lemma 3.18, the only finite energy solutions of the monopole equations on $\nu(K)$ are the flat connections in $\chi(\nu(K))$.*

Proof. On the solid torus $D^2 \times S^1 \cong \nu(K)$ in the homology sphere Y, if the induced metric g_0 has positive scalar curvature, the only finite energy Seiberg-Witten monopoles are the reducibles $\chi(\nu(K))$, whose holonomy around the longitude of K gives a non-trivial circle in $\chi(T^2)$. In fact, the finite energy condition implies that the spinor ψ goes to zero as $s \to \infty$. This implies that the boundary terms in the Weitzenböck formula vanish. Therefore, if the positive scalar curvature satisfies the estimate (45), then the spinor ψ has to vanish identically on $\nu(K)$, because of the estimate

$$|\psi| \leq \max_{\nu(K)} \{0, -s + 4\sqrt{2}|\mu|\}$$

obtained from the Weitzenböck formula.

$\square$

3.8 Deformation retract

Assume that the original metric we choose on $Y = Y(r_0)$ satisfies Lemma 3.18, when restricted to

$$\nu(K) = (D^2 \times S^1) \cup_{T^2} T^2 \times (-r_0, 0].$$

After the surgery, we cannot expect the induced metrics on Y_1 and Y_0 to still have non-negative scalar curvature on $\nu(K)$. In fact, we are performing surgery with low codimension, and it is known by the results of [26] that, in order to preserve the positive scalar curvature under surgery, the codimension should be at least 3.

We discuss here the case of the induced metric g_1 on Y_1. The case of Y_0 is completely analogous. We can assume that the induced metrics on Y_1 still has non-negative scalar curvature on a smaller neighbourhood of the knot,

$$(D^2 \times S^1) \cup_{T^2} T^2 \times (-r_1, 0],$$

with $r_1 < r_0$. Moreover, we can also assume that near $T^2 \times \{-r_0\}$ the induced metric is equal to the flat metric on T^2. Thus, we can consider the metric g_1 extended to the infinite cylinder $T^2 \times (-\infty, -r_0]$ as the flat metric.

On the manifold $\nu(K)$ with the infinite cylindrical end $T^2 \times (-\infty, 0]$ we can then choose a smooth path of metrics $g(\tau)$, with $\tau \in \mathbb{R}^+$, such that $g(\tau) \equiv g_1$ for $\tau \leq r_1$, and $g(\tau) \equiv g$ for $\tau \geq r_0$, where the metric g has non-negative scalar curvature and satisfies Lemma 3.18. We can assume that all the metrics $g(\tau)$ in this family restrict to the flat metric on $T^2 \times (-\infty, -r_0]$.

The parameterized moduli space

$$\cup_\tau \mathcal{M}_{\nu(K)}(g(\tau)) \tag{46}$$

defines a deformation retraction of $\partial_\infty(\mathcal{M}_{\nu(K)}(g_1))$ onto $\chi(\nu(K))$ in $\chi(T^2)$.

Along the cylinder $T^2 \times (-\infty, 0]$ we can write the metrics $g(\tau)$ as $g(\tau) = g_\tau(s) + ds^2$. On the manifold $\nu(K)$ with the infinite cylindrical end $T^2 \times (-\infty, 0]$, we can then consider a metric $\tilde{g}$ which is equal to g_1 on

$$\nu(K) = (D^2 \times S^1) \cup_{T^2} T^2 \times (-r_1, 0]$$

and is equal to $g_{-s}(s) + ds^2$ on $T^2 \times \{s\}$ for $s \in [-r_0, -r_1]$, and finally equals the flat metric on $T^2 \times (-\infty, -r_0]$.

Lemma 3.20. *There exist finitely many irreducible points $x_1, \dots x_N$ in $\chi(T^2, V)$ and small disjoint neighbourhoods U_{x_i} of these points such that the asymptotic values $\partial_\infty(\mathcal{M}_{\nu(K)}(\tilde{g}))$ are contained in*

$$U_{x_1} \cup \cdots \cup U_{x_N} \cup U_{\chi(\nu(K))},$$

where $U_{\chi(\nu(K))}$ is a small open neighbourhood of the circle $\chi(\nu(K))$ which does not intersect the U_{x_i}'s.

Proof. This is a consequence of the existence of the deformation (46).

□

In order to prove the relation

$$\mathcal{M}_{Y,\mu} \cong \mathcal{M}_{Y_1} \cup \bigcup_{\mathfrak{s}_k} \mathcal{M}_{Y_0}(\mathfrak{s}_k),$$

we need a refinement of Lemma 3.20, witha more careful analysis of the asymptotic values. There are two issues that require a more detailed analysis.

First, we need to ensure that, along the deformation retract (46)

$$r : \mathcal{M}_{\nu(K)} \to \chi(\nu(K)) = \mathcal{M}^{red}_{\nu(K)},$$

the irreducibles $\mathcal{M}^*_{\nu(K)}$ concentrate near finitely many point in $\chi_0(T^2, \nu(K))$ and eventually cancel out. In other words, we want to rule out the possibility that components of $\mathcal{M}^*_{\nu(K)}$ form a homotopically non-trivial curve in $\chi_0(T^2, \nu(K))$ that is mapped onto $\chi(\nu(K))$ in the deformation. Such curve corresponds to the example drawn in figure 1. In other words, we want to ensure that the asymptotic values $\partial_\infty(\mathcal{M}^*_{\nu(K)}(\tilde{g}))$ only concentrate near finitely many points in $\chi_0(T^2, \nu(K))$.

Lemma 3.21. *The asymptotic values of the irreducible solutions in $\mathcal{M}^*_{\nu(K)}(\tilde{g})$ only concentrate near finitely many points in $\chi_0(T^2, \nu(K))$, that is*

$$\partial_\infty(\mathcal{M}_{\nu(K)}(\tilde{g})) \subset U_{x_1} \cup \cdots \cup U_{x_N} \cup U_\theta,$$

with the notation of Lemma 3.20, where U_θ is a small contractibe neighbourhood of the "bad point" θ in $\chi(T^2)$.

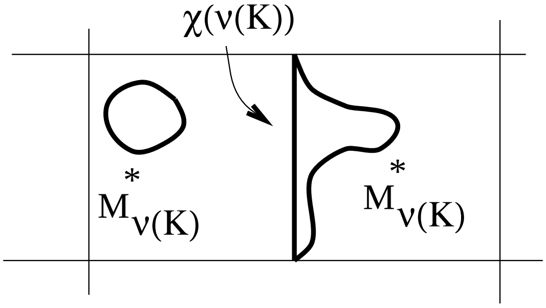

Figure 1: A good and a bad component of $\partial_\infty(\mathcal{M}_{\nu(K)}(\tilde{g}))$

Proof. Suppose given an irreducible solution (A_r, ψ_r) on $Y_1(r)$ obtained by gluing $(A', \psi')\#_{a_1}(A_1, \psi_1)$, where (A_1, ψ_1) is a solution in $\mathcal{M}^*_{\nu(K)}(g_1)$, and a_1 is the asymptotic flat connection on T^2. Let $(A, 0)$ be the element in $\chi(\nu(K))$, $(A, 0) = r(A_1, \psi_1)$, image under the deformation retract. There is an irreducible solution, on $Y_1(r)$ with the metric g, obtained as $(A', \psi')\#_a(A, 0)$. This means that, under the deformation $g(\tau)$ of the metric on Y_1, the irreducible $(A', \psi')\#_{a_1} r_\tau(A_1, \psi_1)$ collides at $\tau = r_0$ with the irreducible $(A', \psi')\#_a(A, 0)$. Notice that, along the deformation, the solution $(A', \psi')\#_a(A, 0)$ is defined for all metrics, since the reducible $(A, 0)$ belongs to the moduli space $\mathcal{M}_{\nu(K)}$ for all choices of the metric, and the metric on V is not changed.

By the local analysis of the universal moduli space (46), we know that the only way that two irreducible solutions on Y_1 can cancel each other out along the deformation

$$\cup_\tau \mathcal{M}_{Y_1}(g(\tau))$$

is in a configuration like the one illustrated in the left part of the figure 2. But since the solution $(A', \psi')\#_a(A, 0)$ continues to exist for all choices of the metric, we have a configuration like one on the right side of the figure 2, which contradicts the fact that the universal moduli space

$$\cup_\tau \mathcal{M}_{Y_1}(g(\tau))$$

is a smooth 1-dimensional manifold.

$\square$

Upon chosing a generic perturbation of the equations on V it is possible to guarantee that the asymptotic values $\partial_\infty(\mathcal{M}^*_V)$ will miss the small neighbourhoods U_{x_i} of the irreducible points. However, even with a generci choice of perturbation on V the asymptotic values $\partial_\infty(\mathcal{M}^*_V)$ might still intersect the neighbourhood U_θ of the bad point. If we want to establish our main gluing result, Theorem 4.2, in the case of the manifolds Y_1 and Y_0, we need an argument that rules out the gluing of solutions in $\mathcal{M}_V$ and solutions in $\mathcal{M}^*_{\nu(K)}$ at the reducible point θ. This will be derived from the obstruction bundle technique, using the essential information on the decay of irreducibles solutions towards the "bad point" θ along $T^2 \times [0, \infty)$, see Proposition 4.12.

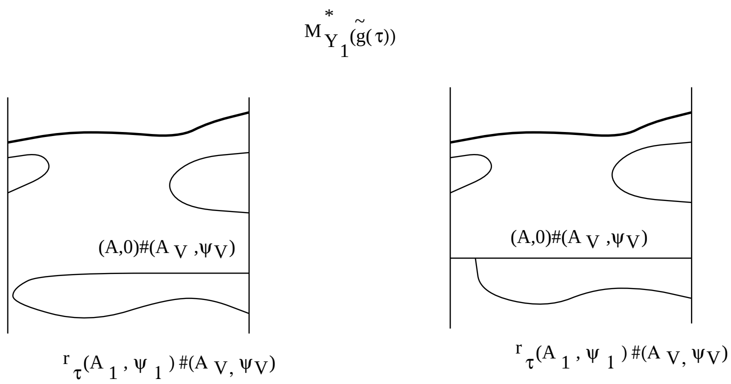

Figure 2: Impossible deformation $\cup_\tau \mathcal{M}_{Y_1}(g(\tau))$

4 Convergence and gluing

In this Section we prove the main technical result needed in this paper. We start with an oriented compact three-manifold without boundary. For our purposes it will be either Y, Y_1, or Y_0: we shall present the argument for the case of the homology sphere Y. The argument is analogous in the case of Y_1 and, up to minor modifications that we shall point out, in the case of Y_0 as well. We have a splitting of Y along the torus T^2 as $Y = V \cup_{T^2} \nu(K)$, with $\partial V = \partial\nu(K) = T^2$. Assume that the metric g on Y is the product metric on a small neighbourhood of T^2. Consider the manifold $Y(r)$ obtained by gluing in a long cylinder, $Y(r) = V \cup_{T^2} ([-r,r] \times T^2) \cup_{T^2} \nu(K)$. We can also consider the two manifolds with an infinite cylindrical end

$$V \cup_{T^2} ([0,\infty) \times T^2), \quad \text{and} \quad ((-\infty,0] \times T^2) \cup_{T^2} \nu(K).$$

As before, we shall refer to these complete manifolds as V and $\nu(K)$ in the following. Our aim is to establish a decomposition of the Seiberg-Witten moduli space for $Y(r)$ in terms of the moduli spaces for V and $\nu(K)$ analysed in the previous sections.

In order to achieve this result we have to show how to choose the Spin^c structures on each of the manifolds. Let $\mathfrak{s}'$ and $\mathfrak{s}''$ be the trivial Spin^c structures on V and $\nu(K)$. These extend to the manifold $Y(r)$ by translation invariance on $T^2 \times [-r,r]$. Let $\mathfrak{s}$ be a Spin^c structure on Y that restricts to $\mathfrak{s}'$ and $\mathfrak{s}''$. This will be trivial since Y is a homology sphere. In the case of Y_0, we shall identify all the possible Spin^c structures that restrict to the trivial one on V and $\nu(K)$.

Let us first introduce covering spaces of the character variety of T^2 that will be useful in the following.

The meridian m' in the boundary $T^2 = \partial V$ generates $H_1(V,\mathbb{Z}) \cong \mathbb{Z} < m' >$, and the longitude l'' in the boundary torus $T^2 = \partial\nu(K)$ generates $H_1(\nu(K),\mathbb{Z}) \cong \mathbb{Z} < l'' >$. We also have surfaces σ and D^2 that generate $H_2(V,T^2,\mathbb{Z})$ and $H_2(\nu(K),T^2,\mathbb{Z})$, respectively, with $\partial\sigma = l'$ and $\partial D^2 = m''$.

