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Bonn

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Preprint Series 2010 (54)

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MPIM 2010-54

DIFFERENTIAL OVERCONVERGENCE

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Abstract. We prove that some of the basic differential functions appearing in the theory of arithmetic differential equations [6], especially some of the basic differential modular forms in that theory, have a remarkable "differential overconvergence" property. One can also go in the opposite direction by using "differentially overconvergent" series to construct "new" differential functions.

1. Introduction

This paper is a continuation of the study of arithmetic differential equations begun in [3, 5]; cf. the Introduction and bibliography of [6]. For the convenience of the reader the present paper is written so as to be logically independent of the above references; we will instead quickly review here the main concepts of that theory and we will only refer to [3, 5, 6] for various results as need.

The purpose of the theory in [3, 5, 6] is to develop an arithmetic analog of ordinary differential equations. Analytic functions $x(t)$ are replaced in our theory by integer numbers $n \in \mathbb{Z}$ (or, more generally, by integers in number fields and their p-adic completions). The derivative operator $x(t) \mapsto \frac{dx}{dt}(t)$ is replaced by a Fermat quotient operator which, on $\mathbb{Z}$, acts as $n \mapsto \delta_p n := \frac{n-n^p}{p}$. Non-linear differential operators $x(t) \mapsto F(x(t), \frac{dx}{dt}(t), ..., \frac{d^r x}{dt^r}(t))$, with F analytic, are replaced by what is being referred to as δ_p-*functions*. The latter have a series of purely arithmetic applications for which we refer to [4, 5, 11]. What we discover in this paper is that some of the most important δ_p-functions appearing in this theory have a remarkable "differential overconvergence" property: they come from "δ_π-*functions*", $\pi = 1-\zeta_p$, where ζ_p denotes, in this paper, a root of unity of order p. We will call this property δ_π-*overconvergence*; cf. the definitions below. Conversely one can use bad reduction phenomena to construct δ_π-functions whose traces, then, are "new", interesting δ_p-functions.

In the rest of the introduction we will define our main concepts and state (in a rough form) our main results. We shall refer to the main body of the paper for detailed statements and for the proofs of our results.

1.1. **Review of notation and terminology** [3, 5, 6]**.** Throughout this paper $p \geq 5$ is a fixed prime and we denote by $R_p = \widehat{\mathbb{Z}}_p^{ur}$ the completion of the maximum unramified extension of $\mathbb{Z}_p$. We set $K_p = R_p[1/p]$ (fraction field of R_p) and $k = R_p/pR_p$ (residue field of R_p); so k is an algebraic closure of $\mathbb{F}_p$. Let π be a root of an Eisenstein polynomial of degree e with coefficients in $\mathbb{Z}_p$. (Recall that $\mathbb{Q}_p(\pi)$ is then a totally ramified extension of $\mathbb{Q}_p$; conversely any finite totally ramified extension of $\mathbb{Q}_p$ is of the form $\mathbb{Q}_p(\pi)$ with π a root of an Eisenstein polynomial with coefficients in $\mathbb{Z}_p$.) In order to simplify some of our exposition we will assume in what follows that

2000 *Mathematics Subject Classification.* 11 F 32, 11 F 85, 11G18.

$\mathbb{Q}_p(\pi)/\mathbb{Q}_p$ is a Galois. (A typical we have in mind for our applications is $\pi = 1-\zeta_p$ in which case $e = p-1$.) Consider the ring $R_\pi := R_p[\pi] = R_p \otimes_{\mathbb{Z}_p} \mathbb{Z}_p[\pi]$. Then R_π is a complete discrete valuation ring with maximal ideal generated by π and with fraction field K_π of degree e over K_p. If $v_p : K_\pi^\times \to \mathbb{R}$ is the p-adic valuation with $v_p(p) = 1$ then $v_p(\pi) = 1/e$. The ring R_π possesses a unique ring automorphism ϕ such that $\phi(\pi) = \pi$ and ϕ lifts the p-power Frobenius of $k = R_\pi/\pi R_\pi$. Clearly ϕ sends R_p into itself and is a lift of the p-power Frobenius of $k = R_p/pR_p$. Also throughout the paper $\hat{}$ denotes p-adic completion. For R_π-algebras the p-adic completion $\hat{}$ is, of course the same as the π-adic completion.

Our substitutes for "differentiation" with respect to p and π respectively are the *Fermat quotient maps* [3] $\delta_p : R_p \to R_p$ and $\delta_\pi : R_\pi \to R_\pi$ defined by

$$\begin{array}{rcl} \delta_p x & := & \frac{\phi(x)-x^p}{p}, \quad x \in R_p, \\ \\ \delta_\pi x & := & \frac{\phi(x)-x^p}{\pi}, \quad x \in R_\pi, \end{array}$$

respectively. In particular, for $x \in R_p$, we have

$$\begin{array}{rcl} \delta_\pi x & = & \frac{p}{\pi}\delta_p x, \\ \\ \delta_\pi^2 x & = & \frac{p^2}{\pi^2}\delta_p^2 x + \left(\frac{p}{\pi^2} - \frac{p^p}{\pi^{p+1}}\right)(\delta_p x)^p, \dots \end{array}$$

Let V be an affine smooth scheme over R_p and fix a closed embedding $V \subset \mathbb{A}^d$ over R_p. (The concepts below will be independent of the embedding.) A function $f_p : V(R_p) \to R_p$ is called a δ_p-*function* (or order $r \geq 0$) if there exists a restricted power series F_p with R_p-coefficients, in $(r+1)d$ variables such that

$$f_p(x) = F_p(x, \delta_p x, \dots, \delta_p^r x), \quad x \in V(R_p) \subset R_p^d. \tag{1.1}$$

Here and later a power series is called *restricted* if its coefficients tend to 0. (If V is not necessarily affine f_p is called a δ_p-function if its restriction to the R_p-points of any affine subset of V is a δ_p-function.) A function $f_\pi : V(R_\pi) \to R_\pi$ is called a δ_π-*function* (or order $r \geq 0$) if there exists a restricted power series F_π with R_π-coefficients, in $(r+1)d$ variables such that

$$f_\pi(x) = F_\pi(x, \delta_\pi x, \dots, \delta_\pi^r x), \quad x \in V(R_\pi) \subset R_\pi^d. \tag{1.2}$$

(If V is not necessarily affine f_π is called a δ_π-function if its restriction to the R_π-points of any affine subset of V is a δ_π-function.)

1.2. **δ_π-overconvergence.** The main concept we would like to explore (and exploit) in this paper is given in the following definition. Let $f_p : V(R_p) \to R_p$ be a δ_p-function. We will say that f_p is δ_π-*overconvergent* if one of the following equivalent conditions is satisfied:

1) There exists an integer $\nu \geq 0$ and a δ_π-function f_π making the diagram below commutative:

$$\begin{array}{ccc} V(R_p) & \stackrel{p^\nu f_p}{\longrightarrow} & R_p \\ \iota \downarrow & & \uparrow Tr \\ V(R_\pi) & \stackrel{f_\pi}{\longrightarrow} & R_\pi \end{array} \tag{1.3}$$

(Here ι stands for the inclusion and Tr stands for the R_π/R_p-trace.)

2) There exists an integer $\nu \geq 0$ and a (necessarily unique) δ_π-function f_π making the diagram below commutative:

$$(1.4)\qquad \begin{array}{ccc} V(R_p) & \stackrel{p^\nu f_p}{\longrightarrow} & R_p \\ \iota \downarrow & & \downarrow \iota \\ V(R_\pi) & \stackrel{f_\pi}{\longrightarrow} & R_\pi \end{array}$$

The equivalence between conditions 1 and 2 above is trivial to check; cf. also Proposition 2.3.

Our terminology above is motivated by the following link with the classical concept of *overconvergence* introduced in the work of Dwork, Monsky, and Washnitzer. Indeed one can show (as we will later) that if $V = \mathbb{A}^d$ is the affine space, say, and $f_p : V(R_p) = R_p^d \to R_p$ is a δ_p-function as in (1.1) which is δ_π-overconvergent and has order $r \leq e-1$ then the series F_p appearing in (1.1) is overconvergent (in the classical sense) in the variables $\delta_p x, ..., \delta_p^r x$ (but of course not necessarily overconvergent in the variables x). Also, if f_p is δ_π-overconvergent of order $r \geq e$ then it does not follow that F_p is overconvergent in the variables $\delta_p x, ..., \delta_p^r x$. Finally note that if f_p has order 0 (which is the same as saying that f_p comes from a global function on the p-adic completion $\hat{V}$ of V) then f_p is automatically δ_π-overconvergent but, of course, F_p will not generally be overconvergent.

1.3. **Main results.** The interaction between δ_p-functions and δ_π-functions turns out to be a two way avenue as follows:

1) *From δ_π-functions to δ_p-functions.* Given a δ_π-function $f_\pi : V(R_\pi) \to R_\pi$ the function f_p defined by the diagram (1.3) with $\nu = 0$ turns out to be a δ_p-function. In this paper we will construct "interesting" δ_π-functions using bad reduction phenomena and then we will apply trace constructions (a *geometric trace* construction and also the R_π/R_p-trace construction in diagram (1.3) which can be referred to as an *arithmetic trace*) to get "new" δ_p-functions. Cf. Theorem 1.1.

2) *From δ_p-functions to δ_π-functions.* In this paper we discover that some of the basic "old" δ_p-functions that played a role in [3, 5, 6] are δ_π-overconvergent. Cf. Theorem 1.2.

We will apply the above considerations mainly to the theory of differential modular forms [5, 6]. To explain this recall the modular curve $X_1(N)_{R_p}$ over R_p with $(N,p) = 1$, $N > 4$. This curve is smooth and carries a line bundle L such that the spaces of sections $H^0(X_1(N)_{R_p}, L^\kappa)$ identify with the spaces of modular forms on $\Gamma_1(N)$ defined over R_p of weight κ; cf. [15], p. 450, where L was denoted by ω. The curve $X_1(N)_{R_p}$ contains two remarkable (disjoint) closed subsets: the *cusp locus* (*cusps*) and the *supersingular locus* (*ss*). On $Y_1(N) = X_1(N) \backslash (cusps)$ the line bundle L identifies with $u_* \Omega^1_{E/Y_1(N)}$ where $u : E \to Y_1(N)$ is the corresponding universal elliptic curve. Next consider an affine open set $X \subset X_1(N)_{R_p}$ and consider the restriction of L to X which we continue to denote by L. We can consider the affine X-scheme $V := Spec\left(\bigoplus_{n \in \mathbb{Z}} L^{\otimes n}\right) \to X$. Then a *$\delta_p$-modular function* (on X, of level N and order r) is simply a δ_p-function $V(R_p) \to R_p$ (of order r). Similarly a *δ_π-modular function* (on X, of level N and order r) is a δ_π-function $V(R_\pi) \to R_\pi$ (of order r). There is a natural concept of weight for a δ_p-modular function or a δ_π-modular function; weights are elements in the ring $\mathbb{Z}[\phi]$ of polynomials in ϕ with

$\mathbb{Z}$-coefficients; cf. the body of the text for the definition of weight. δ_p-modular functions (respectively δ_π-modular functions) possessing weights are called *δ_p-modular forms* (respectively *δ_π-modular forms*). Now, as we shall review in the body of the paper, δ_p-modular functions f (and hence forms) possess *δ_p-Fourier expansions* denoted by $E(f)$ which are restricted power series in variables $\delta_p q, ..., \delta_p^r q$, with coefficients in the ring $R_p((q))\hat{}$.

