PERIODIC BINARY HARMONIC FUNCTIONS ON LATTICES

MIKHAIL ZAIDENBERG

Abstract. A function on a (generally infinite) graph Γ with values in a field K of characteristic 2 will be called *harmonic* if its value at every vertex of Γ is the sum of its values over all adjacent vertices. We consider binary pluri-periodic harmonic functions $f : \mathbb{Z}^s \to \mathbb{F}_2 = \mathrm{GF}(2)$ on integer lattices, and address the problem of describing the set of possible multi-periods $\bar{n} = (n_1, \ldots, n_s) \in \mathbb{N}^s$ of such functions. Actually this problem arises in the theory of cellular automata [MOW, Su1, Su4, GKW]. It occurs to be equivalent to determining, for a certain affine algebraic hypersurface V_s in $\mathbb{A}^s_{\bar{\mathbb{F}}_2}$, the torsion multi-orders of the points on V_s in the multiplicative group $(\bar{\mathbb{F}}_2^\times)^s$. In particular V_2 is an elliptic cubic curve. In this special case we provide a more thorough treatment. A major part of the paper is devoted to a survey of the subject.

In mathematics, our role is more that of servant than master.
Charles Hermite, from reminiscences by Jacques Hadamard.

Contents

Acknowledgements: This article was written partially during the author's visit to the Max-Planck-Institute of Mathematics at Bonn. The author thank this institution for generous support.

1991 Mathematics Subject Classification: 11B39, 11T06, 11T99, 31C05, 37B15, 43A99.

Key words: Chebyshev-Dickson polynomial, Fibonacci polynomial, cellular automaton, graph laplacian, spectrum of a graph, binary function, discrete harmonic function, pluri-periodic function, finite abelian group, lattice, elliptic curve, characteristic two.

Introduction

We use the following notation. We let $\Gamma = (\mathrm{vert}(\Gamma), \mathrm{edg}(\Gamma))$ be a (not necessarily finite) graph without loops and multiple edges, K be a field of characteristic 2, $k = \mathbb{F}_2 = \mathrm{GF}(2)$ the *binary field*; moreover, we write $\bar{k} = \bar{F}_2$ for its algebraic closure.

For K-valued functions $f : \mathrm{vert}(\Gamma) \to K$, we consider two kinds of laplacians: Δ_Γ^+ is averaging over balls of radius 1, respectively Δ_Γ^- is averaging over spheres of radius 1 (cf. e.g., [Ca, Ba]) so that

$$\Delta_\Gamma^+ = \mathrm{id} + \Delta_\Gamma^-, \qquad \text{where} \qquad \Delta_\Gamma^-(f)(v) = \sum_{[v,v'] \in \mathrm{edg}(\Gamma)} f(v')\,, \quad v \in \mathrm{vert}(\Gamma)\,.$$

Actually $\Delta^\pm$ is the $\sigma^\pm$-*cellular automaton* studied e.g. by Martin, Odlyzko and Wolfram [MOW], Sutner [Su1]-[Su4], Goldwasser, Klostermeyer and Ware [GKW], Sarkar, Barua and Ramakrishnan [BR, SB], Hunzikel, Machiavello and Park [HMP] e.a.

A function f on $\mathrm{vert}(\Gamma)$ is called *harmonic* (*antiharmonic*) if $\Delta_\Gamma^+(f) = 0$ ($\Delta_\Gamma^-(f) = 0$, respectively). Thus for f harmonic, $f(v)$ is equal to the sum of values of f over the neighbors of v, whereas for f antiharmonic this sum is always zero. We let

$$\mathrm{Harm}_K^+(\Gamma) = \ker(\Delta_\Gamma^+) \qquad \text{and} \qquad \mathrm{Harm}_K^-(\Gamma) = \ker(\Delta_\Gamma^-) = \ker(\mathrm{id} + \Delta_\Gamma^+)$$

be the corresponding subspaces of the vector space $\mathcal{F}(\Gamma, K)$ of all K-valued functions on $\mathrm{vert}(\Gamma)$. We simply write $\mathrm{Harm}^\pm(\Gamma) = \mathrm{Harm}_k^\pm(\Gamma)$ when dealing with binary functions.

The support $N^\pm = N_\Gamma^\pm(f) = \mathrm{supp}(f)$ of a nonzero binary harmonic (antiharmonic) function f will be called a *nucleus* (*antinucleus*, respectively) of Γ. Note that the constant function 1 is harmonic on an odd graph and antiharmonic on an even one. More generally, the (anti)nuclei can be characterized by the following two properties:

- Every nucleus N^+ of Γ is an odd subgraph, that is each vertex of N^+ is of odd degree within N^+. Whereas every antinucleus N_Γ^- is an even subgraph.
- Every vertex $v \in \mathrm{vert}(\Gamma) \setminus \mathrm{vert}(N_\Gamma^\pm)$ has an even number of neighbors in $N_\Gamma^\pm$.

Nuclei are *even-parity* subgraphs of Γ, see e.g. Amin, Slater and Zhang [ASZ], Klostermeyer [K] and literature therein on parity domination in graphs. The set of all (anti)nuclei in Γ is closed under symmetric difference.

Definition 0.1. We say that Γ is *harmonic* (*antiharmonic*, respectively) if there exists a nonzero binary harmonic (antiharmonic, respectively) function on Γ. A *biharmonic* graph is a graph that is simultaneously harmonic and antiharmonic.

A locally finite graph Γ will be called *even* (*odd*, respectively) if the degree of every vertex v of Γ is even (odd, respectively). Every odd graph is harmonic and every even one is antiharmonic. Moreover, any (anti)harmonic graph can be obtained from an odd (respectively, an even) one by adjoining a certain number of new vertices, each one being joined with the old ones by an even number of edges.

For instance, an infinite plane hexagonal lattice is odd and therefore harmonic, whereas an infinite plane triangular lattice is even and so antiharmonic. Actually the latter one is biharmonic, the nuclei being the maximal inscribed hexagonal lattices. Similarly every integer lattice $\mathbb{Z}^s$ is biharmonic when regarded as a graph with edges parallel to the coordinate axes. In all these examples, none of the (anti)nuclei is finite. However it is easy to find infinite graphs with finite nuclei.

We let $\mathrm{spec}^{\pm}(\Gamma)$ be the spectrum of the laplacian $\Delta^{\pm}_{\Gamma}$ in the algebraic closure $\bar{k}$. Since the laplacian $\Delta^{\pm}_{\Gamma}$ is defined over the binary field k, we have

$$d^{\pm} := \dim_k(\mathrm{Harm}^{\pm}(\Gamma)) = \dim_{\bar{k}}(\mathrm{Harm}^{\pm}_{\bar{k}}(\Gamma))\,.$$

Thus by definition, Γ is harmonic if and only if $0 \in \mathrm{spec}^{+}(\Gamma)$, antiharmonic if and only if $0 \in \mathrm{spec}^{-}(\Gamma)$, and biharmonic if and only if $0, 1 \in \mathrm{spec}^{+}(\Gamma)$. The shift by 1 being an involution on $\bar{k}$, this justifies our terminology.

Our aim is to determine the set of all multi-indices $\bar{n} = (n_1, \ldots, n_s) \in \mathbb{N}^s$ such that the integer lattice $\mathbb{Z}^s$ possesses an $\bar{n}$-periodic nonzero binary harmonic function. For instance, for $\bar{n} = (1, \ldots, 1)$ such a function f on $\mathbb{Z}^s$ must be constant $\equiv 1$, but the constant function 1 on $\mathbb{Z}^s$ is not harmonic, although it is antiharmonic.

Given a Galois covering $\pi : \Gamma' \to \Gamma$ with the Galois group G, there is an isomorphism

$$\pi^* : \mathrm{Harm}^{\pm}(\Gamma) \xrightarrow{\cong} [\mathrm{Harm}^{\pm}(\Gamma')]^G\,, \qquad f \longmapsto f \circ \pi,$$

where the latter vector space consists of all G-stable (anti)harmonic functions on Γ'. Thus our problem reduces to the following one:

Letting C_n be a circular graph with n vertices, we consider a finite abelian group $\mathbb{Z}_{\bar{n}} := \prod_{i=1}^{s} \mathbb{Z}/n_i\mathbb{Z}$ and the associated toric lattice (the Caley graph of $\mathbb{Z}_{\bar{n}}$)

$$\mathbb{T}_{\bar{n}} = C_{n_1} \times \ldots \times C_{n_s}, \qquad \bar{n} = (n_1, \ldots, n_s) \in \mathbb{N}^s\,.$$

Then $\mathbb{Z}^s$ can be viewed as the maximal abelian cover of $\mathbb{T}_{\bar{n}}$ with the Galois group $G = \sum_{i=1}^{s} n_i\mathbb{Z}\vec{e}_i \subseteq \mathbb{Z}^s$. Hence the space $[\mathrm{Harm}^{+}(\mathbb{Z}^s)]^G$ of pluri-periodic binary harmonic functions on $\mathbb{Z}^s$ with periods $n_1\vec{e}_1, \ldots, n_s\vec{e}_s$ can be identified with $\mathrm{Harm}^{+}(\mathbb{T}_{\bar{n}})$. So we would like to determine the set of all harmonic toric lattices $\mathbb{T}_{\bar{n}}$.

In section 1 we deal with (anti)harmonic functions on trees. Following Amin, Slater and Zhang [ASZ], Gravier, Mhalla and Tanner [GMT] we provide in 1.3 an algorithm that computs the dimension $d^{\pm}$ of the vector space $\mathrm{Harm}^{\pm}(\Gamma)$.

In section 2 we give an account for some spectral properties of laplacians on multidimensional grids and toric lattices. We also mention some uniqueness theorems for binary harmonic functions on graphs and an application to the game 'Lights Out'. One of the typical results of this section is as follows (see 2.22.b).

Theorem 0.2. *Given $\bar{n} = (n_1, \ldots, n_s) \in \mathbb{N}^s$, there exists a nonzero $\bar{n}$-periodic binary harmonic function on $\mathbb{Z}^s$ (that is, the toric lattice $\mathbb{T}_{\bar{n}}$ is harmonic) if and only if the affine variety in $\mathbb{A}^s_{\bar{k}}$ with equations*

$$\sum_{i=1}^{s}(x_i + x_i^{-1}) = 1, \qquad x_i^{n_i} = 1, \quad i = 1, \ldots, s\,, \tag{1}$$

is nonempty.

Section 3 is devoted to 2-dimensional grids and lattices. According to the above theorem and taking into account the covering trick, to distinguish the harmonic tori it is enough to determine all bi-torsions $(\mathrm{ord}\, x, \mathrm{ord}\, y) \in \mathbb{N}^2_{\mathrm{odd}}$ [1] of points (x, y) on the elliptic cubic curve E^* in $\mathbb{A}^2_{\bar{k}}$ with equation

$$x + 1/x + y + 1/y = 1\,.$$

[1] We denote by $\mathrm{ord}\,\xi \in \mathbb{N}_{\mathrm{odd}}$ the multiplicative order of an element $\xi \in \bar{k}^{\times}$.

As suggested by Zagier, we consider an equivalence relation on the set $\mathbb{N}_{\mathrm{odd}}$ of all odd natural numbers defined by the connected components of the *partnership graph* $\mathcal{P}^{(1)}$ with the set of vertices $\mathbb{N}_{\mathrm{odd}}$ and the set of edges

$$\{[\operatorname{ord} x, \operatorname{ord} y] \,:\, (x,y) \in E^*\}\,.$$

We indicate some simple properties of this graph. In particular, all its connected components are finite (Theorem 3.11). In Appendix 1 the reader will find an overview of the first 13 connected components of $\mathcal{P}^{(1)}$ calculated by Zagier with PARI.

Finally in Appendix 2 we provide a survey on binary Chebyshev-Dickson and Fibonacci polynomials, as these are closely related to our subject.

We are grateful to Don Zagier for his clarifying suggestions in section 3 and the calculations in Appendix 1. Our thanks also to Roland Bacher, Silvain Gravier, Lenny Makar-Limanov and Pieter Moree for highly useful discussions and pointing out references, to Andrzej Schinzel for disproving a conjectural inequality for the Euler function in 3.14.2, to Gottfried Barthel and Roland Bacher for helpful editorial remarks, and to Andrey Inshakov for his assistance with MAPLE. Gottfried Barthel also helped with some computations for the Euler function.

The aim of the present survey being rather pedagogical, we have to apologize that the list of references is by no means complete, neither our survey follows the chronological order of events.

1. Characteristic functions of graphs and harmonic forests

The following proposition is well known, see e.g., [An], [Su2, 4.1-4.3], cf. also [DG], [OZ]. For the sake of completeness, we give a simple argument that applies in the characteristic 2 case.

For a vertex $v \in \operatorname{vert}(\Gamma)$ (respectively, for an edge $[u,v] \in \operatorname{edg}(\Gamma)$) we let $\Gamma - v$ (respectively, $\Gamma - [u,v]$) be the graph obtained from Γ by deleting v and all incident edges (respectively, the edge $[u,v]$ but not the vertices u and v). We let $\operatorname{adj}(\Gamma)$ be the adjacency matrix of a finite graph Γ. Notice that $\operatorname{adj}(\Gamma)$ coincides with the matrix of the laplacian Δ_Γ^-. We let $\chi_\Gamma(x)$ be the characteristic polynomial of $\operatorname{adj}(\Gamma)$, and denote by $\binom{e}{i}_\Gamma$ the number of i-matchings in Γ i.e., of all possible choices of i non-incident edges among the e edges of Γ.

Proposition 1.1. *For any finite graph Γ with n vertices and e edges, the following hold:*

(a) $\chi_\Gamma(x) = \sum_{i=0}^{[n/2]} \binom{e}{i}_\Gamma x^{n-2i}$. *In particular n and χ_Γ are of the same parity.*

(b) $\forall v \in \operatorname{vert}(\Gamma)$,

$$\chi_\Gamma(x) = x\cdot\chi_{\Gamma-v}(x) + \sum_{[v,v']\in\operatorname{edg}(\Gamma)} \chi_{\Gamma-\{v,v'\}}(x)\,.$$

(c) $\forall [u,v] \in \operatorname{edg}(\Gamma)$,

$$\chi_\Gamma(x) = \chi_{\Gamma-[u,v]}(x) + \chi_{\Gamma-\{u,v\}}(x)\,.$$

(d) $\forall [u,v] \in \operatorname{edg}(\Gamma)$ *with* $\deg u = 1$,

$$\chi_\Gamma(x) = x\cdot\chi_{\Gamma-v}(x) + \chi_{\Gamma-\{u,v\}}(x) \quad \textit{and so,} \quad \chi_\Gamma(0) = \chi_{\Gamma-\{u,v\}}(0)\,.$$

(e) *Given* $u, v, w \in \mathrm{vert}(\Gamma)$ *such that* $\deg u = \deg v = 1$ *and* $[u, w]$, $[v, w] \in \mathrm{edg}(\Gamma)$ *(that is* u, v *are extremal vertices of* Γ *joint with* w*) one has*

$$\chi_\Gamma(x) = x^2 \cdot \chi_{\Gamma - \{u,v\}}(x) \quad \textit{and so,} \quad \chi_\Gamma(1) = \chi_{\Gamma-\{u,v\}}(1)\,.$$

(f) *Given* $u, v, w \in \mathrm{vert}(\Gamma)$ *such that* $\deg u = 1$, $\deg v = 2$ *and* $[u, v]$, $[v, w] \in \mathrm{edg}(\Gamma)$ *(that is* $[u, v]$ *is an extremal edge of* Γ *joint with* w*) one has*

$$\chi_\Gamma(x) = (1 + x^2) \cdot \chi_{\Gamma-\{u,v\}}(x) + x \cdot \chi_{\Gamma-\{u,v,w\}}(x) \quad \textit{and so,} \quad \chi_\Gamma(1) = \chi_{\Gamma-\{u,v,w\}}(1)\,.$$

Proof. The order n symmetric determinant $\det = \sum_{\sigma \in S_n} m_\sigma$, where as usual S_n stands for the n-th symmetric group and $m_\sigma = \pm \prod_{i=1}^n a_{i,\sigma(i)}$, reduces modulo 2 to $\sum_{\sigma^2 = \mathrm{id}} m_\sigma$ by cancelling equal terms m_σ and $m_{\sigma^{-1}}$ with $\sigma^2 \neq \mathrm{id}$. This leads to (a). Now (b) and (c) can be easily deduced from (a). In turn (d), (e) and (f) can be deduced by virtue of (b). □

1.2. Let Γ be a finite forest that is, a disjoint union of trees. It can be reduced, in two different ways, to a rather simple one, by

- iteratively suppressing an extremal vertex (leaf) as in (d). In this way we finally reduce Γ to a forest Γ^-_{red} with only isolated vertices;
- iteratively suppressing a pair of extremal vertices as in (e) or a pair of extremal edges as in (f). Via this procedure, Γ will be finally reduced to a forest Γ^+_{red} with only isolated vertices and isolated edges.

This gives the following result, see [ASZ] or, in any positive characteristic, [GMT, Theorem 4 and Corollary 6]. We recall the notation $d^\pm(\Gamma) = \dim(\mathrm{Harm}^\pm(\Gamma))$.

Corollary 1.3. (a) *A forest* Γ *is harmonic (antiharmonic, respectively) if and only if* Γ^+_{red} *(*Γ^-_{red}*, respectively) contains an isolated edge (an isolated vertex, respectively).*

(b) *Moreover, for any* Γ^+_{red} *(*Γ^-_{red}*, respectively), the number of isolated edges (of isolated vertices, respectively) is* $d^+(\Gamma)$ *(*$d^-(\Gamma)$*, respectively).*

Proof. (a) For a disjoint union $\Gamma = \Gamma' \cup \Gamma''$ of two graphs we have $\chi_\Gamma = \chi_{\Gamma'}\chi_{\Gamma''}$. Since $\det(\Delta^-_\Gamma) = \chi_\Gamma(0)$ and $\det(\Delta^+_\Gamma) = \chi_\Gamma(1)$, we get $\det(\Delta^\pm_\Gamma) = \det(\Delta^\pm_{\Gamma'})\det(\Delta^\pm_{\Gamma''})$. By virtue of (d)-(f), the first reduction preserves $\chi_\Gamma(0)$, and the second one $\chi_\Gamma(1)$, so that $\det(\Delta^\pm_\Gamma) = \det(\Delta^\pm_{\Gamma^\pm_{\mathrm{red}}})$. Thus $\det(\Delta^-_\Gamma) = 1$ if and only if Γ^-_{red} is empty, and $\det(\Delta^+_\Gamma) = 1$ if and only if Γ^+_{red} consists of isolated vertices. This proves (a).

(b) Following our iterative procedure, we can easily see that every (anti)harmonic function on Γ restricts to a (anti)harmonic function on Γ^+_{red} (on Γ^-_{red}, respectively). Moreover we can reconstruct the (anti)harmonic functions on Γ from their restrictions to $\Gamma^\pm_{\mathrm{red}}$, respectively. Indeed, for any isolated vertex v of Γ^-_{red}, the δ-function δ_v on $\mathrm{vert}(\Gamma^-_{\mathrm{red}})$, which takes value 1 at v and 0 at any other vertex, is antiharmonic. At every step, δ_v uniquely extends from a smaller graph to a bigger one preserving antiharmonicity. This results finally in an antiharmonic function $\tilde{\delta}_v$ on Γ.