The morphisms induced by the inclusions give

$$Im(i_1^* : H^1(V,\mathbb{Z}) \to H^1(T^2,\mathbb{Z}))$$

$$= Im(\partial : H_2(V,T^2,\mathbb{Z}) \to H_1(T^2,\mathbb{Z})) = \mathbb{Z} < l' >$$

and

$$Im(i_2^* : H^1(\nu(K),\mathbb{Z}) \to H^1(T^2,\mathbb{Z}))$$

$$= Im(\partial : H_2(\nu(K),T^2,\mathbb{Z}) \to H_1(T^2,\mathbb{Z})) = \mathbb{Z} < m'' > .$$

The manifold Y_0 is obtained by gluing V and $\nu(K)$ along T^2 with the identifications

$$l' \equiv m'' \equiv l,$$

$$m' \equiv l'' \equiv m,$$

where, in the torus $T^2 \hookrightarrow Y_0$, we denote by m the generator of $H_1(Y_0,\mathbb{Z}) = \mathbb{Z} < l >$, and $m = \partial\sigma_1 = \partial D^2$. Thus, we have $H_1(Y_0,\mathbb{Z}) = \mathbb{Z} < m >$ and $H_2(Y_0,\mathbb{Z}) = \mathbb{Z} < \sigma \cup_l D^2 >$.

Let $\chi_0(T^2,V)$ and $\chi_0(T^2,\nu(K))$ denote the quotients of $H^1(T^2,\mathbb{R})$,

$$\chi_0(T^2,V) = H^1(T^2,\mathbb{Z})/(\mathbb{Z} < PD(l') >)$$

$$\chi_0(T^2,\nu(K)) = H^1(T^2,\mathbb{Z})/(\mathbb{Z} < PD(m'') >).$$

The character variety $\chi_0(T^2,Y_0)$ is the quotient of $H^1(T^2,\mathbb{R})$ obtained as

$$\chi_0(T^2,Y_0) = H^1(T^2,\mathbb{R})/(\mathbb{Z} < PD(l) >),$$

where $PD(l)$ is the Poincaré dual of l in T^2, or equivalently as

$$\chi_0(T^2,Y_0) = \chi_0(T^2,V) \equiv \chi_0(T^2,\nu(K)),$$

under the identification $l' \equiv m''$ and $l'' \equiv m'$. The space $\chi_0(T^2,Y_0)$ is a $\mathbb{Z}$-covering of $\chi(T^2) = H^1(T^2,\mathbb{R})/H^1(T^2,\mathbb{Z})$ with fibre

$$H^1(T^2,\mathbb{Z})/(\mathbb{Z} < PD(l) >).$$

In the case of Y or Y_1, we consider the universal cover

$$\chi_0(T^2,Y) = H^1(T^2,\mathbb{R}) = \chi_0(T^2,Y_1).$$

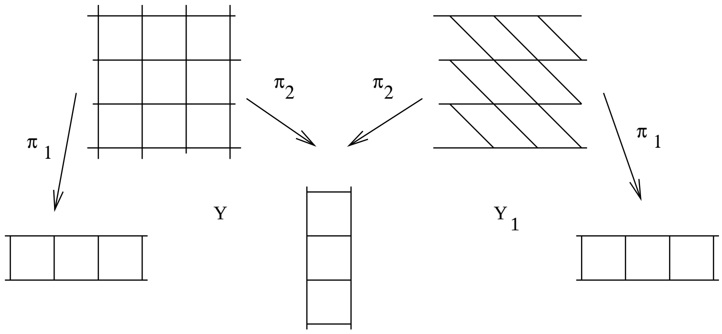

Figure 3: The coverings $\chi_0(T^2, Y)$ and $\chi_0(T^2, Y_1)$

In this case we have covering maps

$$\begin{tikzcd}
 & \chi_0(T^2, Y) \arrow[dl, "\pi_1"'] \arrow[dr, "\pi_2"] & \\
\chi_0(T^2, V) \arrow[dr] & & \chi_0(T^2, \nu(K)) \arrow[dl] \\
 & \chi(T^2) &
\end{tikzcd}$$

where the maps π_i are $\mathbb{Z}$-coverings. The maps π_i for Y and Y_1 are illustrated in figure 3

With this notation understood, we can proceed in comparing the moduli spaces of solutions of the monopole equations on the manifold $(Y(r), \mathfrak{s}_r)$ for large r, and on the manifolds $(V, \mathfrak{s}')$ and $(\nu(K), \mathfrak{s}'')$.

We first show that, upon stretching the neck to infinity, solutions of the monopole equations on $(Y(r), \mathfrak{s}_r)$ induce a pair of finite energy solutions on $(V, \mathfrak{s}')$ and $(\nu(K), \mathfrak{s}'')$.

Lemma 4.1. *Assume that the moduli spaces $\mathcal{M}^*(Y(r), \mathfrak{s}_r)$ are non-empty for $r >> 0$. Suppose given a compact set $K \subset \bigcap_{r>0} Y(r)$. Then a sequence of elements $[A_r, \psi_r]$ of $\mathcal{M}^*(Y(r), \mathfrak{s}_r)$ has a subsequence that converges smoothly on K to either a solution (A', ψ'), with $[A', \psi']$ in $\mathcal{M}^*(V, \mathfrak{s}')$, or to a solution $(A'', 0)$, with $[A'', 0]$ in $\mathcal{M}^{red}(\nu(K), \mathfrak{s}'') = \chi_0(T^2, \nu(K))$.*

Then we prove the following gluing theorem.

Theorem 4.2. *Consider the moduli space $\mathcal{M}_V$ of finite energy monopoles on V, with respect to the trivial* Spin^c*-structure $\mathfrak{s}'$. There are well defined gluing maps on the fibred products*

$$\#_Y : \mathcal{M}_V^* \backslash \partial_\infty^{-1}(U_\theta) \times_{\chi_0(T^2, Y)} \chi(\nu(K)) \to \mathcal{M}_{Y(r)}^*,$$

$$\#_{Y_1} : \mathcal{M}_V^* \backslash \partial_\infty^{-1}(U_\theta) \times_{\chi_0(T^2, Y_1)} \chi(\nu(K)) \to \mathcal{M}_{Y_1(r)}^*,$$

$$\#_{Y_0} : \mathcal{M}_V^* \backslash \partial_\infty^{-1}(U_\theta) \times_{\chi_0(T^2, Y_0)} \chi(\nu(K)) \to \cup_{s_k} \mathcal{M}_{Y_0(r)}^*(\mathfrak{s}_k).$$

Here U_θ is a small neighbourhood of the "bad point" θ in $\chi(T^2)$. The fibre product is taken with respect to the pullbacks under the projections

$$\begin{tikzcd} & \chi_0(T^2, Y) \arrow[dl, "\pi_1"'] \arrow[dr, "\pi_2"] & \\ \chi_0(T^2, V) & & \chi_0(T^2, \nu(K)) \end{tikzcd}$$

of the images of the boundary value map

$$\mathcal{M}_V^* \stackrel{\partial_\infty}{\to} \chi_0(T^2, V)$$

and of the embedding

$$\mathcal{M}^{red}_{\nu(K)} \to \chi_0(T^2, \nu(K)).$$

Similarly, for Y_1 and Y_0 the gluing maps are diffeomorphisms for all $r \geq r_0$. The Spin^c*-structure $\mathfrak{s}_k$ varies in the family of* Spin^c*-structures on Y_0 that restrict to the trivial one on V and $\nu(K)$. The choice of the* Spin^c *structure $\mathfrak{s}_k$ on $Y_0(r)$ in the image of the gluing map is determined by the projection of the asymptotic value $a_\infty \in \chi_0(T^2, Y_0)$ on the fibre of the covering map $\chi_0(T^2, Y_0) \to \chi(T^2)$, as will be made precise in the following.*

In order to prove Theorem 4.2 we need some preliminary analysis of the equations on Y and $Y(r)$.

4.1 Gluing Spin^c-structures

With the notation introduced in the previous section, we now identify the set of Spin^c-structures on Y_0 that restrict to the trivial $\mathfrak{s}'$ and $\mathfrak{s}''$ on V and $\nu(K)$.

Lemma 4.3. *Consider the manifolds $(V, \mathfrak{s}')$ and $(\nu(K), \mathfrak{s}'')$, with $\mathfrak{s}'$ and $\mathfrak{s}''$ the trivial* Spin^c*-structures. Then, the family of* Spin^c*-structures on Y_0 that restrict to $\mathfrak{s}$ and $\mathfrak{s}''$ on V and $\nu(K)$ can be identified with the fibre*

$$H^1(T^2, \mathbb{Z})/(\mathbb{Z} < PD(l) >)$$

of the covering map

$$\chi_0(T^2, Y_0) \to \chi(T^2).$$

Proof. The Spin^c structure $\mathfrak{s}$ on Y_0 is obtained by gluing the trivial line bundle over V and $\nu(K)$ with a non-trivial transition function given by a gauge transformation λ over T^2 that does not extend to V and $\nu(K)$. The class $[\lambda]$ in $H^1(T^2, \mathbb{Z})/\mathbb{Z} < PD(l) >$ counts exactly, up to a factor 2, the Chern class of the line bundle. In fact, let $[\lambda] = a < PD_{T^2}(m) >$. The

Chern class of the line bundle is given by $c_1(\mathfrak{s}) = bPD_{Y_0}(m)$, since m is the generator of $H_1(Y_0, \mathbb{Z})$. We have

$$b = \int_\sigma \frac{1}{\pi} F_A = \frac{1}{\pi} \int_l \lambda^{-1} d\lambda = 2a. \tag{47}$$

In other words, we can think of the set of Spin^c structures on Y_0 as identified with the elements of $2\ H^2(Y_0, \mathbb{Z})$. In particular, in the exact sequence

$$H^1(Y_0, \mathbb{Z}) \stackrel{i^*}{\to} H^1(V, \mathbb{Z}) \oplus H^1(\nu(K), \mathbb{Z}) \stackrel{j^*}{\to} H^1(T^2, \mathbb{Z})$$

$$\stackrel{\Delta}{\to} 2\ H^2(Y_0, \mathbb{Z}) \stackrel{i^*}{\to} 2\ H^2(V, \mathbb{Z}) \oplus 2\ H^2(\nu(K), \mathbb{Z}),$$

the Spin^c structures that reduce to the trivial one on V and $\nu(K)$ are given by $Ker(i^*) \cong Im(\Delta)$. Thus, they can be identified with the elements of $H^1(T^2, \mathbb{Z})/Ker(\Delta) \cong H^1(T^2, \mathbb{Z})/Im(j^*)$. These are the elements of

$$H^1(T^2, \mathbb{Z})/(\mathbb{Z} < PD(l) >),$$

that is the elements of the fibre of the map $\chi_0(T^2, Y_0) \to \chi(T^2)$, under the identification $\frac{1}{2} c_1(det\ \mathfrak{s}_k) = k \in\ H^1(T^2, \mathbb{Z})/(\mathbb{Z} < PD(l) >)$.

$\square$

This lemma will be relevant in the gluing theorem. In fact, gluing along an asymptotic value in $\chi_0(T^2, Y_0)$ that lies in the k-th level in the fibre over $\chi(T^2)$ gives rise to solutions over Y_0 with respect to the choice of the Spin^c structure $\mathfrak{s}_k$.

4.2 Monopoles on a 3-manifold with a long cylinder

The purpose of this subsection is to show that, upon stretching the neck $[-r, r] \times T^2$ to infinity, solutions $S_r = (A_r, \psi_r)$ on $Y(r)$ approach a pair of finite energy solutions $S' = (A', \psi')$, and $S'' = (A'', 0)$ on the two manifolds with infinite cylinders V and $\nu(K)$, respectively.

We need to introduce some *ad hoc* assumptions on the class $\mathcal{P}$ of perturbations, so that it behaves nicely under the splitting $r \to \infty$. We consider perturbations of the monopole equations as in (6), induced by the perturbations of the Chern-Simons-Dirac functional (5) previously discussed. Thus, we have a choice of complete L^2 bases $\{\nu_i(r)\}_{i=1}^\infty$ and $\{\mu_j(r)\}_{j=1}^\infty$ of the imaginary valued 1-forms. Notice that such choice depends on the metric on $Y(r)$. There is no harm in assuming that the following pointwise estimates are satisfied:

$$\sup_{T^2 \times [-r,r]} |\nu_i(r)| \le \sup_{T^2 \times [-r_0,r_0]} |\nu_i(r_0)|$$

$$\sup_{T^2 \times [-r,r]} |\mu_j(r)| \le \sup_{T^2 \times [-r_0,r_0]} |\mu_j(r_0)|,$$

for all $r \geq r_0$, and for all i and j.

In the arguments that follow, we want to ensure that, in the stretching process, the perturbation remains mostly supported on the knot complement, so that in the limit we get unperturbed equations on the solid torus and perturbed equations on the knot complement. In order to achieve this result we introduce a suitable rescaling, so that we can make the perturbation exponentially small on the long cylinder and on $\nu(K)$. Consider smooth cutoff functions $f_r(s)$ defined for $s \in [-r, r]$, with values in $[0,1]$,

$$f_r(s) = e^{-\delta(s+r)},$$

for $-r + \epsilon \leq s \leq r - \epsilon$, patched smoothly to $f_r \equiv 1$ on $V \cup_{T^2} T^2 \times [-r, -r+\epsilon]$ and to $f_r \equiv e^{-2\delta r}$ on $\nu(K) \cup_{T^2} T^2 \times [r-\epsilon, r]$. The weight δ satisfies

$$\delta \geq \frac{1}{2} min\{\lambda_{a_\infty} | a_\infty \in \chi(T^2) \backslash U_\theta\}.$$

The elements $\{f_r \nu_i(r)\}_{i=1}^\infty$ and $\{f_r \mu_j(r)\}_{j=1}^\infty$ still give complete bases, and in the limit $r \to \infty$, they induce elements $\{\nu_i(\infty)\}_{i=1}^\infty$ and $\{\mu_j(\infty)\}_{j=1}^\infty$ exponentially small, with a decay rate of at least δ, along the cylindrical end of V. With a slight abuse of notation we refer to the corresponding class of perturbations on $Y(r)$ with the same notation $\mathcal{P}_\delta$ used for the exponentially decaying perturbations on the non-compact manifold V with an infinite cylindrical end.