Our first main result is a construction of some interesting "new" δ_p-modular forms as R_π/R_p-traces of some δ_π-modular forms. In their turn, these δ_π-modular forms will be constructed using the bad reduction of modular curves. Here is the result (in which X is assumed to be disjoint from the supersingular locus):

Theorem 1.1. *Let $f = \sum a_n q^n$ be a classical normalized newform of weight 2 and level $\Gamma_0(Np)$ over $\mathbb{Z}$. Assume $a_p = 1$ and let $\pi = 1 - \zeta_p$. Then there exists a δ_p-modular form $f_p^\sharp$ of level N, order 1, and weight 0 which is δ_π-overconvergent and whose δ_p-Fourier expansion satisfies the following congruence mod p:*

$$E(f_p^\sharp) \equiv \left(\sum_{(n,p)=1} \frac{a_n}{n} q^n \right) - \left(\sum_{n \geq 1} a_n q^{np} \right) \frac{\delta_p q}{q^p}.$$

Cf. Proposition 4.11 in the paper. Note that the condition $a_p = 1$ is equivalent to the condition that the elliptic curve attached to f via the Eichler-Shimura construction have split multiplicative reduction at p. The δ_p-modular form $f_p^\sharp$ in Theorem 1.1 should be viewed as a bad reduction analogue of the δ_p-modular forms $f^\sharp = f_p^\sharp$ of level N, order ≤ 2, and weight 0 that were attached in [7] to classical normalized newforms $f = \sum a_n q^n$ of weight 2 and level $\Gamma_0(N)$ over $\mathbb{Z}$. For such an f on $\Gamma_0(N)$ that does not have CM (in the sense that the elliptic curve attached to it via the Eichler-Shimura construction does not have CM) the forms $f_p^\sharp$ have order exactly 2 and were shown in [11] to have δ_p-Fourier expansions satisfying the following congruence mod p:

$$E(f_p^\sharp) \equiv \left(\sum_{(n,p)=1} \frac{a_n}{n} q^n \right) - a_p \left(\sum_{m \geq 1} a_m q^{mp} \right) \frac{\delta_p q}{q^p} + \left(\sum_{m \geq 1} a_m q^{mp^2} \right) \cdot \left(\frac{\delta_p q}{q^p} \right)^p.$$

Similar results are available for f on $\Gamma_0(N)$ having CM; cf. [7, 11]. Unlike the forms $f_p^\sharp$ for f on $\Gamma_0(Np)$ the forms $f_p^\sharp$ for f on $\Gamma_0(N)$ were defined for *any* X (not necessarily disjoint from the supersingular locus).

Our second main result is a construction of δ_π-modular forms from certain δ_p-modular forms. Indeed, a key role in the theory in [5, 1, 6] was played by certain δ_p-modular forms denoted by $f_p^1, f_p^2, f_p^3, ...$ of weights $-1-\phi, -1-\phi^2, -1-\phi^3, ...$ and by a δ_p-modular form denoted by f_p^∂ of weight $\phi - 1$ (where the former are defined whenever X is disjoint from the cusps while the latter is only defined if X is disjoint from both the cusps and the supersingular locus). The definition of these forms will be reviewed in the body of the paper. Our second main result (cf. Theorems 5.1, 5.3, and 5.5 in the body of the paper) is the following:

Theorem 1.2. *Assume $v_p(\pi) \geq \frac{1}{p-1}$. Then the δ_p-functions $f_p^\partial, f_p^1, f_p^2, f_p^3, ...$ are δ_π-overconvergent. Also $f_p^\sharp$ is δ_π-overconvergent for any classical normalized newform f of weight 2 and level $\Gamma_0(N)$ over $\mathbb{Z}$.*

By the way the forms $f_p^1, f_p^2, f_p^3, ...$ and f_p^∂ "generate" (in a sense explained in [5, 1, 6]) all the so called *isogeny covariant* δ_p-modular forms (in the sense of loc.cit.).

We refer to loc.cit. for the role of these forms in the theory and for the significance of the theory itself (in relation, for instance, to the construction in δ_p-*geometry* of the quotient of the modular curve by the action of the Hecke correspondences); reviewing this background here would take as too far afield and is not necessary for the understanding of our second main result above.

1.4. **Summary of the main forms.** We end our discussion by summarizing (cf. the table below) the main δ_π-overconvergent δ_p-modular forms appearing in this paper.

form	attached to	order	weight	domain X
f_p^r	$r \geq 1$	r	$-1-\phi^r$	X disjoint from $(cusps)$
$f_p^\sharp$	f on $\Gamma_0(N)$	1 or 2	0	X arbitrary
$f_p^\sharp$	f on $\Gamma_0(Np)$	1	0	X disjoint from (ss)
f_p^∂		1	$\phi-1$	X disjoint from $(cusps)$ and (ss)

1.5. **Plan of the paper.** We begin, in section 2, by revisiting our main set theoretic concepts above from a scheme theoretic viewpoint; δ_p-functions and δ_π-functions will appear as functions on certain formal schemes called p-jet spaces and π-jet spaces respectively; cf. [3, 4]. We shall review some of the properties of the latter and we shall analyze the concept of δ_π-overconvergence in some detail. Section 3 is mainly devoted to reviewing some basic aspects of modular parameterization and bad reduction of modular curves, following [13, 12, 15]; so this section is exclusively concerned with "non-differential" matters. In section 4 we go back to arithmetic differential equations: we will use modular parameterizations and bad reduction of modular curves to construct certain δ_π-modular forms and eventually the "new" δ_p-modular forms in Theorem 1.1. In section 5 we prove δ_π-overconvergence of some of the basic δ_p-functions of the theory, in particular we prove Theorem 1.2.

1.6. **Acknowledgment.** While writing this paper the first author was partially supported by NSF grant DMS-0852591 and by the Max Planck Institut fur Mathematik in Bonn.

2. δ_π-OVERCONVERGENCE

As expained in the Introduction we begin in this section by presenting δ_p-functions and δ_π-functions from a scheme-theoretic viewpoint (which is equivalent to the set-theoretic viewpoint of the Introduction). The scheme-theoretic viewpoint is less direct than the set-theoretic one but is the correct viewpoint when it comes to proofs so will be needed in the sequel. We then introduce and examine the concept of δ_π-overconvergence in a general setting.

2.1. **p-jet spaces and π-jet spaces** [3]**.** Let $C_p(X,Y)\in\mathbb{Z}[X,Y]$ be the polynomial with integer coefficients

$$C_p(X,Y):=\frac{X^p+Y^p-(X+Y)^p}{p}.$$

A p-*derivation* from a ring A into an $A-$algebra B, $\varphi:A\to B$, is a map $\delta_p:A\to B$ such that $\delta_p(1)=0$ and

$$\begin{array}{rcl}\delta_p(x+y) & = & \delta_p x+\delta_p y+C_p(x,y)\\ \delta_p(xy) & = & x^p\cdot\delta_p y+y^p\cdot\delta_p x+p\cdot\delta_p x\cdot\delta_p y,\end{array}$$

for all $x,y\in A$. Given a $p-$derivation we always denote by $\phi:A\to B$ the map $\phi(x)=\varphi(x)^p+p\delta_p x$; then ϕ is a ring homomorphism. A δ_p-*prolongation sequence* is a sequence $S^*=(S^n)_{n\geq 0}$ of rings S^n, $n\geq 0$, together with ring homomorphisms (still denoted by) $\varphi:S^n\to S^{n+1}$ and $p-$derivations $\delta_p:S^n\to S^{n+1}$ such that $\delta_p\circ\varphi=\varphi\circ\delta_p$ on S^n for all n. We view S^{n+1} as an S^n-algebra via φ. A morphism of δ_p-prolongation sequences, $u^*:S^*\to\tilde{S}^*$ is a sequence $u^n:S^n\to\tilde{S}^n$ of ring homomorphisms such that $\delta_p\circ u^n=u^{n+1}\circ\delta_p$ and $\varphi\circ u^n=u^{n+1}\circ\varphi$. Let W be the ring of polynomials $\mathbb{Z}[\phi]$ in the indeterminate ϕ. For $w=\sum a_i\phi^i$ (respectively for w with $a_i\geq 0$), S^* a δ_p-prolongation sequence, and $x\in(S^0)^\times$ (respectively $x\in S^0$) we can consider the element $x^w:=\prod_{i=0}^r\varphi^{r-i}\phi^i(x)^{a_i}\in(S^r)^\times$ (respectively $x^w\in S^r$).

Recall the ring $R_p:=\hat{\mathbb{Z}}_p^{ur}$, completion of the maximum unramified extension of the ring of p-adic integers $\mathbb{Z}_p$. Then R_p has a unique $p-$derivation $\delta_p:R_p\to R_p$ given by

$$\delta_p x=(\phi(x)-x^p)/p,$$

where $\phi:R_p\to R_p$ is the unique lift of the $p-$power Frobenius map on $k=R_p/pR_p$. One can consider the δ_p-prolongation sequence R_p^* where $R_p^n=R_p$ for all n. By a δ_p-*prolongation sequence over* R_p we understand a prolongation sequence S^* equipped with a morphism $R_p^*\to S^*$. From now on all our δ_p-prolongation sequences are assumed to be over R_p.

Let now π be a root of an Eisentein polynomial with $\mathbb{Z}_p$-coefficients and let $C_\pi(X,Y)\in\mathbb{Z}_p[\pi][X,Y]$ be the polynomial

$$C_\pi(X,Y):=\frac{X^p+Y^p-(X+Y)^p}{\pi}=\frac{p}{\pi}C_p(X,Y).$$

A π-*derivation* from an $\mathbb{Z}_p[\pi]$-algebra A into an $A-$algebra B, $\varphi:A\to B$, is a map $\delta_\pi:A\to B$ such that $\delta_\pi(1)=0$ and

$$\begin{array}{rcl}\delta_\pi(x+y) & = & \delta_\pi x+\delta_\pi y+C_\pi(x,y)\\ \delta_\pi(xy) & = & x^p\cdot\delta_\pi y+y^p\cdot\delta_\pi x+\pi\cdot\delta_\pi x\cdot\delta_\pi y,\end{array}$$

for all $x,y\in A$. Given a $\pi-$derivation we always denote by $\phi:A\to B$ the map $\phi(x)=\varphi(x)^p+\pi\delta_\pi x$; then ϕ is a ring homomorphism. A δ_π-*prolongation sequence* is a sequence $S^*=(S^n)_{n\geq 0}$ of $\mathbb{Z}_p[\pi]$-algebras S^n, $n\geq 0$, together with $\mathbb{Z}_p[\pi]$-algebra homomorphisms (still denoted by) $\varphi:S^n\to S^{n+1}$ and $\pi-$derivations $\delta_\pi:S^n\to S^{n+1}$ such that $\delta_\pi\circ\varphi=\varphi\circ\delta_\pi$ on S^n for all n. A morphism of δ_π-prolongation sequences, $u^*:S^*\to\tilde{S}^*$ is a sequence $u^n:S^n\to\tilde{S}^n$ of $\mathbb{Z}_p[\pi]$-algebra homomorphisms such that $\delta_\pi\circ u^n=u^{n+1}\circ\delta_\pi$ and $\varphi\circ u^n=u^{n+1}\circ\varphi$. Let W

be, again, the ring of polynomials $\mathbb{Z}[\phi]$ in the indeterminate ϕ. For $w = \sum a_i\phi^i$ (respectively for w with $a_i \geq 0$), S^* a δ_π-prolongation sequence, and $x \in (S^0)^\times$ (respectively $x \in S^0$) we can consider the element $x^w := \prod_{i=0}^r \varphi^{r-i}\phi^i(x)^{a_i} \in (S^r)^\times$ (respectively $x^w \in S^r$).

As above we may consider $R_\pi = R_p[\pi]$ and the $\pi-$derivation $\delta_\pi : R_\pi \to R_\pi$ given by

$$\delta_\pi x = (\phi(x) - x^p)/\pi.$$

One can consider the δ_π-prolongation sequence R_π^* where $R_\pi^n = R_\pi$ for all n. By a *δ_π-prolongation sequence over* R_π we understand a prolongation sequence S^* equipped with a morphism $R_\pi^* \to S^*$. From now on all our δ_π-prolongation sequences are assumed to be over R_π.