On the other hand, given an antiharmonic function h on Γ, it is uniquely determined by the restriction $h \mid \Gamma^-_{\mathrm{red}}$. This restriction can be decomposed in the basis of δ-functions $\big(\delta_v \,:\, v \in \mathrm{vert}(\Gamma^-_{\mathrm{red}})\big)$ in $\mathcal{F}(\Gamma^-_{\mathrm{red}}, k)$. Hence $\left(\tilde{\delta}_v \,:\, v \in \mathrm{vert}(\Gamma^-_{\mathrm{red}})\right)$ form a basis of $\mathrm{Harm}^-(\Gamma)$.

Similarly, given an isolated edge $[u,v]$ of Γ^+_{red}, $\delta_{[u,v]} = \delta_u + \delta_v$ is a harmonic function on Γ^+_{red}. At each step it extends uniquely to a harmonic function on a bigger graph, and finally to a function $\tilde{\delta}_{[u,v]} \in \mathrm{Harm}^+(\Gamma)$. These functions form a basis of $\mathrm{Harm}^+(\Gamma)$. This shows (b). $\square$

Remark 1.4. The analysis of (anti)harmonicity of unicyclic graphs can be reduced in the same way to that of cyclic graphs C_n [Su2, §4]. As for the latter one, see section 2.1 below.

2. Chebyshev-Dickson-Fibonacci polynomials and harmonicity

2.1. **1-dimensional case.** We refer the reader to Appendix 2 for a survey on the Chebyshev-Dickson polynomials T_n (E_n) of the first (second) kind and the Fibonacci polynomials F_n. We also need the following notation.

2.1. For $n \in \mathbb{N}_{\mathrm{odd}}$, the order and the suborder of 2 modulo n are, respectively,

$$f(n) = \mathrm{ord}_n\, 2 = \min\{j \,:\, 2^j \equiv 1 \mod n\}$$

and

$$f_0(n) = \mathrm{sord}_n\, 2 = \min\{j \,:\, 2^j \equiv \pm 1 \mod n\}\,.$$

Thus $f(n)/f_0(n) \in \{1,2\}$. Moreover,

$$f(n) = 2f_0(n) \iff \exists j \in \mathbb{N} \,:\, 2^j \equiv -1 \mod n\,.$$

Letting $q = 2^{f_0(n)}$ we note that $n \mid (q-1)$ if $f_0(n) = f(n)$ and $n \mid (q+1)$ otherwise. Anyhow, n divides exactly one of $q-1$ and $q+1$. Further, $f(2^r-1) = f_0(2^r-1) = r$ $\forall r \geq 3$ (but $f_0(3) = 1$, $f(3) = 2$) and $f_0(2^r+1) = r = f(2^r+1)/2$ $\forall r \geq 1$, see Appendix B in [MOW].

2.2. We notice that $\Delta^-_{C_n} = \tau + \tau^{-1}$, where $\tau \in \mathrm{End}(\mathbb{A}^n_k)$ is the cyclic right shift, and $\Delta^-_{P_n} = \tau_l + \tau_r$, where τ_l (τ_r) $\in \mathrm{End}(\mathbb{A}^n_k)$ is the left (right) shift. Hence the adjacency matrices of the graphs C_n and P_n are, respectively,

$$\mathrm{adj}(C_n) = \begin{pmatrix} 0 & 1 & 0 & \dots & 0 & 0 & 1 \\ 1 & 0 & 1 & \dots & 0 & 0 & 0 \\ 0 & 1 & 0 & \ddots & 0 & 0 & 0 \\ \vdots & \vdots & \ddots & \ddots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \ddots & 0 & 1 & 0 \\ 0 & 0 & 0 & \dots & 1 & 0 & 1 \\ 1 & 0 & 0 & \dots & 0 & 1 & 0 \end{pmatrix}, \quad \mathrm{adj}(P_n) = \begin{pmatrix} 0 & 1 & 0 & \dots & 0 & 0 & 0 \\ 1 & 0 & 1 & \dots & 0 & 0 & 0 \\ 0 & 1 & 0 & \ddots & 0 & 0 & 0 \\ \vdots & \vdots & \ddots & \ddots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \ddots & 0 & 1 & 0 \\ 0 & 0 & 0 & \dots & 1 & 0 & 1 \\ 0 & 0 & 0 & \dots & 0 & 1 & 0 \end{pmatrix}$$

with the characteristic polynomials $\chi_{C_n} = T_n$ and $\chi_{P_n} = E_n$, respectively. For n odd we have $\mathrm{spec}(\tau) = \mu_n$, where μ_n stands for the cyclic group of n-th roots of unity in $\bar{k}$. According to the spectral mapping theorem, $\mathrm{spec}(\Delta^-_{C_n}) = \{\xi + \xi^{-1} \,:\, \xi \in \mu_n\}$ and $\mathrm{spec}(\Delta^+_{C_n}) = \{1+\xi+\xi^{-1} \,:\, \xi \in \mu_n\}$. Moreover, for the circular graphs C_n the following results hold, see e.g. [Su3, 4.1, 6.1], [SB, 2.1], [HJ, 3.3.8].

Proposition 2.3. (a) *$\forall n \geq 3$, C_n is antiharmonic (that is $\Delta^-_{C_n}$ is non-invertible). Whereas C_n is harmonic (i.e., $\Delta^+_{C_n}$ is non-invertible) if and only if $n \equiv 0 \mod 3$.*

(b) *The minimal polynomial of $\Delta^-_{C_n}$ is F_k if $n = 2k$ and xR_k, where $R_k = \sqrt{F_n}$, if $n = 2k+1$.*

(c) *The polynomial xR_k having simple roots, for every $n \in \mathbb{N}_{\mathrm{odd}}$ the matrix* $\mathrm{adj}(C_n)$ *is similar over $\bar{k}$ to the diagonal matrix* $\mathrm{diag}(\zeta^i + \zeta^{-i} \,:\, i = 0, \ldots, n)$*, where $\zeta \in \mu_n$ is a primitive n-th root of unity.*

(d) *Consequently, $\forall n \in \mathbb{N}_{\mathrm{odd}}$, $(\Delta^{\pm}_{C_n})^q = \Delta^{\pm}_{C_n}$, where $q = 2^{f_0(n)}$.*

(e) *The kernel of $\Delta^+_{C_{3k}}$ is two-dimensional, spanned by the vector* $(1, 1, 0, 1, 1, 0, \ldots)$ *and its shift. If n is even then the kernel of $\Delta^-_{C_n}$ is also two-dimensional, spanned by the vector* $(1, 0, 1, 0, 1, 0, \ldots)$ *and its shift. For n odd this kernel is one-dimensional, spanned by* $(1, 1, 1, 1, 1, \ldots)$*. Hence* $d^+(C_{3k}) = 2 = d^-(C_{2k})$ *and* $d^-(C_{2k-1}) = 1\ \forall k \geq 1$.

(f) *Respectively, the nuclei of C_{3k} are the cyclic shifts of* $N^+ = \{v_i \,:\, i \not\equiv 0 \mod 3\}$*, the antinuclei of C_{2k} are the cyclic shifts of* $N^- = \{v_i \,:\, i \not\equiv 0 \mod 2\}$*, whereas $N^- = C_n$ is the only antinucleus of C_n, $\forall n = 2k - 1$.*

Similarly, for the paths P_n we have the following results, see e.g. [MOW], [Su3], [BR, 4.4], [SB, 3.3-3.4].

Proposition 2.4. (a) $\exists(\Delta^-_{P_{n-1}})^{-1} \iff n \in \mathbb{N}_{\mathrm{odd}}$, *and* $\exists(\Delta^+_{P_{n-1}})^{-1} \iff n \not\equiv 0 \mod 3$.

(b) *The minimal polynomial of $\Delta^-_{P_n}$ is E_n.*

(c) $\forall n \in \mathbb{N}_{\mathrm{odd}}$, *$\Delta^-_{P_n}$ admits a generalized inverse* $\kappa_{P_n} \in \mathrm{End}(\mathcal{F}(P_n, k))$ *such that* $\Delta^-_{P_n} \kappa_{P_n} \Delta^-_{P_n} = \Delta^-_{P_n}$.

(d) $\forall n \in \mathbb{N}_{\mathrm{odd}}$, $\mathrm{ord}(\Delta^-_{P_{n-1}}) = 2e_n - 2$, *where* $e_n = \min\{j \in \mathbb{N} \,:\, (\Delta^-_{C_n})^j = \Delta^-_{C_n}\}$, *$e_n \in \mathbb{N}_{\mathrm{even}}$ is such that $(e_n - 1) \mid (q - 1)$ for $q = 2^{f_0(n)}$.*

(e) $\forall n \in \mathbb{N}_{\mathrm{odd}}$, $(\Delta^+_{P_{n-1}})^{2q} = (\Delta^+_{P_{n-1}})^2$. *Furthermore, if $n \in \mathbb{N}_{\mathrm{odd}}$ and $n \not\equiv 0 \mod 3$ then* $\mathrm{ord}(\Delta^+_{P_{n-1}}) \mid (2q - 2)$*, where $q = 2^{f_0(n)}$.*

(f) *The only nucleus of P_{3k-1} is* $N^+ = \{v_i \,:\, i \not\equiv 0 \mod 3\}$*, and the only antinucleus of P_{2l-1} is* $N^- = \{v_i \,:\, i \equiv 1 \mod 2\}$*. Hence* $d^+(P_{3k-1}) = d^-(P_{2l-1}) = 1$ $\forall k, l \in N$.

Remark 2.5. For every $n \in \mathbb{N}_{\mathrm{odd}}$, both $(\Delta^-_{P_{n-1}})^{-1}$ and the generalized inverse κ_{P_n} for $\Delta^-_{P_n}$ are explicitly found in [SB].

2.2. **Spectra of products.**

2.6. Letting E, E' be vector spaces over a field K and $e_1, \ldots, e_m$ ($e'_1, \ldots, e'_n$, respectively) be a basis of E (E', respectively), we represent every $X = \sum_{i,j} x_{i,j} e_i \otimes e'_j \in E \otimes E'$ by the matrix (or *pattern*) $X = (x_{i,j}) \in \mathrm{Mat}_{m,n}(K)$. Following [BR, §5], [SB, §4], for any two square matrices $A \in \mathrm{Mat}_{m,m}(K)$ and $B \in \mathrm{Mat}_{n,n}(K)$ we consider the Sylvester derivation

$$\delta_{A,B} \in \mathrm{End}(E \otimes E'), \quad X \longmapsto AX + XB^t, \qquad \text{with the matrix} \quad C = A \otimes 1 + 1 \otimes B\,.$$

The following lemma is well known [Ga, VIII.3] and holds for arbitrary fields. We provide a simple argument in the characteristic 2 case for $K = \bar{k}$.

Lemma 2.7. *(a) In the notation of 2.6 we have*

$$\chi_C(x) = \mathrm{Res}_y\left(\chi_A(x + y), \chi_B(y)\right) \quad \text{and} \quad \mathrm{spec}(C) = \mathrm{spec}(A) + \mathrm{spec}(B)$$

(the Minkowski sum in $\bar{k}^{\,2}$).

[2]We recall that for a commutative semigroup Π and for two subsets $\Lambda_1, \Lambda_2 \subseteq \Pi$ their Minkowski sum is $\Lambda_1 + \Lambda_2 = \{\lambda_1 + \lambda_2 \,:\, \lambda_i \in \Lambda_i,\ i = 1, 2\}$.

(b) C is invertible if and only if the characteristic polynomials χ_A and χ_B are coprime.

Proof. Let $A = S_A + N_A$ be the Jordan decomposition of $A \in \mathrm{End}(\mathbb{A}^n_k)$, with $S_A N_A = N_A S_A$, where $S_A, N_A \in \mathrm{End}(\mathbb{A}^n_k)$, S_A is semi-simple and N_A is nilpotent. Then $A^q = S_A^q = S_A$ for certain $q = 2^r$, $r > 0$.

We fix $q = 2^r$ so that $A^q = S_A$, $B^q = S_B$ and $C^q = S_C$. Since $A \otimes 1$ and $1 \otimes B$ commute, we have

$$S_C = C^q : X \longmapsto A^q X + X(B^t)^q = S_A X + X S_B$$

i.e., $S_C = S_A \otimes 1 + 1 \otimes S_B$. If the bases (e_i), (e'_j) as in 2.6 are diagonalizing for S_A, S_B, respectively, with $S_A(e_i) = \lambda_i e_i$ and $S_B(e'_j) = \mu_j e'_j$, then $(e_i \otimes e'_j)$ is a diagonalizing basis for S_C with $S_C(e_i \otimes e'_j) = (\lambda_i + \mu_j) e_i \otimes e'_j$. For any two polynomials $p = \prod_{i=1}^m (x + \lambda_i)$ and $q = \prod_{j=1}^m (x + \mu_j)$ we have [vdW]

$$\mathrm{Res}_y\,(p(x+y), q(y)) = \prod_{1 \le i \le m, 1 \le j \le n} (x + \lambda_i + \mu_j)\ .$$

Since $\chi_A = \chi_{S_A}$ etc., the assertions follow easily. $\square$

Remark 2.8. If $C^q = S_C$ then $(p(C))^q = p(C^q) = p(S_C)$ $\forall p \in \bar{k}[x]$. It follows that $p^q_{\min}(C) = 0$, where

$$p_{\min}(x) = \prod_{\gamma = \lambda + \mu,\, \lambda \in \mathrm{spec}(A),\, \mu \in \mathrm{spec}(B)} (x + \gamma)\,.$$

2.9. The Cartesian product $\Gamma = \Gamma_1 \times \Gamma_2$ of two graphs Γ_1, Γ_2 is defined via

$$\mathrm{vert}(\Gamma) = \mathrm{vert}(\Gamma_1) \times \mathrm{vert}(\Gamma_2), \quad \mathrm{edg}(\Gamma) = [\mathrm{vert}(\Gamma_1) \times \mathrm{edg}(\Gamma_2)] \cup [\mathrm{vert}(\Gamma_2) \times \mathrm{edg}(\Gamma_1)]\,.$$

In particular, the *$m \times n$-grid* is the product $P_{m,n} = P_m \times P_n$, and the *toric $m \times n$-lattice* is the product $T_{m,n} = C_m \times C_n$.

Fixing an ordering of the m (n) vertices of Γ_1 (Γ_2), we may regard any K-valued function on $\Gamma_1 \times \Gamma_2$ as an $m \times n$-matrix X with entries in K. The laplacian $\Delta^{\pm}_{\Gamma}$ acts on X via

$$\Delta^{\pm}_{\Gamma} : X \longmapsto \mathrm{adj}(\Gamma_1)^{\pm} \cdot X + X \cdot \mathrm{adj}(\Gamma_2) = \mathrm{adj}(\Gamma_1) \cdot X + X \cdot \mathrm{adj}(\Gamma_2)^{\pm}\,,$$

where $A^- = A$ and $A^+ = A + 1$.

We let $\mathrm{spec}^{\pm}(\Gamma) = \mathrm{spec}(\Delta^{\pm}_{\Gamma}) \subseteq \bar{k}$ and $\chi_{\Gamma} = \chi_{\mathrm{adj}(\Gamma)}$. From 2.7 we deduce such a corollary, see e.g. [Ba, Lemma 8].

Corollary 2.10. (a) *The spectrum* $\mathrm{spec}^-(\Gamma)$ *of the product* $\Gamma = \Gamma_1 \times \Gamma_2$ *of two graphs is the Minkowski sum of the spectra* $\mathrm{spec}^-(\Gamma_i)$, $i = 1, 2$. *Moreover*

$$\chi_{\Gamma}(x) = \mathrm{Res}_y\left(\chi_{\Gamma_1}(x+y), \chi_{\Gamma_2}(y)\right)\ .$$

Whereas

$$\mathrm{spec}^+(\Gamma) = 1 + \mathrm{spec}^-(\Gamma_1) + \mathrm{spec}^-(\Gamma_2)\,.$$

(b) *Consequently, Γ is antiharmonic if and only if the characteristic polynomials χ_{Γ_1}, χ_{Γ_2} are not coprime, and is harmonic if and only if the polynomials χ_{Γ_1}, $\chi^+_{\Gamma_2}$ are not coprime.*

2.3. **2-dimensional grids and tori.** The following is immediate from 2.10, see [BR], [Su3], [HMP].

Proposition 2.11. (a) *The grid $P_{m-1,n-1}$ is antiharmonic (respectively, harmonic) if and only if the Chebyshev-Dickson polynomials E_{m-1} and E_{n-1} (respectively, E_{m-1} and E^+_{n-1}) are not coprime. Furthermore $P_{m-1,n-1}$ is antiharmonic if and only if* $\gcd(m,n) \neq 1$.

(b) $\det(\Delta^+_{P_{m-1,n-1}}) = \mathrm{Res}_x(E_{m-1}, E^+_{n-1})$.

(c) $\forall m, n \geq 3$, *the toric lattice $\mathbb{T}_{m,n}$ is antiharmonic and, moreover, is an even graph. Furthermore $\mathbb{T}_{m,n}$ is harmonic if and only if the polynomials T_m and T^+_n are not coprime.*

Corollary 2.12. (a) $\forall k, l \in \mathbb{N}$, *the grids $P_{2k-1,3l-1}$ and $P_{3k-1,2l-1}$ are harmonic. The grid $P_{m-1,n-1}$ different from any one of these is harmonic if and only if the system*

$$u + u^{-1} + v + v^{-1} = 1 = u^m = v^n \tag{2}$$

admits a solution $(u, v) \in (\bar{k}^\times)^2$.

(b) *We have*

$$\mathrm{spec}(\Delta^-_{\mathbb{T}_{m,n}}) = \left\{u + u^{-1} + v + v^{-1} \,:\, u \in \mu_m,\, v \in \mu_n\right\},$$

respectively,

$$\mathrm{spec}(\Delta^+_{\mathbb{T}_{m,n}}) = \left\{1 + u + u^{-1} + v + v^{-1} \,:\, u \in \mu_m,\, v \in \mu_n\right\}.$$

Thus $\mathbb{T}_{m,n}$ is harmonic if and only if the system (2) admits a solution, if and only if either $mn \equiv 0 \bmod 3$ or $P_{m-1,n-1}$ is harmonic.

(c) $\forall m, n \equiv 0 \bmod 5$, *both the grid $P_{m-1,n-1}$ and the toric lattice $\mathbb{T}_{m,n}$ are harmonic.*

(d) *If $\mathbb{T}_{m,n}$, respectively, $P_{m-1,n-1}$ is harmonic then so is $\mathbb{T}_{km,ln}$, respectively, $P_{km-1,ln-1}$* $\forall k, l \in \mathbb{N}$.

(e) $\forall q = 2^a$, $\forall q' = 2^b$, *$\mathbb{T}_{m,n}$ is harmonic if and only if $\mathbb{T}_{qm,q'n}$ is, and $P_{m-1,n-1}$ is harmonic if and only if either $P_{qm-1,q'n-1}$ is, or one of the following holds:* $m \equiv 0 \bmod 2$, $n \equiv 0 \bmod 3$ *or* $m \equiv 0 \bmod 3$, $n \equiv 0 \bmod 2$.