The convergence result we prove in this section depends essentially on a uniform pointwise bound on the solutions (A_r, ψ_r) in $\mathcal{M}_{Y(r)}$ which is independent of r. The argument for the manifold $Y_1(r)$ is the same. The case of the manifold $Y_0(r)$ is also analogous, whenever Y_0 is endowed with a Spin^c structure that restricts to the trivial Spin^c structures on V and $\nu(K)$.

Lemma 4.4. *Suppose we are given solutions (A_r, ψ_r) of the perturbed Seiberg-Witten equations (6) on the compact 3-manifold $Y(r)$, with a perturbation satisfying the decaying properties described above. Then we have the pointwise bounds*

$$|\psi_r(y)| \leq s(Y),$$

for all $y \in Y(r)$, and

$$|F_{A_r}(y)| \leq C(s(Y))^2,$$

where $s(Y)$ is a constant which does not depend on r. Moreover, suppose we are given a compact set K of the form $V \cup_{T^2} [0, r_0] \times T^2$ or $\nu(K) \cup_{T^2} [-r_0, 0] \times T^2$ in $Y(r)$ with $r > r_0$. Then we have a bound

$$\|\nabla_{A_r} \psi_r\|_{L^2(K)} \leq C' s(Y)^2 Vol(K),$$

for all $y \in Y(r)$.

Proof. Consider $s(Y(r)) = \max_{y\in Y(r)}\{-s(y)+C,0\}$, where $s(y)$ is the scalar curvature and C is a constant depending only on the perturbation, $C = C(U,V)$ as in Lemma 2.3, with $P = (U,V) \in \mathcal{P}_\delta$. Notice that, by the assumption on the behaviour of the perturbation under the stretching $r \to \infty$, we can assume that C is independent of r. The minimum of the scalar curvature also remains constant upon stretching the cylinder $T^2 \times [-r,r]$, so that we have $s(Y(r)) = s(Y)$ for all $r > 0$.

The Weitzenböck formula provides a uniform bound on the spinors in terms of the scalar curvature as in Lemma 2.3, namely at a point y where $|\psi_r(y)|$ achieves a maximum we have either $\psi_r(y) = 0$ or $|\psi_r(y)|^2 \leq -s(y)+C$. The bound

$$\|\nabla_{A_r}\psi_r\| \leq C's(Y)^2 Vol(K)$$

also follows from the Weitzenböck formula [21], and from the exponential decay of ψ along the cylinder, so that the boundary term in the integration by parts of the term $\nabla^*_{A_r}\nabla_{A_r}\psi_r$ is sufficiently small for r sufficiently large. The pointwise bound for the curvature form F_{A_r} follows from the bound on $|\psi_r|$ and from the equations. □

Remark 4.5. *The pointwise bound on the curvature obtained in Lemma 4.4 implies the estimate* $|c_1(det\ \mathfrak{s})| \leq \frac{Cs(Y)^2}{4\pi}$. *Thus, only finitely many of the* Spin^c *structures* $\mathfrak{s}_k$ *on* Y_0 *determine a non-empty* $\mathcal{M}_{Y_0}(\mathfrak{s}_k)$.

Now we can establish some results in the topology of smooth convergence on compact sets. With the notation of the previous section, we have the following Lemma.

Lemma 4.6. *Assume that the metric on* $Y(r)$ *is chosen according to Lemma 3.18, and the perturbation is chosen as specified in the beginning of this section. Suppose the moduli spaces* $\mathcal{M}^*_{Y(r)}(\mathfrak{s}_r)$ *are non-empty for* $r >> 0$. *Fix a compact set* $K \subset \bigcap_{r>0} Y(r)$. *Suppose we are given a sequence of elements* $\alpha_r = (A_r,\psi_r)$ *that represent gauge classes* $[\alpha_r]$ *in* $\mathcal{M}^*_{Y(r)}(\mathfrak{s}_r)$.

- *(a) There exist gauge transformations* λ_r *on* $Y(r)$, *such that there is a subsequence of* $\lambda_r\alpha_r$ *that converges smoothly on* K *to either a solution* $\alpha' = (A',\psi')$ *with* $[A',\psi']$ *in* $\mathcal{M}_V(\mathfrak{s}')$, *or to a solution* $\alpha'' = (a''_\infty, 0)$, *with* $[a''_\infty, 0]$ *in*

$$\mathcal{M}^{red}_{\nu(K)}(\mathfrak{s}'') = \chi(\nu(K)) \subset \chi_0(T^2,\nu(K)).$$

- *(b) The solutions* $\lambda_r\alpha_r$ *restricted to the cylinder* $[-r,r]\times T^2$ *converge smoothly on compact sets to a constant flat connection* a_∞ *on* T^2.

- *(c) Let* $\partial_\infty\alpha' = a'_\infty$ *be the asymptotic limit, that is, an element of* $\chi_0(T^2,V)$. *Then there exist two gauge transformations* λ' *and* λ'' *on* T^2 *that extend to* V *and* $\nu(K)$ *respectively, such that we have* $\lambda''a''_\infty = \lambda'a'_\infty$ *in* $\chi_0(T^2,Y) = H^1(T^2,\mathbb{R})$.

(d) *In the case of Y_0, we obtain similarly $\lambda'' a''_\infty = \lambda' a'_\infty$ in $\chi_0(T^2, Y_0)$. The gauge transformation $(\lambda')^{-1}\lambda''$ over T^2 determines a cohomology class in $H^1(T^2,\mathbb{Z})/\mathbb{Z} < PD(l) >$. This is the element of $H^1(T^2,\mathbb{Z})/\mathbb{Z} < PD(l) >$ uniquely associated to the* Spin^c *structure $\mathfrak{s}$ on Y_0.*

Proof. Decompose the manifold $Y(r)$ as

$$V \cup_{T^2\times\{-r\}} T^2 \times [-r, 0] \text{ and } \nu(K) \times_{T^2\times\{r\}} T^2 \times [0, r].$$

Suppose given a fixed compact set $K = V \cup_{T^2} T^2 \times [-r_0, r_0]$ in $Y(r)$. We show that a sequence of elements $[A_r, \psi_r]$ of $\mathcal{M}(Y(r), \mathfrak{s}_r)$ has a subsequence that converges smoothly on K to a solution of the equations. The same result holds for compact sets K of the form $T^2 \times [-r_0, r_0] \cup_{T^2} \nu(K)$. The bound of Lemma 4.4 shows that there is a uniform bound for the L^2 norms

$$\|\psi_r\|_{L^2(K)} \leq s(Y) Vol(K),$$

$$\|F_{A_r}\|_{L^2(K)} \leq C s(Y) Vol(K).$$

This implies an L^2_1 bound on the connections

$$\|A_r - A_0\|_{L^2_1(K)} \leq \tilde{C} s(Y) Vol(K).$$

The bound

$$\|\nabla_{A_r}\psi_r\| \leq C' s(Y)^2 Vol(K)$$

of Lemma 4.4, together with the L^2 bound on the spinors, implies a bound on the L^2_1-norms of the spinors by the elliptic estimate.

There are gauge transformations λ_r in the connected component of the gauge group $\mathcal{G}_{Y(r)}$, such that the forms $\lambda_r A_r - A_0$ are co-closed. Thus the elliptic estimate for the operator $d + d^*$ gives a bound on the L^2_2-norms of the connections. A bootstrapping argument bounds the higher Sobolev norms.

Thus, upon passing to a subsequence, we have elements (A_{r_i}, ψ_{r_i}) that converge smoothly on K to a solution of the equations. This defines a solution (A', ψ') on V with the Spin^c structure $\mathfrak{s}'$ extended by translation invariance on the infinite end $T^2 \times [0, \infty)$. The case of $\nu(K)$ is analogous. On $\nu(K)$, by Lemma 3.19, a finite energy solution will necessarily be reducible.

Thus, we have shown that a family of solutions on the manifolds $(Y(r), \mathfrak{s}_r)$ converges smoothly on compact sets to a pair of solutions on $(V, \mathfrak{s}')$ and $(\nu(K), \mathfrak{s}'')$. The following estimate on the energy on the tube $[-r, r] \times T^2$ shows that the asymptotic value of (A', ψ')

matches the flat connection $(a''_\infty, 0)$. In other words, we show that in the process of stretching the neck no part of the energy remains concentrated on the cylinder.

We need to prove the following fact. Suppose the moduli spaces $\mathcal{M}_{Y(r)}(\mathfrak{s}_r)$ are non-empty for $r >> 0$. Then the restriction to the cylinder $[-1,1]\times T^2$ in $Y(r)$ of a sequence of solutions $x_r \in \mathcal{M}_{Y(r)}(\mathfrak{s}_r)$ converges to a translation invariant solution of equations (17).

Consider the elements $x_r = (A_r, \psi_r)$ restricted to the cylinder $[-r,r]\times T^2$. Up to a gauge transformation, they can be written in the form

$$x_r(s) = (a_r(s), \alpha_r(s), \beta_r(s)).$$

There is a uniform bound, independent of r for the variation of the functional f along the cylinder,

$$|f(a_r(r), \alpha_r(r), \beta_r(r)) - f(a_r(-r), \alpha_r(-r), \beta_r(-r))| \leq C.$$

In fact, we have pointwise estimates, independent of r, as in Lemma 4.4. This gives a uniform bound on the L^2 norms of ψ_r and A_r on any fixed section $T^2 \times \{r_0\}$ of the cylinder,

$$\|\lambda_r(a_r(\pm r), \alpha_r(\pm r), \beta_r(\pm r))\|_{L^2(T^2)} \leq C.$$

This implies that the values of the functional at $\pm r$ are also uniformly bounded, independently of r,

$$|f(a_r(\pm r), \alpha_r(\pm r), \beta_r(\pm r))| \leq C.$$

The variation of the functional is given by

$$f(a_r(r), \alpha_r(r), \beta_r(r)) - f(a_r(-r), \alpha_r(-r), \beta_r(-r)) =$$

$$\int_{-r}^{r} \|\nabla f(a_r(s), \alpha_r(s), \beta_r(s))\|^2 ds,$$

hence the functional f is monotone along the cylinder. Thus we can find a subsequence $(a_{r_k}(s), \alpha_{r_k}(s), \beta_{r_k}(s))$ such that

$$\int_{-1}^{1} \|\nabla f(a_{r_k}(s), \alpha_{r_k}(s), \beta_{r_k}(s))\|^2 ds \leq 1/k.$$

Thus, for k large enough, we have that $\|\nabla f\|$ is smaller than ϵ on the interval $[-1,1]$. Moreover, we know, by assumption on the class of perturbations considered, that the perturbation term $P_r(x_r(s), s)$ is bounded by an estimate

$$\|P_r(x_r(s), s)\|^2_{L^2(T^2\times\{s\})} \leq Ce^{-\delta(s+r)}.$$

Thus, using the estimate (25), we obtain that $x_r(s) = (a_r(s), \alpha_r(s), \beta_r(s))$ is close to a critical point a_∞. Thus, up to gauge transformations, the sequence of solutions (A_r, ψ_r) has a subsequence (A_{r_i}, ψ_{r_i}) that converges smoothly on compact sets to a pair $((A', \psi'), (a''_\infty, 0))$. In the asymptotic limit we get

$$\lim_{s\to\infty} \lambda'(A', \psi') = \lambda' a'_\infty = \lambda'' a''_\infty.$$

In the case of the manifold Y_0, by the result of Lemma 4.3, we have

$$[(\lambda')^{-1}\lambda''] = c_1(\mathfrak{s})$$

in $H^1(T^2, \mathbb{Z})/\mathbb{Z} < PD(l) >$, under the identifications

$$[\lambda] \in H^1(T^2, \mathbb{Z})/\mathbb{Z} < PD(l) >$$

$$[\lambda] = a < PD_{T^2}(m) >$$

$$c_1(\mathfrak{s}) = bPD_{Y_0}(m)$$

$$\int_\sigma \frac{1}{\pi} F_A = \frac{1}{\pi} \int_l \lambda^{-1} d\lambda$$

used in Lemma 4.3.

$\square$

This completes the proof of the convergence part of the gluing theorem for generators. In fact, we have seen that, a gauge class in the moduli space $\mathcal{M}_{Y(r)}(\mathfrak{s})$, with a long neck $r >> 0$, determines an element in

$$\mathcal{M}^*_V \times_{\chi_0(T^2,\mathbb{Z})} \chi(\nu(K)).$$

The reducible case

$$\mathcal{M}^{red}_V \times_{\chi_0(T^2,\mathbb{Z})} \chi(\nu(K))$$

will be discussed separately in Lemma 5.2.