We note that if $S^* = (S^n)_{n\geq 0}$ is a δ_p-prolongation sequence such that each S^n is flat over R_p then the sequence $S^* \otimes_{R_p} R_\pi = (S^n \otimes_{R_p} R_\pi)_{n\geq 0}$ has a natural structure of δ_π-prolongation sequence. Indeed letting $\phi : S^n \to S^{n+1}$ denote, as usual, the ring homomorphisms $\phi(x) = x^p + p\delta_p x$ one can extend these ϕs to ring homomorphisms $\phi : S^n \otimes_{R_p} R_\pi \to S^{n+1} \otimes_{R_p} R_\pi$ by the formula $\phi(x\otimes y) = \phi(x)\otimes\phi(y)$ where $\phi : R_\pi \to R_\pi$ is given, as usual, by $\phi(y) = y^p + \pi\delta_\pi y$. Then one can define π-derivations $\delta_\pi : S^n \otimes_{R_p} R_\pi \to S^{n+1} \otimes_{R_p} R_\pi$ by $\delta_\pi(z) = (\phi(z) - z^p)/\pi$ for $z \in S^n \otimes_{R_p} R_\pi$. With these δ_πs the sequence $S^* \otimes_{R_p} R_\pi$ is a δ_π-prolongation sequence.

For any affine R_p-scheme of finite type $X = Spec\ A$ there exists a (unique) δ_p-prolongation sequence, $A^* = (A^n)_{n\geq 0}$, with $A^0 = A$ such that for any δ_p-prolongation sequence B^* and any R_p-algebra homomorphism $u : A \to B^0$ there exists a unique morphism of δ_p-prolongation sequences $u^* : A^* \to B^*$ with $u^0 = u$. We define the *p-jet spaces* $J_p^n(X)$ of X as the formal schemes $J_p^n(X) := Spf\ \hat{A}^n$. This construction immediately globalizes to the case X is not necessarily affine (such that the construction commutes, in the obvious sense, with open immersions). For X smooth over R_p the ring of δ_p-functions $X(R_p) \to R_p$ naturally identifies with the ring of global functions $\mathcal{O}(J_p^n(X))$: under this identification any function $f \in \mathcal{O}(J_p^n(X))$ gives rise to a δ_p-function $X(R_p) \to R_p$ sending any point $P \in X(R_p)$, $P : Spec\ R_p \to X$ into the R_p-point of the affine line $\mathbb{A}^1_{R_p}$ defined by

$$Spf\ R_p \stackrel{P^n}{\to} J_p^n(X) \stackrel{f}{\to} \hat{\mathbb{A}}^1_{R_p};$$

here P^n is the morphism induced from P via the universality property of the p-jet space. If X is a group scheme over R_p then

$$f : J_p^n(X) \to \hat{\mathbb{G}}_{a,R_p} = \hat{\mathbb{A}}^1_{R_p}$$

is a group homomorphism into the additive group of the line if and only if the corresponding map $X(R_p) \to R_p$ is a group homomorphism; such an f is called a *δ_p-character* of X.

As a prototypical example if $X = \mathbb{A}^N_{R_p} = Spec\ R_p[x]$ is the affine space (where x is an N-tuple of variables) then $J_p^n(X) = Spf\ R_p[x, \delta_p x, ..., \delta_p^n x]\hat{}$ (where $\delta_p x, ..., \delta_p^n x$ are new N-tuples of variables).

We will need, in this paper, a slight generalization of the above constructions as follows; cf. [5]. First note that the p-jet spaces $J_p^n(X)$ only depend on the p-adic completion of X and not on X. This immediately implies that one can introduce

p-jet spaces $J_p^n(\mathcal{X})$ attached to formal p-adic schemes $\mathcal{X}$ over R_p which are locally p-adic completions of schemes of finite type over R_p; the latter association is functorial in $\mathcal{X}$.

Similarly, for any affine R_π-scheme of finite type $Y = Spec\ A$ there exists a (unique) δ_π-prolongation sequence, $A^* = (A^n)_{n\geq 0}$, with $A^0 = A$ such that for any δ_π-prolongation sequence B^* and any R_π-algebra homomorphism $u : A \to B^0$ there exists a unique morphism of δ_π-prolongation sequences $u^* : A^* \to B^*$ with $u^0 = u$. We define the *π-jet spaces* $J_\pi^n(Y)$ of Y as the formal schemes $J_\pi^n(Y) := Spf\ \hat{A^n}$. This construction immediately globalizes to the case Y is not necessarily affine (such that the construction commutes, in the obvious sense, with open immersions). Again, for Y smooth over R_π the ring of δ_π-functions $Y(R_\pi) \to R_p$ naturally identifies with the ring of global functions $\mathcal{O}(J_\pi^n(Y))$. If Y is a group scheme over R_π then $f : J_\pi^n(Y) \to \hat{\mathbb{A}}^1_{R_\pi}$ is a group homomorphism into the additive group of the line if and only if the corresponding map $Y(R_\pi) \to R_\pi$ is a group homomorphism; such an f is called a *δ_π-character* of Y.

As a prototypical example if $Y = \mathbb{A}^N_{R_\pi} = Spec\ R_\pi[x]$ is the affine space then $J_\pi^n(Y) = Spf\ R_\pi[x, \delta_\pi x, ..., \delta_\pi^n x]\hat{}$ (where $\delta_\pi x, ..., \delta_\pi^n x$ are new N-tuples of variables).

As in the case of p-jet spaces, note that the π-jet spaces $J_\pi^n(Y)$ only depend on the π-adic completion of Y and not on Y. This immediately implies that one can introduce π-jet spaces $J_\pi^n(\mathcal{Y})$ attached to formal π-adic schemes $\mathcal{Y}$ over R_π which are locally π-adic completions of schemes of finite type over R_π; the latter association is functorial in $\mathcal{Y}$.

For any scheme X/R_p we write $X_{R_\pi} := X \otimes_{R_p} R_\pi$. Let X/R_p be a smooth affine scheme. The δ_p-prolongation sequence $(\mathcal{O}(J^n(X)))_{n\geq 0}$ induces a structure of δ_π-prolongation sequence on the sequence $(\mathcal{O}(J^n(X)) \otimes_{R_p} R_\pi)_{n\geq 0}$. By the universality property of the δ_π-prolongation sequence $(\mathcal{O}(J_\pi^n(X_{R_\pi})))_{n\geq 0}$ we get a canonical morphism of δ_π-prolongation sequences

$$\mathcal{O}(J_\pi^n(X_{R_\pi})) \to \mathcal{O}(J_p^n(X)) \otimes_{R_p} R_\pi. \tag{2.1}$$

The following is trivial to prove by induction:

Lemma 2.1. *For any $n \geq 1$ there exists a polynomial $F_n \in R_\pi[t_1, ..., t_n]$ without constant term with the property that for any $f \in \mathcal{O}(X)$ we have*

$$\delta_\pi^n f \mapsto \frac{p^n}{\pi^n} \delta_p^n f + \pi^{\max\{e-n,0\}} F_n(\delta_p f, ..., \delta_p^{n-1} f) \tag{2.2}$$

under the map (2.1).

In particular, for instance,

$$\begin{array}{rcl} \delta_\pi f & \mapsto & \frac{p}{\pi}\delta_p f \\ \delta_\pi^2 f & \mapsto & \frac{p^2}{\pi^2}\delta_p^2 f + \left(\frac{p}{\pi^2} - \frac{p^p}{\pi^{p+1}}\right)(\delta_p f)^p. \end{array} \tag{2.3}$$

Note that for $1 \leq n \leq e-1$ and $f \in \mathcal{O}(X)$ the image of $\delta_\pi^n f$ in $\mathcal{O}(J_p^n(X)) \otimes_{R_p} R_\pi$ is always in the ideal generated by π. Also note that for $f \in \mathcal{O}(X)$ the image of $\delta_\pi^e f$ in $\mathcal{O}(J_p^n(X)) \otimes_{R_p} R_\pi$ is *not* always in the ideal generated by π; indeed the image of $\delta_\pi^e p$ in $\mathcal{O}(J_p^n(X)) \otimes_{R_p} R_\pi$ belongs to $R_\pi^\times$. For X not necessarily affine we get a morphism (2.1) and a canonical morphism of π-adic formal schemes

$$J_p^n(X) \otimes_{R_p} R_\pi \to J_\pi^n(X_{R_\pi}). \tag{2.4}$$

Note that the map (2.1) is an isomorphism if $n = 0$. For $n \geq 1$ the map (2.1) is not surjective and its reduction mod p is not injective. Nevertheless, we have:

Proposition 2.2. *The map (2.1) is injective.*

We will usually view the map (2.1) as an inclusion.

Proof. Indeed it is enough to prove this for X affine and sufficiently small. So let us assume that X has étale coordinates i.e. there is an étale map $R[x] \to \mathcal{O}(X)$ with x a tuple of variables. Then by the local product property of π-jet spaces [3], Proposition 1.4, (2.1) becomes the natural map

$$\mathcal{O}(X_{R_\pi})[\delta_\pi x, ..., \delta_\pi^n x]\hat{} \to \mathcal{O}(X)[\delta_p x, ..., \delta_p^n x]\hat{} \otimes_{R_p} R_\pi. \tag{2.5}$$

Now let L be the fraction field of the π-adic completion of $\mathcal{O}(X_{R_\pi})$. (The latter is an integral domain by the smoothness of X/R_p.) Then the left hand side of (2.5) embeds into $L[[\delta_\pi x, ..., \delta_\pi^n x]]$ while the right hand side of (2.5) embeds into $L[[\delta_p x, ..., \delta_p^n x]]$ (the latter because R_π is finite over R_p). Finally we claim that we have a natural isomorphism

$$L[[\delta_\pi x, ..., \delta_\pi^n x]] \simeq L[[\delta_p x, ..., \delta_p^n x]] \tag{2.6}$$

that induces (2.1); this of course will end the proof that (2.1) is injective. To prove the claim note that there is natural homomorphism

$$L[\delta_\pi x, ..., \delta_\pi^n x] \to L[\delta_p x, ..., \delta_p^n x] \tag{2.7}$$

which is trivially seen by induction to be surjective by the formulae (2.2). Since the rings in (2.7) have both dimension n it follows that (2.7) is an isomorphism. Since (2.7) maps the ideal $(\delta_\pi x, ..., \delta_\pi^n x)$ into (and hence onto) the ideal $(\delta_p x, ..., \delta_p^n x)$ we get an isomorphism like in (2.6) and we are done. $\square$

Let now $Tr : R_\pi \to R_p$ be the (R_p-linear) trace map. We may consider the R_p-linear map

$$1 \otimes Tr : \mathcal{O}(J_p^n(X)) \otimes_{R_p} R_\pi \to \mathcal{O}(J_p^n(X)) \otimes_{R_p} R_p = \mathcal{O}(J_p^n(X)). \tag{2.8}$$

Composing (2.1) with (2.8) we get an R_p-linear *arithmetic trace* map:

$$\tau_\pi : \mathcal{O}(J_\pi^n(X_{R_\pi})) \to \mathcal{O}(J_p^n(X)). \tag{2.9}$$

(Later we will encounter another type of trace maps which will be referred to as *geometric trace* maps.)

Proposition 2.3. *Let X be a smooth scheme over R_p and $f \in \mathcal{O}(J_p^n(X))$. The following conditions are equivalent:*

1) f times a power of p belongs to the image of the trace map (2.9).

2) f times a power of p belongs to the image of the inclusion map (2.1).

Proof. The fact that condition 2 implies condition 1 is trivial.

In order to check that condition 1 implies condition 2 let Σ be the Galois group of $\mathbb{Q}_p(\pi)/\mathbb{Q}_p$ (and hence also of K_π/K_p) and let us consider the action of Σ on $\mathcal{O}(J_p^r(X)) \otimes_{R_p} R_\pi$ via the action on the second factor. We will prove that Σ acts on the image of $\mathcal{O}(J_\pi^r(X_{R_\pi})) \to \mathcal{O}(J_p^r(X)) \otimes_{R_p} R_\pi$; this will of course end the proof of the Proposition. Let $\sigma \in \Sigma$ and $\frac{\sigma\pi}{\pi} =: u \in \mathbb{Z}_p[\pi]^\times$.