(f) *In particular $\forall a, b \geq 0$, $\mathbb{T}_{q,q'}$ and $P_{q-1,q'-1}$ are not harmonic.*

Proof. $P_{m-1,n-1}$ is harmonic if and only if $E_{m-1}(z) = E_{n-1}(z+1) = 0$ for some $z \in \bar{k}$. These equations are satisfied by $z = 0$ (respectively, $z = 1$) if and only if $m \equiv 0 \bmod 2$, $n \equiv 0 \bmod 3$ (respectively, $m \equiv 0 \bmod 3$, $n \equiv 0 \bmod 2$), see 5.4.a,e. Suppose further that $z \neq 0, 1$. Letting

$$z = u + u^{-1},\ z + 1 = v + v^{-1}, \quad \text{where} \quad u, v \in \bar{k}^\times,$$

by virtue of (9) in 5.1 and 5.4.a we obtain

$$E_{m-1}(z) = E_{m-1}(u + u^{-1}) = 0 = E^+_{n-1}(z) = E_{n-1}(v + v^{-1})$$

$$\iff \quad T_m(u + u^{-1}) = u^m + u^{-m} = 0 = T_n(v + v^{-1}) = v^n + v^{-n}.$$

This shows (a). The same argument proves (b). The assertions (c), (d) and (e) follow from (b) by virtue of 5.4.e and, in turn, imply (f). $\square$

In order to find all harmonic toric lattices it is enough, by virtue of 2.12.e, to restrict to $\mathbb{T}_{m,n}$ with $(m,n) \in \mathbb{N}^2_{\text{odd}}$. The following facts are established in [GKW, Theorem 14], see also [HMP, 5.1].

Proposition 2.13. *$\forall q = 2^r$, $r \geq 1$, the toric lattices $\mathbb{T}_{q-1,q-1}$, $\mathbb{T}_{q-1,q+1}$ and $\mathbb{T}_{q+1,q+1}$ are harmonic except for $\mathbb{T}_{1,1}$ and $\mathbb{T}_{7,7}$.*

Proof. Letting, according to 5.2.c,d,

$$A_q = \text{roots}(T_{q-1}) = \{0\} \cup \{z \in \mathbb{F}_q^\times \,:\, \text{Tr}_{\mathbb{F}_q}(z^{-1}) = 0\}\,,$$

$$B_q = \text{roots}(T_{q+1}) = \{0\} \cup \{z \in \mathbb{F}_q^\times \,:\, \text{Tr}_{\mathbb{F}_q}(z^{-1}) = 1\}$$

and

$$A_q^+ = \text{roots}(T_{q-1}^+) = 1 + A_q, \qquad B_q^+ = \text{roots}(T_{q+1}^+) = 1 + B_q$$

we have $A_q \cup B_q = \mathbb{F}_q$ and $A_q \cap B_q = \{0\}$. Indeed by 5.2.g, $T_{q+1} + T_{q-1} = x^{q+1}$ and $T_{q+1}T_{q-1} = x^{2q} + x^2 = x^2(x^{q-1}+1)^2$. So the zeros of the product $T_{q+1}T_{q-1}$ fill in $\mathbb{F}_q$, while 0 is the only common zero of T_{q+1} and T_{q-1}. Hence $\text{card}(A_q) = q/2$ and $\text{card}(B_q) = q/2+1$. It follows that $A_q \cap B_q^+ \neq \emptyset$ and $B_q \cap B_q^+ \neq \emptyset$. Thus the polynomials T_{q-1}, T_{q+1}^+, respectively, T_{q+1}, T_{q+1}^+ are not coprime. In view of 2.11.c, the toric lattices $\mathbb{T}_{q-1,q+1}$ and $\mathbb{T}_{q+1,q+1}$ are harmonic $\forall q = 2^r$, $r \geq 1$.

Suppose further that $r \geq 2$ and $\mathbb{T}_{q-1,q-1}$ is not harmonic, that is $A_q \cap A_q^+ = \emptyset$. Then $A_q^+ \subseteq B_q \setminus \{0\}$. Actually $A_q^+ = B_q \setminus \{0\}$ as these sets have the same cardinality. Thus $\text{roots}(T_{q-1}^+) = \text{roots}(F_{q+1})$. More precisely,

$$F_{q+1} = (x+1)T_{q-1}^+ \quad \Longleftrightarrow \quad x^q + F_{q-1} = (x+1)^2 F_{q-1}^+ \quad \Longleftrightarrow \quad x^q + 1 = F_{q-1}^+ + x^2 F_{q-1}\,.$$

For every $z \in \mathbb{F}_q \setminus \mathbb{F}_2$ we obtain $z+1 = F_{q-1}(z+1) + z^2 F_{q-1}(z)$. Equivalently, by virtue of 5.4.d,

$$(z+1)\left(1 + \text{Tr}_{\mathbb{F}_q}((z+1)^{-1})\right) = z^3 \text{Tr}_{\mathbb{F}_q}(z^{-1})\,. \tag{3}$$

From (3) we deduce the following alternative.

• Either

$$z^3 = z+1 \quad \Longrightarrow \quad z \in \mathbb{F}_8 \setminus \mathbb{F}_2 \subset \mathbb{F}_q \setminus \mathbb{F}_2 \quad \Longrightarrow \quad r \equiv 0 \mod 3\,,$$

and then $1 + \text{Tr}_{\mathbb{F}_q}((z+1)^{-1}) = \text{Tr}_{\mathbb{F}_q}(z^{-1})$,

• or $1 + \text{Tr}_{\mathbb{F}_q}((z+1)^{-1}) = \text{Tr}_{\mathbb{F}_q}(z^{-1}) = 0$ and so, $F_{q-1}(z) = 0$ $\forall z \in \mathbb{F}_q \setminus \mathbb{F}_8$.

Henceforth, if $\mathbb{F}_q \supseteq \mathbb{F}_8$ and $\mathbb{F}_q \neq \mathbb{F}_8$ then

$$\text{card}(A_q \setminus \{0\}) = q/2 - 1 \geq q - 8 \quad \Longrightarrow \quad q \leq 14 \quad \Longrightarrow \quad q = 8\,,$$

which is a contradiction. If $\mathbb{F}_q \not\supseteq \mathbb{F}_8$ then, by the same argument as above, $F_{q-1}(z) = 0$ $\forall z \in \mathbb{F}_q \setminus \mathbb{F}_2$. Hence

$$\text{card}(A_q \setminus \{0\}) = q/2 - 1 \geq q - 2 \quad \Longrightarrow \quad q \leq 2\,,$$

which again gives a contradiction.

Therefore $\mathbb{F}_q = \mathbb{F}_8$. Indeed, for $q = 2^3$ we have $A_q^+ = B_q \setminus \{0\}$ and so, the toric lattice $\mathbb{T}_{q-1,q-1} = \mathbb{T}_{7,7}$ is not harmonic, as stated. □

Remarks 2.14. 1. By virtue of 2.12.b,

$$r \equiv 0 \mod 2 \iff q-1 \equiv 0 \mod 3 \iff 0 \in A_q \cap A_q^+ \cap B_q\,,$$

hence both $\mathbb{T}_{q-1,q-1}$ and $\mathbb{T}_{q-1,q+1}$ are harmonic, and

$$r \equiv 1 \mod 2 \iff q+1 \equiv 0 \mod 3 \iff 0 \in A_q \cap B_q \cap B_q^+\,,$$

hence both $\mathbb{T}_{q-1,q+1}$ and $\mathbb{T}_{q+1,q+1}$ are harmonic.

2. The polynomials $h_1(x) = x^2+x+1$ and $h_2(x) = x^4+x+1$ satisfy $h_i(x+1) = h_i(x)$, $i = 1,2$. They divide the Fibonacci polynomials $F_{q\pm1}$ (and hence also $T_{q\pm1}$) in the following cases:

$$h_1 \mid F_{q-1} \iff r \equiv 0 \mod 4 \quad \text{and} \quad h_1 \mid F_{q+1} \iff r \equiv 2 \mod 4\,,$$

$$h_2 \mid F_{q-1} \iff r \equiv 0 \mod 8 \quad \text{and} \quad h_2 \mid F_{q+1} \iff r \equiv 4 \mod 8\,.$$

2.15. The above theory can be naturally extended to the laplacians $\Delta_\Gamma^\pm$ on $\mathcal{F}(\Gamma,\mathbb{Z})$ and on $\mathcal{F}(\Gamma,\mathbb{F}_p)$ for all primes $p > 2$, see e.g., [MOW], [GMT], [HMP]. We say that Γ is *p-(anti)harmonic* if $\ker(\Delta_\Gamma^\pm)$ has a positive dimension $d_p^\pm$ in $\mathcal{F}(\Gamma,\mathbb{F}_p)$. For s-dimensional grids, and especially for 2-dimensional square grids, the following is proved in [HMP, §§4-5].

Proposition 2.16. (a) *For $\bar n = (n_1,\dots,n_s) \in \mathbb{N}^s$, the grid $P_{\bar n}$ is p-harmonic if and only if*

$$\det(\Delta^+_{P_{\bar n}}) = \prod_{(i_1,\dots,i_s),\, 1\le i_j\le n_j} \left(1 - \sum_{j=1}^{s}\left(\varsigma_{2(n_j+1)}^{i_j} + \varsigma_{2(n_j+1)}^{-i_j}\right)\right) \equiv 0 \mod p\,,$$

where $\varsigma_n = e^{\frac{2\pi i}{n}} \in \mathbb{C}$ [3].

(b) *$\forall n \ge 3$ there exists a prime p such that the square grid $P_{n-1,n-1}$ and the toric lattice $\mathbb{T}_{n,n}$ are p-harmonic.*

(c) *$P_{n-1,n-1}$ ($\mathbb{T}_{n,n}$, respectively) is p-harmonic for every prime p if and only if $n \equiv 0 \mod 5$ or $n \equiv 0 \mod 6$ ($n \equiv 0 \mod 5$ or $n \equiv 0 \mod 3$, respectively).*

(d) *If $l > 5$ and p are primes such that p is a primitive root modulo l then both the square grid $P_{l-1,l-1}$ and the toric lattice $\mathbb{T}_{l,l}$ are not p-harmonic.*

(e) *For every prime p with at most two exceptions, the set I_p of all primes l such that the square grid $P_{l-1,l-1}$ (the toric lattice $\mathbb{T}_{l,l}$, respectively) is not p-harmonic, is infinite.*

(f) *The square grid $P_{n-1,n-1}$ and the toric lattice $\mathbb{T}_{n,n}$ with $n = (p\pm1)/2$ are p-harmonic for every prime $p > 23$.*

For the proof of (a), (b), (d), (f) see [HMP], 4.6, 4.4, 4.3, 4.7 and 5.4, respectively. The proof of (e) in [HMP] is based on (d) and on a result of Heath-Brown [HB] on Artin's conjecture of primitive roots. [4]

---

[3]The above product being an integer.

[4]This conjecture suggests that every integer $n \neq -1$ which is not a square, is a primitive root modulo l for an infinite set, say, I_n of primes l. The result of Heath-Brown *loc.cit.* says that the property in Artin's conjecture holds for all primes with at most 2 exceptions, and for all square-free integers with at most 3 exceptions. However, so far no concrete example of a prime satisfying the conjecture has been found, see [HB, Mo, Mu].

2.17. For a graph Γ we let as before $d^{\pm}(\Gamma) = \dim(\ker(\Delta^{\pm}_{\Gamma})) = \dim(\mathrm{Harm}^{\pm}(\Gamma))$. The kernels of the grid laplacians $\Delta^{\pm}_{P_{m,n}}$ admit the following description, see [Su2], [Su3, 3.6, 3.10], [BR, 3.6, 4.1] and 2.30 below.

Proposition 2.18. (a) $\forall p \in k[x]$, $\dim(\ker(p(\Delta^{-}_{P_m}))) = \deg(\gcd(p, E_m))$.

(b) $\ker(\Delta^{\pm}_{P_{m,n}}) \cong \ker(E_m(\Delta^{\pm}_{P_n})) \cong \ker(E_n(\Delta^{\pm}_{P_m}))$.

(c) *Furthermore,* $\ker(\Delta^{-}_{P_{m-1,n-1}}) \cong \ker(E_{\gcd(m,n)-1}(\Delta^{-}_{P_{n-1}}))$.

(d) *Consequently,* $d^{-}(P_{m-1,n-1}) = \gcd(m,n) - 1$ *and* $d^{+}(P_{m-1,n-1}) = \deg(\gcd(E_{m-1}, E^{+}_{n-1}))$.

(e) $\forall m = 2^r$, $\forall n = 2^k p$, *where* $r \geq 1$ *and* $p \in \mathbb{N}_{\mathrm{odd}}$, *one has* [5]

$$d^{+}(P_{m-1,n-1}) = d^{+}(P_{n-1}) = \begin{cases} 0 & \textit{if} \quad p \not\equiv 0 \mod 3, \ (\textit{and so } P_{m-1,n-1} \textit{ is not harmonic}), \\ 2^{k+1} & \textit{if} \quad p \equiv 0 \mod 3 \quad \textit{and} \quad k < r-1, \\ m-1 & \textit{otherwise}. \end{cases}$$

(f) *Moreover* $\min\{n \geq m \,:\, d^{+}(P_{m-1,n}) = m-1\} = \frac{3}{2}m - 1$.

Examples 2.19. The path P_2 and the grids $P_{2,2n-1}$, $n \geq 1$, $P_2 \times C_n$ and $\mathbb{T}_{3,n}$, $n \geq 3$, are harmonic, whereas $P_{2,2n}$, $n \geq 1$, are not. The grid $P_{2,3}$ has the nuclei

$$\begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \qquad \begin{pmatrix} 0 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 1 & 0 & 1 \\ 1 & 0 & 1 \end{pmatrix}.$$

Thus $d^{+}(P_{2,3}) = 2$. Similarly, the grid $P_{2,2n-1}$ ($P_{2,4n-1}$, respectively) has a nucleus

$$\begin{pmatrix} 1 & 0 & 1 & 0 & 1 & 0 & \ldots & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 & \ldots & 1 \end{pmatrix}, \quad \text{respectively,} \quad \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & \ldots & 0 \\ 1 & 1 & 1 & 0 & 1 & 1 & 1 & 0 & \ldots & 1 \end{pmatrix}.$$

2.4. ***n*-dimensional case.** From 2.10 we deduce by recursion the following, cf. [SB].

Proposition 2.20. (a) $\mathrm{spec}^{-}(\prod_{i=1}^{s} \Gamma_i) = \sum_{i=1}^{s} \mathrm{spec}^{-}(\Gamma_i)$ *and* $\mathrm{spec}^{+}(\prod_{i=1}^{s} \Gamma_i) = 1 + \sum_{i=1}^{s} \mathrm{spec}^{-}(\Gamma_i)$.

(b) *For the product* $\Gamma = \prod_{i=1}^{s} \Gamma_i$ *of* s_1 *harmonic and* $s - s_1$ *antiharmonic graphs we have* $(s_1 + 1) \bmod 2 \in \mathrm{spec}^{+}(\Gamma)$. *Consequently,* Γ *is harmonic if* s_1 *is odd and* Γ *is antiharmonic otherwise. If at least one of the factors* Γ_i *is biharmonic then so is* Γ.

(c) $\forall f_i \in \mathrm{Harm}^{+}_{\bar{k}}(\Gamma_i)$, $i = 1, \ldots, s_1$, $\forall g_j \in \mathrm{Harm}^{-}_{\bar{k}}(\Gamma_j)$, $j = s_1 + 1, \ldots, s$, *the function* $h = (\bigotimes_{i=1}^{s_1} f_i) \otimes (\bigotimes_{j=s_1+1}^{s} g_j) \in \mathcal{F}(\Gamma, \bar{k})$ *is harmonic for* s_1 *odd and antiharmonic for* s_1 *even.*

(d) *If* N_i^{+} *is a nucleus of* Γ_i, $i = 1, \ldots, s_1$, *and* N_j^{-} *is an antinucleus of* Γ_j, $j = s_1 + 1, \ldots, s$, *then* $N = \prod_{i=1}^{s_1} N_i^{+} \times \prod_{j=s_1+1}^{s} N_j^{-}$ *is a nucleus of* $\Gamma = \prod_{i=1}^{s} \Gamma_i$ *for* s_1 *odd and an antinucleus of* Γ *for* s_1 *even.*

2.21. We keep the notation

$$\mathbb{T}_{\bar{n}} = \prod_{i=1}^{s} C_{n_i}, \qquad P_{\bar{n}} = \prod_{i=1}^{s} P_{n_i} \quad \text{and} \quad \overline{n-1} = (n_1 - 1, \ldots, n_s - 1).$$

For the next results see e.g. [Su1], [SB, §5-6], [HMP, §3]).

[5] See also [Su3, 5.2] for the case $m = 3 \cdot 2^r$, $n = 2^k p$.

Proposition 2.22. (a) *For any graph* Γ *and for every* $n \in \mathbb{N}$,

$$\chi_{\Gamma\times P_{n-1}}(x) = \mathrm{Res}_y\left(\chi_\Gamma(x+y), F_n(y)\right), \qquad \chi_{\Gamma\times C_{n-1}}(x) = \mathrm{Res}_y\left(\chi_\Gamma(x+y), T_n(y)\right)$$

and

$$\mathrm{spec}^{\pm}(\Gamma\times C_n) = \{\lambda \in \bar{k}^\times \;:\; \lambda+\lambda^{-1} \in \mathrm{spec}^{\pm}(\Gamma)\}\,.$$

Hence $\Gamma\times C_n$ *is harmonic if and only if* $1+\lambda+\lambda^{-1} \in \mathrm{spec}^+(\Gamma)$ *for some* $\lambda \in \mu_n$.

(b) *We have*

$$\mathrm{spec}^+(\mathbb{T}_{\bar n}) = \{1 + \sum_{i=1}^{s}(\xi_i+\xi_i^{-1}) \;:\; \xi_i \in \mu_{n_i},\; i=1,\dots,s\}\,.$$

Thus $\mathbb{T}_{\bar n}$ *is harmonic if and only if the system (1) in 0.2 has a solution* $(x_1,\dots,x_s) \in (\bar{k}^\times)^s$.

(c) *If* $\Gamma\times C_n$ *is harmonic then so is* $\Gamma\times C_{ln}$ *for every* $l \geq 1$.

(d) *If* $\mathbb{T}_{\bar n}$ *is harmonic then so is* $\mathbb{T}_{\bar n'}\times\mathbb{T}_{\bar m}$, $\forall \bar m \in \mathbb{N}^t$, $\forall \bar n' = (l_1n_1,\dots,l_sn_s) \in \mathbb{N}^s$, *where* $l_i \geq 1\,\forall i=1,\dots,s$.

(e) *If* $\Gamma\times C_{2n}$ $(n\geq 3)$ *is harmonic then so is* $\Gamma\times C_n$. *Consequently,* $\Gamma\times C_{2^r}$ $(r\geq 2)$ *is harmonic if and only if so is* Γ.

(f) $\forall \bar n = (2^{r_1},\dots,2^{r_s})$, *the toric lattice* $\mathbb{T}_{\bar n}$ *is not harmonic.*

Proof. (a) follows from 2.2 and 2.10, and implies (b) by recursion. The covering $\Gamma\times C_{ln}\to\Gamma\times C_n$ with the Galois group $\mathbb{Z}/l\mathbb{Z}$ induces the injections

$$\pi^* : \mathrm{Harm}^{\pm}(\Gamma\times C_n) \hookrightarrow \mathrm{Harm}^{\pm}(\Gamma\times C_{ln})\,, \qquad f \longmapsto f\circ\pi\,.$$

This proves the harmonicity of $\mathbb{T}_{\bar n'}$ in (d), whereas that of the product $\mathbb{T}_{\bar m}\times\mathbb{T}_{\bar n}$ follows from 2.20.b. The proof of (e) uses (a) and the fact that $\phi : \bar k \to \bar k$, $x \longmapsto x^2$ is an automorphism. (f) follows from (e) by recursion. □

Examples 2.23. For any antiharmonic graph Γ and $\forall n \geq 3$, $\forall l \geq 1$, the products $\Gamma\times C_n$ and $\Gamma\times P_{2l-1}$ are antiharmonic , whereas $\Gamma\times C_{3l}$ and $\Gamma\times P_{3l-1}$ are harmonic. If Γ is harmonic then so are the products $\Gamma\times C_n$ and $\Gamma\times P_{2l-1}$, whereas $\Gamma\times C_{3l}$ and $\Gamma\times P_{3l-1}$ are antiharmonic. See also [SB, §§5,6] for the (anti)harmonicity of the hypercubic grids $P_{\bar n}$ and of the products $P_{\bar n}\times\mathbb{T}_{\bar m}$.