4.3 Gluing map for 3-dimensional monopoles

In this section we prove the gluing theorem stated in 4.2. We prove the gluing theorem under the hypothesis that the gluing takes place away from the lattice of "bad points" in $H^1(T^2, \mathbb{R})$ corresponding to the reducible point θ in the character variety $\chi(T^2)$. Later in this section we justify the assumption that gluing at the bad point can be avoided. Moreover, we state the gluing theorem for the manifold $Y(r)$ with a choice of the metric satisfying Lemma 3.18. The case of Y_1 and Y_0, with the deformation (46) of the induced metrics, will also be discussed later in this section.

We define a pre-gluing operation, where we splice together solutions in $\mathcal{M}^*_V$ and $\chi(\nu(K))$ with matching asymptotic values, via a smooth cutoff function. This produces an approximate solution $(A', \psi')\#_r(a''_\infty, 0)$ of the monopole equations on $Y(r)$.

We then recall an essential result of Capell-Lee-Miller [3] on the decomposition of small eigenvalues of the linerization $H_{(A',\psi')\#_r(a''_\infty,0)}$ at the approximate solution in terms of elements in $Ker(H_{(A',\psi')})$, $Ker(H_{(a''_\infty,0)})$ and $Ker(Q_{a_\infty}) = H^1(T^2,\mathbb{R})$.

The results of [3] allow us to construct the obstruction bundle for the gluing. We prove that the obstruction vanishes, hence the expected gluing result holds.

We can choose smooth cutoff functions $\rho_r^\pm(s)$ with values in $[0,1]$. The function ρ_r^+ is defined for $s\in[0,\infty)$, and ρ_r^- is defined for $s\in(-\infty,0]$, satisfying $\rho_r^+(s)\equiv 1$ for $s\le r-1$ and $\rho_r^+(s)\equiv 0$ for $s\ge r+1$, and $\rho_r^-(s)\equiv 1$ for $s\ge -r+1$ and $\rho_r^-(s)\equiv 0$ for $s\le -r-1$. Define the pre-gluing map with values in $\mathcal{B}(Y(r))$, or in the case of Y_0 with values in $\cup_{\mathfrak{s}_k}\mathcal{B}(Y_0(r),\mathfrak{s}_k)$, by setting

$$(A',\psi')\#_r^0(a''_\infty,0) = \begin{cases} \lambda'(A',\psi') & \text{on } V \\ a_\infty + \rho_r^+(s)\lambda'(a'(s),\psi'(s)) + & \\ \rho_r^-(-s)\lambda''(a''_\infty,0) & r-1\le s\le r+1 \\ \lambda''(a''_\infty,0) & on\ \nu(K) \end{cases}$$

Here $(a'(s),\psi'(s))$ is gauge equivalent to (A',ψ'), in a temporal gauge along $s\in[r-1,\infty)$.

An approximate solution is by definition an element in the image of this pre-gluing map.

We denote by $\mathcal{U}(a_\infty,r)$ the image of the pregluing map

$$\#_r^0 : \mathcal{M}_V^*(a_\infty)\times[a''_\infty,0]\to\mathcal{B}(Y(r)),$$

where

$$\mathcal{M}_V^*(a_\infty) = \partial_\infty^{-1}(a_\infty)\subset\mathcal{M}_V^*.$$

Consider the operators

$$H_{(A',\psi')\#_r^0(a''_\infty,0)}(f,\alpha,\phi) = \begin{cases} L_{(A',\psi')\#_r^0(a''_\infty,0)}(\alpha,\phi) + G_{(A',\psi')\#_r^0(a''_\infty,0)}(f) \\ G^*_{(A',\psi')\#_r^0(a''_\infty,0)}(\alpha,\phi) \end{cases}$$

$$H_{(A',\psi')}(f,\alpha,\phi) = \begin{cases} L_{(A',\psi')}(\alpha,\phi) + G_{(A',\psi')}(f) \\ G^*_{(A',\psi')}(\alpha,\phi) \end{cases}$$

$$H_{(a''_\infty,0)}(f,\alpha,\phi) = \begin{cases} L_{(a''_\infty,0)}(\alpha,\phi) + G_{(a''_\infty,0)}(f) \\ G^*_{(a''_\infty,0)}(\alpha,\phi) \end{cases},$$

where the operator L is defined as in (30). We think of $H_{(A',\psi')\#_r^0(a''_\infty,0)}$ as acting on the elements (α,ϕ) in the L^2 tangent space of the configuration space over the closed manifold $Y(r)$. The operators $H_{(A',\psi')}$ and $H_{(a''_\infty,0)}$ are the operators defined in the deformation complex (29), acting on L^2 forms and spinors. We still denote with $H_{(A',\psi')}$ and $H_{(a''_\infty,0)}$ the operators defined in the deformation complex (31) acting on the extended L^2 spaces of connections and spinors, over V and $\nu(K)$ respectively.

We use the following result, which follows from the work of Capell, Lee, and Miller [3].

Proposition 4.7. *The dimension $N(r,o(1))$ of the span of eigenvectors of the operator $H_{(A',\psi')\#_r^0(A'',\psi'')}$ with eigenvalues satisfying $\mu(r)\to 0$ as $r\to\infty$ is given by*

$$N(r,o(1)) = \dim Ker_{L^2}(H_{(A',\psi')}) + \dim Ker_{L^2}(H_{(a''_\infty,0)}) + \dim Ker(Q_{a_\infty}).$$

The dimension $N(r,r^{-(1+\epsilon)})$ of the span of eigenvectors of the operator $H_{(A',\psi')\#_r^0(a''_\infty,0)}$ with eigenvalues $\mu < r^{-(1+\epsilon)}$ is given by

$$N(r,r^{-(1+\epsilon)}) = \dim Ker_{L^2}(H_{(A',\psi')}) + \dim Ker_{L^2}(H_{(a''_\infty,0)}) + \dim \ell_1\cap\ell_2,$$

where $\ell_1\cap\ell_2$ is the intersection of the two Lagrangian submanifolds determined by the extended L^2 solutions of $H_{(A',\psi')}(\alpha,\phi)=0$ and $H_{(a''_\infty,0)}(\alpha,\phi)=0$ in the tangent space $H^1(T^2,\mathbb{R}) = Ker(Q_{a_\infty})$ of $\chi_0(T^2)$.

Proof. Notice that the second part of the statement follows directly from Theorem A of [3]. In order to prove the first claim it is sufficient to check that elements of $H^1(T^2,\mathbb{R}) = Ker(Q_{a_\infty})$ give rise to approximate eigenfunctions on $Y(r)$ with slowly decaying eigenvalues, that is, with eigenvalues $\mu(r)$ satisfying $\mu(r)\to 0$ at most like $1/r$. The claim then follows from Proposition 6.B of [3].

Suppose given an element $\xi\in Ker(Q_{a_\infty})$. It is easy to check that, if $\chi(s)$ is a cutoff function supported in $[r/2-\epsilon, 3r/2+\epsilon]$ satisfying $\chi(s)\equiv 1$ on $[r/2,r]$, we have the estimate

$$\frac{\|(\partial_s + Q_{a_\infty})\chi\xi\|_{L^2(Y(r))}}{\|\chi\xi\|_{L^2(Y(r))}} \leq \frac{C}{r}.$$

This implies a similar estimate for the operator $H_{(A',\psi')\#_r^0(a''_\infty,0)}$ on $Y(r)$, for $r\geq r_0$ large enough, and this proves the claim.

□

Proposition 4.7 yields the following.

Corollary 4.8. *There are no fast decaying eigenvalues, that is, in our problem $N(r,r^{-(1+\epsilon)})=0$. However, there is a two dimensional family of eigenvectors of the linearization $H_{(A',\psi')\#_r^0(a''_\infty,0)}$ at the approximate solution $(A',\psi')\#_r^0(a''_\infty,0)$, which correspond to slowly decaying eigenvalues, satisfying $\mu(r)\to 0$ at most like $1/r$. The two dimensional family is parametrized by the elements of $H^1(T^2,\mathbb{R}) = Ker(Q_{a_\infty})$.*

Proof. We have

$$\dim Ker_{L^2}(H_{(A',\psi')}) = \dim Ker_{L^2}(H_{(a''_\infty,0)}) = 0.$$

Moreover, for a generic choice of the perturbation of the monopole equations on V, the Lagrangian subspaces ℓ_1 and ℓ_2 intersect transversely.

Thus, we have $N(r,o(1)) = \dim Ker(Q_{a_\infty})$ and $N(r,r^{-(1+\epsilon)})=0$.

□

In order to show that the approximate solutions in $\mathcal{U}(a_\infty, r)$ can be deformed to actual solutions of the monopole equations on $Y(r)$, we construct the obstruction bundle associated to the span of eigenvectors corresponding to the small eigenvalues. We briefly recall the construction of the obstruction bundle [16], [27].

Consider the extended Hessian operator H_x defined in (7). Choose $\mu = \mu(r) > 0$ such that $\mu(r)$ is not an eigenvalue of $H_{x_r} = H_{(A',\psi')\#^0_r(a''_\infty,0)}$, for all pre-glued elements $x_r = (A',\psi')\#^0_r(a''_\infty, 0)$ in $\mathcal{U}(a_\infty, r)$. Consider the projection maps $\Pi(\mu(r), x_r)$ onto the span of the eigenvectors of H_{x_r} with eigenvalues smaller than $\mu(r)$.

For $x = (A, \psi)$ let us introduce the notation

$$\Sigma(x) = \begin{pmatrix} *F_A - \sigma(\psi,\psi) \\ \partial\!\!\!/_A \psi \end{pmatrix} + P(A,\psi),$$

with

$$P(A,\psi) = \begin{pmatrix} \sum_j \frac{\partial U}{\partial \tau_j} \mu_j \\ \sum_j \frac{\partial V}{\partial \zeta_j} \nu_j . \psi \end{pmatrix}$$

a perturbation in the class $\mathcal{P}_\delta$ on $Y(r)$. Moreover we define the non-linear term

$$N_{A,\psi}(a,\phi) = \begin{pmatrix} \sigma(\phi,\phi) \\ a.\phi \end{pmatrix} + \mathcal{N}P_{A,\psi}(a,\phi),$$

where the non-linear part $\mathcal{N}P_{A,\psi}$ of the perturbation is

$$\mathcal{N}P_{A,\psi} = P((A,\psi) + (a,\phi)) - \mathcal{D}P_{A,\psi}(a,\phi),$$

with $\mathcal{D}P_{A,\psi}$ defined in (7),

$$\mathcal{D}P_{A,\psi}(a,\phi) = \begin{pmatrix} \sum_{i,j} \frac{\partial^2 U}{\partial \tau_i \partial \tau_j} \tau_i(a) \mu_j \\ \sum_j \frac{\partial V}{\partial \zeta_j} \nu_j . \phi + \sum_{i,j} \frac{\partial^2 V}{\partial \zeta_i \partial \zeta_j} \zeta_i(\psi,\phi) \nu_j . \psi \end{pmatrix}.$$

Suppose given an element (a, ϕ) on $Y(r)$ such that $x_r + (a, \phi)$ is a solution of the monopole equations on $Y(r)$. Then (a, ϕ) satisfies

$$H_{x_r}(a,\phi) + N_{x_r}(a,\phi) + \Sigma(x_r) = 0,$$

where N and Σ are defined as above.

We can introduce the operator $\Delta_{x_r} = H_{x_r} H^*_{x_r}$. For $\mu(r)$ small enough we can assume that (a, ϕ) is in $Ker(\Pi(\mu(r), x_r))$, and we can write $(a, \phi) = H^*_{x_r}(b, \xi)$. Then the condition that $x_r + (a, \phi)$ is a solution of (6) can be written as the system of two equations

$$\Pi(\mu(r), x_r)(NH^*_{x_r}(b,\xi) + \Sigma(x_r)) = 0 \tag{48}$$

$$\Delta_{x_r}(b,\xi) + (1 - \Pi(\mu(r), x_r))(NH^*_{x_r}(b,\xi) + \Sigma(x_r)) = 0. \tag{49}$$

The second equation can be written as the fixed point problem

$$(b,\xi) = -\Delta^{-1}_{x_r}(1 - \Pi(\mu(r), x_r))(NH^*_{x_r}(b,\xi) + \Sigma(x_r)).$$

Lemma 4.9. *There exists a positive constant $C > 0$, such that the following holds. Assume that we have*

$$\mu(r) > \sqrt{2C}\epsilon(r)^{1/4} \tag{50}$$

and

$$\epsilon(r) > \frac{C'}{C}e^{-\delta r}, \tag{51}$$

for all $r \geq r_0$. If the pre-glued solution x_r satisfies

$$\|\Sigma(x_r)\|_{L^2(Y(r))} \leq C'e^{-\delta r},$$

for some $\delta > 0$, then the map

$$\mathcal{T}_r(b,\xi) = -\Delta_{x_r}^{-1}(1 - \Pi(\mu(r), x_r))(NH_{x_r}^*(b,\xi) + \Sigma(x_r))$$

maps the ball

$$B_{\epsilon(r)} = \{(b,\xi)|\ \|(b,\xi)\|^2_{L^2(Y(r))} \leq \epsilon(r)\}$$

to itself and is a contraction on $B_{\epsilon(r)}$.