Claim 1. ϕ and σ commute on R_π. Indeed $\phi \circ \sigma$ and $\sigma \circ \phi$ have the same effect on R_p and on π.

Claim 2. $\phi \circ \sigma = \sigma \circ \phi$ as maps from $\mathcal{O}(J_p^i(X)) \otimes_{R_p} R_\pi$ to $\mathcal{O}(J_p^{i+1}(X)) \otimes_{R_p} R_\pi$. Indeed it is enough to check this for $X = Spec\ R_p[x]$ the affine space, x a tuple of variables. So it is enough to check that ϕ and σ commute as maps from $R_\pi[x, \delta_p x, ..., \delta_p^i x]\hat{}$ to $R_\pi[x, \delta_p x, ..., \delta_p^{i+1} x]\hat{}$. This is clear because ϕ and σ commute on R_π and on each tuple $\delta_p^j x$.

Claim 3. $\sigma \circ \delta_\pi = \frac{1}{u} \cdot \delta_\pi \circ \sigma$ as maps from $\mathcal{O}(J_p^i(X)) \otimes_{R_p} R_\pi$ to $\mathcal{O}(J_p^{i+1}(X)) \otimes_{R_p} R_\pi$. This follows trivially from Claim 2.

Now to conclude it is enough to show that for any $1 \leq i \leq r$, and any $f \in \mathcal{O}(X)$ we have that $\sigma(\delta_\pi^i f)$ is obtained by evaluating a polynomial P_i with R_π-coefficients at $\delta_\pi f, ..., \delta_\pi^i f$. We proceed by induction on i. The case $i = 1$ is clear. Assume our assertion is true for i. Then

$$\begin{aligned} \sigma \delta_\pi^{i+1} f &= \sigma \delta_\pi (\delta_\pi^i f) \\ &= \frac{1}{u} \delta_\pi (\sigma(\delta_\pi^i f)) \\ &= \frac{1}{u} \delta_\pi (P_i(\delta_\pi f, ..., \delta_\pi^i f)) \end{aligned}$$

and we are done. □

Definition 2.4. A function $f \in \mathcal{O}(J_p^n(X))$ is called δ_π-*overconvergent* if it satisfies one of the equivalent conditions in Proposition 2.3.

Remark 2.5. The set of δ_π-overconvergent elements of $\mathcal{O}(J_p^n(X))$ is a subring containing all the elements of the form $\delta_p^i f$ with $i \leq n$ and $f \in \mathcal{O}(X)$. In particular if X is affine then the subring of δ_π-overconvergent elements of $\mathcal{O}(J_p^n(X))$ is p-adically dense in $\mathcal{O}(J_p^n(X))$ and it is sent into itself by any R_p-derivation $\mathcal{O}(J_p^n(X)) \to \mathcal{O}(J_p^n(X))$.

Remark 2.6. Under the identification of δ_p-functions (respectively δ_π-functions) with elements of the ring $\mathcal{O}(J_p^n(X))$ (respectively $\mathcal{O}(J_\pi^n(X_{R_\pi}))$) the definition of δ_π-overconvergence above corresponds to the definition of δ_π-overconvergence given in the Introduction.

Remark 2.7. Let us note that δ_π-overconvergence is preserved by precomposition with regular maps. Indeed, if $u : Y \to X$ is a morphism of smooth R_p-schemes and if $\lambda \cdot f$ is in the image of (2.1) for some $\lambda \in R_\pi$ and some $f \in \mathcal{O}(J_p^r(X))$ then if f is identified with the corresponding map $f : X(R_p) \to R_p$ it follows that $\lambda \cdot f \circ u$ is in the image of $\mathcal{O}(J_\pi^n(Y_{R_\pi})) \to \mathcal{O}(J_p^n(Y)) \otimes_{R_p} R_\pi$. (Here $f \circ u$ is identified with $u^* f$ where u^* is the naturally induced map $\mathcal{O}(J_p^r(X)) \to \mathcal{O}(J_p^r(Y))$.

The next Proposition shows that the trace map τ_π in (2.1), although not injective, is "as close as possible" to being so.

Proposition 2.8. *The map*

$$\mathcal{O}(J_\pi^n(X_{R_\pi})) \to \bigoplus_{i=0}^{e-1} \mathcal{O}(J_p^n(X)), \quad f \mapsto (\tau_\pi(f), \tau_\pi(\pi f), ..., \tau_\pi(\pi^{e-1} f)) \tag{2.10}$$

is injective.

Proof. Indeed if the image of f in $\mathcal{O}(J_p^n(X)) \otimes_{R_p} R_\pi$ is $\sum_{i=0}^{e-1} f_i \otimes \pi^i$ and the image of f via the map (2.10) is 0 then we get $\sum_{i=0}^{e-1} Tr(\pi^{i+j}) f_i = 0$ for all $j = 0, ..., e-1$.

Now $det(Tr(\pi^{i+j}) \neq 0$ which implies $f_0 = ... = f_{e-1} = 0$ hence, by Proposition 2.2, $f = 0$. $\square$

Example 2.9. Consider the multiplicative group $\mathbb{G}_{m,R_p} = Spec\ R_p[x, x^{-1}]$ and the standard δ_p-character

$$\psi_p \in \mathcal{O}(J^1_p(\mathbb{G}_{m,R_p})) = R_p[x, x^{-1}, \delta_p x]\hat{}$$

in [3] defined by

$$\psi_p := \text{``}\frac{1}{p} \log \left(\frac{\phi(x)}{x^p} \right)\text{''} := \sum_{n \geq 1} (-1)^{n-1} \frac{p^{n-1}}{n} \left(\frac{\delta_p x}{x^p} \right)^n.$$

Assume $v_p(\pi) \geq \frac{1}{p-1}$, e.g. $\pi = 1 - \zeta_p$. Then clearly $p\psi_p = \pi\psi_\pi$ where ψ_π is the δ_π-character $\psi_\pi \in \mathcal{O}(J^1_\pi(\mathbb{G}_{m,R_\pi}))$ defined by

$$\psi_\pi := \sum_{n \geq 1} (-1)^{n-1} \frac{\pi^{n-1}}{n} \left(\frac{\delta_\pi x}{x^p} \right)^n. \tag{2.11}$$

(which is well defined because if $v_p(\pi) \geq \frac{1}{p-1}$ then $v_p(\pi^{n-1}/n)$ is ≥ 0 and $\to \infty$ as $n \to \infty$). So ψ_p is δ_π-overconvergent. Moreover

$$\tau_\pi(\psi_\pi) = Tr\left(\frac{1}{\pi}\right) \cdot p\psi_p.$$

By the way, if $\pi = 1 - \zeta_p$ then $Tr(\frac{1}{\pi}) = \frac{p-1}{2}$.

The above global concepts and remarks have a local counterpart as follows. Let q be a variable and $\delta^i_\pi q$, $\delta^i_p q$ corresponding variables. Then exactly as above we have that the natural map

$$R_\pi((q))[\delta_\pi q, ..., \delta^n_\pi q]\hat{} \to R_\pi((q))[\delta_p q, ..., \delta^n_p q]\hat{} \tag{2.12}$$

is injective. We shall view this map as an inclusion. On the other hand there is a natural *trace map*

$$\tau_\pi : R_\pi((q))[\delta_\pi q, ..., \delta^n_\pi q]\hat{} \to R_\pi((q))[\delta_p q, ..., \delta^n_p q]\hat{} \stackrel{Tr}{\to} R_p((q))[\delta_p q, ..., \delta^n_p q]\hat{}, \tag{2.13}$$

where the first map is the inclusion (2.12) and the second map Tr is induced by the trace map $Tr : R_\pi \to R_p$ on the coefficients of the series. As in the global case we have that:

Proposition 2.10. *For a series f in $R_p((q))[\delta_p q, ..., \delta^n_p q]\hat{}$ the following conditions are equivalent:*

1) f times a power of p belongs to the image of the trace map (2.13).

2) f times a power of p belongs to the image of the inclusion map (2.12).

So as in the global case we can make the following:

Definition 2.11. A series in $R_p((q))[\delta_p q, ..., \delta^n_p q]\hat{}$ is δ_π-*overconvergent* if it satisfies one of the equivalent conditions in Proposition 2.10.

Example 2.12. Assume $v_p(\pi) \geq \frac{1}{p-1}$, e.g. $\pi = 1 - \zeta_p$. Then the series

$$\Psi_p := \text{``}\frac{1}{p} \log \left(\frac{\phi(q)}{q^p} \right)'' := \sum_{n \geq 1} (-1)^{n-1} \frac{p^{n-1}}{n} \left(\frac{\delta_p q}{q^p} \right)^n \in R_p((q))[\delta_p q]\hat{} \tag{2.14}$$

is δ_π-overconvergent. Indeed we can write $p\Psi_p = \pi\Psi_\pi$ where the series

$$\Psi_\pi := \sum_{n\geq 1}(-1)^{n-1}\frac{\pi^{n-1}}{n}\left(\frac{\delta_\pi q}{q^p}\right)^n;$$

is in $R_\pi((q))[\delta_\pi q]\hat{}$ because $v_p(\pi^{n-1}/n)$ is ≥ 0 and $\to \infty$.

2.2. **δ_π-overconvergence versus classical overconvergence.** We would like to compare the concept of δ_π-overconvergence introduced above with the classical concept of overconvergence as it was introduced in the work of Dwork, Monsky, and Washnitzer. The material in this subsection will not be used in the rest of the paper. Let us recall the classical concept of *overconvergence* of power series or, more generally the concept of *overconvergence of series with respect to a subset of variables.*

Definition 2.13. Let x and y be tuples of variables and $F \in R_p[x,y]\hat{} \subset R_p[[x,y]]$ a restricted power series, $F = \sum a_{\alpha,\beta}x^\alpha y^\beta$ (where α, β are multiindices). Then F is called *overconvergent in the variables* y if there exist positive real numbers C_1, C_2 such that for all α, β one has $v_p(a_{\alpha,\beta}) \geq C_1|\beta| - C_2$.

Here $|\beta|$ is, of course, the sum of the components of β.

Then we have the following:

Proposition 2.14. *Let $F \in \mathcal{O}(J_p^r(\mathbb{A}^d)) = R_p[x, \delta_p x, ..., \delta_p^n x]\hat{}$ where x is a d-tuple of variables. Assume $n \leq e-1$. Assume furthermore that F is δ_π-overconvergent. Then F is overconvergent in the variables $\delta_p x, ..., \delta_p^n x$.*

Proof. By hypothesis $p^\nu F$ is in the image of

$$R_\pi[x, \delta_\pi x, ..., \delta_\pi^n x]\hat{} \to R_p[x, \delta_p x, ..., \delta_p^n x]\hat{} \otimes_{R_p} R_\pi$$

for some ν. We may assume $\nu = 0$. Write

$$F(x, \delta_p x, ..., \delta_p^n x) = \sum_{\alpha_0,...,\alpha_n} a_{\alpha_0...\alpha_n} x^{\alpha_0}(\delta_\pi x)^{\alpha_1}...(\delta_\pi x)^{\alpha_n},$$

with $a_{\alpha_0...\alpha_n} \in R_\pi$. By (2.2) one can find polynomials $G_i \in R_\pi[t_1, ..., t_i]$ such that

$$\delta_\pi^i x = \pi \cdot G_i(\delta_p x, ..., \delta_p^i x), \quad 1 \leq i \leq e-1.$$

We get that

$$F(x, \delta_p x, ..., \delta_p^n x) = \sum a_{\alpha_0...\alpha_n}\pi^{\alpha_1+...+\alpha_n}x^{\alpha_0}(G_1(\delta_p x))^{\alpha_1}...(G_n(\delta_p x, ..., \delta_p^n x))^{\alpha_n}.$$

Let $D_n := \max\{deg(G_1), ..., deg(G_n)\}$. Then clearly the coefficient of the monomial

$$x^\alpha(\delta_p x)^{\beta_1}...(\delta_p^n x)^{\beta_n}$$

in F is going to be divisible by

$$\pi^{\left[\frac{\beta_1+...+\beta_n}{D_n}\right]}$$

in R_π and we are done. □

Remark 2.15. Proposition 2.14 fails if we do not asssume the order n is strictly less than the ramification index e. Here is a typical example. Let $\pi = \sqrt{p}$; so $e = 2$ and $\delta_\pi^2 x = p(\delta_p^2 x) + u(\delta_p x)^p$, $u = 1 - p^{(p-1)/2}$. Let $a_n \in R_p$, $v_p(a_n) \to \infty$, $v_p(a_n) \leq n^\epsilon$, $0 < \epsilon < 1$, let x be *one* variable, and let

$$F = F(x, \delta_p x, \delta_p^2 x) := \sum a_n(p(\delta_p^2 x) + u(\delta_p x)^p)^n \in R_p[x, \delta_p x, \delta_p^2 x]\hat{}.$$

Then (the map $R_p \to R_p$ defined by) F is δ_π-overconvergent because

$$F = \sum a_n(\delta_\pi^2 x)^n \in R_\pi[x, \delta_\pi x, \delta_\pi^2 x]\hat{}.$$

On the other hand $F(x, \delta_p x, \delta_p^2 x)$ is not overconvergent in $\delta_p x, \delta_p^2 x$. Indeed if this were the case then

$$F(0, y, 0) = \sum a_n u^n y^{np}$$

would be overconvergent in the variable y which is clearly not the case.