2.5. Symmetrization.

2.24. Let K be a field with $\mathrm{char}(K)=2$. If $\alpha:\Gamma\to\Gamma$ is an involution then for any nonzero $f\in\mathcal{F}(\Gamma,K)$, either $f\circ\alpha = f$ or the average $g = f+f\circ\alpha$ is again nonzero and is α-stable: $g\circ\alpha = g$. Anyhow, if $\mathrm{Harm}^{\pm}_K(\Gamma)\neq\{0\}$ then also $[\mathrm{Harm}^{\pm}_K(\Gamma)]^\alpha\neq\{0\}$. Moreover, if $F=\mathrm{Fix}(\alpha)\neq\emptyset$ then for any $f\in[\mathcal{F}(\Gamma,K)]^\alpha$,

$$\Delta^{\pm}_\Gamma(f)|F = \Delta^{\pm}_F(f|F)\,, \tag{4}$$

and so the restriction of an α-symmetric (anti)harmonic function f to F is again (anti)harmonic. Furthermore, if $f|F\equiv 0$ then also $f|(\Gamma\ominus F)\in\mathrm{Harm}^{\pm}(\Gamma\ominus F)$. Thus if Γ is (anti)harmonic then so is at least one of the graphs F and $\Gamma\ominus F$. In case that F is a 'separation wall' for Γ, from the above discussion we deduce the following result.

Lemma 2.25. *Let* α *be an involution of* Γ *such that* $F=\mathrm{Fix}(\alpha)$ *separates* Γ *that is,* $\Gamma\ominus F = \Gamma^+\cup\Gamma^-$, *where* Γ^+ *and* Γ^- *are two disjoint subgraphs of* Γ *with* $\alpha(\Gamma^\pm)=\Gamma^\mp$. *If* Γ *is (anti)harmonic then so is at least one of the graphs* F *and* $\Gamma^\pm$.

Corollary 2.26. (a) *If $\Gamma \times P_{n-1}$ is (anti)harmonic then so is $\Gamma \times C_n$. Vice versa, if $\Gamma \times C_n$ is (anti)harmonic then so is at least one of the graphs Γ and $\Gamma \times P_{n-1}$.*

(b) *(cf.* [SB, 6.1]*) Consequently, if the grid $P_{\overline{n-1}} = \prod_{i=1}^s P_{n_i-1}$ is harmonic, where $\bar{n} = (n_1, \ldots, n_s)$ then so is the toric lattice $\mathbb{T}_{\bar{n}}$. In particular for every $\bar{n} = (2^{r_1} - 1, \ldots, 2^{r_s} - 1)$ the grid $P_{\bar{n}}$ is not harmonic.*

Vice versa, if $\mathbb{T}_{\bar{n}}$ is harmonic and $\prod_{i=1}^s n_i \not\equiv 0 \mod 3$ then $P_{\overline{n-1}}$ is harmonic too.

(c) *If $\Gamma \times P_{n-1}$ is harmonic then so is $\Gamma \times P_{ln-1}$ $\forall l \geq 1$.*

(d) *If $\Gamma \times P_{2n+1}$ is (anti)harmonic then so is at least one of the graphs Γ and $\Gamma \times P_n$.*

Proof. (a) follows from 2.25. To show (b), letting f be a nonzero (anti)harmonic function on $\Gamma \times P_{n-1}$ symmetric w.r.t. the reflection

$$\alpha : P_{n-1} \to P_{n-1}, \quad v_i \longmapsto v_{(n-1-i) \mod (n-1)},$$

the extension of f by zero to $\Gamma \times C_n \supseteq \Gamma \times P_{n-1}$ is again (anti)harmonic, as required. The converse in (b) follows from 2.25. Iterating this argument yields the first and the last assertions of (c). The second one follows by 2.22.d.

To show (d) we take l copies $P_{n-1}^{(i)}$, $i = 1, \ldots, l$ of P_{n-1}. For a nucleus N of $\Gamma \times P_{n-1}$, we consider its copy N_1 in $\Gamma \times P_{n-1}^{(1)}$, the mirror image N_2 of N_1 in $\Gamma \times P_{n-1}^{(2)}$, the mirror image N_3 of N_2 in $\Gamma \times P_{n-1}^{(3)}$, etc. Taking also new vertices $v_1', \ldots, v_{l-1}'$ and representing $\Gamma \times P_{ln-1}$ as ordered 'connected sum' of the graphs

$$\Gamma \times P_{n-1}^{(1)}, \quad \Gamma \times \{v_1'\}, \quad \Gamma \times P_{n-1}^{(2)}, \quad \Gamma \times \{v_2'\}, \quad \ldots, \quad \Gamma \times \{v_{l-1}'\}, \quad \Gamma \times P_{n-1}^{(l)}$$

we obtain a nucleus $N' = \bigcup_{i=1}^l N_i$ of $\Gamma \times P_{ln-1}$. □

Remarks 2.27. 1. Starting with $N = P_2$ the proof of (d) gives a nucleus of P_{3l-1} $\forall l \geq 1$.

2. Instead of taking average of f over an involution, one can consider the average of f over the shifts on $\mathbb{Z}_{\bar{n}}$. Suppose that for some $i \in \{1, \ldots, s\}$, the subgroup $\mathbb{Z}_{\bar{n}^{(i)}}$ of $\mathbb{Z}_{\bar{n}}$ is not harmonic, where $\bar{n}^{(i)} = (n_1, \ldots, n_{i-1}, n_{i+1}, \ldots, n_s)$. Then for any $f \in \mathrm{Harm}^+(\mathbb{Z}_{\bar{n}})$, the average $f + f_{\vec{e}_i} + \ldots + f_{(n_i-1)\vec{e}_i}$ of f over the shifts by the subgroup $\mathbb{Z}_{n_i} \subseteq \mathbb{Z}_{\bar{n}}$ must be zero. This means that the intersection of each nucleus N^+ of $\mathbb{Z}_{\bar{n}}$ with every 'line' $l_{p,i} = \{p, p + \vec{e}_i, \ldots, p + (n_i - 1)\vec{e}_i\}$, $p \in \mathbb{Z}_{\bar{n}}$, has even cardinality.

2.28. For a vertex v of a graph Γ we let

$$a_v^+ = \delta_v + \sum_{[u,v] \in \mathrm{edg}(\Gamma)} \delta_u \in \mathcal{F}(\Gamma, k)\,. \tag{5}$$

Actually the toric lattice $\mathbb{T}_{\bar{n}}$ represents the Caley graph of the group $\mathbb{Z}_{\bar{n}} = \prod_{i=1}^s \mathbb{Z}/n_i\mathbb{Z}$ with its standard generators $(e_i)_{i=1,\ldots,s}$. Every involution α' of $\mathbb{T}_{\bar{n}}$ with a fixed point is conjugated with an involutive automorphism $\alpha : \mathbb{Z}_{\bar{n}} \to \mathbb{Z}_{\bar{n}}$ stabilizing a_e^+: $a_e^+ \circ \alpha = a_e^+$. Moreover $\alpha(e_i) = \pm e_{\sigma(i)}$, where $\sigma \in S_s$ is a product of independent transpositions such that $n_i = n_{\sigma(i)}$ $\forall i = 1, \ldots, s$. The induced action of α on $\mathcal{F}(\mathbb{T}_{\bar{n}}, K)$ commutes with $\Delta = \Delta^+_{\mathbb{T}_{\bar{n}}}$:

$$(\Delta f) \circ \alpha = \Delta(f \circ \alpha) \quad \forall f \in \mathcal{F}(\mathbb{T}_{\bar{n}}, K)\,.$$

Hence $\Delta^k(\delta_e)|F = \Delta^k_F(\delta_e|F)$, where $F = \mathrm{Fix}(\alpha)$. In particular, if $\mathbb{Z}_{\bar{n}}$ is not harmonic then so is F. Choosing α appropriately, we arrive at the same conclusion as in 2.22.d.

2.6. **Uniqueness sets.** Let K be a field of characteristic 2. A subset $U \subseteq \mathrm{vert}(\Gamma)$ is called a *uniqueness set* for $\mathrm{Harm}^{\pm}_K(\Gamma)$ if every function $f \in \mathrm{Harm}^{\pm}_K(\Gamma)$ that vanishes on U vanishes identically. Thus every (anti)harmonic function f on Γ is uniquely determined by its restriction $f|U$.

The boundary of a bounded plane domain is a uniqueness set for the classical harmonic functions. In our discrete setting, it may happen that just a part of the boundary (or of the interior) serves as a uniqueness set for binary harmonic functions. Let us give several examples.

Examples 2.29. 1. An extremal vertex of the linear string P_n is a uniqueness set for $\mathrm{Harm}^{\pm}_K(P_n)$. Every pair of neighborhooding vertices of the circular graph C_n is a uniqueness set for $\mathrm{Harm}^{\pm}_K(C_n)$.

2. More generally, $\Gamma \times \{v_1\}$ and $\Gamma \times \{v_i, v_{i+1}\}$, $2 \le i \le n-2$, are uniqueness sets for the (anti)harmonic functions on $\Gamma \times P_n$, and $\Gamma \times \{v_i, v_{i+1}\}$ is that on $\Gamma \times C_n$.

3. The set of all extremal vertices of a finite forest Γ is a uniqueness set for the (anti)harmonic functions on Γ. On the other hand, the reduction Γ^+_{red} (Γ^-_{red}, respectively) as in 1.2 regarded as a subgraph of Γ is a uniqueness set for harmonic (respectively, antiharmonic) functions on Γ, see the proof of 1.3.

4. Every side of a triangle Π_n inscribed in a triangular plane lattice is a uniqueness set for $\mathrm{Harm}^{\pm}_K(\Pi_n)$.

5. The exterior circle is a uniqueness set for the conic lattice $C_n(m)$ made of m concentric plane copies of C_n joint one to another by radial edges, the last copy being also joint with a new vertex at their common center.

2.7. **Periodic harmonic extension.** The idea behind 2.22.a and 2.18 is as follows, cf. [Su3, BR].

2.30. To any function $f \in \mathcal{F}(\Gamma \times C_n, K)$ one associates a sequence $f_1, \ldots, f_n \in \mathcal{F}(\Gamma, K)$, where $f_i = f|(\Gamma \times \{v_i\})$. Letting $\Delta^{\pm} = \Delta^{\pm}_{\Gamma \times C_n}$ we obtain

$$(\Delta^{\pm} f)_i = f_{(i-1) \bmod n} + \Delta^{\pm}_{\Gamma}(f_i) + f_{(i+1) \bmod n}, \qquad i = 1, \ldots, n\,.$$

Therefore $f \in \mathrm{Harm}^{\pm}(\Gamma \times C_n)$ if and only if

$$f_{(i+1) \bmod n} = f_{(i-1) \bmod n} + \Delta^{\pm}_{\Gamma}(f_i) \qquad \forall i = 1, \ldots, n\,. \tag{6}$$

Starting with an arbitrary pair $u_0 = (f_0, f_1) \in V := [\mathcal{F}(\Gamma, k)]^2$ and applying successively the automorphism $J_\Gamma = J^{\pm}_\Gamma = \begin{pmatrix} 0 & 1 \\ 1 & \Delta^{\pm}_\Gamma \end{pmatrix} \in \mathrm{Aut}(V)$ we extend u_0 to a function f on $\Gamma \times C_n$ so that

$$(f_1, f_2) = u_1 = J_\Gamma(u_0) = J_\Gamma(f_0, f_1), \ldots, (f_n, f_{n+1}) = u_n = J_\Gamma(u_{n-1}) = J_\Gamma(f_{n-1}, f_n)\,.$$

This extension f is (anti)harmonic provided that it is periodic. The latter holds if and only if $J^n_\Gamma(u_0) = u_0$. By recursion we obtain

$$J^n_\Gamma = \begin{pmatrix} F_{n-1}(\Delta^{\pm}_\Gamma) & F_n(\Delta^{\pm}_\Gamma) \\ F_n(\Delta^{\pm}_\Gamma) & F_{n+1}(\Delta^{\pm}_\Gamma) \end{pmatrix}.$$

Thus

$$J^n_\Gamma(u_0) = u_0 \quad \Longleftrightarrow \quad \begin{cases} F_{n-1}(\Delta^{\pm}_\Gamma) f_0 + F_n(\Delta^{\pm}_\Gamma) f_1 = f_0 \\ F_n(\Delta^{\pm}_\Gamma) f_0 + F_{n+1}(\Delta^{\pm}_\Gamma) f_1 = f_1\,. \end{cases}$$

In particular $(0, f_1) \in \ker(\mathrm{id} + J^n_\Gamma) \quad \Longleftrightarrow \quad f_1 \in \ker(F_n(\Delta^{\pm}_\Gamma)) \cap \ker(\mathrm{id} + F_{n-1}(\Delta^{\pm}_\Gamma))$.

Hence $\Gamma \times C_n$ is (anti)harmonic if and only if $1 \in \operatorname{spec}(J_\Gamma^n)$, or equivalently, if there exists $\lambda \in \operatorname{spec}(J_\Gamma) \cap \mu_n$. We have

$$\operatorname{spec}(J_\Gamma^\pm) = \{\lambda \in \bar{k}^\times \,:\, \lambda + \lambda^{-1} \in \sigma_\Gamma^\pm\}\,.$$

Thus $\Gamma \times C_n$ is (anti)harmonic if and only if there is $\lambda \in \mu_n$ such that $\lambda + \lambda^{-1} \in \sigma_\Gamma^\pm$. This proves the second assertion in 2.22.a.

Remark 2.31. If $\Gamma \times C_n$ is harmonic then, according to the symmetrization and uniqueness principles, every (anti)harmonic function on $\Gamma \times C_n$ with $f_0 = f_1$ is necessarily symmetric i.e., $f_i = f_{(-i+1) \bmod n}$ for all $i = 1, \ldots, n$.

We let $m = \operatorname{card}(\operatorname{vert}(\Gamma))$. Since $0 \notin \operatorname{spec}(J_\Gamma^\pm)$, $J_\Gamma^\pm \in \mathbf{GL}_k(2m)$ has finite order i.e., $\exists n \in \mathbb{N} \,:\, J^n = 1 \quad \Longleftrightarrow \quad F_{n-1}(\Delta_\Gamma^\pm) + \mathrm{id} = 0 = F_n(\Delta_\Gamma^\pm)$. The previous discussion leads to the following result.

Proposition 2.32. $\forall n \geq 3$, $d^\pm(\Gamma \times C_n) \leq \dim_k(V) = 2\operatorname{card}(\operatorname{vert}(\Gamma))$. *The equality holds if and only if* $n \equiv 0 \mod \operatorname{ord} J_\Gamma^\pm$.

Examples 2.33. 1. For $\Gamma = C_3$, $\operatorname{ord}(\Delta_\Gamma^+) = 2$ and $\operatorname{ord}(J_\Gamma^+) = 6$. So $d^+(\mathbb{T}_{3,n}) \leq 6$ $\forall n \geq 1$ and $d^+(\mathbb{T}_{3,n}) = 6 \quad \Longleftrightarrow \quad n \equiv 0 \bmod 6$. The cyclic shifts in the vertical direction of the harmonic patterns

$$h_1 = \begin{pmatrix} 1 & 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 \end{pmatrix}, \quad h_2 = \begin{pmatrix} 0 & 1 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 \end{pmatrix}$$

form a basis of $\operatorname{Harm}^+(\mathbb{T}_{3,6})$. The 2-sheeted covering $\pi : \mathbb{T}_{3,6} \to \mathbb{T}_{3,3}$ yields 4-dimensional subspace $\pi^*(\operatorname{Harm}^+(\mathbb{T}_{3,3})) \subseteq \operatorname{Harm}^+(\mathbb{T}_{3,6})$.

2. Likewise, for $\Gamma = C_5$, $\operatorname{ord}(\Delta_\Gamma^+) = 3$ and $\operatorname{ord}(J_\Gamma^+) = 15$, so $d^+(\mathbb{T}_{5,n}) \leq 10$ $\forall n \geq 1$ and $d^+(\mathbb{T}_{5,n}) = 10 \quad \Longleftrightarrow \quad n \equiv 0 \bmod 15$. The 3-sheeted covering $\pi : \mathbb{T}_{5,15} \to \mathbb{T}_{5,5}$ gives rise to 8-dimensional subspace $\pi^*(\operatorname{Harm}^+(\mathbb{T}_{5,5})) \subseteq \operatorname{Harm}^+(\mathbb{T}_{5,15})$.

2.34. Similarly, for the path P_{n-1}, the function $f = (f_1, \ldots, f_{n-1}) \in \mathcal{F}(\Gamma \times P_{n-1}, K)$ is (anti)harmonic if and only if (6) holds for all $i = 1, \ldots, n-1$ with $f_0 = f_n = 0$. By recursion, the latter holds if and only if $f_k = F_k(\Delta_\Gamma^\pm) f_1$ $\forall k = 2, \ldots, n$. Thus $f_1 \in \mathcal{F}(\Gamma, K)$ extends to $f \in \operatorname{Harm}^\pm(\Gamma \times P_{n-1}, K)$ if and only if $F_{n+1}(\Delta_\Gamma^\pm)(f_1) = 0$. Hence $\operatorname{Harm}^\pm(\Gamma \times P_{n-1}, K) \cong \ker(E_n(\Delta_\Gamma^\pm))$. This shows 2.18.b.

2.8. **Doubling the periods.**

2.35. In this subsection we consider an injective k-endomorphism $\delta : \mathcal{F}(\mathbb{Z}^2, k) \to \mathcal{F}(\mathbb{Z}^2, k)$, which sends a function

$$f = \begin{pmatrix} \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & r & s & t & \cdots \\ \cdots & u & v & w & \cdots \\ \cdots & x & y & z & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \end{pmatrix}$$

into

$$\delta(f) = \begin{pmatrix} \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & r & r+s & s & s+t & t & \cdots \\ \cdots & r+u & 0 & s+v & 0 & t+w & \cdots \\ \cdots & u & u+v & v & v+w & w & \cdots \\ \cdots & u+x & 0 & v+y & 0 & w+z & \cdots \\ \cdots & x & x+y & y & y+z & z & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \end{pmatrix}.$$

For instance, δ sends the harmonic function

$$h_0 = \begin{pmatrix} \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & 1 & 1 & 1 & 1 & 1 & \cdots \\ \cdots & 1 & 1 & 1 & 1 & 1 & \cdots \\ \cdots & 0 & 0 & 0 & 0 & 0 & \cdots \\ \cdots & 1 & 1 & 1 & 1 & 1 & \cdots \\ \cdots & 1 & 1 & 1 & 1 & 1 & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \end{pmatrix}$$

to the harmonic function [6]

$$\delta(h_0) = \begin{pmatrix} \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & 1 & 0 & 1 & 0 & 1 & \cdots \\ \cdots & 1 & 0 & 1 & 0 & 1 & \cdots \\ \cdots & 0 & 0 & 0 & 0 & 0 & \cdots \\ \cdots & 1 & 0 & 1 & 0 & 1 & \cdots \\ \cdots & 1 & 0 & 1 & 0 & 1 & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \end{pmatrix}$$

replacing the bi-period $\bar{n}(h_0) = (1,3)$ by the bi-period $\bar{n}(\delta(h_0)) = (2,3)$.