Proof. We have an estimate

$$\|\mathcal{T}_r(b,\xi)\| \leq \frac{1}{\mu(r)^2}\|NH_{x_r}^*(b,\xi) + \Sigma(x_r)\| \leq \frac{C}{\mu(r)^2}\|(b,\xi)\|^2 + \frac{1}{\mu(r)^2}\|\Sigma(x_r)\|.$$

Thus, we obtain

$$\|\mathcal{T}_r(b,\xi)\| \leq \frac{C\epsilon(r)}{\mu(r)^2} + \frac{C'e^{-\delta r}}{\mu(r)^2}.$$

The conditions (50) and (51) then imply that the expression above is bounded by $\epsilon(r)^{1/2}$, since we have

$$\frac{C\epsilon(r)^{1/2}}{\mu(r)^2} < \frac{1}{2}$$

and

$$\frac{C'e^{-\delta r}}{\mu(r)^2\epsilon(r)^{1/2}} < \frac{C'e^{-\delta r}}{2C\epsilon(r)} < \frac{1}{2}.$$

Thus the first statement follows.

Similarly, by verifying that the quadratic term satisfies an estimate

$$\|N(a,\phi) - N(\tilde{a},\tilde{\phi})\| \leq C(\|(a,\phi)\| + \|(\tilde{a},\tilde{\phi})\|)\|(a,\phi) - (\tilde{a},\tilde{\phi})\|,$$

we obtain that $\mathcal{T}_r$ is a contraction, cf. the similar argument in [16]. The constant of the contraction is bounded by $\frac{C\epsilon(r)^{1/2}}{\mu(r)^2}$. $\square$

If the equation (49) admits a unique solution (b,ξ), then the condition that $x_r + (\alpha,\phi)$ is a solution of the monopole equations on $Y(r)$ can be rephrased as the condition that (48) is satisfied, with (b,ξ) the unique solution of (49). Thus, we can define the obstruction bundle as follows [27].

Over $\mathcal{U}(a_\infty, r)$ define a bundle with fibre the span of small eigenvalues of the linearization H_{x_r}, $x_r \in \mathcal{U}(a_\infty, r)$. By Proposition 4.7, we can identify the fibre is $H^1(T^2,\mathbb{R})$. There is a canonical section $s_{\mu(r)}$ given by

$$s_{\mu(r)}(x_r) = \Pi(\mu(r), x_r)(NH^*_{x_r}(b,\xi) + \Sigma(x_r)),$$

where (b,ξ) is the unique solution of (49). An approximate solution x_r in $\mathcal{U}(a_\infty, r)$ can be deformed to an actual solution of the monopole equations on $Y(r)$ if and only if we have $x_r \in s^{-1}_{\mu(r)}(0)$.

Lemma 4.10. *For a choice of*

$$\mu(r) = r^{-(1+\epsilon)}$$

and

$$\epsilon(r) = \sqrt{\frac{C}{2}}\,\frac{1}{r^{(1/4+\epsilon)}},$$

the conditions (50) and (51) of Lemma 4.9 are satisfied, and the canonical section $s_{\mu(r)}$ of the obstruction bundle is identically vanishing. This implies the gluing theorem stated in Theorem 4.2.

Proof. The estimate

$$\|\Sigma(x_r)\|_{L^2(Y(r))} \le C' e^{-\delta r},$$

on the error term follows from the exponential decay estimate proved in Proposition 3.5. By Lemma 4.9 we know that the map $\mathcal{T}_r$ maps the ball $B_{\epsilon(r)}$ to itself and is a contraction, hence the equation (49) has a unique solution and the obstruction bundle is well defined. Moreover, for $\mu(r) \le r^{-(1+\epsilon)}$, the projection $\Pi(\mu(r), x_r)$ is identically zero, because of Corollary 4.8.

□

4.4 Obstruction to the gluing at the reducible point

We justify our initial assumption that only the gluing of $\mathcal{M}^*_V$ and $\mathcal{M}^{red}_{\nu(K)}$ with asymptotic values away from the lattice of bad points θ contributes to produce solutions on $Y(r)$, for large r.

In fact, we consider the same analysis of the obstruction bundle used in the previous section, and we prove that there is a non-trivial obstruction to the gluing of solutions in

$$\partial_\infty^{-1}(\theta) \subset \mathcal{M}^*_V.$$

The key observation is the decay estimate proved in Corollary 3.15. If $x(s) = (a(s), \alpha(s), \beta(s))$ is an element in $\partial_\infty^{-1}(\theta)$, then Corollary 3.15 implies that we have

$$\|x(s)\|_{L^2(T^2\times\{s\})} \leq \frac{c}{s}.$$

Thus, the error term $\Sigma(x_r)$ is estimated as

$$\|\Sigma(x_r)\|_{L^2(Y(r))} \leq \left(\int_{r-1}^{r+1} \|x(s)\|^2_{L^2(T^2\times\{s\})} ds\right)^{1/2}$$

$$\leq \left(\frac{2c^2}{r^2-1}\right)^{1/2} \sim \frac{C'}{r},$$

for large $r \geq r_0$.

Thus, we can formulate the analogue of Lemma 4.9 in this case.

Lemma 4.11. *There exists a positive constant $C > 0$, such that the following holds. Assume that we have*

$$\mu(r) > \sqrt{2C}\epsilon(r)^{1/4} \tag{52}$$

and

$$\epsilon(r) > \frac{C'}{C}\frac{1}{r}, \tag{53}$$

for all $r \geq r_0$. If the pre-glued solution x_r satisfies

$$\|\Sigma(x_r)\|_{L^2(Y(r))} \leq \frac{C'}{r},$$

for some $\delta > 0$, then the map

$$\mathcal{T}_r(b,\xi) = -\Delta_{x_r}^{-1}(1 - \Pi(\mu(r), x_r))(NH^*_{x_r}(b,\xi) + \Sigma(x_r))$$

maps the ball

$$B_{\epsilon(r)} = \{(b,\xi) |\ \|(b,\xi)\|^2_{L^2(Y(r))} \leq \epsilon(r)\}$$

to itself and is a contraction on $B_{\epsilon(r)}$.

The proof is completely analogous to the case of Lemma 4.9, using conditions (52) and (53) instead of (50) and (51). This, combined with the obstruction bundle technique described in the previous section, yields the following result.

Proposition 4.12. *For a choice of*

$$\mu(r) = B\ \frac{1}{r^{1/4}}$$

and

$$\epsilon(r) = A\ \frac{1}{r},$$

with $A > C'/C$ and $B > \sqrt{2C}$, the conditions (50) and (51) of Lemma 4.9 are satisfied. The canonical section $s_{\mu(r)}$ is a map

$$s_{\mu(r)} : \mathcal{U}(\theta, r) \to \mathbb{C}^3,$$

where we identify

$$\mathcal{U}(\theta, r) \cong \partial_\infty^{-1}(\theta) \subset \mathcal{M}_V^*.$$

For a generic choice of the perturbation of the equations on V, the section $s_{\mu(r)}$ is transverse to the zero section, hence generically we have

$$s_{\mu(r)}^{-1}(0) = \emptyset.$$

This implies that there is a non-trivial obstruction to gluing solutions in

$$\mathcal{U}(\theta, r) \cong \partial_\infty^{-1}(\theta) \subset \mathcal{M}_V^*$$

to form solutions on $Y(r)$.

Proof. The first statement follows as in the the case of irreducible asymptotic values. Since we need a slow decaying $\mu(r) \sim r^{-1/4}$ in the construction of the obstruction bundle, we can no longer guarantee that the section $s_{\mu(r)}$ is identically vanishing. The fibre of the obstruction bundle is $Ker(Q_\theta) \cong \mathbb{C}^3$.

Consider the universal obstruction bundle $\tilde{\mathcal{E}}$, with fibre $Ker(Q_\theta) = \mathbb{C}^3$ over the universal pre-gluing space $\tilde{\mathcal{U}}(\theta, r)$ identified with

$$\{(P, [A', \psi'] \mid P \in \mathcal{P}_\delta,\ [A', \psi'] \in \partial_\infty^{-1}(\theta) \subset \mathcal{M}_{V,P}^*\}.$$

The universal obstruction section $\tilde{s}_{\mu(r)}$ is transverse to zero. In fact, if we just vary $V \mapsto V + \epsilon\tilde{V}$, the corresponding variation of the correction term $\Pi_{\mu(r)}\Sigma(x_r)$ in the image of $\tilde{s}_{\mu(r)}$ is given by

$$\epsilon \left(\sum_j \frac{\partial \tilde{V}}{\partial \zeta_j} \nu_j.\xi(s) + \sum_{i,j} \frac{\partial^2 \tilde{V}}{\partial \zeta_i \partial \zeta_j} \zeta_i(\xi, \rho_r^+ \psi') \nu_j.\rho_r^+(s)\psi'(s) \right).$$

Let ϕ_k with $k = 1, 2$ be a basis of $\mathbb{C}^2 = ker(\bar{\partial}_\theta^* + \bar{\partial}_\theta)$. If $\xi(s) \neq 0$ and η is an imaginary valued 1-form, we have

$$\langle \phi_k, \eta(s).\xi(s) \rangle \neq 0.$$

Thus, by varying arbitrarily $\tilde{V}$ we obtain that the linearization of the section $\tilde{s}_{\mu(r)}$ is surjective.

The pre-gluing space $\mathcal{U}(\theta, r)$ is zero-dimensional, hence for generic choice of the perturbation we have

$$s_{\mu(r)}^{-1}(0) = \emptyset.$$

The result then follows from the obstruction bundle technique.

□

Now we can also discuss the second problem addressed in the analysis of the boundary values $\partial_\infty(\mathcal{M}^*_{\nu(K)})$ in the case of the induced metric g_1 on $\nu(K)$ which does not satisfy Lemma 3.18.

We proved in Lemma 3.21 that the image $\partial_\infty(\mathcal{M}^*_{\nu(K)}(\tilde{g}))$ only concentrates near finitely many irreducible points $x_1, \dots, x_N$ in $\chi_0(T^2, \nu(K))$, and possibly also in a small neighbourhood of the "bad point" θ. By chosing a generic perturbation of the equations on V, we can guarantee that the image $\partial_\infty(\mathcal{M}^*_V)$ misses the neighbourhoods U_{x_i}. The same obstruction bundle argument used in the proof of Proposition 4.12 applies to show that there is a non-trivial obstruction to the gluing of solutions in $\mathcal{M}_V$ and solutions in $\mathcal{M}^*_{\nu(K)}$ at the reducible point θ. Thus we have the following result.

Corollary 4.13. *Upon chosing the metric $\tilde{g}$ on $\nu(K)$, as in Lemma 3.21, the gluing result stated in Theorem 4.2 hold for the manifolds Y_1 and Y_0.*

5 The geometric triangle

Choose metrics for Y and Y_1, with a long cylinder $[-r, r] \times T^2$, such that the induced metrics in the torus neighbourhood $\nu(K)$ have non-negative scalar curvature, strictly positive away from the boundary, as in Lemma 3.18. Then, as proved in Lemma 3.19, the induced metric on $Y_0 = V \cup_{f_0} T^2 \times [-r, r] \cup D^2 \times S^1$ can be chosen so that the asymptotic values of the irreducible part $\partial_\infty(\mathcal{M}^*_{\nu(K)})$ only concentrate near $\chi(\nu(K))$ in $\chi(T^2)$. Thus, the gluing theorem for monopoles on three-manifolds, together with the above discussion, provides the structure of the Seiberg-Witten moduli spaces for the closed Riemannian manifolds Y, Y_1, and Y_0.

Theorem 5.1. *Let Y be an integral homology sphere with an embedded knot K. Let Y_1 and Y_0 be obtained by 1 and 0 surgeries on K, respectively. Assume the metrics are chosen as discussed above, and let $P \in \mathcal{P}_\delta$ be a generic perturbation exponentially decaying along the cylindrical end of V. Then, the Seiberg-Witten moduli spaces have the following structure:*

$$\mathcal{M}^*_{Y,\ P} = \mathcal{M}^*_{V,\ P} \times_{\chi_0(T^2, Y)} \{v = 0\},$$

$$\mathcal{M}^*_{Y_1,\ P} = \mathcal{M}^*_{V,\ P} \times_{\chi_0(T^2, Y_1)} \{u + v = -1\},$$

$$\mathcal{M}^*_{Y_0,\ P}(\mathfrak{s}_k) = \mathcal{M}^*_{V,\ P} \times_{\chi_0(T^2, Y_0)} \{u = k\}.$$

Here the coordinates u and v on $H^1(T^2, \mathbb{R})$ are defined by the property that, under the projection to $\chi(T^2)$, they satisfy the following condition. For $[A] \in \chi(T^2)$, $v([A])$ is the holonomy around the meridian m and $u([A])$ is the holonomy around the longitude l. The Spin^c*-structures $\mathfrak{s}_n (n \in \mathbb{Z})$ are all the* Spin^c *structures on Y_0 that restrict to the trivial* Spin^c *structures on V and $\nu(K)$.*

We have seen in Proposition 4.12 that the gluing of the irreducible part $\mathcal{M}^*_V$ with $\mathcal{M}^{red}_{\nu(K)}$ only happens away from the lattice of "bad points". Thus, the only gluing that happens at the reducible is the gluing of $\mathcal{M}^{red}_V$ with $\mathcal{M}^{red}_{\nu(K)}$, as explained in the following.