2.3. **p-jets and π-jets of formal groups.** In what follows we recall from [6], section 4.4, the construction of p-jets of formal groups and we also introduce the π-jet analogue of that construction.

Start with a formal group law $\mathcal{F} \in S[[T_1, T_2]]$ (in one variable T) over $S = \mathcal{O}(X)$, where X is a smooth affine R_p-scheme. For $r \geq 1$ we let $S_p^r := \mathcal{O}(J_p^r(X))$. Let $\mathbf{T}$ be the pair of variables T_1, T_2. One has a natural p-prolongation sequence

$$(S_p^r[[\mathbf{T}, \delta_p \mathbf{T}, ..., \delta_p^r \mathbf{T}]])_{r \geq 0}$$

(where $\delta_p \mathbf{T}, \delta_p^2 \mathbf{T}, ...$ are pairs of new variables). Then the $r+1$-tuple

$$\mathcal{F}, \delta_p \mathcal{F}, ..., \delta_p^r \mathcal{F}$$

defines a commutative formal group in $r+1$ variables $T, \delta_p T, ..., \delta_p^r T$. Setting $\mathbf{T} = 0$ in the above series, and forgetting about the first of them, we obtain an r-tuple of series

$$F_1 := \{\delta_p \mathcal{F}\}_{|\mathbf{T}=0}, ..., F_r := \{\delta_p^r \mathcal{F}\}_{|\mathbf{T}=0}.$$

This r-tuple belongs to $S_p^r[\delta_p \mathbf{T}, ..., \delta_p^r \mathbf{T}]\hat{}$ and defines a group

$$(\hat{\mathbb{A}}^r_{S_p^r}, [+]) \tag{2.15}$$

in the category of p-adic formal schemes over S_p^r. Now let

$$l(T) = \sum_{n \geq 1} a_n T^n \in (S \otimes \mathbb{Q})[[T]]$$

be the logarithm of $\mathcal{F}$. Recall that $na_n \in S$ for all n. Define

$$L_p^r := \frac{1}{p}\{\phi^r(l(T))\}_{|T=0} \in (S_p^r \otimes \mathbb{Q})[[\delta_p T, ..., \delta_p^r T]]. \tag{2.16}$$

Then L_p^r actually belong to $S_p^r[\delta_p T, ..., \delta_p^r T]\hat{}$ and define group homomorphisms

$$L_p^r : (\hat{\mathbb{A}}^r_{S_p^r}, [+]) \to (\hat{\mathbb{A}}^1_{S_p^r}, +) = \hat{\mathbb{G}}_{a, S_p^r}.$$

For all the facts above we refer to [6], pp. 123-125.

Now let $S_\pi^r := \mathcal{O}(J_\pi^r(X_{R_\pi})) \subset S_p^r \otimes_{R_p} R_\pi$; cf. Proposition 2.2. We have the following

Proposition 2.16.

1) For some integer $n(r) \geq 1$, $p^{n(r)} F_r$ belongs to the image of the natural homomorphism

$$S_\pi^r[\delta_\pi \mathbf{T}, ..., \delta_\pi^r \mathbf{T}]\hat{} \to (S_p^r \otimes_R R[\pi])[\delta_p \mathbf{T}, ..., \delta_p^r \mathbf{T}]\hat{}.$$

2) If $v_p(\pi) \geq \frac{1}{p-1}$, e.g. if $\pi = 1 - \zeta_p$, then $L_\pi^r := \frac{p}{\pi} L_p^r$ belongs to the image of the natural homomorphism

$$S_\pi^r[\delta_\pi T, ..., \delta_\pi^r T]\hat{} \to (S_p^r \otimes_R R[\pi])[\delta_p T, ..., \delta_p^r T]\hat{}.$$

Proof. Since $\phi^r(T) \equiv T^{p^r}$ mod π in $R_\pi[\delta_\pi T, ..., \delta_\pi^r T]$ we have $\{\phi^r(T)\}_{|T=0} \equiv 0$ mod π in the same ring. Set $G_{r,\pi} = \frac{1}{\pi}\{\phi^r(T)\}_{|T=0}$. We claim that for any $F \in S[[T]]$ with $F(0) = 0$ we have

$$p^N\{\delta_p^r F\}_{|T=0} \in S_\pi^r[\delta_\pi T, ..., \delta_\pi^r T]\hat{}$$

for some N. Indeed since some power of p times $\delta_p^r F$ is a polynomial with $\mathbb{Z}$-coefficients in $F, \phi(F), ..., \phi^r(F)$ it is enough to show that $\{\phi^r(F)\}_{|T=0}$ is a restricted power series in $\delta_\pi T, ..., \delta_\pi^r T$ for any r. But $\{\phi^r(F)\}_{|T=0}$ is a power series with S_π^r-coefficients in $\{\phi^i(T)\}_{|T=0} = \pi G_{i,\pi}$, $i \leq r$, and our claim is proved. The same argument works for T replaced by and tuple of variables; this ends the proof of assertion 1. To check assertion 2 note that

$$L_\pi^r = \sum_{n\geq 1} \phi(na_n)\frac{\pi^{n-1}}{n} G_{r,\pi}^n$$

and we are done because $na_n \in S$ and $v_p(\pi^{n-1}/n)$ is ≥ 0 and $\to \infty$. □

2.4. **Conjugate operators.** Recall from [6], Proposition 3.43, that if X/R_p is a smooth affine scheme and $\partial : \mathcal{O}(X) \to \mathcal{O}(X)$ is an R_p-derivation then for each $r \geq 1$ there is a unique R_p-derivation

$$\partial_r : \mathcal{O}(J_p^r(X)) \to \mathcal{O}(J_p^r(X)) \tag{2.17}$$

with the properties that ∂_r vanishes on $\mathcal{O}(J_p^{r-1}(X))$ and $\partial_r \circ \delta_p^r = \phi^r \circ \partial$ on $\mathcal{O}(X)$. (By uniqueness this construction extends, in its obvious sheafified version, to the case when X is not necessarily affine.) The operators ∂_r are a special case of what is called in [6] *conjugate operators* and were introduced, in a special case by Barcau in [1]. By Remark 2.5 $\partial_r : \mathcal{O}(J_p^r(X)) \to \mathcal{O}(J_p^r(X))$ sends the ring of δ_π-overconvergent elements into itself. We would like to slightly strengthen this as follows. Note that ∂_r uniquely extend to R_π-derivations $\partial_r : \mathcal{O}(J_p^r(X)) \otimes_{R_p} R_\pi \to \mathcal{O}(J_p^r(X)) \otimes_{R_p} R_\pi$. View $\mathcal{O}(J_\pi^r(X_{R_\pi}))$ as a subring of $\mathcal{O}(J_p^r(X)) \otimes_{R_p} R_\pi$. Then we have:

Proposition 2.17. $\partial_r(\mathcal{O}(J_\pi^r(X_{R_\pi})) \subset \mathcal{O}(J_\pi^r(X_{R_\pi}))$.

Proof. Since $\mathcal{O}(J_\pi^r(X_{R_\pi}))$ contains, as a dense subring, the $\mathcal{O}(J_\pi^{r-1}(X_{R_\pi}))$-algebra generated by all the elements of the form $\delta_\pi^r f$ with $f \in \mathcal{O}(X)$ it is enough to show that $\partial_r \delta_\pi^r f \in \mathcal{O}(J_\pi^r(X_{R_\pi}))$ for $f \in \mathcal{O}(X)$. Since $\delta_\pi^r f$ is a polynomial with R_π-coefficients in $\delta_p f, ..., \delta_p^r f$ it is enough to show that $\partial_r \delta_p^r f \in \mathcal{O}(J_\pi^r(X_{R_\pi}))$. But the latter equals $\phi^r \partial f$ which is obviously in $\mathcal{O}(J_\pi^r(X_{R_\pi}))$. □

3. Review of bad reduction

This section is entirely "non-differential" and represents a review of essentially well-known facts about bad reduction of modular curves. These facts will play a role later in the paper.

3.1. **Modular parameterization.** Consider the following classes of objects:

(1) Normalized newforms

$$f = \sum_{n\geq 1} a_n q^n \tag{3.1}$$

of weight 2 on $\Gamma_0(M)$ over $\mathbb{Q}$; in particular $a_1 = 1$, $a_n \in \mathbb{Z}$.

(2) Elliptic curves A over $\mathbb{Q}$ of conductor M.

Say that f in (1) and A in (2) correspond to each other if there exists a morphism

$$\Phi : X_0(M) \to A \tag{3.2}$$

over $\mathbb{Q}$ such that the pull back to $X_0(M)$ of some 1-form on A over $\mathbb{Q}$ corresponds to f and $L(A, s) = \sum a_n n^{-s}$. We have the following fundamental result:

Theorem 3.1.
i) For any f as in (1) there exists an A as in (2) which corresponds to f.
ii) For any A as in (2) there exists f as in (1) which corresponds to A.

The first part of the theorem is due to Eichler, Shimura, and Carayol; cf. [17] for an exposition of this theory and references. The second part of the Theorem is the content of the Tanyiama-Shimura conjecture proved, in its final form, in [2].

From now on we fix $M = Np$ with $(N, p) = 1$, $p \geq 5$, $N \geq 5$, and we fix f and A corresponding to each other, as in Theorem 3.1. We further assume $a_p = 1$ or, equivalently, A has split multiplicative reduction at p.

Recall from [17], p.282, that $a_{mn} = a_m a_n$ for $(m, n) = 1$, $a_{\ell^r} a_\ell = a_{\ell^{r+1}} + \ell a_{\ell^{r-1}}$ for $\ell \not| Np$, and $a_{\ell^r} = a_\ell^r$ for $\ell | Np$; in particular $a_{p^r} = 1$ for all r.