Proposition 2.36. *The endomorphism δ stabilizes the subspace $\mathrm{Harm}^+(\mathbb{Z}^2)$ and generically doubles bi-periods. More precisely, if $f \in \mathrm{Harm}^+(\mathbb{Z}^2)$ is a bi-periodic binary harmonic function with bi-period $\bar{n}(f) = (n_1, n_2)$ different from a shift of h_0 or ${}^t h_0$, then $n_1, n_2 > 1$ and $\delta(f) \in \mathrm{Harm}^+(\mathbb{Z}^2)$ has bi-period $\bar{n}(\delta(f)) = 2\bar{n}(f) = (2n_1, 2n_2)$.*

Proof. It is easily seen that $\delta(f)$ is harmonic if so is f. If $f \neq 0$ is constant in vertical or horizontal direction then it is a shift of one of the functions h_0 or ${}^t h_0$, which has been excluded. Thus $n_1, n_2 > 1$, the function f on $\mathbb{Z}^2$ is non-constant in vertical direction, and so $\delta(f)$ possesses a nonzero line

$$\begin{pmatrix} \cdots & u+x & 0 & v+y & 0 & w+z & \cdots \end{pmatrix}.$$

If $(m_1, 0)$ is a period of $\delta(f)$ then necessarily m_1 is even and $(m_1/2, 0)$ is a period of f, so $(m_1, 0) = (2n_1, 0)$ is a minimal such period. By symmetry $\bar{n}(\delta(f)) = (2n_1, 2n_2)$, as stated. □

Remarks 2.37. 1. Actually δ provides linear injections $\mathrm{Harm}^+(\mathbb{T}_{n_1,n_2}) \hookrightarrow \mathrm{Harm}^+(\mathbb{T}_{2n_1,2n_2})$.

2. Doubling of just one of the periods n_1, n_2 is impossible in general, as Example 2.33.1 above shows.

[6] Respectively, ${}^t h_0$ to $\delta({}^t h_0) = {}^t \delta(h_0)$.

3. A similar doubling is equally applied in higher dimensions, over any field K of characteristic 2. Namely, for any $f \in \mathcal{F}(\mathbb{Z}^s, K)$ and $u \in \mathbb{Z}^s$ we let $\delta(f)(u) = f(v)$ if $u = 2v$ has all coordinates even, otherwise

$$\delta(f)(u) = \sum_{2v \in \mathrm{neighb}(u)} f(v)\,,$$

where $2v \in \mathrm{neighb}(u)$ iff $v \in \mathbb{Z}^s$ and the coordinates of $u - 2v$ are equal to 0 or ± 1.

2.9. **'Lights Out' game on graphs.** The game 'Lights Out' on a finite graph Γ consists in the following [Pe], [Su4]. Each vertex of Γ can be in one of the two states 'on' or 'off'. A move consists in changing the state of a vertex and, simultaneously, of all its neighbors. The goal is to get, after a sequence of moves, all states 'off'. An initial position will be called *pattern*. A pattern is *winning* if there exists a sequence of moves terminating at a zero pattern. The graph Γ is called *winning* if the game on Γ wins starting with an arbitrary pattern. Most of the following results are well known, see e.g., [Su5].

Proposition 2.38. (a) *A finite graph Γ is winning if and only if Γ is not harmonic.*

(b) *Every nucleus N^+ of Γ yields a linear relation for the functions $\{a_v^+\}_{v\in\mathrm{vert}(\Gamma)}$:*

$$\Delta_\Gamma^+(\delta_{N^+}) = \sum_{v\in N^+} a_v^+ = 0\,,$$

where δ_N is the characteristic function of N and a_v^+ is as in (5).

(c) *The space of winning patterns $V_\Gamma = \mathrm{span}\,(a_v^+ \,:\, v \in \mathrm{vert}(\Gamma))$ in $\mathcal{F}(\Gamma, k)$ is the orthogonal complement to $\mathrm{Harm}^+(\Gamma)$ w.r.t. the standard bilinear form $\langle x, y\rangle$ on $\mathcal{F}(\Gamma, k)$.*

(d) *Every antiharmonic pattern on Γ is winning.*

(e) *For every Γ, the all-on pattern is winning.*

Proof. (a) A pattern can be considered as a binary function on Γ. The move at a vertex v corresponds to the shift by a_v^+ in $\mathcal{F}(\Gamma, k)$. Thus the game on Γ is winning if and only if the group of translations generated by $\big(t_{a_v^+} \,:\, v \in \mathrm{vert}(\Gamma)\big)$ acts transitively on $\mathcal{F}(\Gamma, k)$. The latter holds if and only if the functions a_v^+, where $v \in \mathrm{vert}(\Gamma)$, span $\mathcal{F}(\Gamma, k)$, if and only if the matrix $I + \mathrm{adj}(\Gamma)$ of Δ_Γ^+ with columns (a_v^+) is non-degenerate, or, equivalently, $0 \notin \mathrm{spec}^+(\Gamma)$, as stated.

By definition, $\delta_N \in \mathrm{Harm}^+(\Gamma)$ if and only if N is a nucleus of Γ. This yields (b). Now (c) follows from (b). The proper subspaces $\mathrm{Harm}^+(\Gamma)$ and $\mathrm{Harm}^-(\Gamma)$ of the laplacian Δ_Γ^- being orthogonal, (d) follows by virtue of (c). Further by (c), $1 \in V_\Gamma$ if and only if $1 \perp \mathrm{Harm}^+(\Gamma)$. The latter holds indeed because for any $h \in \mathrm{Harm}^+(\Gamma)$, the nucleus $h^{-1}(1)$ being an odd graph, by the handshaking theorem it has an even number of vertices. □

Remarks 2.39. 1. In view of 2.38.c the harmonic functions on Γ are linear invariants of the game 'Lights Out'. That is,

$$\langle h, f + a_v^+\rangle = \langle h, f\rangle \qquad \forall v \in \mathrm{vert}(\Gamma),\ \forall f \in \mathcal{F}(\Gamma, k),\ \forall h \in \mathrm{Harm}^+(\Gamma)\,.$$

By virtue of 2.38.a, Γ is winning if and only if it does not admit a nonzero linear invariant.

2. 2.38.e is Sutner's Garden-of-Eden theorem [Su5]. The desired transformation of the all-one pattern into the all-zero one is achieved via moves at the vertices of an

odd-domination subgraph N of Γ. The latter means that every vertex v of Γ must have in N an odd number of neighbors including v itself if $v \in N$, so that $1 = \sum_{v\in N} a_v^+ \in V_\Gamma$. Given any graph Γ, Sutner's theorem actually proves the existence of an odd-domination subgraph of Γ (see also [An]).

3. Counting harmonic toric 2-lattices and counting points on an elliptic cubic curve

3.1. Constructing harmonic tori from polynomials.

Proposition 3.1. *Given a polynomial $p(x) \in k[x]$ with $p(0) = 1$ and a root $z \in \mathbb{F}_q$ of p, one can construct a harmonic toric lattice $\mathbb{T}_{\bar n}$, and every such lattice appears that way.*

Proof. Indeed, z can be written in a unique way as $z = \zeta + \zeta^{-1}$, where $\zeta \in \mathbb{F}_{q^2}$. There is a unique decomposition

$$p = 1 + \sum_{0<\alpha_1<\alpha_2<\ldots<\alpha_s} T_{\alpha_i}\,,$$

and $p(z) = 0$ yields

$$\sum_{0<\alpha_1<\alpha_2<\ldots<\alpha_s} T_{\alpha_i}(\zeta + \zeta^{-1}) = \sum_{0<\alpha_1<\alpha_2<\ldots<\alpha_s} (\zeta^{\alpha_i} + \zeta^{-\alpha_i}) = 1\,.$$

Thus letting $x_i = \zeta^{\alpha_i}$ gives a solution of (1) with

$$n_i = \operatorname{ord} x_i = \frac{n}{\gcd(n, \alpha_i)}\,, \qquad i = 1, \ldots, s\,,$$

where $n = \operatorname{ord}\zeta = \operatorname{ford} z$.

In this way we obtain all harmonic toric lattices $\mathbb{T}_{\bar n}$. Indeed, given $\bar n = (n_1, \ldots, n_s)$ and $m = \operatorname{lcm}(n_1, \ldots, n_s)$, we let $q = 2^{f(m)}$, and we fix a primitive $(q-1)$-st root of unity $\zeta \in \mu_{q-1}$. Any solution $\bar x = (x_1, \ldots, x_s) \in (\mathbb{F}_q^\times)^s$ of (1) can be written as $x_i = \zeta^{\alpha_i}$, where $\alpha_i n_i \equiv 0 \mod (q-1)$, $i = 1, \ldots, s$. Letting $z = \zeta + \zeta^{-1} \in \mathbb{F}_q$, the first equation in (1) is equivalent to $p(z) = 0$, where $p = 1 + \sum_{i=1}^s T_{\alpha_i} \in k[x]$. □

Remark 3.2. Letting above $\zeta = \xi^c$, where $\gcd(c, q-1) = 1$, we obtain $z = T_c(z')$ and $x_i = \xi^{c\alpha_i}$, where $z' = \xi + \xi^{-1} \in \mathbb{F}_q$ is a root of the polynomial $p_c = 1 + \sum_{i=1}^s T_{c\alpha_i}$.

Example 3.3. For every $n = 2s+1$ odd and for every s-tuple $\bar n = (n, \ldots, n)$, the hypercubic toric lattice $\mathbb{T}_{\bar n}$ is harmonic. Indeed, if $\zeta \in \bar k$ is a primitive n-th root of unity then $x_j = \zeta^j$, $j = 1, \ldots, s$, gives a solution of (1) with $n_j = n \,\forall j$. In particular $\mathbb{T}_{(5,5)}$, $\mathbb{T}_{(7,7,7)}$, $\mathbb{T}_{(11,11,11,11,11)}$ etc. are harmonic. Therefore by 2.22.d, so is $\mathbb{T}_{\bar n}$ provided that $n_i \equiv 0 \mod 5$ for at least 2 values of i, or $n_i \equiv 0 \mod 7$ for at least 3 values of i, etc. However, C_5, $\mathbb{T}_{(7,7)}$ and $\mathbb{T}_{(11,11)}$ are not harmonic, see 5.1.a, 3.7 and Appendix 1 below.

3.2. Partners.

3.4. Let E be the affine plane cubic with equation

$$(1 + x + y)(1 + xy) = 1\,, \tag{7}$$

and set $E^* = E \setminus \{(0,0)\}$. By virtue of 2.12.b the toric 2-lattice $\mathbb{T}_{m,n}$ is harmonic if and only if the curve $E^*(\bar k)$ possesses a point (x, y) with $\operatorname{ord} x \mid m$ and $\operatorname{ord} y \mid n$.

We are interested in the infinite table $\mathcal{E}$ composed of all pairs $(m,n) \in \mathbb{N}^2$ such that the lattice $\mathbb{T}_{m,n}$ is harmonic, or equivalently, such that $x^m = y^n = 1$ for some point $(x,y) \in E^*(\bar{k})$. We consider also the subtable

$$\mathcal{E}_0 = \{(\operatorname{ord} x, \operatorname{ord} y) \,:\, (x,y) \in E^*(\bar{k})\} \subseteq \mathcal{E}\,.$$

Thus $\mathcal{E}_0$ is the set of all bi-torsions of points on E^*, or in other words, the set of all minimal bi-periods of double periodic binary harmonic functions on $\mathbb{Z}^2$. Notice that $\mathcal{E}_0$ contains a set of primitive generators of $\mathcal{E}$ viewed as a module over the multiplicative semigroup $\mathbb{N}^2$ (cf. 2.22.d). We call m and n *partners* if $(m,n) \in \mathcal{E}_0$. For instance (47, 178481) is a pair of partners found by Zagier.

The following lemma is a reformulation of Theorem 5.2 in [HMP]; the latter also covers the case of square grids over $\mathbb{F}_3$.

Proposition 3.5. *The table of partners $\mathcal{E}_0$ is infinite.*

Proof. If $\mathcal{E}_0$ were finite there would exist a prime p such that

$$p > M = \max\{m \,:\, (m,n) \in \mathcal{E}_0 \text{ for some } n \in \mathbb{N}\} \geq 5\,.$$

We have $d \equiv 1 \bmod p \ \forall d \mid (2^p - 1)$. Indeed, for every prime divisor l of $2^p - 1$,

$$2^p \equiv 1 \mod l \implies \operatorname{ord}_l 2 = p \implies l \equiv 1 \mod p\,.$$

It follows that $d \equiv 1 \bmod p$ and so, $d > p > M$ if $d > 1$.

However the toric lattice $\mathbb{T}_{q-1,q-1}$ being harmonic by 2.13, we must have $(q-1, q-1) = (kd_1, ld_2)$ for some d_1, d_2, k, $l \in \mathbb{N}$ such that $(d_1, d_2) \in \mathcal{E}_0$ and $d_1 > 1$. Since $d_1 > M$ this yields a contradiction. □

Somewhat more precise information can be deduced by using the Hasse-Weil formula.

3.3. **Hasse-Weil formula.** The cubic curve E as in 3.4 has 3 points at infinity: $(1:0:0)$, $(0:1:0)$ and $(1:1:0)$. Hence the projective closure $\bar{E}$ of E is a smooth elliptic curve. For $q = 2^r$ we let $\bar{s}_r$ ($s_r = \bar{s}_r - 4$, respectively) be the number of points on $\bar{E}(\mathbb{F}_q)$ ($E^*(\mathbb{F}_q)$, respectively).

Lemma 3.6. *We have $s_r = q\left(1 - (\alpha_+^r + \alpha_-^r)\right) - 3$, where $\alpha_\pm = (-1 \pm \sqrt{-7})/4$ are the complex roots of the polynomial $2t^2 + t + 1$. Moreover, the Hasse inequalities hold:*

$$(\sqrt{q} - 1)^2 \leq \bar{s}_r \leq (\sqrt{q} + 1)^2\,.$$

Proof. Since $\bar{s}_1 = 4$, the Hasse-Weil formula [Ko, Ch. V, §1, Exercise 7] gives in our case:

$$\sum_{r=1}^{\infty} \frac{\bar{s}_r}{r} t^r = \log \zeta_{\bar{E}}(t) = \log \frac{1 + t + 2t^2}{(1-t)(1-2t)} = \sum_{r=1}^{\infty} \frac{t^r}{r} \left(1 + 2^r(1 - (\alpha_+^r + \alpha_-^r))\right)\,.$$

Now the assertions follow easily. □

From 3.6 and [HMP, §5, Remark] we deduce the following results.

Corollary 3.7. (a) $\forall q = 2^r \geq 16$, $E^*(\mathbb{F}_q) \neq \emptyset$.

(b) *All Mersenne primes $q - 1 = 2^p - 1$ with $p > 3$, and all Fermat primes $q + 1 = 2^{2^l} + 1$ with $l \geq 1$, are self-partners i.e., $(q \pm 1,\, q \pm 1) \in \mathcal{E}_0$. Whereas for $p = 2, 3$ one has $(3,3)$, $(7,7) \notin \mathcal{E}_0$.*

Proof. (a) follows by virtue of 3.6.

(b) According to 2.13, the toric lattice $\mathbb{T}_{q\pm1,q\pm1}$ is harmonic $\forall q = 2^r$, except for $\mathbb{T}_{1,1}$ and $\mathbb{T}_{7,7}$. Thus for all those $m = n = q \pm 1$, (2) has a solution (ξ, η). If $q \pm 1$ is prime and $\xi, \eta \neq 1$ then $\operatorname{ord} \xi = \operatorname{ord} \eta = q \pm 1$ that is, $(q \pm 1, q \pm 1) \in \mathcal{E}_0$. This proves the first assertion.

Since $x = 1$ or $y = 1$ for every point $(x, y) \in E^*(4)$ then $(\operatorname{ord} x, \operatorname{ord} y) = (1, 3)$ or $(3, 1)$ and so $(3, 3) \notin \mathcal{E}_0$. Neither $(7, 7) \in \mathcal{E}_0$ since $s_3 = 0$. □

So far only 5 Fermat primes and at most 43 Mersenne primes were found, see e.g., [LLMP, 7.3], [Me], [Wa].

The computer findings in Appendix 1 suggest the following conjecture, cf. 2.13:

Conjecture 3.8. $\forall q = 2^r$ $(r \geq 6)$, $q-1$ and $q+1$ are partners and auto-partners that is, $(q \pm 1, q \pm 1) \in \mathcal{E}_0$ and $(q + 1, q - 1) \in \mathcal{E}_0$.

The latter does not hold for $r = 5$. Indeed $(31, 33) \in \mathcal{E} \setminus \mathcal{E}_0$, see 2.13 and Appendix 1.

Examples 3.9. 1. $(5, n) \in \mathcal{E}$ if and only if $n \equiv 0 \bmod 3$ or $n \equiv 0 \bmod 5$. In particular $(5, 5) \in \mathcal{E}_0$ is a self-partner, and there is no further partner of 5.

Indeed, as $(1, 3) \in \mathcal{E}$ then $(k, 3l) \in \mathcal{E}$ for every $k, l \geq 1$, in particular for $k = 5$. Further, 5 is a primitive self-partner since for any primitive 5-th root of unity $\zeta \in \mu_5$, the pair $(x, y) = (\zeta, \zeta^2)$ satisfies (1) with $s = 2$, $m = n = 5$. Consequently by virtue of 2.22.a, $(5k, 5l) \in \mathcal{E}$ $\forall k, l \geq 1$ and so $(5, 5l) \in \mathcal{E}$ $\forall l \geq 1$.

Conversely, if $(x, y) \in E^*$ and $(\operatorname{ord} x, \operatorname{ord} y) = (5, n)$ then $x \in \mathbb{F}_{16} \setminus \mathbb{F}_4$ satisfies $x^5 = 1$ and $y = x^2$, x^{-2} satisfy

$$y^2 + (z + 1)y + 1 = 0, \quad \text{where} \quad z = x + x^{-1}\,.$$

Thus $n = \operatorname{ord} y = 5$.

2. $(7, n) \in \mathcal{E}$ if and only if $n \equiv 0 \bmod 3$, and 9 is the only partner of 7.

Indeed, $(7, 3k) \in \mathcal{E}$ $\forall k \geq 1$ because $(1, 3) \in \mathcal{E}$. If $(x, y) \in E^*$ and $(\operatorname{ord} x, \operatorname{ord} y) = (7, n)$ then $x \in \mathbb{F}_8 \setminus \mathbb{F}_2$. We have $(7, n) \in \mathcal{E}_0 \implies f_0(n) = f_0(7) = 3 \implies n \mid (2^3 \pm 1) \implies n \in \{7, 9\}$. But $n \neq 7$ as $s_3 = 0$ and so, $E^*(\mathbb{F}_{2^7}) = \emptyset$. Hence $n = 9$.

Since $(1, 3)$, $(7, 9) \in \mathcal{E}_0$, the latter set properly contains the set of all primitive generators of $\mathcal{E}$ over $\mathbb{N}^2$.

3.4. **Partnership graph.** We observe that:

- For every $(m, n) \in \mathcal{E}_0$, both m and n are odd (cf. 2.22.c).
- Every odd $n \in \mathbb{N}$ has a partner, and the number of these partners is finite.
- $(m, n) \in \mathcal{E}_0 \iff (n, m) \in \mathcal{E}_0$.