Proposition 5.2. *The gluing for reducibles is obtained by the following decomposition*

$$\mathcal{M}^{red}_{Y,\ P} = \mathcal{M}^{red}_{V,\ P} \times_{\chi_0(T^2,Y)} \{v = 0\},$$
$$\mathcal{M}^{red}_{Y_1,\ P} = \mathcal{M}^{red}_{V,\ P} \times_{\chi_0(T^2,Y_1)} \{u + v = -1\},$$
$$\mathcal{M}^{red}_{Y_0,\ P}(\mathfrak{s}_k) = \mathcal{M}^{red}_{V,\ P} \times_{\chi_0(T^2,Y_0)} \{u = k\}.$$

In particular, $\mathcal{M}^{red}_{Y,\ P}$ and $\mathcal{M}^{red}_{Y_1,\ P}$ consist of a unique point, whose image in $\chi(T^2)$ under the restriction map is the bad point θ. In the case of Y_0 with a non-trivial Spin^c *structure $\mathfrak{s}_k$, the set $\mathcal{M}^{red}_{Y_0,\ P}(\mathfrak{s}_k)$ is empty, and for $\mathfrak{s}_0$ it consists on the circle $\{u = 0\}$ in the cylinder $\chi_0(T^2, Y_0)$.*

Proof. We already know [14] [16] that there is a unique reducible point θ in $\mathcal{M}_Y$ and a unique reducible θ_1 in $\mathcal{M}_{Y_1}$. In the case of Y, the solution θ is obtained by gluing along the unique intersection point between the lines

$$\{v = 0\} = \pi_2^{-1}(\chi(\nu(K)) \subset \chi_0(T^2, Y)$$

and

$$\{u = 0\} = \pi_1^{-1}(\chi(V)) \subset \chi_0(T^2, Y),$$

in the covering $\chi_0(T^2, Y) = H^1(T^2, \mathbb{R})$. The case of Y_1 is analogous: the solution θ_1 is obtained by gluing along the unique intersection point between the lines

$$\{u + v = -1\} = \pi_2^{-1}(\chi(\nu(K))) \subset \chi_0(T^2, Y_1)$$

and

$$\{u = 0\} = \pi_1^{-1}(\chi(V)) \subset \chi_0(T^2, Y_1).$$

In the case of Y_0, if we consider a cohomologically trivial perturbation P of the Seiberg-Witten equations, then, in the case of the trivial Spin^c-structure $\mathfrak{s}_0$, there is a circle of reducibles obtained as the image under the gluing map of the circles

$$\chi(\nu(K)) = \{u = 0\} = \chi(V)$$

in the covering $\chi_0(T^2, Y_0) \to \chi(T^2)$. There are no reducible solutions on Y_0 with a non-trivial Spin^c-structure $\mathfrak{s}_k$ for a generic choice of a cohomologically trivial perturbation P. In fact, the intersection of the line $\{u = 0\} = \chi(V)$ with the line $\{u = k\} = \chi(\nu(K))$ in $\chi_0(T^2, Y_0)$ is empty.

$\square$

In order to get rid of the circle of reducible solutions in the case $< c_1(\mathfrak{s}), [Y_0] >= 0$, we need to use a cohomologically non-trivial perturbation. The details of this case will be analysed later in a following section of this paper.

The gluing according to Theorem 5.1 and Proposition 5.2 is illustrated in the top row and bottom left part of Figure 4. In the next section we define the surgery perturbation μ and explain the deformed gluing illustrated in the bottom right part of Figure 4.

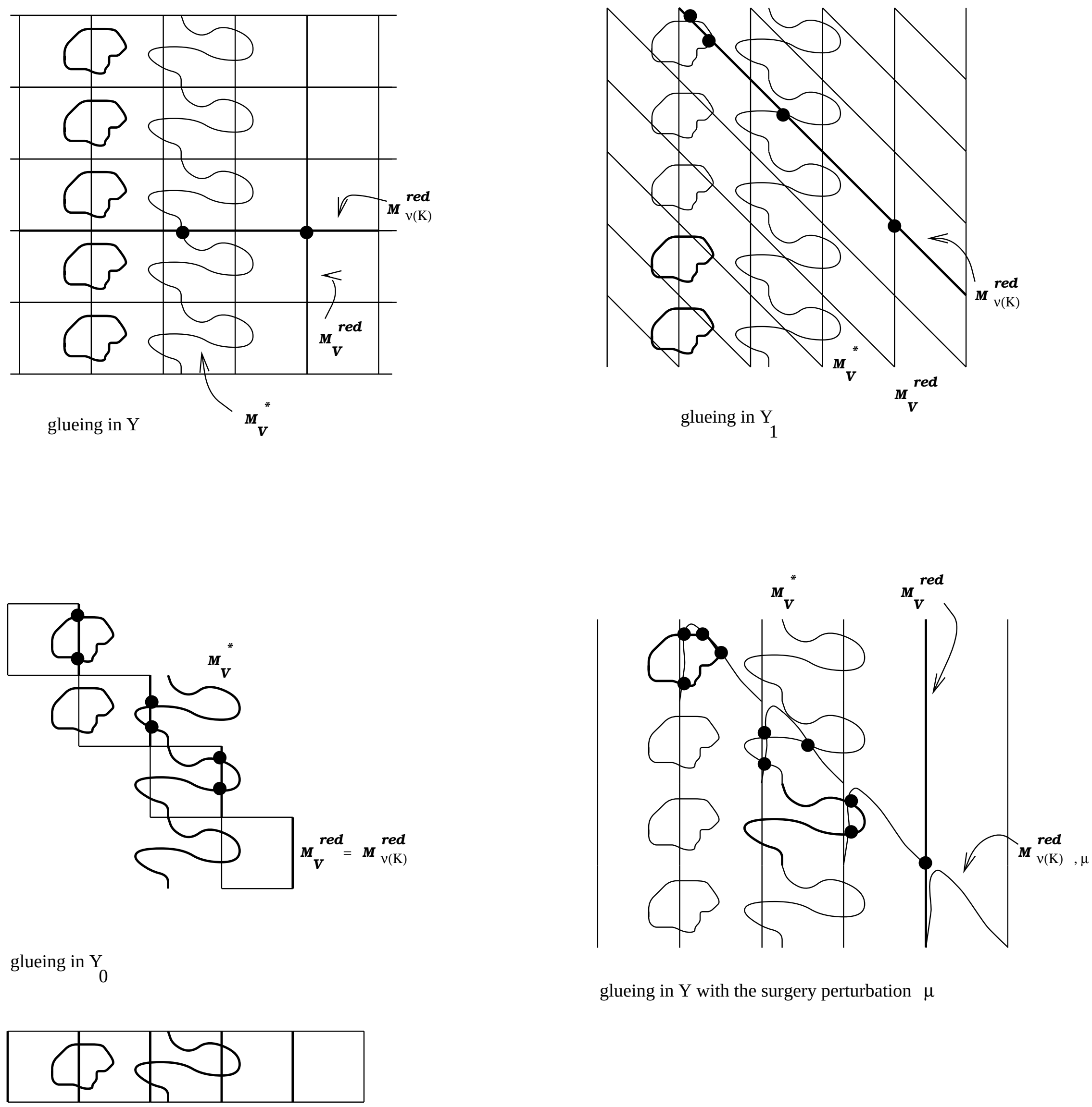

Figure 4: The geometric triangles and their deformation

6 Deforming the geometric triangle

We fix a perturbation P of the equations on V, exponentially decaying along the cylindrical end, as described in a previous section. This is enough to ensure that the moduli space $\mathcal{M}_{V,P}$ is cut out transversely. The asymptotic values $\partial_\infty(\mathcal{M}_{V,P})$ form a compact one-dimensional submanifold in

$$\chi_0(T^2, V) = H^1(T^2, \mathbb{Z})/(Im(H^1(V, \mathbb{Z}) \to H^1(T^2, \mathbb{Z}))).$$

In this section we define a suitable perturbation of the equations on the solid torus $\nu(K)$, that "simulates the effect of surgery". A similar analysis is used in instanton theory [2]. In the case of the manifolds Y_1 and Y_0, we consider the decomposition of the moduli spaces $\mathcal{M}_{Y_1}$

and $\mathcal{M}_{Y_0}$ obtained in Theorem 5.1. The use of a generic perturbation on Y, Y_1 or Y_0, as described previously, exponentially small on the long cylinder and on $\nu(K)$, is understood. The explicit use of the notation is suppressed in the following. For the manifold Y, we introduce a further perturbation μ of the curvature equation, compactly supported in $\nu(K)$, such that the resulting moduli space $\mathcal{M}_{Y,\mu}$ decomposes according to the formula

$$\mathcal{M}_{Y,\mu} \cong \mathcal{M}_{Y_1} \cup \bigcup_{\mathfrak{s}_k} \mathcal{M}_{Y_0}(\mathfrak{s}_k).$$

The main purpose of the following sections is to establish this formula.

6.1 Perturbation on $\nu(K)$

We want to introduce a suitable perturbation of the curvature equation, supported in the solid torus $D^2 \times S^1$, that simulates the effect of surgery. Choose a compactly supported 2-form μ representing the generator of $H^2_{cpt}(D^2 \times S^1)$, defined as in Lemma 3.18, such that we have

$$\int_{D^2 \times \{pt\}} \mu = 1$$

for any point on S^1. Under the isomorphism $H^2_{cpt}(\nu(K)) \cong H_1(\nu(K))$, given by Poincaré duality, this form corresponds to the generator $[\mu] = PD_{\nu(K)}(l)$. The class of μ in $H^2(D^2 \times S^1)$ is trivial, and we can write $\mu = d\nu$, where ν is a 1-form satisfying $\int_{S^1 \times \{pt\}} \nu = 1$, i.e. $\nu = PD_{T^2}(l)$. Choose on $\nu(K)$ a metric as in Lemma 3.18.

Fix a $U(1)$-connection A_0. For any $U(1)$-connection A, define T_A to be

$$T_A(z) = -i \int_{\{z \in D^2\} \times S^1} (A - A_0).$$

For any given $\epsilon > 0$, we can choose a function $f : \mathbb{R} \to \mathbb{R}$ with the following properties.

(a) f is continuously differentiable on $\mathbb{R}$ and satisfies the periodicity $f(t+1) = f(t)$;

(b) the derivative f' has range $f'(t) \in [0,1]$ for all $t \in [0,1]$, and satisfies $f'(0) = 0$;

(c) the following estimate holds:

$$\sup_{t \in [\epsilon, 1]} |f'(t) - t| < \epsilon.$$

Then we can perturb the Seiberg-Witten equations in the following way

$$\begin{cases} F_A = *\sigma(\psi,\psi) + f'(T_A)\mu \\ \partial\!\!\!/_A(\psi) = 0 \end{cases} . \tag{54}$$

With respect to the chosen metric on $\nu(K)$, with sufficiently large positive scalar curvature on the support of μ as specified in Lemma 3.18, the only solutions of the perturbed monopole equations are reducibles $(A, 0)$, that satisfy

$$F_A = f'(T_A)\mu. \tag{55}$$

Lemma 6.1. *Suppose the holonomy for the solution A of (55) around the longitude l (parallel to the knot K) is* $\exp(\pi i u)$. *Then u $(mod\ 2)$ is independent of the choice of the longitude l. For a reducible solution of the perturbed equations, the holonomy around the meridian m is then* $\exp(\pi i v)$, *where v satisfies the relation $v = f'(u)$.*

Proof. Let Θ be any vector field on $D^2 \times S^1 = \nu(K)$, transverse to K. Then $i_\Theta(F_A)$ has no component along K, this implies that T_A is constant on $D^2 \times \{pt\}$. Therefore, u $(mod\ 2)$ is independent of the choice of the longitude l, that is u is constant over $D^2 \times \{pt\}$

$$F_A = f'(u)\mu.$$

The holonomy around $m = \partial(D^2 \times \{pt\})$ is given by $exp(\pi i v)$ where

$$v = \int_{D^2 \times \{pt\}} F_A = \int_{D^2} f'(u)\mu = f'(u).$$

$\square$

Lemma 6.2. *The perturbation $f'(T_A)\mu$ is cohomologically trivial, that is*

$$[f'(T_A)\mu] = 0 \in H^2(\nu(K)),$$

for all $U(1)$-connections A. Thus, restriction to the boundary T^2 defines a boundary value map $\partial_\infty : \mathcal{M}_{\nu(K),\mu} \to \chi_0(T^2, \nu(K))$ that is an embedding.

With these preliminary results in place, we can prove the following decomposition of the moduli space $\mathcal{M}_{Y,\mu}$.

Theorem 6.3. *With the introduction of a suitable perturbation as discussed above, we have the following relation between the critical sets of the Chern-Simons-Dirac functional on the manifolds Y, Y_1, and Y_0 respectively.*

$$\mathcal{M}_{Y,\mu} = \mathcal{M}_{Y_1} \bigcup_{n\in\mathbb{Z}} \mathcal{M}_{Y_0}(\mathfrak{s}_n),$$

provoded that, in the case of the trivial Spin^c *structure $\mathfrak{s}_0$, we consider the moduli space $\mathcal{M}_{Y_0}(\mathfrak{s}_0)$ of gauge classes of solutions of the equations perturbed with $P + \rho$, with P in the class $\mathcal{P}$ and ρ a co-closed 1-form representing a non-trivial cohomology class $[*\rho] \in H^2(Y_0, \mathbb{R})$. The class $[*\rho]$ satisfies*

$$[*\rho] = \frac{\epsilon}{2} PD_{Y_0}(m),$$

where $\epsilon > 0$ is the small parameter used in the construction of the surgery perturbation.