3.2. **Model of $X_1(Np)$ over $\mathbb{Z}[1/N, \zeta_p]$.** Recall that the modular curve $X_1(Np)$ over $\mathbb{C}$ has a model (still denoted by $X_1(Np)$ in what follows) over $\mathbb{Z}[1/N, \zeta_p]$ considered in [15], p. 470; this is a version of a curve introduced by Deligne and Rapoport [12] and the two curves become canonically isomorphic over $\mathbb{Z}[1/N, \zeta_N, \zeta_p]$ if ζ_N is a fixed primitive N-th root of unity. Recall some of the main properties of $X_1(Np)$. First $X_1(Np)$ is a regular scheme proper and flat of relative dimension 1 over $\mathbb{Z}[1/N, \zeta_p]$ and smooth over $\mathbb{Z}[1/Np, \zeta_p]$. Also the special fiber of $X_1(Np)$ over $\mathbb{F}_p$ is a union of two smooth projective curves I and I' crossing transversally at a finite set Σ of points. Furthermore I is isomorphic to the Igusa curve $I_1(N)$ in [15], p. 160, so I is the smooth compactification of the curve classifying triples (E, α, β) with E an elliptic curve over a scheme of characteristic p, and $\alpha : \mu_N \to E$, $\beta : \mu_p \to E$ are embeddings (of group schemes). Similarly I' is the smooth compactification of the curve classifying triples (E, α, b) with E an elliptic curve over a scheme of characteristic p, and $\alpha : \mu_N \to E$, $b : \mathbb{Z}/p\mathbb{Z} \to E$ are embeddings. Finally Σ corresponds to the supersingular locus on the corresponding curves.

3.3. **Neron model of A over R_π.** Let $\pi = 1 - \zeta_p$ and consider a fixed embedding of $\mathbb{Z}[\zeta_N, \zeta_p, 1/N]$ into R_π (hence of $\mathbb{Z}[\zeta_N, 1/N]$ into R_p.]

Let A_{R_π} be the Neron model of $A_{K_\pi} := A \otimes_{\mathbb{Q}} K_\pi$ over R_π; cf. [20], p. 319. Then the π-adic completion $(A^0_{R_\pi})\hat{}$ of the connected component $A^0_{R_\pi}$ of A_{R_π} is isomorphic to the π-adic completion $(\mathbb{G}_m)\hat{}$ of $\mathbb{G}_m = Spec\ R_\pi[x, x^{-1}]$. Indeed by [20], Theorem 5.3, p. 441, A_{K_π} is isomorphic over K_π to a Tate curve E_q/K_π with $q \in \pi R_\pi$. By [20], Corollary 9.1, p. 362, $A^0_{R_\pi}$ is the smooth locus over R_π of a projective curve defined by the minimal Weierstrass equation of A_{K_π}. Now the defining Weierstrass equation of the Tate curve ([20], p. 423) is already minimal (cf. [20], Remark 9.4.1, p. 364). The isomorphism $(A^0_{R_\pi})\hat{} \simeq (\mathbb{G}_m)\hat{}$ then follows from the formulae of the Tate parameterization [20], p. 425.

On the other hand recall that the modular curve $X_1(N)$ over $\mathbb{C}$ has a natural smooth projective model (still denoted by $X_1(N)$) over $\mathbb{Z}[1/N]$ such that

$$Y_1(N) := X_1(N) \backslash (cusps)$$

parameterizes pairs (E, α) consisting of elliptic curves E with an embedding $\alpha : \mu_N \to E$. The morphism $X_1(Np) \to X_1(N)$ over $\mathbb{C}$ induces a morphism

$$\epsilon : X_1(Np)_{R_\pi} \backslash \Sigma \to X_1(N)_{R_\pi} \backslash (ss)$$

over R_π, where (ss) is the supersingular locus in the closed fiber of $X_1(N)_{R_\pi}$. Indeed the morphism $X_1(Np) \to X_1(N) \to J_1(N)$ over $\mathbb{C}$ (where $J_1(N)$ is the Jacobian of $X_1(N)$ over $\mathbb{C}$ and $X_1(N) \to J_1(N)$ is the Abel-Jacobi map defined by ∞) induces a morphism from $X_1(Np)_{R_\pi} \backslash \Sigma$ into the Jacobian $J_1(N)_{R_\pi}$ of $X_1(N)_{R_\pi}$ (by the Neron property, because the latter Jacobian is an abelian scheme and hence is the Neron model of its generic fiber). But the image of $X_1(Np)_{R_\pi} \backslash \Sigma \to J_1(N)_{R_\pi}$ is clearly contained in the image of the Abel-Jacobi map $X_1(N)_{R_\pi} \to J_1(N)_{R_\pi}$ which gives a morphism $X_1(Np)_{R_\pi} \backslash \Sigma \to X_1(N)_{R_\pi}$ and hence the desired morphism $\epsilon : X_1(Np)_{R_\pi} \backslash \Sigma \to X_1(N)_{R_\pi} \backslash (ss)$. Let $X \subset X_1(N)_{R_p} \backslash (ss)$ be an affine open set, $X_{R_\pi} := X \otimes_{R_p} R_\pi \subset X_1(N)_{R_\pi} \backslash (ss)$ its base change to R_π, and $X_! := \epsilon^{-1}(X_{R_\pi})$. Denote by $\mathcal{X}_{R_\pi}$ the π-adic completion of X_{R_π}. Also note that the π-adic completion of $X_!$ has two connected components; let $\mathcal{X}_!$ be the component whose reduction mod π is contained in $I \backslash \Sigma$. We get a morphism $\epsilon : \mathcal{X}_! \to \mathcal{X}_{R_\pi}$.

3.4. **Igusa curve and lift to characteristic zero.** It will be useful to recall one of the possible constructions of the Igusa curve I. Let L be the line bundle on $X_1(N)_{R_p}$ such that the sections of the powers of L identify with the modular forms of various weights on $\Gamma_1(N)$; cf. [15] p. 450 where L was denoted by ω. Let $E_{p-1} \in H^0(X_1(N)_{R_p}, L^{p-1})$ be the normalized Eisenstein form of weight $p-1$ and let (ss) be the supersingular locus on $X_1(N)_{R_p}$ (i.e. the zero locus of E_{p-1}). (Recall that E_{p-1} is normalized by the condition that its Fourier expansion has constant term 1.) Take an open covering (X_i) of X such that L is trivial on each X_i and we let x_i be a basis of L on X_i. Then $E_{p-1} = \varphi_i x_i^{p-1}$ where $\varphi_i \in \mathcal{O}(X_i)$. Set $x_i = u_{ij} x_j$, $u_{ij} \in \mathcal{O}^\times(X_{ij})$, $X_{ij} = X_i \cap X_j$. Consider the R_π-scheme $X_{!!}$ obtained by gluing the schemes $X_{!!i} := Spec\ \mathcal{O}(X_{i,R_\pi})[t_i]/(t_i^{p-1} - \varphi_i)$ via $t_i = u_{ij}^{-1} t_j$ (where $X_{i,R_\pi} := X_i \otimes_{R_p} R_\pi$). Note that $t_i^{p-1} - \varphi_i$ are monic polynomials whose derivatives are invertible in $\mathcal{O}(X_{i,R_\pi})[t_i]/(t_i^{p-1} - \varphi_i)$. Denote in the discussion below by an upper bar the functor $\otimes k$. Note that the scheme $\overline{X}_{!!} = X_{!!} \otimes k$ is isomorphic to $\overline{\mathcal{X}}_! = \mathcal{X}_! \otimes k$; indeed $\overline{X}_{!!}$ is clearly birationally equivalent to I (cf. [15], pp. 460, 461) and is the integral closure of $\overline{X}$ in the fraction field of $\overline{X}_{!!}$. We claim that:

Proposition 3.2. *The isomorphism $\overline{X}_{!!} \simeq \overline{\mathcal{X}}_!$ lifts uniquely to an isomorphism $(X_{!!})\hat{} \simeq \mathcal{X}_!$.*

Proof. Indeed this follows immediately by applying the standard Lemma 3.3 below to $S := \mathcal{O}(\mathcal{X}_i)$, $\mathcal{X}_i = \widehat{X}_i$, $S_! = \mathcal{O}(\mathcal{X}_{!i})$, $\mathcal{X}_{!i} = \epsilon^{-1}(\mathcal{X}_i)$. □

Lemma 3.3. *Let $S \to S_!$ be a morphism of flat π-adically complete R_π-algebras, let $f \in S[t]$ be a monic polynomial and assume we have an isomorphism $\overline{\sigma} : \overline{S}[t]/(\overline{f}) \to \overline{S}_!$ such that df/dt is invertible in $\overline{S}[t]/(\overline{f})$. Then $\overline{\sigma}$ lifts uniquely to an isomorphism $\sigma : S[t]/(f) \to S_!$.*

Proof. The homomorphism σ exists and is unique by Hensel's Lemma; it is an isomorphism because $\overline{\sigma}$ is one and π is a non-zero divisor in both S and $S_!$. □

3.5. **Review of diamond operators.** Recall from [15] that $G := (\mathbb{Z}/p\mathbb{Z})^\times$ acts on the covering $X_1(Np) \to X_1(N)$ over $\mathbb{Z}[1/N, \zeta_p]$ via the diamond operators $\langle d\rangle_p$, $d \in G$; this action preserves the Igusa curve I and induces on I the usual diamond operators. In particular $I/G \to X_1(N)_{\mathbb{F}_p}$ is an isomorphism. So G acts on the covering $\epsilon : \mathcal{X}_! \to \mathcal{X}_{R_\pi}$ and hence on the isomorphic covering $(\mathcal{X}_{!!})\hat{} \to \mathcal{X}_{R_\pi}$; cf. Proposition 3.2. It is easy to explicitly find the latter action. Indeed any G-action on a covering $(X_{!!i})\hat{} \to X_{i,R_\pi}$ must have the form

$$d \cdot t_i = \zeta_{p-1}^{\chi(d)} t_i, \quad d \in G, \tag{3.3}$$

for some homomorphism $\chi : G \to \mathbb{Z}/(p-1)\mathbb{Z}$, where ζ_{p-1} is a primitive root of unity of order $p-1$. Now we claim that χ must be an isomorphism. Indeed if χ was not surjective then the G-action on the Igusa curve I would be such that $I/G \to X_1(N)_{\mathbb{F}_p}$ has degree > 1, a contradiction.

3.6. **Classical and p-adic modular forms.** We end by reviewing some more terminology and facts, to be used later, about classical modular forms and their relation with the p-adic modular forms of Serre and Katz. Let M be any positive integer. (In applications we write $M = Np^\nu$, $(N,p) = 1$.) In what follows a *classical modular form* over a ring B, of weight κ, on $\Gamma_1(M)$ will be understood in the sense of [12, 16, 15] as a rule that attaches to any B-algebra C and any triple consisting of an elliptic curve E/C, an embedding $\mu_{M,C} \to E[M]$, and an invertible one form on E an element of C satisfying the usual compatibility rules and the usual holomorphy condition for the Fourier expansion (evaluation on the Tate curve). We denote by

$$M(B,\kappa,M) = M(B,\kappa,\Gamma_1(M))$$

the B-module of all these forms. We denote by

$$M(B,\kappa,\Gamma_0(M))$$

the submodule of those forms which are invariant under the usual diamond operators. In particular any newform as in (3.1) is an element of $M(\mathbb{Z}, 2, \Gamma_0(Np))$; cf. [13], p.113. Also by [14], p. 21, the spaces $M(R_p, \kappa, Np^\nu)$ embed into Katz's ring of generalized p-adic modular forms $\mathbb{W} = \mathbb{W}(N, R_p)$ parameterizing trivialized elliptic curves E over p-adically complete R_p-algebras, equipped with an embedding $\mu_N \subset E[N]$; if $f \in M(R_p, \kappa, Np^\nu)$ then f, as an element of $\mathbb{W}$, satisfies $\lambda \cdot f = \lambda^\kappa f$ for $\lambda \in \mathbb{Z}_p^\times$, $\lambda \equiv 1 \bmod p^\nu$. (Here $\lambda \cdot f$ denotes the action of $\mathbb{Z}_p^\times$ on $\mathbb{W}$ induced by changing the trivialization.) If f is actually in $M(R_p, \kappa, \Gamma_0(Np^\nu))$ then $\lambda \cdot f = \lambda^\kappa f$ for $\lambda \in \mathbb{Z}_p^\times$. In particular any newform f as in (3.1) on $\Gamma_0(Np)$ defines an element (still denoted by f) of $\mathbb{W}$ such that $\lambda \cdot f = \lambda^2 f$, $\lambda \in \mathbb{Z}_p^\times$. By [14], p.21, any f as in (3.1) on $\Gamma_0(Np)$ is a p-adic modular form of weight 2 over R_p *in the sense of Serre*, i.e. it is a p-adic limit in $\mathbb{W}$ (or equivalently in $R_p[[q]]$) of classical modular forms over R_p of weight $\kappa_n \in \mathbb{Z}$ on $\Gamma_1(N)$ with $\kappa_n \equiv 2 \bmod p^n(p-1)$. So if $f = \sum a_n q^n$ is as in (3.1) on $\Gamma_0(Np)$ then, by [14], p. 36, $\sum a_n q^{np}$ is also a p-adic modular form of weight 2 in the sense of Serre. In particular the reduction mod p of $\sum a_n q^{np}$ is the expansion of a modular form over k on $\Gamma_1(N)$ of weight $\equiv 2 \bmod p-1$. Finally recall from [15] that the Serre operator $\theta := q\frac{d}{dq} : k[[q]] \to k[[q]]$ increases weights of classical modular forms over k by $p+1$. We conclude that the image in $k[[q]]$ of

$$\sum_{(n,p)=1} \frac{a_n}{n} q^n \in R_p[[q]]$$

is the expansion of a modular form over k on $\Gamma_1(N)$ of weight $\equiv 0$ mod $p-1$.