Thus the partnership defines an equivalence relation on $\mathbb{N}_{\text{odd}}$. Answering a question of the author, Zagier proposed the following 3.10, 3.11 and 3.12 below. Our proof of 3.11 based on 5.9 is somewhat different from the original one.

We let below $\operatorname{div}(n)$ ($\operatorname{div}^*(n)$, respectively) be the set of all (proper) divisors of $n \in \mathbb{N}$. For $q = 2^r$ we write for short $\operatorname{div}(q \pm 1)$ meaning $\operatorname{div}(q - 1) \cup \operatorname{div}(q + 1)$.

Definition 3.10. We let $\mathcal{P}^{(1)}$ be the infinite graph with loops such that $\operatorname{vert}(\mathcal{P}^{(1)}) = \mathbb{N}_{\text{odd}}$ and $[m, n] \in \operatorname{edg}(\mathcal{P}^{(1)}) \iff (m, n) \in \mathcal{E}_0$. We call $\mathcal{P}^{(1)}$ the *partnership graph.*

Theorem 3.11. *All connected component of* $\mathcal{P}^{(1)}$ *are finite.*

Proof. We let $\mathcal{V}_r$ be the subgraph of $\mathcal{P}^{(1)}$ with vertices in the finite set

$$V_r = \{n \in \mathbb{N}_{\rm odd} \,:\, n \mid (2^r \pm 1)\}\,.$$

Given $n \in \mathbb{N}_{\rm odd}$, we let $\mathcal{P}^{(1)}(n)$ be the connected component of $\mathcal{P}^{(1)}$ which contains the vertex n. We claim that the function $f_0(n)$ is constant on each connected component of $\mathcal{P}^{(1)}$. In particular $\mathcal{P}^{(1)}(n) \subseteq \mathcal{V}_{f_0(n)}$. The level sets V_r of f_0 being finite, this proves the theorem.

To show the claim we note that, due to 2.12.a, $[m,n] \in \operatorname{edg}(\mathcal{P}^{(1)})$ if and only if $\xi+\xi^{-1} = 1+\eta+\eta^{-1}$ for some primitive roots $\xi \in \mu_m$ and $\eta \in \mu_n$. According to 5.10.a,

$$f_0(m) = \deg(\xi+\xi^{-1}) = \deg(\eta+\eta^{-1}) = f_0(n)$$

and so, the claim follows. □

Notation 3.12. We denote by $S(m,n)$ the set of all solutions (ξ,η) of (7) of type (m,n) that is, with $\xi \in \mu_m$ (respectively, $\eta \in \mu_n$) being a primitive m-th (respectively, n-th) root of unity. We label the edges $[m,n] \in \operatorname{edg}(\mathcal{P}^{(1)})$ with $s(m,n) = \frac{1}{2}\operatorname{card}(S_{m,n}) \in \mathbb{N}$, with one exception: instead of the edge $[1,3]$ we introduce two directed edges, $[1 \to 3]$ labeled by 1 and $[3 \to 1]$ labeled by 2. Clearly,

$$s_r = 2 \sum_{m,n \in \operatorname{div}(q-1)} s(m,n), \qquad \forall r \in \mathbb{N}\,.$$

Moreover this labeling possesses the following properties.

Proposition 3.13. *$\forall n \in \mathbb{N}_{\rm odd}$ and $\forall q = 2^r$, $r \geq 3$,*

(a) $\sum_{m\in\mathbb{N}_{\rm odd}} s(m,n) = \varphi(n)$ [7].
(b) $\sum_{d\in\operatorname{div}(n),\, m\in\mathbb{N}_{\rm odd}} s(d,m) = n$.
(c) $\sum_{n\in V_r} \varphi(n) = 2\sum_{m,n\in V_r,\, m\neq n} s(m,n) + \sum_{n\in V_r} s(n,n)$.
(d) $2\sum_{d,d'\in\operatorname{div}(q\pm1),\, d\neq d'} s(d,d') + \sum_{d\in\operatorname{div}(q\pm1)} s(d,d) = 2q$.
(e) $s(q-1,q-1)+s(q+1,q+1)+2s(q-1,q+1) \geq 2\left(\varphi(q-1)+\varphi(q+1)-q\right)$.

Proof. (a) holds because for every $(\zeta,\eta) \in S(m,n)$, the pairs (η,η^{-1}) and (ζ,ζ^{-1}) uniquely correspond to each other. Since $\sum_{d|n}\varphi(d) = n$, (b) follows from (a). Summing up (a) over the edges of $\mathcal{V}_r$ yields (c). It is easily seen that $d \in \operatorname{div}(q\pm1) \iff f_0(d) \mid r \iff d \in V_s$ for some $s \in \operatorname{div}(r)$. Hence

$$\sum_{s|r}\sum_{n\in V_s}\varphi(n) = \sum_{n\in\operatorname{div}(q\pm1)}\varphi(n) = \sum_{n|(q-1)}\varphi(n) + \sum_{n'|(q+1)}\varphi(n') = 2q\,.$$

Thus the summation of (c) over the set $\operatorname{div}(r)$ yields (d).

(e) By virtue of 3.13.a, $s(d,q-1)+s(d,q+1) \leq \varphi(d)$. Moreover

$$\begin{aligned}\varphi(q-1)+\varphi(q+1) &= s(q-1,q-1)+s(q+1,q+1)+2s(q-1,q+1)\\ &\quad + \sum_{d\in\operatorname{div}^*(q\pm1)} \left(s(d,q-1)+s(d,q+1)\right)\,.\end{aligned}$$

Hence

$$s(q-1,q-1)+s(q+1,q+1)+2s(q-1,q+1) \geq \left(\varphi(q-1) - \sum_{d\in\operatorname{div}^*(q-1)}\varphi(d)\right)$$

[7]Hereafter φ stands for the Euler function.

$$+\left(\varphi(q+1)-\sum_{d'\in\operatorname{div}^*(q+1)}\varphi(d')\right)=2\left(\varphi(q-1)+\varphi(q+1)-q\right).$$

□

Remarks 3.14. 1. If (ζ,η) is a solution of (7) of type (m,n) then (ζ^2,η^2) is as well such a solution. Thus the Galois group $\mathrm{Gal}(\mathbb{F}(\zeta))$ acts freely on $S(m,n)$ and so, its order $f(n)$ divides $s(m,n)$. Letting $k(m,n)=s(m,n)/f(n)$, from 3.13.a we obtain the equality

$$\sum_m k(m,n)=g(n)\,.$$

Indeed, there are $\varphi(n)=f(n)g(n)$ primitive n-th roots of unity.

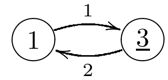
2. For $q=2^r$ the inequality $\varphi(q-1)+\varphi(q+1)>q$ does not hold in general, although it holds at least for all $r\le 150$ (a MAPLE checking)[8]. Moreover, according to A. Schinzel[9], $\varphi(2^r-1)+\varphi(2^r+1)$ can be $<<2^r/(\log\log r_t)^{7/24}$ for infinitely many r, although $\varphi(2^r-1)+\varphi(2^r+1)\ge 2^r/\log\log r\ \forall r\in\mathbb{N}$. This can be seen as follows. Let r_t be the least common multiple of all numbers $(p-1)/2$,where p runs through primes $\equiv 3 \mod 4$ less than t. By Euler's Theorem and the quadratic reciprocity law, p divides $2^{(p-1)/2}-(-1)^{(p+1)/4}$, hence $2^{r_t}-1$ is divisible by the product of all primes $\equiv 7 \mod 8$ less than t, while $2^{r_t}+1$ is divisible by the product of all primes $\equiv 3 \mod 8$ less than t. Using the Mertens formula for primes in arithmetic progressions, we obtain that $\varphi(2^{r_t}\pm 1)<<1/\log t<<1/(\log\log r_t)^{1/4}$. A slight modification of the argument increases the exponent 1/4 to 7/24, which gives the claim. But a gap between the lower and the upper bound remains. Inded from a theorem of Erdös [Er], and from the inequality between phi and sigma functions [HW, Theorem 329] it follows that $\varphi(2^r-1)+\varphi(2^r+1)>>2^r/\log\log r$.

3. Similarly to $\mathcal{P}^{(1)}$, one might consider an infinite hypergraph $\mathcal{P}=\cup_{s\ge1}\mathcal{P}^{(s)}$ with set of vertices $\mathbb{N}_{\mathrm{odd}}$ such that $(n_0,\dots,n_s)$ is an s-simplex of $\mathcal{P}^{(s)}$ if and only if $(n_0,\dots,n_s)$ is the multi-order of a point on the affine hypersurface as in (1) (with s replaced by $s+1$). Evidently, every s-tuple $(n_1,\dots,n_s)\in\mathbb{N}_{\mathrm{odd}}^s$ is a face of an s-simplex in $\mathcal{P}^{(s)}$.

4. Appendix 1: Connected components of the partnership graph

The first 13 connected components of the partnership graph $\mathcal{P}^{(1)}$ are shown below. They were found by Zagier with PARI. The labeling of the edges is according to 3.12. We recall (see the proof of 3.11) that the value of $f_0(n)$ equals r for every vertex n of $\mathcal{V}_r$, and this determines $\mathcal{V}_r$. The value of $f(n)$ equals r if n is not underlined, and $2r$ otherwise. The edges $[m,n]$ (the loops $[n,n]$, respectively) correspond to harmonic toric lattices $\mathbb{T}_{m,n}$ ($\mathbb{T}_{n,n}$, respectively) with $(m,n)\in\mathcal{E}_0$.

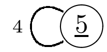
$(\mathcal{V}_1)$



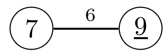
[8] Computations with Pari/GP done by Gottfried Barthel (a letter to the author) confirm the inequality in the range $r\le 275$.

[9] A letter to the author.

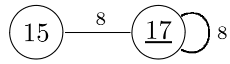
$(\mathcal{V}_2)$



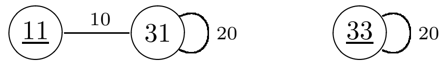
$(\mathcal{V}_3)$



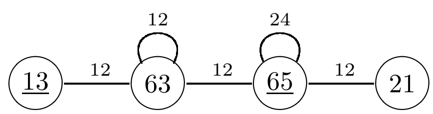
$(\mathcal{V}_4)$



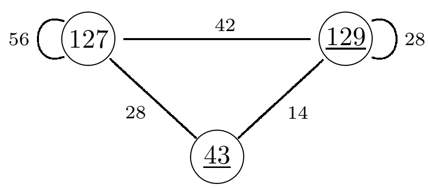
$(\mathcal{V}_5)$



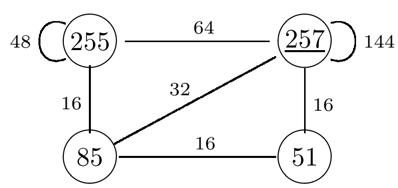
$(\mathcal{V}_6)$



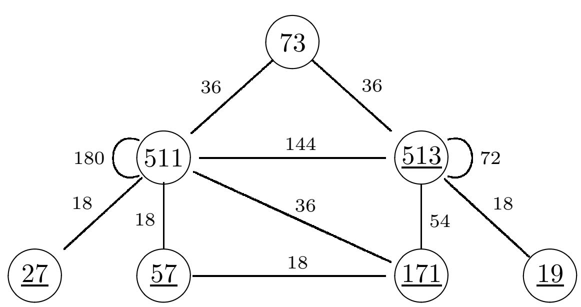
$(\mathcal{V}_7)$



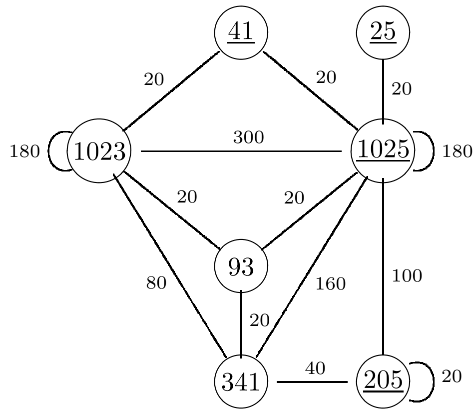
$(\mathcal{V}_8)$



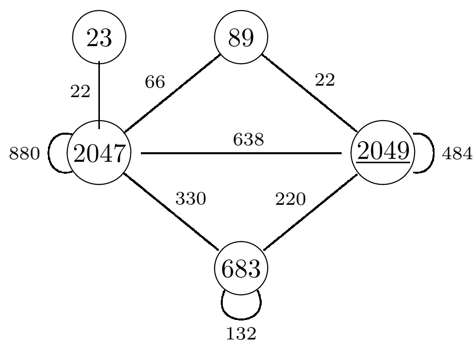
$(\mathcal{V}_9)$



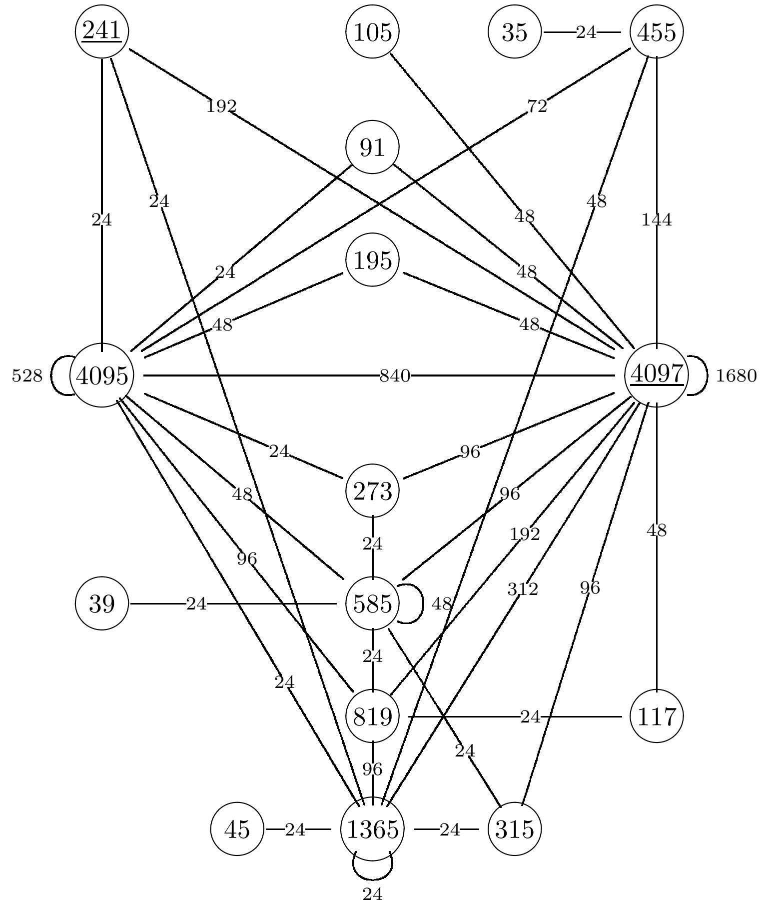
$(\mathcal{V}_{10})$



$(\mathcal{V}_{11})$



$(\mathcal{V}_{12})$



We observe that the graph $\mathcal{V}_{12}$ is not planar, in contrast to $\mathcal{V}_r$ with $r \leq 11$.

These computations suggest the following

Conjecture 4.1. $\mathcal{V}_r$ is connected $\forall r \neq 5$. In other words, the connected components of $\mathcal{P}^{(1)}$ are $\mathcal{V}_r$ for $r \neq 5$ and the two components of $\mathcal{V}_5$.

5. Appendix 2: Chebyshev-Dickson and Fibonacci polynomials

5.1. **Chebyshev-Dickson and Fibonacci polynomials.** These polynomials T_n, E_n and F_n provide an important tool for analysis of harmonicity. Indeed, as we have seen in 2.2, T_n, respectively, E_n is the characteristic polynomial of the laplacian $\Delta^-_{C_n}$, respectively, $\Delta^-_{P_n}$, where C_n stands for the circular graph with n vertices and P_n denotes the path of length n. We give an account of some of their properties in 5.1-5.4 below according to [LMT, Ch. 2], [Su3], [BR], [GKW], see also references therein.

Definition 5.1. Consider the linear recurrence

$$p_{n+1} = xp_n + p_{n-1}, \qquad \text{where} \quad p_i \in k[x] \quad \forall i \geq 0\,. \tag{8}$$

Thus

$$\begin{pmatrix} p_n \\ p_{n+1} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & x \end{pmatrix}^n \begin{pmatrix} p_0 \\ p_1 \end{pmatrix}.$$

The *Chebyshev-Dickson polynomials* of the first, respectively, second kind $T_n,\ E_n \in k[x]$ and the *Fibonacci polynomials* $F_n \in k[x]$ are defined via (8) by the initial conditions

$$\begin{pmatrix} T_0 \\ T_1 \end{pmatrix} = \begin{pmatrix} 0 \\ x \end{pmatrix}, \qquad \begin{pmatrix} E_0 \\ E_1 \end{pmatrix} = \begin{pmatrix} 1 \\ x \end{pmatrix}, \qquad \begin{pmatrix} F_0 \\ F_1 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad \text{respectively.}$$

Thus $\deg T_n = \deg E_n = \deg F_{n+1} = n$, the polynomials $T_{2n},\ E_{2n},\ F_{2n+1}$ are even and $T_{2n+1},\ E_{2n+1},\ F_{2n}$ are odd. They are related via

$$T_n = xF_n = xE_{n-1}, \qquad \text{where} \quad E_{-1} = 0\,. \tag{9}$$

So any property of one of the sequences (T_n), (E_n), (F_n) is enjoyed by the other two up to evident changes. Notice that $F_n(1)$ is the n-th Fibonacci number modulo 2.

The following identities hold, see e.g. [Ri, LMT], [BR, §4], [WP].

Proposition 5.2. *$\forall m, n \in \mathbb{N}$, $\forall q = 2^r$, $r \geq 1$, we have*

(a) $F(z) = z(z^2 + xz + 1)^{-1}$ *is the generating function of the sequence* (F_n).

(b)

$$E_n(x) = \sum_{i=0}^{[n/2]} \binom{n-i}{i} x^{n-2i} \mod 2 = \sum_{j=0,\dots,n,\ j\equiv n \bmod 2} \binom{n+j}{n-j} x^j \mod 2 \equiv U_n\left(\frac{x}{2}\right) \mod 2\,,$$

where $U_n(\cos x) = \frac{\sin nx}{\sin x}$ *stands for the Chebyshev polynomial of the second kind over* $\mathbb{R}$. [10]

(c) $F_{q-1} + F_{q+1} = xF_q = x^q$. *Furthermore,*

$$F_{q+1}(x) = x^q + F_{q-1}(x) = x^q \left(1 + \sum_{i=0}^{r-1} x^{-2^i}\right)^2 .$$

(d) $\forall z \in \mathbb{F}_q^\times$,

$$F_{q-1}(z) = z\mathrm{Tr}^2_{\mathbb{F}_q}(z^{-1}), \quad F_{q+1}(z) = z\left(1 + \mathrm{Tr}_{\mathbb{F}_q}(z^{-1})\right)^2 \quad \textit{and} \quad F_{q-1}(z) + F_{q+1}(z) = z\,.$$

(e) $E_{m+n} = E_m E_n + E_{m-1}E_{n-1}$.

(f) $E_{2n} = xE_nE_{n-1} + 1 = E_n^2 + E_{n-1}^2$, $E_{2n+1} = xE_n^2$ *and* $\sum_{i=1}^n E_{2i} = E_n^2$.