Proof. This follows from the previous discussions and Theorem 5.1. From Theorem 5.1 and Proposition 5.2 we know that, with a choice of a generic perturbation $P \in \mathcal{P}$ understood, we have

$$\mathcal{M}_Y = \mathcal{M}_V \times_{\chi_0(T^2,Y)} \{v = 0\}.$$

Now choose ϵ small enough, and construct the corresponding function $f : \mathbb{R} \to \mathbb{R}$ that defines the perturbation $f(T_A)\mu$, as in Lemma 6.1. We see that, away from the bad point θ, solutions of the equations (54) can be identified with

$$\mathcal{M}^*_{Y,\,\mu} \cong \mathcal{M}^*_V \times_{\chi_0(T^2,Y)} \{\ \textit{either}\ u + v = -1,\ \textit{or}\ u = 0\}.$$

Solutions on $(Y_0, \mathfrak{s}_0)$ are obtained in the gluing by considering the subset

$$\mathcal{M}^*_V \times_{H^1(T^2,\mathbb{R})} \{v = f'(u) | 0 < u \leq \epsilon\}$$

of the fibred product

$$\mathcal{M}^*_{Y,\,\mu} \cong \mathcal{M}^*_V \times_{\chi_0(T^2,Y)} \{\ \textit{either}\ u + v = -1,\ \textit{or}\ u = 0\}.$$

For ϵ small enough, the intersection points

$$\pi_1^*(\partial_\infty \mathcal{M}^*_V) \cap \{v = f'(u) | 0 < u \leq \epsilon\}$$

in $H^1(T^2, \mathbb{R})$ can be deformed to

$$\partial_\infty(\mathcal{M}^*_V) \cap \{(\epsilon/2, v) | 0 < v \leq 1\}$$

in $\chi_0(T^2, Y_0)$. In the above, π_1 is the projection $\pi_1 : \chi_0(T^2, Y) \to \chi_0(T^2, V)$. This corresponds to the gluing

$$\mathcal{M}^*_{Y_0}(\mathfrak{s}_0, \rho) \cong \mathcal{M}^*_V \times_{\chi(T^2,Y_0)} \{(\epsilon/2, v) | 0 < v \leq 1\}.$$

which gives the perturbed moduli space with an additional perturbation by a co-closed 1-form ρ satisfying

$$[*\rho] = \frac{\epsilon}{2} PD_{Y_0}(m)$$

in $H^2(Y_0, \mathbb{R})$, with m the generator of $H_1(Y_0, \mathbb{Z})$. □

Notice that different choices of ϵ small give rise to cobordant moduli spaces of irreducible solutions on $(Y_0, \mathfrak{s}_0)$.

7 Relative grading

In this section we show that the grading of the Floer complex $C_*(Y, \mu)$, defined with respect to the unique reducible point $[\nu, 0]$, induces compatible gradings on the Floer complexes $C_*(Y_1)$ and $C_*(Y_0, \mathfrak{s}_k)$, for a non-trivial Spin^c-structure $\mathfrak{s}_k$.

7.1 Lagrangians in $H^1(T^2,\mathbb{R})$

Consider the space $\chi_0(T^2,Y) = H^1(T^2,\mathbb{R})$, endowed with the symplectic structure $(a,b) \mapsto \int_{T^2} a \wedge b$. Notice that this symplectic form is the restriction of the form $(x,y) \mapsto \langle x, \sigma y \rangle$ used in [6], where $x = (a,\alpha_1,\beta_1,f_1,g_1)$ and $y = (b,\alpha_2,\beta_2,f_2,g_2)$, with (α_i,β_i) the spinor components and (f_i,g_i) in Λ^0. The matrix σ is defined so that the operator $L_{(a_\infty,0)}$ of (30) on the cylinder $T^2 \times [-r,r]$ is of the form

$$\sigma \left(\frac{\partial}{\partial t} + Q_{a_\infty} + \begin{pmatrix} 0 & -d_{T^2} & *d_{T^2} \\ -d^*_{T^2} & 0 & 0 \\ -*d_{T^2} & 0 & 0 \end{pmatrix} \right).$$

The choice of a Lagrangian subspace ℓ in $\chi_0(T^2,Y)$ determines a Lagrangian subspace (which we still denote ℓ) in $\chi_0(T^2,Y) \oplus \mathbb{R}^2$ spanned by ℓ and $(0,0,0,1)$.

In the following it will be useful to consider the following Lagrangian submanifolds of $H^1(T^2,\mathbb{R})$. Consider

$$\ell_1 = \pi_2^*(\partial_\infty(\mathcal{M}_{\nu(K)})) = \{(u,v) \in \mathbb{R}^2 | u+v=-1\},$$

where

$$\pi_2 : H^1(T^2,\mathbb{R}) \cong \chi_0(T^2,Y_1) \to \chi_0(T^2,\nu(K))$$

is the covering map. Consider also the lines

$$\ell_k = \{(k,v) | v \in \mathbb{R}\},$$

for any fixed $k \in \mathbb{Z}$, where we have

$$\cup_{k\in\mathbb{Z}} \ell_k = \pi_2^*(\partial_\infty(\mathcal{M}_{\nu(K)})),$$

with respect to the covering map

$$\pi_2 : H^1(T^2,\mathbb{R}) \to \chi_0(T^2,Y_0).$$

Moreover, given the broken line

$$\ell_\mu = \pi_2^*(\partial_\infty(\mathcal{M}_{\nu(K)})) = \{(u,v) \in \mathbb{R}^2 | v = f'(u)\},$$

with

$$\pi_2 : H^1(T^2,\mathbb{R}) \cong \chi_0(T^2,Y) \to \chi_0(T^2,\nu(K)),$$

we can form a continuous piecewise regular Lagrangian submanifold of $H^1(T^2,\mathbb{R})$, as illustrated in Figure 5, by taking

$$\ell_\mu^* = \ell_\mu \cup_{k\in\mathbb{Z}} \{(k,v) | -k-1 \le v \le -k\}.$$

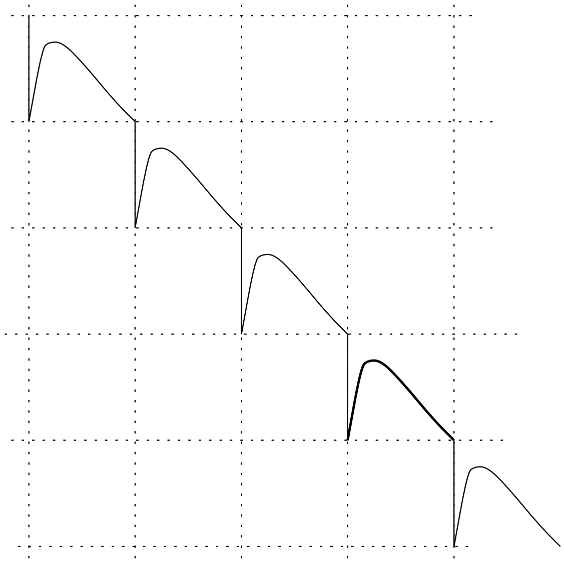

Figure 5: The piecewise smooth Lagrangian ℓ_μ^*

Let us recall the following setup, as in [4].

The choice of two Lagrangians $\ell_\pm$ in $\chi_0(T^2, Y)$ defines boundary conditions for the operators that linearise the monopole equations on the manifolds with boundary $V(r) = V \cup_{T^2} T^2 \times [0, 2r]$ and $\nu(K)(r) = \nu(K) \cup_{T^2} T^2 \times [0, 2r]$. More precisely, for a sufficiently large $r \geq r_0$, the gluing theorem gives a splitting $(A, \psi) = (A', \psi')\#_r(a, 0)$, and we can consider the operators

$$L_{(A',\psi'),\ell_+} : L_1^2(P_+ \oplus \ell_+) \to L^2$$

and

$$L_{(A',\psi'),\ell_-} : L_1^2(P_- \oplus \ell_-) \to L^2,$$

where $P_\pm$ are APS boundary conditions [1].

Given a path $\{(A'(\tau), \psi'(\tau)) | \tau \in [0,1]\}$, and a corresponding path $\{(a(\tau), 0) | \tau \in [0,1]\}$, we can consider the corresponding operators $L_{(A'(\tau),\psi'(\tau))}$, $L_{(a(\tau),0)}$, and $L_{(A'(\tau),\psi'(\tau))\#_r(a(\tau),0)}$.

We recall the endpoint condition for pairs of Lagrangians goven in [4].

Definition 7.1. *A family of pairs of Lagrangians $(\ell_+(\tau), \ell_-(\tau))$ satisfies the endpoint conditions with respect to the operators $L_{(A'(\tau),\psi'(\tau))}$, and $L_{(a(\tau),0)}$ if the pairs $(\ell_+(0), \ell_-(0))$ and $(\ell_+(1), \ell_-(1))$ represent respectively the asymptotic values of extended L^2-solutions of*

$$(L_{(A'(0),\psi'(0))}, L_{(a(0),0)}) \text{ and } (L_{(A'(1),\psi'(1))}, L_{(a(1),0)}).$$

We can now recall the Capell-Lee-Miller decomposition of the spectral flow (Theorem C of [4]) in our setting.

Proposition 7.2. *Suppose given intervals $\tau \in [\tau_i, \tau_{i+1}]$, with $i = 0, \dots, N$. Assume that the path $(a(\tau), 0)$ avoids the singular point θ in the chracter variety of T^2. Assume that on each interval $[\tau_i, \tau_{i+1}]$ the operator $Q_{a(\tau)}$ has no non-zero eigenvalues in the range $(-\delta, \delta)$. Moreover, fix $r_0 > 0$ such that, for $r \geq r_0$ the operator $L_{(A'(\tau),\psi'(\tau))\#_r(a(\tau),0)}$ has no eigenvalues equal to $\pm 1/r^2$. Choose families of Lagrangians $\ell_\pm(\tau, i)$, on each interval $\tau \in [\tau_i, \tau_{i+1}]$, that satisfy the endpoint conditions at τ_i and τ_{i+1}, and with the condition $\ell_\pm(\tau_{i+1}, i) \equiv \ell_\pm(\tau_{i+1}, i+1)$ for all $i = 0, \dots, N$. Choose $\epsilon > 0$ such that at each point τ_i there are no non-zero eigenvalues of $L_{(A'(\tau_i),\psi'(\tau_i)),\ell_+(\tau_i)}$ and of $L_{(a(\tau),0),\ell_-(\tau_i)}$ in the interval $[-\epsilon, \epsilon]$. Under these hypotheses the spectral flow decomposes as*

$$\begin{array}{c} \frac{1}{r^2} SF(L_{(A'(\tau),\psi'(\tau))\#_r(a(\tau),0)}) = \\ \sum_{i=1}^N \left(\epsilon SF(L_{(A'(\tau),\psi'(\tau)),\ell_+(\tau,i)}, V(r)) + \epsilon SF(L_{(a(\tau),0),\ell_-(\tau,i)}, \nu(K)(r)) \right) \\ + \sum_{i=1}^N Maslov(\ell_+(\tau,i), \ell_-(\tau,i)) + \frac{1}{2}(\dim Ker(Q_{a(\tau_N)}) - \dim Ker(Q_{a(\tau_0)})) \end{array} .$$

For the definition of the ϵ-spectral flow ϵSF, see [4].

7.2 Maslov indices

Let (A', ψ') be a solution that defines a class in $\mathcal{M}_V$. We define a grading $\deg_V(A', \psi')$.

Definition 7.3. *Let (A', ψ') be a solution on V representing a class $[A', \psi']$ in $\mathcal{M}_V$. Let $a_\infty = \partial_\infty([A', \psi'])$ be the asymptotic value in $\chi(T^2, V)$. Fix an origin θ_0 in the lattice of bad points $H^1(T^2, \mathbb{Z})$ in $\chi_0(T^2, Y)$. Let ℓ^*_μ be the piecewise smooth Lagrangian defined above, containing the point θ_0. There exists a unique homotopy class of gauge transformations $[\lambda] \in H^1(T^2, \mathbb{Z})/H^1(T^2, V)$ such that the element $\lambda(A', \psi')$ has asymptotic value λa_∞ in $\chi_0(T^2, Y)$ that lies on ℓ^*_μ. We define the grading $\deg_V(A', \psi')$ to be*

$$\deg_V(A', \psi') := \epsilon SF(L_{\lambda(A'(\tau),\psi'(\tau)),\ell(\tau)}) + Maslov(\ell(\tau), \ell^*_\mu). \tag{56}$$

Here $\ell(\tau)$ is a family of Lagrangians satisfying the endpoint condition, and $\lambda(A'(\tau), \psi'(\tau))$ is a path connecting $\lambda(A', \psi')$ to the reducible solution $(\theta_0, 0)$ in the configuration space on V. We assume that the elements $\lambda(A', \psi')$ are cut off at some large r_0, cf. [4], [6] .