We end by recalling a few basic facts about Hecke operators. Throughout the discussion below the divisors of a given non-zero integer are always taken to be positive, the greatest common divisor of two non-zero integers m, n is denoted by (m, n), and we use the convention $(m, n) = n$ for $m = 0$, $n \neq 0$. Fix again a positive integer M and let $\epsilon_M : \mathbb{Z}_{>0} \to \{0, 1\}$ be the "trivial primitive character" mod M defined by $\epsilon_M(A) = 1$ if $(A, M) = 1$ and $\epsilon_M(A) = 0$ otherwise.

For each integers $n \geq 1$, $\kappa \geq 2$ and any ring C define the operator $T_{\kappa,M}(n) : C[[q]] \to C[[q]]$ by the formula

$$T_{\kappa,M}(n)f = \sum_{m\geq 0} \left(\sum_{A|(n,m)} \epsilon_M(A)A^{\kappa-1}a_{\frac{mn}{A^2}} \right) q^m.$$

Recall (cf., say, [17]) that if $f = \sum_{m\geq 0} a_m q^m \in \mathbb{C}[[q]]$ is the Fourier expansion of a form in $M(\mathbb{C}, \kappa, \Gamma_0(M))$, $\kappa \geq 2$, then the series $T_{\kappa,M}(n)f$ is the Fourier expansion of the corresponding Hecke operator on f. Note that if $M = Np^\nu$, $(N, p) = 1$, $(n, p) = 1$ then $T_{\kappa,N} = T_{\kappa,M}$ as operators on $C[[q]]$. Now if f is as in (3.1) then $T_{2,Np}(n)f = a_n f$ for all $n \geq 1$; so, for $(n, p) = 1$ we have $T_{2,N}(n)f = a_n f$. On the other hand, going back to an arbitrary $f = \sum a_m q^m \in C[[q]]$, we have

$$T_{\kappa,N}(p)f = \sum_m a_{mp}q^m + p^{\kappa-1}\sum_m a_m q^{pm},$$

$$T_{\kappa,Np}(p)f = \sum_m a_{mp}q^m.$$

So $T_{\kappa,N}(p) \equiv T_{\kappa,Np}(p)$ mod p as operators on $C[[q]]$. Specializing again to $f \in \mathbb{Z}[[q]]$ as in (3.1) on $\Gamma_0(Np)$ we have $T_{2,Np}(p)f = a_p f = f$ so we get $T_{2,N}(p)f \equiv f$ mod p in $\mathbb{Z}[[q]]$.

4. δ_p-modular forms arising from bad reduction

In this section we return to "differential matters". We will use bad reduction of the modular curve $X_1(Np)$ at p to construct certain δ_π-functions on this curve. These functions will then induce (via a *geometric trace* construction) certain new interesting δ_π-*modular forms* on the modular curve $X_1(N)$. By further applying the *arithmetic trace* from R_π down to R_p we will obtain certain new δ_p-modular forms on $X_1(N)$. We will then analyze the δ_π-*Fourier expansions* (respectively δ_p-Fourier expansions) of these forms. On our way of doing this we will review the concepts of δ_p-*modular form* and δ_p-*Fourier expansion* following [5, 6].

4.1. δ_p-modular forms and δ_π-modular forms. Let L be the line bundle on $X_1(N)_{R_p}$ such that the spaces of sections $H^0(X_1(N)_{R_p}, L^{\otimes\kappa})$ identify with the spaces $M(R_p, \kappa, N)$ of classical modular forms over R_p of weight κ on $\Gamma_1(N)$; cf. [15] p. 450 where L was denoted by ω.

Let $X \subset X_1(N)_R$ an affine open subset. (In [6, 10, 9] we always assumed that X is disjoint from the cusps; we will not assume this here because we find it convenient to cover a slightly more general case.) The restriction of L to X will still be denoted by L. Consider the X-scheme

$$V := Spec\left(\bigoplus_{n\in\mathbb{Z}} L^{\otimes n}\right). \tag{4.1}$$

By a δ_p-*modular function of order* r *on* X [10] we understand an element of the ring $M_p^r := \mathcal{O}(J_p^r(V))$. If we set, as usual, $V_{R_\pi} := V \otimes_{R_p} R_\pi$ then by a δ_π-*modular function of order* r *on* X we will understand an element of $M_\pi^r := \mathcal{O}(J_\pi^r(V_{R_\pi}))$. The formation of these rings is functorial in X. Also if L is trivial on X with basis x then M_p^r identifies with $\mathcal{O}(J_p^r(X))[x, x^{-1}, \delta_p x, ..., \delta_p^r x]\hat{}$ and M_π^r identifies with $\mathcal{O}(J_\pi^r(X))[x, x^{-1}, \delta_\pi x, ..., \delta_\pi^r x]\hat{}$. Recall the ring $W := \mathbb{Z}[\phi]$ of polynomials in ϕ; it will play in what follows the role of ring of *weights*. By a δ_p-*modular form of order* r *and weight* $w \in W$ *on* X we mean a δ_p-modular function $f \in M_p^r$ such that for each i, $f \in \mathcal{O}(J_p^r(X_i)) \cdot x_i^w$; cf. [10]. We denote by $M_p^r(w)$ the R_p-module of δ_p-modular forms of order r and weight w on X. For $w = 0$ we set $S_p^r = M_p^r(0) = \mathcal{O}(J_p^r(X))$. By a δ_π-*modular form of order* r *and weight* w *on* X we will mean a δ_π-modular function $f \in M_\pi^r$ such that for each i, $f \in \mathcal{O}(J_\pi^r(X_{i,R_\pi})) \cdot x_i^w$. We denote by $M_\pi^r(w)$ the R_π-module of δ_π-modular forms of order r and weight w on X. For $w = 0$ we set $S_\pi^r = M_\pi^r(0) = \mathcal{O}(J_\pi^r(X_{R_\pi}))$. In view of (2.1) and (2.9) we have natural R_π-algebra homomorphisms

$$M_\pi^r \to M_p^r \otimes_{R_p} R_\pi \tag{4.2}$$

preserving weights, i.e. inducing R_π-linear maps

$$M_\pi^r(w) \to M_p^r(w) \otimes_{R_p} R_\pi, \quad w \in W.$$

Also we have R_p-linear trace maps

$$\tau_\pi : M_\pi^r \to M_p^r \tag{4.3}$$

that preserve weights i.e. induce maps

$$\tau_\pi : M_\pi^r(w) \to M_p^r(w), \quad w \in W. \tag{4.4}$$

In particular we have R_π-algebra homomorphisms

$$S_\pi^r \to S_p^r \otimes_{R_p} R_\pi$$

and R-linear trace maps

$$\tau_\pi : S_\pi^r \to S_p^r.$$

When applied to the scheme V, Definition 2.4 translates into the following:

Definition 4.1. A δ_p-modular function $f \in M_p^r$ is called δ_π-*overconvergent* if one of the following equivalent conditions is satisfied:
1) f times a power of p belongs to the image of the map (4.2);
2) f times a power of p belongs to the image of the map (4.3).

4.2. **δ_π-modular forms from δ_π-functions on $\mathcal{X}_!$.** Let $X \subset X_1(N)_{R_p}$ be disjoint from the supersingular locus (ss) (but necessarily from ($cusps$) !). There is a canonical way of constructing δ_π-modular forms of weights $0, -1, ..., -p+2$ on X from δ_π-functions on $\mathcal{X}_!$. Indeed we will construct natural *geometric trace* maps

$$\tau_\kappa : \mathcal{O}(J_\pi^r(\mathcal{X}_!)) \to M_\pi^r(-\kappa), \quad \kappa = 0, ..., p-2, \tag{4.5}$$

as follows. The isomorphism $(X_{!!})\hat{} \simeq \mathcal{X}_!$ in Proposition 3.2 induces an isomorphism $J_\pi^r(\mathcal{X}_!) \simeq J_\pi^r(X_{!!})$. Since $X_{!!i} := Spec\ \mathcal{O}(X_{i,R_\pi})[t_i]/(t_i^{p-1} - \varphi_i)$ is étale over X_{i,R_π} and since the formation of π-jet spaces commutes with étale maps it follows that we have an identification

$$\mathcal{O}(J_\pi^r(X_{!!i})) = \mathcal{O}(J_\pi^r(X_{i,R_\pi}))[t_i]/(t_i^{p-1} - \varphi_i). \tag{4.6}$$

Let us denote the class of t_i in the latter ring again by t_i and let the image of $\alpha \in \mathcal{O}(J^r_\pi(\mathcal{X}_!)) \simeq \mathcal{O}(J^r_\pi(X_{!!}))$ in $\mathcal{O}(J^r_\pi(X_{!!i}))$ be $\sum_{\kappa=0}^{p-2} \alpha_{\kappa,i} t_i^\kappa$, $\alpha_{\kappa,i} \in \mathcal{O}(J^r_\pi(X_{i,R_\pi}))$. Then define

$$\tau_{\kappa,i}\alpha := \alpha_{\kappa,i} x_i^{-\kappa} \in \mathcal{O}(J^r_\pi(X_{i,R_\pi})) \cdot x_i^{-\kappa}.$$

Note that from the equalities

$$\sum_{\kappa=0}^{p-2} \alpha_{\kappa,j} t_j^\kappa = \sum_{\kappa=0}^{p-2} \alpha_{\kappa,i} t_i^\kappa = \sum_{\kappa=0}^{p-2} \alpha_{\kappa,i} u_{ij}^{-\kappa} t_j^\kappa$$

it follows that $\alpha_{\kappa,i} = u_{ij}^\kappa \alpha_{\kappa,j}$ hence $\tau_{\kappa,i}\alpha = \tau_{\kappa,j}\alpha$ for all i and j. So the latter give rise to well defined elements $\tau_\kappa\alpha \in M^r_\pi(-\kappa)$ which ends the construction of the map (4.5).

Proposition 4.2. *The map*

$$\mathcal{O}(J^r_\pi(\mathcal{X}_!)) \to \bigoplus_{\kappa=0}^{p-2} M^r_\pi(-\kappa), \quad \alpha \mapsto (\tau_0\alpha, ..., \tau_{p-2}\alpha) \tag{4.7}$$

is an isomorphism.

Proof. Injectivity is clear from construction. Surjectivity immediately follows by reversing the construction of the trace maps above. $\square$.