(g) $\forall n \geq t \geq 0$, $T_{n+t} + T_{n-t} = T_nT_t$. *In particular for* $n \equiv t \bmod 2$, $T_n + T_t = T_{\frac{n+t}{2}}T_{\frac{n-t}{2}}$.

(h) $T_m \circ T_n = T_n \circ T_m = T_{mn}$.

(i) $T_{qn}(x) = T_n(x^q) = T_n^q(x)$.

Proof. If $\Gamma = P_n$ is a linear graph with n vertices and $e = n-1$ edges then $\binom{e}{i}_\Gamma = \binom{n-i}{i}$. So (b) follows by virtue of 1.1.a and 2.2, whereas 1.1.c yields (e). In turn (e) implies (f), (g). Now (c) follows by recursion and implies (d). The assertions (a), (h) and (i) can be deduced, by virtue of (b), from the analogous identities for the usual Chebyshev polynomials over $\mathbb{R}$, see e.g., [LMT, Ri]. $\square$

5.3. As before, $\mathrm{ord}\,\xi \in \mathbb{N}_{\mathrm{odd}}$ denotes the multiplicative order of an element $\xi \in \bar{k}^\times$. In the next proposition we indicate certain divisibility properties and factorization of the Chebyshev-Dickson and Fibonacci polynomials according to [LMT, Ch. 2], [Su3], [GKW], [BR, §4] and [WP].

[10] Attention: our enumeration of classical polynomials does not coincide with those used in MAPLE. It is so chosen in order to write the identities in a more elegant way.

Proposition 5.4. (a) $\forall n \geq 0$, $\forall \xi \in \bar{k}^\times$, $T_n(\xi + \xi^{-1}) = \xi^n + \xi^{-n}$.

(b) $\forall \xi \in \bar{k}^\times$, $\operatorname{ord} \xi = \min\{n > 0 \,:\, T_n(\xi + \xi^{-1}) = 0\}$.

(c) $\forall n \in \mathbb{N}_{\text{odd}}$ *and for any primitive n-th root of unity $\zeta \in \mu_n$, we have*

$$T_n(x) = x \prod_{i=1}^{(n-1)/2} (x + \zeta^i + \zeta^{-i})^2 \,.$$

Consequently, $z = \xi + \xi^{-1}$ runs over the roots of T_n when ξ runs over μ_n. [11]

(d) $E_{n-1}(0) = 0 \iff n \equiv 0 \bmod 2$, $E_{n-1}(1) = 0 \iff n \equiv 0 \bmod 3$ *and* $(x^2 + x + 1) \mid E_{n-1} \iff n \equiv 0 \bmod 5$.

(e) $\forall m, n \in \mathbb{N}$, $\gcd(T_m, T_n) = T_{\gcd(m,n)}$ *and* $\gcd(E_{m-1}, E_{n-1}) = E_{\gcd(m,n)-1}$.

(f) $T_d \mid T_n \iff E_{d-1} \mid E_{n-1} \iff d \mid n$.

5.2. **Irreducible factors of Fibonacci polynomials.**

5.5. Every $z \in \bar{k}$ can be written in a unique way as $z = \zeta + \zeta^{-1}$, where ζ and ζ^{-1} are the roots of $f_z(x) = x^2 + zx + 1 \in \bar{k}[x]$. By virtue of 5.4.c, every irreducible polynomial $\tau \in k[x]$ divides one of the F_n [Su3, 3.1]. Namely, if $\tau \neq x$ and $\tau(\zeta + \zeta^{-1}) = 0$ then $\tau \mid F_n$ with $n = \operatorname{ord} \zeta$.

Remark 5.6. An element $z \in \mathbb{F}_q^\times$ $(q = 2^r)$ can be written as $z = (u^2 + u)^{-1}$ for some $u \in \mathbb{F}_q \setminus \mathbb{F}_2$ if and only if it can be written as $z = \xi + \xi^{-1}$ for some $\xi \in \mathbb{F}_q^\times$, see 5.2.d and 5.4.c.

These observations lead to the following definition [GKW, Su3].

Definition 5.7. Letting $\operatorname{Irr}[x]$ be the set of all irreducible polynomials in $k[x]$, we remind that for $\tau \in \operatorname{Irr}[x]$, $\operatorname{ord} \tau = \operatorname{ord} z \in \mathbb{N}_{\text{odd}}$, whenever z is a root of τ. For $\tau \neq x$ we define its *Fibonacci order* [12]

$$\operatorname{ford} \tau = \operatorname{ford} z = \min\{n > 0 \,:\, \tau \mid F_n\} = \operatorname{ord} \zeta \in \mathbb{N}_{\text{odd}}, \quad \text{where} \quad z = \zeta + \zeta^{-1} \,.$$

5.8. For $n \in \mathbb{N}_{\text{odd}}$, $x^n - 1 = \prod_{d|n} \Phi_d$ is a product of cyclotomic polynomials

$$\Phi_d(x) = \prod_{\tau \in \operatorname{Irr}[x],\, \operatorname{ord} \tau = d} \tau = \prod_{1 \leq i \leq d-1,\, \gcd(i,d)=1} (x - \zeta^i) \,, \tag{10}$$

where $\zeta \in \mu_d$ is a primitive d-th root of unity. Hence $\deg \Phi_d = \varphi(n)$, and Φ_d is a product of $g(d) = \varphi(d)/f(d)$ distinct irreducible factors of the same degree $f(d)$ [LN, 2.47]. By virtue of (10), Φ_d is auto-reciprocal, that is $\Phi_d^* = \Phi_d$, where

$$* : p(x) \longmapsto x^{\deg p} p(x^{-1})$$

is an involutive automorphism of the multiplicative semigroup

$$\Pi[x] = \{p \in k[x] \,:\, p(1) = 1\} \,.$$

It occurs that either all irreducible factors of Φ_d are auto-reciprocal or none of them is, depending on d. More precisely, the following happens.

Lemma 5.9. *Let $\zeta \in \mu_d$ be a primitive root of unity of odd order d. We denote by $\tau(x) = \prod_{j=0}^{r-1} (x - \zeta^{2^j}) \in k[x]$ its minimal polynomial of degree $r = f(d)$. The following conditions are equivalent:*

---

[11] By making use of 5.2.h, one can deduce in the same way the roots of T_n for any even n.

[12] $\operatorname{ford} \tau$ is called *Fibonacci index* of τ in [GKW], and *depth* of τ in [Su3].

(i) *τ is auto-reciprocal, or palindrome.*
(ii) $2^{f_0(d)} \equiv -1 \mod d$.
(iii) $\deg\zeta = 2\deg(\zeta+\zeta^{-1})$. [13]
(iv) $\mathrm{Tr}_{k(\zeta+\zeta^{-1})}(\zeta+\zeta^{-1})^{-1} = 1$.

Proof. τ being irreducible of degree ≥ 2, we have $\tau(1) = 1$. But 1 is the only fixed point of the involution $z \longmapsto z^{-1}$ on $\bar{k}^\times$. In case that $\tau = \tau^*$ this involution acts on the roots of τ, hence $f(d) = \deg\tau$ is even.

The roots of τ being the conjugates of ζ in $\bar{k}$, τ is auto-reciprocal if and only if ζ and ζ^{-1} are conjugated. By virtue of 2.1 this yields the equivalence (i)$\Longleftrightarrow$(ii). The condition (iii) holds if and only if the polynomial f_z as in 5.5 above is irreducible, where $z = \zeta+\zeta^{-1}$. Thus the equivalence (iii)$\Longleftrightarrow$(iv) follows, see e.g. [McE, 8.13]. To show the remaining equivalence (i)$\Longleftrightarrow$(iii) we consider the Laurent polynomial $g(x) = \tau(x)\tau(x^{-1}) \in k[x,x^{-1}]$. It is *auto-reciprocal* that is, $g(x) = g(x^{-1})$ or, equivalently, $g(x) = h(x+x^{-1})$ for some $h \in k[z]$ of degree r. As $h(\zeta+\zeta^{-1}) = g(\zeta) = 0$, (iii) holds if and only if h is reducible.

Supposing (i) we have $r = 2s$, where $s \in \mathbb{N}$, and $\tau(x^{-1}) = x^{-r}\tau(x)$. Thus $h(x+x^{-1}) = \left(x^{-s}\tau(x)\right)^2 =: \tilde{g}^2(x)$. The Laurent polynomial $\tilde{g}$ being auto-reciprocal, it follows that $\tilde{g}(x) = \tilde{h}(x+x^{-1})$, where $\tilde{h} \in k[x]$, $\deg\tilde{h} = s$ and $h = \tilde{h}^2$. Clearly, $\tilde{h}$ is the minimal polynomial of $\zeta+\zeta^{-1}$. This yields (iii).

Conversely, let (iii) holds i.e., $h = h_1h_2$ is reducible, where $h_i \in k[x]$, $\deg h_i = r_i \geq 1$, $i = 1,2$ and $r_1+r_2 = r$. Letting $g_i(x) = x^{r_i}h_i(x+x^{-1}) \in k[x]$ we have $\deg g_i = 2r_i$, $i = 1,2$. Furthermore

$$\tau(x)\tau^*(x) = x^r\tau(x)\tau(x^{-1}) = x^r h(x+x^{-1}) = g_1(x)g_2(x)\,.$$

Since τ, $\tau^* \in \mathrm{Irr}[x]$, up to interchanging g_1 and g_2 we obtain $g_1 = \tau$, $g_2 = \tau^*$. Hence $r = 2r_1 = 2r_2$ is even and

$$\tau(x) = g_1(x) = x^{r/2}h_1(x+x^{-1}) = x^r\tau(x^{-1}) = \tau^*(x)\,.$$

Thus $\tau = \tau^*$, so (i) holds. □

Corollary 5.10. (a) *For every primitive d-th root of unity $\zeta \in \mu_d$ we have* $\deg\zeta = f(d)$ *and* $\deg(\zeta+\zeta^{-1}) = f_0(d)$.
(b) *Let $\tau(x) = \sum_{i=0}^r \varepsilon_i x^i$ be the minimal polynomial of ζ, and let $\tau(x)\tau(x^{-1}) = 1+\sum_{j=1}^r \delta_j(x^j+x^{-j})$. Then the minimal polynomial $\eta(x)$ of $\zeta+\zeta^{-1}$ is*

$$\eta(x) = \begin{cases} 1+\sum_{j=1}^s \varepsilon_{s-j}T_j(x) \\ \\ 1+\sum_{j=1}^r \delta_j T_j(x) \end{cases} \qquad \textit{with} \qquad \deg\eta = \begin{cases} s = r/2 & \textit{if} \quad \tau = \tau^*, \\ r & \textit{otherwise}\,. \end{cases}$$

Following [GKW], [Su3, 3.2], [SB] we list below some important features of irreducible factors of the Fibonacci polynomials.

Proposition 5.11. *For every $\tau \in \mathrm{Irr}^*[x]$ with* $\operatorname{ford}\tau = d$*, the following hold.*
(a) $\deg\tau = f_0(d)$ *i.e., the splitting field of τ is $\mathbb{F}_q$ with $q = 2^{f_0(d)}$.*
(b) $\tau \mid F_n \iff d \mid n \iff F_d \mid F_n$.

[13] Or, equivalently, $[\mathbb{F}_2(\zeta) : \mathbb{F}_2(\zeta+\zeta^{-1})] = 2$.

(c) $d \mid (q-1)$ *(and so* $\tau \mid F_{q-1}$*) if and only if the linear term of* τ *vanishes, if and only if the polynomial* $f_z(x) = x^2 + zx + 1 \in \mathbb{F}_q[x]$ *splits over* $\mathbb{F}_q$*, where* $z \in \mathbb{F}_q$ *is a root of* τ*. Otherwise* $d \mid (q+1)$ *(and so* $\tau \mid F_{q+1}$*).*

(d) $\forall n = 2k+1 \in \mathbb{N}_{\text{odd}}$,

$$F_n = \prod_{\tau \in \mathrm{Irr}^*[x],\, \mathrm{ford}\, \tau | n} \tau^2 = R_k^2\,,$$

where $R_k = F_{k+1} + F_k$, $\deg R_k = k$, R_k *is square-free and contains the monomial* x^{k-1}*. The splitting field of* F_n *is* $\mathbb{F}_q$*, where* $q = 2^{f_0(n)}$*.*

(e) $\forall q = 2^r$, $r \geq 1$*, the splitting field of* $F_{q\pm 1}$ *is* $\mathbb{F}_q$*. Moreover*

$$F_{q-1}F_{q+1} = \prod_{\tau \in \mathrm{Irr}^*[x],\, \deg\ \tau | r} \tau^2 = (x^{q-1}+1)^2\,.$$

(f) *For every odd prime* p *and for every irreducible factor* τ *of* F_p*,*

$$\deg \tau = \begin{cases} f(p) & \textit{if} \quad f(p) \equiv 1 \mod 2, \\ f(p)/2 & \textit{otherwise}\,. \end{cases}$$

In particular $F_p = \tau^2$ *if and only if either* $f(p) = p-1$ *or* $f(p) = \frac{p-1}{2} \equiv 1 \mod 2$*. This cannot happen if* $p \equiv \pm 1 \mod 8$, $p > 1$*.*

Proof. (a) follows by virtue of 5.10.a, and (b) follows from 5.4.b,f. If z is a root of τ then by (a), $z \in \mathbb{F}_q^\times$ and so, by virtue of 5.2.d, $F_{q-1}(z) = 0 \iff \mathrm{Tr}_{\mathbb{F}_q}(z^{-1}) = 0$ and $F_{q+1}(z) = 0 \iff \mathrm{Tr}_{\mathbb{F}_q}(z^{-1}) = 1$. By (b), $d \mid (q-1)$ in the former case and $d \mid (q+1)$ in the latter one. This yields (c). As $T_n = xF_n$, (d) and (e) follow easily from 5.2.f and 5.4.c. For the proof of (f), see [GKW]. □

Remark 5.12. We let $\mathrm{Irr}_1[x] = \{\tau \in \mathrm{Irr}[x] \,:\, \tau = x^{\deg \tau} + x^{\deg \tau - 1} + \ldots\}$. Thus $\tau \in \mathrm{Irr}_1[x]$ if and only if $\tau \in \mathrm{Irr}[x]$ and $\mathrm{Tr}_{k(z)}(z) = 1$ for any root $z = \zeta + \zeta^{-1}$ of τ. By virtue of (d) above, an odd number of irreducible factors of R_k belong to $\mathrm{Irr}_1[x]$. Hence for every $n \in \mathbb{N}_{\text{odd}}$, there exists $d \mid n$ and a primitive d-th root of unity $\zeta \in \mu_d$ such that

$$\mathrm{Tr}_{k(z)}(\zeta + \zeta^{-1}) = \sum_{i=0}^{f_0(d)-1} (\zeta^{2^i} + \zeta^{-2^i}) = 1\,.$$

5.13. For any $n \in \mathbb{N}_{\text{odd}}$, the following analog of the cyclotomic polynomial Φ_n (see 10) were introduced in [Su3]:

$$\rho_n = \prod_{\tau \in \mathrm{Irr}^*[x],\, \mathrm{ford}\, \tau = n} \tau^2\,.$$

The following properties of these polynomials were established in [Su3].

Proposition 5.14. (a) $\deg \rho_n = \varphi(n)$*. Furthermore,* ρ_n *has* $\frac{\varphi(n)}{2f_0(n)}$ *irreducible factors, all of the same degree* $f_0(n)$*, the same multiplicity* 2 *and with the same linear term.*

(b) $\forall q = 2^r$, $r \geq 0$*,*

$$F_n = \prod_{d|n} \rho_d \qquad \textit{and} \qquad F_{qn} = x^{q-1}F_n^q = x^{q-1}\prod_{d|n} \rho_d^q\,.$$

(c) *By the Möbius inversion formula,*

$$\rho_n = \prod_{d|n} F_{n/d}^{\mu(d)} .$$

5.3. **+-involution.** The automorphism group $\mathrm{Aut}(k[x])$ is isomorphic to $\mathbb{Z}/2\mathbb{Z}$ and consists of the identity and the involution

$$\rho : k[x] \to k[x], \qquad p(x) \longmapsto p^+(x) := p(x+1) .$$

Following [GKW] we call p^+ the *conjugate* of p. This notion plays an important role in the analysis of harmonicity of plane grids, see Section 2. The ring of invariants $k^+[x] := k[x]^\rho = \ker(\delta)$, where $\delta = \rho + \mathrm{id} \in \mathrm{End}(k[x])$, consists of all *self-conjugate* polynomials $p = p^+$. Clearly, ρ preserves degree and irreducibility.

The following proposition extends Lemma 15 in [GKW].

Proposition 5.15. (a) $\mathrm{im}(\delta) = \ker(\delta) = \mathrm{vect}\,(\delta(x^n) \,:\, n \in \mathbb{N}_{\mathrm{odd}})$.

(b) *We have* $\deg \delta(p) = \deg p - 1$ $\forall p \in k[x]$ *with* $\deg p \in \mathbb{N}_{\mathrm{odd}}$*, and* $\deg p \in \mathbb{N}_{\mathrm{even}}$ $\forall p \in k^+[x]$.

(c) $k[x] = k^+[x] \oplus k_{\mathrm{odd}}[x]$*, where* $k_{\mathrm{odd}}[x] \subseteq k[x]$ *is the subspace of odd polynomials.*

(d) $\delta(fg) = f\delta(g) + g\delta(f) + \delta(f)\delta(g)$, $\forall f, g \in k[x]$.

(e) *In particular* $\delta(fg) = f\delta(g)$ $\forall f \in \ker(\delta), \forall g \in k[x]$.

(f) $\delta(p^2) = (\delta(p))^2$, $\forall p \in k[x]$. *Consequently,* $k^+[x] = \ker(\delta)$ *is stable under the endomorphism* $p \longmapsto p^2$.

Remark 5.16. 1. By (a) and 5.22.c below, every polynomial $g \in k^+[x]$ is of the form

$$g = p + p^+ = \sum_{0 \le k_1 < \ldots < k_n} \left(x^{2k_i+1} + (x+1)^{2k_i+1}\right) = g_1(x^2 + x)$$

for some odd $p \in k[x]$ and some $g_1 \in k[x]$.

Similarly every polynomial f satisfying $f^+ = f + 1$ is of the form $f = x + g$ for some $g \in k^+[x]$. For instance, by virtue of 5.2.c this is so for the polynomials $f_q = F_{q-1} + F_{q+1}^+$ $\forall q = 2^r$.

We let $\mathrm{Irr}^+[x] = \mathrm{Irr}[x] \cap k^+[x] \subseteq k[x]$ be the set of all irreducible self-conjugate polynomials.

Examples 5.17. 1. The polynomials in $\mathrm{Irr}^+[x]$ of degree ≤ 8 are the following ones:

$$x^2 + x + 1, \quad x^4 + x + 1, \quad x^6 + x^5 + x^3 + x^2 + 1,$$
$$x^8 + x^6 + x^5 + x^3 + 1, \quad x^8 + x^6 + x^5 + x^4 + x^3 + x + 1 .$$

2. $F_5 = (x^2 + x + 1)^2$ is the only self-conjugate Fibonacci polynomial [GKW].

5.18. For any $q = 2^r$, we let $\mathrm{Tr}_q(x) = \sum_{i=0}^{r-1} x^{2^i}$, so that $\mathrm{Tr}_q(x+y) = \mathrm{Tr}_q(x) + \mathrm{Tr}_q(y)$. We let $h_r(x) = x^q + x + 1 \in k[x]$ and $\tilde{h}_r(x) = 1 + \mathrm{Tr}_q(x)$. The following proposition can be checked readily, see [LN, Theorem 3.80] for (b) and [LN, Exercise 3.90] for (c).