Theorem 7.4. *The grading of the Floer complex $C_*(Y, \mu)$ induces a consistent choice of gradings of $C_*(Y_1)$ and $C_*(Y_0, \mathfrak{s}_k)$.*

Proof. Let $[A, \psi]$ be an element of $\mathcal{M}_{Y_1} \hookrightarrow \mathcal{M}_{Y,\mu}$. For $r \geq r_0$ we have the splitting $(A, \psi) = (A', \psi')\#_r(a, 0)$. The grading $\deg_{Y,\mu}(A, \psi)$ is defined as the relative Morse index with respect to the reducible point $[\nu, 0]$ on Y, that is, as the spectral flow of $L_{A(\tau),\psi(\tau)}$ along a path $(A(\tau),\psi(\tau))$ connecting the point $[A,\psi]$ to $[\nu,0]$. According to Proposition 7.2, we can write the grading as follows

$$\begin{aligned}\deg_{Y,\mu}(A,\psi) &= \epsilon SF(L_{(A'(\tau),\psi'(\tau)),\ell(\tau)}) + Maslov(\ell(\tau),\ell_\mu^*) \\ &\quad + \epsilon SF(L_{(a(\tau),0),\ell_\mu^*}) \\ &= \deg_V(A',\psi') + \epsilon SF(L_{(a(\tau),0),\ell_\mu^*}).\end{aligned} \tag{57}$$

Notice that, if we denote with τ_i the points where ℓ_μ^* is singular, the Maslov index is given by the quantity

$$Maslov(\ell(\tau),\ell_\mu^*) = \sum_{i=1}^N Maslov(\ell_i(\tau),\ell_\mu^*),$$

where each $\ell_i(\tau)$ connects endpoints $\{(k,-k)\}$ and $\{(k,-k-1)\}$ of the vertical segments of ℓ_μ^*, or endpoints $\{(k,-k-1)\}$ and $\{(k+1,-k-1)\}$ of a connected component of ℓ_μ. Under a generic choice of the perturbation on V we can guarantee that the Lagrangians ℓ_i at the points τ_i where ℓ_μ^* is singular are transverse to ℓ_μ and to the vertical segment $\ell_k = \{(k,v)| -k-1 \leq v \leq -k\}$. Moreover, we can assume that $\ell(\tau) = \cup_i \ell_i(\tau)$ is a smooth Lagrangian submanifold, with prescribed tangent spaces ℓ_i at the points τ_i.

Thus, again by Proposition 7.2, we can rewrite (57) as

$$= \epsilon SF(L_{(A'(\tau),\psi'(\tau)),\ell(\tau)}) + Maslov(\ell(\tau),\ell_1) + \epsilon SF(L_{(a(\tau),0),\ell_1}).$$

Notice, moreover, that we have the equality

$$Maslov(\ell(\tau),\ell_\mu^*) = Maslov(\ell(\tau),\ell_1).$$

In fact, according to the normalisation rule for the Maslov index illutrated in Figure 6, we can see that the contribution of each vertex $\{(k,-k-1)\}$ to the total Maslov index

$$Maslov(\ell(\tau),\ell_\mu^*) = \sum_{i=1}^N Maslov(\ell_i(\tau),\ell_\mu^*)$$

consists of a pair of $+1$ and -1 or of 0 and 0. Moreover, the contribution to the Maslov index of every intersection of $\ell(\tau)$ with the vertical lines $\ell_k = \{(k,v)| -k-1 \leq v \leq -k\}$ is cancelled by a nearby intersection with ℓ_m with the opposite sign. Thus the only intersection points of $\ell(\tau)$ and ℓ_μ^* that contribute non-trivially to the Maslov index are the points that correspond to intersections with the line ℓ_1, as can be seen in the example drawn in Figure 6. This implies that also the quantities $\epsilon SF(L_{(a(\tau),0),\ell_1})$ and $\epsilon SF(L_{(a(\tau),0),\ell_\mu^*})$ coincide. We denote this quantity as $\deg_{\nu(K)}(a,0)$.

Thus, we obtained the following equality

$$\deg_{Y,\mu}(A,\psi) = \deg_V(A',\psi') + \deg_{\nu(K)}(a,0) = \deg_{Y_1}(A,\psi).$$

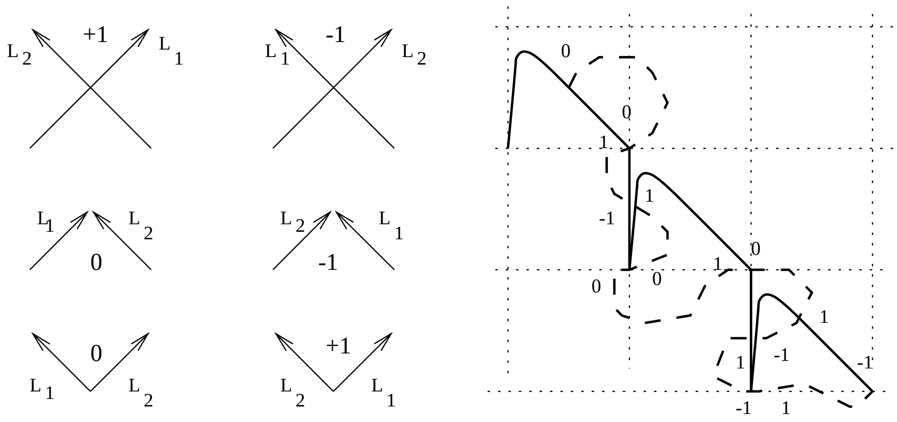

Figure 6: Normalisation of the Maslov index and an example of $Maslov(\ell(\tau), \ell^*_\mu)$

Analogously, consider two elements $[A_1, \psi_1]$ and $[A_2, \psi_2]$ in $\mathcal{M}_{Y_0, \mathfrak{s}_k}$. Let a_1 and a_2 be the corresponding asymptotic values in $\chi_0(T^2, V)$ under the map ∂_∞, and we still denote by a_1 and a_2 their unique preimages in $\chi_0(T^2, Y)$ that lie on the curve ℓ^*_μ. These are two points on the same vertical segment ℓ_k. The relative grading is given by the spectral flow of the operator $L_{A(\tau),\psi(\tau)}$ along a path $(A(\tau), \psi(\tau))$ connecting the points $[A_1, \psi_1]$ and $[A_2, \psi_2]$. This can be decomposed as follows under the splitting $(A_1, \psi_1) = (A'_1, \psi'_1)\#_r(a_1, 0)$ and $(A_2, \psi_2) = (A'_2, \psi'_2)\#_r(a_2, 0)$:

$$\deg_{Y_0,\mathfrak{s}_k}(A_1, \psi_1) - \deg_{Y_0,\mathfrak{s}_k}(A_2, \psi_2) =$$
$$\epsilon SF(L_{(A'(\tau),\psi'(\tau)),\ell_+(\tau)}) + Maslov(\ell_+(\tau), \ell_-(\tau)) + \epsilon SF(L_{(a(\tau),0),\ell_-(\tau)}) .$$

We can write the Maslov index as

$$Maslov(\ell_+(\tau), \ell_-(\tau)) = Maslov(\ell_+(\tau), \ell_k) - Maslov(\ell_-(\tau), \ell_k),$$

where ℓ_k is the vertical segment $\ell_k = \{(k, v) | -k-1 \leq v \leq -k\}$. We can observe that, under the identification $\mathcal{M}_{Y_0,\mathfrak{s}_k} \hookrightarrow \mathcal{M}_{Y,\mu}$, the Maslov indices satisfy

$$Maslov(\ell_+(\tau), \ell_k) = Maslov(\ell_+(\tau), \ell_\mu),$$

and

$$Maslov(\ell_-(\tau), \ell_k) = Maslov(\ell_-(\tau), \ell_\mu).$$

In fact, a starting point with $+1$ contribution can be replaced by a pair of a starting point with 0 contribution and a crossing of the line ℓ_μ with $+1$ contribution, or, conversely, a pair of a starting point with 0 contribution and a crossing with $+1$ contribution can be replaced by a starting point with $+1$ contribution as in Figure 7.

Now complete the Lagrangians $\ell_\pm(\tau)$ to Lagrangians $\tilde{\ell}_\pm(\tau)$ that connect the points a_1 and a_2 in $\chi_0(T^2, Y)$ to the point θ_0, with the prescribed endpoint conditions at the τ_i where ℓ^*_μ is singular. In particular, if τ_1 corresponds to the point $(k, -k-1)$, the endpoint condition

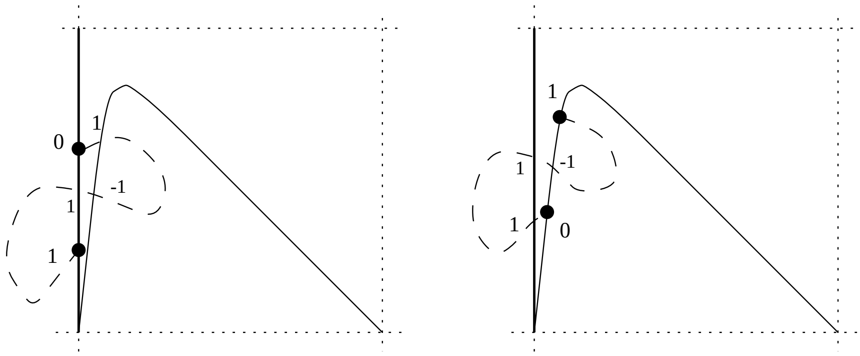

Figure 7: The Maslov indices $Maslov(\ell_{\pm}(\tau), \ell_k)$ and $Maslov(\ell_{\pm}(\tau), \ell_{\mu})$

implies that the Lagrangian manifolds $\tilde{\ell}_{\pm}$ have the same tangent space at τ_1. We can choose the extensions $\tilde{\ell}_{\pm}$ such that for all $\tau \geq \tau_1$ we have $\tilde{\ell}_{+}(\tau) \equiv \tilde{\ell}_{+}(\tau) \equiv \ell(\tau)$. Assume that τ_0 corresponds to the point a_2 and that $\epsilon > 0$ is such that in the intervals $[\tau_0 - \epsilon, \tau_0)$ and $(\tau_1, \tau_1 + \epsilon]$ no intersection of the Lagrangians $\tilde{\ell}_{\pm}$ with ℓ^*_{μ} occurs. By the rules of normalization of the Maslov index, and by direct inspection of the various cases, we can verify that, independently of how the extensions $\tilde{\ell}_{\pm}(\tau)$ are chosen in the interval $[\tau_0, \tau_1]$, the following equalities of Maslov indices hold:

$$\{Maslov(\ell_{+}(\tau), \ell_{\mu})) - Maslov(\ell_{-}(\tau), \ell_{\mu})\}_{\tau \in (\tau_0 - \epsilon, \tau_0]} =$$
$$\{Maslov(\tilde{\ell}_{+}(\tau), \ell^*_{\mu}) - Maslov(\tilde{\ell}_{-}(\tau), \ell^*_{\mu})\}_{\tau \in (\tau_0 - \epsilon, \tau_1 + \epsilon)}.$$

This implies that we have obtained

$$Maslov(\ell_{+}(\tau), \ell_{-}(\tau)) = Maslov(\tilde{\ell}_{+}(\tau), \ell^*_{\mu}) - Maslov(\tilde{\ell}_{-}(\tau), \ell^*_{\mu}).$$

We can also complete the path $(A(\tau), \psi(\tau))$ connecting (A_1, ψ_1) to (A_2, ψ_2) to a path connecting (A_2, ψ_2) to $(\nu, 0)$ so that we have the relation between the relative gradings

$$\deg_{Y_0, \mathfrak{s}_k}(A_1, \psi_1) - \deg_{Y_0, \mathfrak{s}_k}(A_2, \psi_2) = \deg_{Y, \mu}(A_1, \psi_1) - \deg_{Y, \mu}(A_2, \psi_2).$$

Here the relative grading $\deg_{Y_0, \mathfrak{s}_k}(A_1, \psi_1) - \deg_{Y_0, \mathfrak{s}_k}(A_2, \psi_2)$ refers to an integer lift of the $\mathbb{Z}_{2k}$-valued relative grading of $C_*(Y_0, \mathfrak{s}_k)$. We discuss the integer lift $C_{(*)}(Y_0, \mathfrak{s}_k)$ of $C_*(Y_0, \mathfrak{s}_k)$ in [19]. The equivalence of the relative gradings above, gives us a preferred way of fixing the grading of $C_{(*)}(Y_0, \mathfrak{s}_k)$, by setting

$$\deg_{Y_0, \mathfrak{s}_k}(A, \psi) := \deg_{Y, \mu}(A, \psi),$$

under the identification $\mathcal{M}_{Y_0, \mathfrak{s}_k} \hookrightarrow \mathcal{M}_{Y, \mu}$.

$\square$

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A.L. Carey and B.L. Wang, Department of Pure Mathematics, University of Adelaide, Adelaide SA 5005
acarey@maths.adelaide.edu.au
bwang@maths.adelaide.edu.au

M. Marcolli, Department of Mathematics, Massachusetts Institute of Technology, 2-275, Cambridge MA 02139, USA
matilde@math.mit.edu