Proposition 5.19. (a) $\forall q = 2^r$, $r \ge 0$, *we have*

$$h_r(x) = \tilde{h}_r(x(x+1)) = 1 + \mathrm{Tr}_{\mathbb{F}_q}(x) + \mathrm{Tr}^2_{\mathbb{F}_q}(x) \in k^+[x]$$

and

$$\tilde{h}_{2r}(x) = 1 + \mathrm{Tr}_{q^2}(x) = 1 + \mathrm{Tr}_q(x) + \mathrm{Tr}_q^q(x) = h_r(\mathrm{Tr}_q(x)) \in k^+[x] .$$

(b) *The decompositions of* $\tilde{h}_r$, h_r *into irreducible factors over* $\mathbb{F}_q$ *are, respectively,*

$$\tilde{h}_r(x) = \prod_{j=1}^{q/2}(x+\beta_j) \qquad \textit{and} \qquad h_r(x) = \prod_{j=1}^{q/2}(x^2+x+\beta_j)\,,$$

where $\beta_j \in \mathbb{F}_q$ *runs over the affine subspace* $\{z \in \mathbb{F}_q \,:\, \mathrm{Tr}_{\mathbb{F}_q}(z) = 1\}$.

(c) $\forall u, v \in \bar{k}$, $h_r(u+v) = h_r(u)+h_r(v)+1$. *The splitting field of* h_r *is* $\mathbb{F}_{q^2}$. *Moreover* h_r *has simple roots that fill in the* r*-dimensional affine subspace*

$$\{z \in \mathbb{F}_{q^2} \,:\, z^q = z+1\} = \{z \in \mathbb{F}_{q^2} \,:\, \rho(z) = \rho_q(z)\} \subseteq \mathbb{F}_{q^2}\,,$$

where $\rho : z \longmapsto z+1$ *and* $\rho_q : z \longmapsto z^q$. *This subspace is parallel to* $h_r^{-1}(1) = \mathbb{F}_q$ *and stable under the Frobenius automorphism and under the involutions* ρ *and* ρ_q.

$\forall e \in \mathbb{N}$ and $\forall m \in \mathbb{N}_{\mathrm{odd}}$, we let $D_2(2^e m) = e$.

Proposition 5.20. (a) $h_r \circ h_s = h_s \circ h_r = h_{r+s}+h_r+h_s$. *Consequently,* $h_{2s} = h_s \circ h_s$. *More generally,* $\forall q = 2^r$ *and* $\forall s \in \mathbb{N}$,

$$h_{qs} = \underbrace{h_s \circ \ldots \circ h_s}_{q}.$$

(b) $h_s \mid h_r \iff r = ms$, *where* $m \in \mathbb{N}_{\mathrm{odd}}$. *In particular* $h_1 \mid h_r \iff r \in \mathbb{N}_{\mathrm{odd}}$.

(c)

$$\gcd(h_r, h_s) = \begin{cases} h_{\gcd(r,s)} & \textit{if} \quad D_2(r) = D_2(s), \\ 1 & \textit{otherwise.} \end{cases}$$

Similarly,

$$\gcd(\tilde{h}_r, \tilde{h}_s) = \begin{cases} \tilde{h}_{\gcd(r,s)} & \textit{if} \quad D_2(r) = D_2(s), \\ 1 & \textit{otherwise.} \end{cases}$$

Proof. The proof of (a) is easy and can be omitted. To show (b) we assume that $h_s \mid h_r$, and we let $q = 2^r$, $q' = 2^s$. By virtue of 5.19.b, $\mathbb{F}_{q'^2} \subseteq \mathbb{F}_{q^2}$, hence $s \mid r$. If $r = ms$ then for any root z of h_s,

$$z^q = z^{q'^m} = \begin{cases} z & \text{if } m \text{ is even,} \\ z^{q'} & \text{otherwise.} \end{cases}$$

Thus $h_s \mid h_r$ implies that $m \in \mathbb{N}_{\mathrm{odd}}$. The converse is easy.

(c) Assume that h_s and h_r have a common root $z \in \bar{k}$ of degree δ. By virtue of 5.19.c, $\mathbb{F}_{2^\delta} \subseteq \mathbb{F}_{2^{2r}} \cap \mathbb{F}_{2^{2s}}$, hence $\delta \mid 2\gcd(r,s)$. Since $z^{2^r} = z^{2^s} = z+1 \neq z$, we have $z \in (\mathbb{F}_{2^{2r}} \setminus \mathbb{F}_{2^r}) \cap (\mathbb{F}_{2^{2s}} \setminus \mathbb{F}_{2^s})$. It follows that δ does not divide $\gcd(r,s)$ i.e., $\delta = 2\delta'$, where $\delta' \mid \gcd(r,s)$.

Letting $r = m\delta'$ and $\tilde{q} = 2^{\delta'}$ we obtain

$$z+1 = z^{2^r} = z^{\tilde{q}^m} = \begin{cases} z & \text{if } m \text{ is even,} \\ z^{\tilde{q}} & \text{otherwise.} \end{cases}$$

Hence $m \in \mathbb{N}_{\mathrm{odd}}$. Similarly, $s = n\delta'$, where $n \in \mathbb{N}_{\mathrm{odd}}$, and so, $D_2(r) = D_2(s) = D_2(\delta')$. Moreover, $h_{\delta'}(z) = 0$. It follows that $\gcd(h_r, h_s) \mid h_{\gcd(r,s)}$. Vice versa, by virtue of (b), $h_{\gcd(r,s)} \mid \gcd(h_r, h_s)$. Therefore (c) follows. $\square$

5.21. For a polynomial $\tau = x^r + a_{r-1}x^{r-1} + \ldots \in \mathrm{Irr}[x]$ we denote $\mathrm{Tr}(\tau) = a_{r-1} = \mathrm{Tr}_{\mathbb{F}_q}(z)$, where $q = 2^r$ and $z \in \mathbb{F}_q$ is a root of τ. We let

$$\mathrm{Irr}_i[x] = \{\tau \in \mathrm{Irr}[x] \,:\, \mathrm{Tr}(\tau) = i\}, \quad i = 0, 1,$$

and

$$\mathrm{Irr}^-[x] = \{\tau\tau^+ \,:\, \tau \in \mathrm{Irr}[x] \setminus \mathrm{Irr}^+[x]\}\,.$$

Proposition 5.22. (a) *Every irreducible factor of h_r belongs to $\mathrm{Irr}^+[x]$, and every $\tau \in \mathrm{Irr}^+[x]$ divides one of the h_r, $r \geq 1$.*

(b) *Every polynomial $\tau \in \mathrm{Irr}^+[x]$ of degree $2r$ admits a decomposition*

$$\tau(x) = \prod_{i=0}^{r-1}(x^2 + x + \beta^{2^i})\,,$$

where $\beta \in \mathbb{F}_q$ ($q = 2^r$) and $\mathrm{Tr}_{\mathbb{F}_q}(\beta) = 1$.

(c) *The map*

$$\alpha : k[y] \to k^+[x] = k[x(x+1)], \qquad q(y) \longmapsto p(x) := q(x(x+1))\,,$$

is an isomorphism of k-algebras.

(d) $\mathrm{Irr}(k^+[x]) = \alpha(\mathrm{Irr}[x]) = \mathrm{Irr}^+[x] \bigcup \mathrm{Irr}^-[x]$. *Moreover* $\alpha(\mathrm{Irr}_1[x]) = \mathrm{Irr}^+[x]$ *and* $\alpha(\mathrm{Irr}_0[x]) = \mathrm{Irr}^-[x]$.

Proof. An element $z \in \bar{k}$ is a root of one of the polynomials h_r, $r \geq 1$, if and only if z and $z+1$ are conjugated, if and only if the minimal polynomial τ of z is stable under the involution $\rho : z \longmapsto z+1$ i.e., $\tau \in \mathrm{Irr}^+[x]$. Hence (a) follows.

(b) is immediate from 5.15.b and 5.19.b.

To show (c) it is enough to establish that α is surjective. For every $f = \prod_{i=1}^n \tau_i \in k^+[x]$ of positive degree, the involution ρ acts on the set $\{\tau_i\}_{i=1,\ldots,n}$ of all irreducible factors of f. Since $f = f^+ = \prod_{i=1}^n \tau_i^+$ then either $\tau_i \in \mathrm{Irr}^+[x]$ or $\tau_i^+ = \tau_j \neq \tau_i$ $\forall i = 1, \ldots, n$.

For every $\tilde{\tau} = \prod_{i=0}^{r-1}(x + z^{2^i}) \in \mathrm{Irr}[x]$,

$$(\tilde{\tau}\tilde{\tau}^+)(x) = \prod_{i=0}^{r-1}(x + z^{2^i})(x + 1 + z^{2^i}) = \prod_{i=0}^{r-1}(x^2 + x + \beta^{2^i}) = \alpha(\tau)(x)\,,$$

where $\beta = z(z+1) \in \mathbb{F}_q$, $q = 2^r$, and $\tau \in k[x]$.

We will show below that for every $\tau \in \mathrm{Irr}^+[x]$, $\tau = \alpha(\tilde{\tau})$ for some $\tilde{\tau} \in \mathrm{Irr}[x]$. Thus $f \in k[x(x+1)]$. Hence $k^+[x] = k[x(x+1)]$. Now (c) follows.

(d) Let $\beta \in \mathbb{F}_q$, where $q = 2^r$, be a root of a polynomial $\tau \in \mathrm{Irr}_0[x]$ of degree $r \geq 1$. Then $\mathrm{Tr}_{\mathbb{F}_q}(\beta) = 0$, hence $\beta = z(z+1)$ for some $z \in \mathbb{F}_q$ [LN, 2.80]. Therefore

$$\tau(x(x+1)) = \prod_{j=0}^{r-1}(x^2 + x + z^{2^j}(z+1)^{2^j})) = \prod_{j=0}^{r-1}(x + z^{2^j})\prod_{j=0}^{r-1}(x + 1 + z^{2^j}) = (\tilde{\tau}\tilde{\tau}^+)(x)\,,$$

where $\tilde{\tau}(x) = \prod_{j=0}^{r-1}(x + z^{2^j}) \in \mathrm{Irr}[x]$ as $\deg z = r = \deg\, \tilde{\tau}$.

Suppose further that $\tilde{\tau} = \tilde{\tau}^+$. Then $z^{2^i} = z + 1$ for some $i \in \{1, \ldots, r-1\}$. Hence $\beta^{2^i} = \beta$, and so $\deg \beta \leq i < r$, a contradiction. This proves the last equality in (d) (cf. [LN, Exercise 3.86]).

For every $\tau \in \mathrm{Irr}^+[x]$ of degree $\deg \tau = 2r > 0$, the involution ρ acts on the set $\{z_j = z^{2^j}\}_{j=0,\ldots,2r-1}$ of roots of τ, where z is one of these roots. If $\rho(z) = z+1 = z^{2^i} = z_i$

then $\rho(z_j) = z_j + 1 = z_{(i+j) \mod 2r}$, $j = 0, \ldots, 2r-1$. As $\rho^2 = \mathrm{id}$ we have $j + 2i \equiv j \mod 2r$ $\forall j$, hence $i = r$. Therefore

$$\tau(x) = \prod_{j=0}^{2r-1} (x + z_j) = \prod_{j=0}^{r-1} (x + z_j)(x + z_j + 1) = \tilde{\tau}(x(x+1)) = \alpha(\tilde{\tau})(x)\,,$$

where

$$\tilde{\tau}(x) = \prod_{j=0}^{r-1} (x + \beta_j) = \prod_{j=0}^{r-1} (x + \beta^{2^j}) \in k[x]$$

with $\beta_j = z_j(z_j+1)$ and $\beta = z(z+1) \in \mathbb{F}_q$, $q = 2^r$. Since $\deg \beta = r = (\deg z)/2$, we have $\tilde{\tau}(x) \in \mathrm{Irr}[x]$. Moreover $\mathrm{Tr}_{\mathbb{F}_q}(\beta) = 1$ since the polynomial $z^2 + z + \beta \in \mathbb{F}_q[x]$ is irreducible over $\mathbb{F}_q$. Hence $\tilde{\tau} \in \mathrm{Irr}_1[x]$. This shows that $\alpha(\mathrm{Irr}_1[x]) = \mathrm{Irr}^+[x]$. Now (d) follows. $\square$

Remarks 5.23. 1. ([LN, Exercise 3.87]) For any odd prime p such that $\mathrm{ord}_p\, 2 = p-1$, $\tau(x) = \frac{x^p+1}{x+1} \in \mathrm{Irr}_1[x]$ and so, $\alpha(\tau) \in \mathrm{Irr}^+[x]$.

2. $h_r \in \mathrm{Irr}^+[x] \quad \Longleftrightarrow \quad r \in \{1, 2\}$.
3. $\forall r = 2^a$, h_r has 2^{r-a-1} irreducible factors of the same degree $2r = 2^{a+1}$.
4. We have $\tilde{h}_4(x) = 1 + \mathrm{Tr}_{2^4}(x) = \tau\tau^+$, where τ, $\tau^+ \in \mathrm{Irr}[x]$, $\tau = x^4 + x^3 + 1 \neq \tau^+$.

References

[ASZ] A. T. Amin, P. J. Slater, Guo-Hui Zhang, *Parity dimension for graphs—a linear algebraic approach*, Linear Multilinear Algebra 50 (2002), 327–342.

[An] B. Andrásfai, *Cellular automata in trees*, Finite and infinite sets, Vol. I, II (Eger, 1981), 35–45, Colloq. Math. Soc. János Bolyai, 37, North-Holland, Amsterdam, 1984.

[Ba] R. Bacher, *Valeur propre minimale du laplacien de Coxeter pour le groupe symétrique*, J. Algebra 167 (1994), 460–472.

[BR] R. Barua, S. Ramakrishnan, *σ-game, σ^+-game and two-dimensional additive cellular automata*, Theoret. Comput. Sci. 154 (1996), 349–366.

[Ca] P. Cartier, *Géométrie et analyse sur les arbres*, Séminaire Bourbaki, 24ème année (1971/1972), Exp. No. 407, 123–140. Lecture Notes in Math. 317, Springer, Berlin, 1973.

[DG] D. Drucker, D. M. Goldschmidt, *Graphical evaluation of sparse determinants*, Proc. Amer. Math. Soc. 77 (1979), 35–39.

[Er] P. Erdös, *On the sum $\sum_{d|2^n-1} d^{-1}$*, Israel J. Math. 9 (1971), 43–48.

[Ga] F. R. Gantmacher, *The theory of matrices*, Vol. 1 (transl. from Russian). AMS Chelsea Publishing, Providence, RI, 1998.

[GKW] J. Goldwasser, W. Klostermeyer, H. Ware, *Fibonacci Polynomials and Parity Domination in Grid Graphs*, Graphs and Combinatorics 18 (2002), 271–283.

[GMT] S. Gravier, M. Mhalla, E. Tannier, *On a modular domination game*, Theoret. Comput. Sci. 306 (2003), 291–303.

[HW] Hardy, Wright, *An introduction to the theory of numbers*, Fifth ed. The Clarendon Press, Oxford University Press, New York, 1979.

[HB] D. R. Heath-Brown, *Artin's conjecture for primitive roots*, Quart. J. Math. (2) 37 (1986), 27–38; MR1804616.

[HJ] R. A. Horn, C. R. Johnson, *Topics in matrix analysis*, Cambridge University Press, Cambridge, 1994.

[HMP] M. Hunziker, A. Machiavelo, J. Park, *Chebyshev polynomials over finite fields and reversibility of σ-automata on square grids*, Theoret. Comput. Sci. 320 (2004), 465–483.

[J] D. Jungnickel, *Finite fields. Structure and arithmetics*, Bibliographisches Institut, Mannheim, 1993.

[K] W. F. Klostermeyer, *Parity domination: results and problems*, Graph Theory Notes N. Y. 44 (2003), 35–40.

[KG] W. F. Klostermeyer, J. L. Goldwasser, *Nullspace-primes and Fibonacci polynomials*, Fibonacci Quart. 40 (2002), 323–327.

[Ko] N. Koblitz, *p-adic numbers, p-adic analysis, and zeta-functions*, Graduate Texts in Mathematics, 58. Springer-Verlag, New York, 1984.

[LLMP] A. K. Lenstra, H. W. Lenstra, Jr., M. S. Manasse, J. M. Pollard, *The factorization of the ninth Fermat number*, Math. Comp. 61 (1993), no. 203, 319–349.

[LMT] R. Lidl, G. L. Mullen, G. Turnwald, *Dickson polynomials*, Pitman Monographs in Pure and Applied Mathematics, 65. Longman Scientific and Technical, Harlow; John Wiley and sons, Inc., New York, 1993.

[LN] R. Lidl, H. Niederreiter, *Introduction to finite fields and their applications*, Cambridge University Press, Cambridge, 1994.

[MOW] O. Martin, A. M. Odlyzko, S. Wolfram, *Algebraic properties of cellular automata*, Comm. Math. Phys. 93 (1984), 219–258.

[McE] R. J. McEliece, *Finite fields for computer scientists and engineers*, Kluwer Academic Publishers, Boston, MA, 1987.

[Me] *Mersenne Primes: History, Theorems and Lists*, http://www.utm.edu/research/primes/mersenne/.

[Mo] P. Moree, *Artin's primitive root conjecture - a survey*, math.NT/0412262 (2004), 30p.

[Mu] M. R. Murty, *Artin's conjecture for primitive roots*, Math. Intelligencer 10 (1988), 59–67.

[OZ] S. Orevkov, M. Zaidenberg, *On the number of singular points of plane curves*, Geometry of Algebraic Varieties, Proc. Conf. Algebraic Geometry, Saitama University, March 16–18, 1995, 156-177.

[Pe] D. Pelletier, *Merlin's magic square*, Amer. Math. Monthly 94 (1987), 143-150.

[Ri] T. J. Rivlin, *Chebyshev polynomials. From approximation theory to algebra and number theory*, John Wiley Sons, Inc., New York, 1990.

[SB] P. Sarkar, R. Barua, *Multidimensional σ-automata, π-polynomials and generalised S-matrices*, Theoret. Comput. Sci. 197 (1998), 111–138.

[Su1] K. Sutner, *On σ-automata*, Complex Systems 2 (1988), 1–28.

[Su2] K. Sutner, *Additive automata on graphs*, Complex Systems 2 (1988), 649–661.

[Su3] K. Sutner, *σ-automata and Chebyshev-polynomials*, Theoret. Comput. Sci. 230 (2000), 49–73.

[Su4] K. Sutner, *The σ-game and cellular automata*, Amer. Math. Monthly 97 (1990), 24–34.

[Su5] K. Sutner, *Linear cellular automata and the Garden-of-Eden*, Math. Intelligencer 11 (1989), 49–53.

[vdW] B. L. van der Waerden, *Algebra*, Vol. I, II. Springer-Verlag, New York, 1991.

[Wa] Zhe-Xian Wan, *Lectures on finite fields and Galois rings*, World Scientific Publishing Co., Inc., River Edge, NJ, 2003.

[WP] W. A. Webb, E. A. Parberry, *Divisibility properties of Fibonacci polynomials*, Fibonacci Quart. 7 (1969), 457–463.

Université Grenoble I, Institut Fourier, UMR 5582 CNRS-UJF, BP 74, 38402 St. Martin d'Hères cédex, France

E-mail address: `zaidenbe@ujf-grenoble.fr`