On the other hand it will be useful to have a criterion saying when a δ_π-function on $\mathcal{X}_!$ "comes from" a δ_π-modular form on X of weight 0, i.e. from a δ_π-function on X. Indeed recall the $G = (\mathbb{Z}/p\mathbb{Z})^\times$-action on $\mathcal{X}_!$ induced by the diamond operators. This action induces a G-action on $\mathcal{O}(J^r_\pi(\mathcal{X}_!))$ for all $r \geq 1$. Then we have:

Proposition 4.3. *The ring $\mathcal{O}(J^r_\pi(\mathcal{X}_!))^G$ of G-invariant elements of $\mathcal{O}(J^r_\pi(\mathcal{X}_!))$ equals $\mathcal{O}(J^r_\pi(X_{R_\pi}))$.*

Proof. This follows immediately from the identification (4.6) and the fact that G acts on t_i by the formula (3.3) where χ is an isomorphism. $\square$

4.3. **δ_π-character composed with modular parameterization.** We assume, unless otherwise specified, that $\pi = 1 - \zeta_p$ and we fix, as usual an embedding $\mathbb{Z}[1/N, \zeta_N, \zeta_p] \to R_\pi$. Also recall our fixed elliptic curve A with modular parametrization (3.2) and the modular form f in (3.1). We continue to consider $X \subset X_1(N)_{R_p}$ an affine open set disjoint from (ss). We shall freely use the notation in our section on bad reduction. By the Néron property [20], p. 319, we get a morphism $\Phi : X_! \to A_{R_\pi}$ over R_π. We get an induced morphism from $\mathcal{X}_!$ into the connected component $(A^0_{R_\pi})\hat{} \simeq (\mathbb{G}_m)\hat{}$. This morphism $\Phi^0 : \mathcal{X}_! \to (\mathbb{G}_m)\hat{}$ induces a morphism $\Phi^1 : J^1_\pi(\mathcal{X}_!) \to J^1_\pi(\mathbb{G}_m)$. Now take the standard δ_π-character $\psi_\pi \in \mathcal{O}(J^1_\pi(\mathbb{G}_m))$, cf. (2.11), identified with a morphism $\psi_\pi : J^1_\pi(\mathbb{G}_m) \to (\mathbb{A}^1_{R_\pi})\hat{}$. By composition we get an induced morphism $f^\sharp := \psi_\pi \circ \Phi^1 : J^1_\pi(\mathcal{X}_!) \to (\mathbb{A}^1_{R_\pi})\hat{}$. This morphism can be identified with an element

$$f^\sharp_\pi \in \mathcal{O}(J^1_\pi(\mathcal{X}_!)). \tag{4.8}$$

(Here f in $f^\sharp_\pi$ refers to the newform $f = \sum a_n q^n$ (3.1).) Now, since f is a form on $\Gamma_0(Np)$ it follows that $\Phi : X_! \to A_{R_\pi}$ is invariant under the diamond operators $\langle d \rangle_p$, $d \in G$. This implies that $f^\sharp_\pi$ is G-invariant. By Proposition 4.3 it follows that

$$f^\sharp_\pi \in \mathcal{O}(J^1_\pi(X_{R_\pi})) = M^1_\pi(0) = S^1_\pi \tag{4.9}$$

i.e. $f_\pi^\sharp$ is a δ_π-modular form of weight 0. Consequently its image via the corresponding map (4.4) defines a δ_p-modular form of weight 0,

$$\tau_\pi f_\pi^\sharp \in M_p^1(0) = S_p^1. \tag{4.10}$$

4.4. **δ_p-Fourier expansions and δ_π-Fourier expansions.** The R_p-point ∞ on $X_1(N)_{R_p}$ induces δ_π-*Fourier expansion maps*

$$E : \mathcal{O}(J_\pi^r(\mathcal{X}_!)) \to R_\pi((q))[\delta_\pi q, ..., \delta_\pi^r q]\hat{}.$$

Indeed to construct such a map we may assume X contains ∞; but in this case the map arises because $X_{!!} \to X_{R_\pi}$ is étale so the inverse image of ∞ by this map is a disjoint union of R_π-points.

On the other hand there are δ_π-*Fourier expansion maps*

$$E : M_\pi^r \to R_\pi((q))[\delta_\pi q, ..., \delta_\pi^r q]\hat{}. \tag{4.11}$$

compatible, in the obvious sense, with the previous ones and with the δ_p-Fourier expansion maps in [6, 10]

$$E : M_p^r \to R_p((q))[\delta_p q, ..., \delta_p^r q]\hat{}. \tag{4.12}$$

We recall [6] the δ_p-Fourier *expansion principle* according to which for any w the map

$$E : M_p^r(w) \to R_p((q))[\delta_p q, ..., \delta_p^r q]\hat{}$$

is injective and has a torsion free cokernel.

Remark 4.4. The maps (4.11) and (4.12) commute with the trace maps $\tau_\pi : M_\pi^r \to M_p^r$ and $\tau_\pi : R_\pi((q))[\delta_\pi q, ..., \delta_\pi^r q]\hat{} \to R_p((q))[\delta_p q, ..., \delta_p^r q]\hat{}$, in the sense that $E \circ \tau_\pi = \tau_\pi \circ E$.

Remark 4.5. Clearly if $f \in M_p^r$ of δ_π-overconvergent then its δ_p-Fourier expansion $E(f)$ is δ_π-overconvergent. Later we will prove the δ_π-overconvergence of a number of remarkable δ_p-modular functions. By the present remark we will also get that their δ_p-Fourier expansions are δ_π-overconvergent. However the δ_π-overconvergence of these expansions can also be proved directly.

The next Proposition establishes a link between the δ_π-Fourier expansions of δ_π-functions on $\mathcal{X}_!$ and δ_π-Fourier expansions of their geometric traces. Recall the series $E_{p-1}(q) := E(E_{p-1}) \in R_p[[q]]$ and the fact that $E_{p-1}(q) \equiv 1 \bmod p$ in $R_p[[q]]$ [16]. So the series $E_{p-1}(q)$ has a unique $(p-1)$-root $\epsilon(q) \in R_p[[q]]$ such that $\epsilon(q) \equiv 1$ mod p in $R_p[[q]]$.

Proposition 4.6. *If $\alpha \in \mathcal{O}(J_\pi^1(\mathcal{X}_!))$ then its δ_π-Fourier expansion is given by*

$$E(\alpha) = \sum_{\kappa=0}^{p-2} E(\tau_\kappa \alpha)\epsilon(q)^\kappa.$$

Proof. Shrinking X we may assume $X = X_i$ for some i. From $E_{p-1} = \varphi_i x_i^{p-1}$ we get

$$E_{p-1}(q) = E(\varphi_i)E(x_i)^{p-1} = E(t_i)^{p-1}E(x_i)^{p-1}.$$

So $E(t_i x_i) = c \cdot \epsilon(q)$, $c \in R_p^\times$, $c^{p-1} = 1$. Now the birational isomorphism between the Igusa curve I and $\overline{X}_{!!}$ sends $t_i x_i$ into the form a in [15], p. 460-461, and the

Fourier expansion in $k[[q]]$ of the form a at ∞ is 1. It follows that $c=1$. We get

$$\begin{aligned} E(\alpha) &= \sum_{\kappa=0}^{p-2} E(\alpha_{\kappa,i})E(t_i)^{\kappa} \\ &= \sum_{\kappa=0}^{p-2} E(\alpha_{\kappa,i})E(x_i)^{-\kappa}E(x_i)^{\kappa}E(t_i)^{\kappa} \\ &= \sum_{\kappa=0}^{p-2} E(\tau_{\kappa}\alpha)\epsilon(q)^{\kappa}. \end{aligned}$$

□

Proposition 4.7. *If $f^{\sharp}_{\pi} \in \mathcal{O}(J^1_{\pi}(\mathcal{X}_!))$ is attached to $f=\sum a_n q^n$ an in (4.9) then its δ_{π}-Fourier expansion $E(f^{\sharp}_{\pi}) \in R_{\pi}[[q]][\delta_{\pi}q]\hat{}$ has the form:*

$$\begin{aligned} E(f^{\sharp}_{\pi}) &= \frac{1}{\pi}(\phi-p)\sum_{n\geq 1}\frac{a_n}{n}q^n \\ &= \frac{1}{\pi}\left[\left(\sum_{n\geq 1}\frac{a_n}{n}(q^p+\pi\delta_{\pi}q)^n\right)-p\left(\sum_{n\geq 1}\frac{a_n}{n}q^n\right)\right] \\ &= \frac{1}{\pi}\left[\left(\sum_{n\geq 1}\frac{a_n}{n}(q^p+p\delta_{p}q)^n\right)-p\left(\sum_{n\geq 1}\frac{a_n}{n}q^n\right)\right]. \end{aligned} \tag{4.13}$$

Proof. Entirely similar to the proof of Theorem 6.3 in [7] □

Remark 4.8. The series in the right hand side of Equation 4.13 are a priori elements of

$$K_{\pi}[[q,\delta_{\pi}q]] = K_{\pi}[[q,\delta_p q]].$$

The Lemma says in particular that these series are actually in $R_{\pi}[[q]][\delta_{\pi}q]\hat{}$. One can also check the latter directly.

Proposition 4.9. *The form $\tau_{\pi}f^{\sharp}_{\pi}$ in (4.10) satisfies the following identity in the ring $R_p[[q]][\delta_p q]\hat{}$:*

$$E(\tau_{\pi}f^{\sharp}_{\pi}) = \frac{p-1}{2}\left[\left(\sum_{n\geq 1}\frac{a_n}{n}(q^p+p\delta_p q)^n\right)-p\left(\sum_{n\geq 1}\frac{a_n}{n}q^n\right)\right].$$

Proof. This follows from Proposition 4.7 by using $Tr(\frac{1}{\pi})=\frac{p-1}{2}$. □

One can get a more explicit picture mod π (respectively mod p) as follows.

Proposition 4.10. *The form $f^{\sharp}_{\pi}$ in (4.9) satisfies the following congruence mod π in the ring $R_{\pi}[[q]][\delta_{\pi}q]\hat{}$:*

$$E(f^{\sharp}_{\pi}) \equiv \left(\sum_{n\geq 1} a_n q^{np}\right)\cdot\frac{\delta_{\pi}q}{q^p} - \left(\sum_{n\geq 1} a_n q^{np^2}\right)\cdot\left(\frac{\delta_{\pi}q}{q^p}\right)^p.$$

Proof. Using Proposition 4.7 and the fact that $a_{mn}=a_m a_n$ for $(m,n)=1$ and $a_{p^i}=1$ for all i [17], p.282, one gets immediately that

$$E(f^{\sharp}_{\pi})_{|\delta_{\pi}q=0} = -\frac{p}{\pi}\sum_{(m,p)=1}\frac{a_m}{m}q^m \equiv 0 \quad mod \ \ \pi.$$

Also the coefficient of the monomial $q^{p(n-1)}\delta_{\pi}q$ in $E(f^{\sharp}_{\pi})$ equals a_n. Finally fix $i\geq 2$; the coefficient of the monomial $q^{p(n-i)}(\delta_{\pi}q)^i$ in $E(f^{\sharp}_{\pi})$ equals

$$c_{i,n} := \frac{\pi^{i-1}}{i!}(n-1)(n-2)...(n-i+1)a_n \in K_{\pi}.$$

If $i<p$ clearly $v_p(c_{i,n})>0$. If $i>p$ or if $i=p$ and $(n,p)=1$ then

$$v_p((n-1)(n-2)...(n-i+1))\geq 1$$

and since

$$v_p\left(\frac{\pi^{i-1}}{i!}\right)\geq \frac{i-1}{p-1}-\frac{i}{p-1}=-\frac{1}{p-1}.$$

we get $v_p(c_{i,n})>0$. Finally, assume $i=p$ and $p|n$. Then

$$c_{i,n}\equiv \frac{\pi^{p-1}}{p}a_n\equiv -a_n \quad mod \quad \pi$$

because

$$\begin{aligned} p &= \pi^{p-1}(1+\zeta_p)(1+\zeta_p+\zeta_p^2)...(1+\zeta_p+...+\zeta_p^{p-2}) \\ &\equiv \pi^{p-1}(p-1)! \quad mod \quad \pi \\ &\equiv -\pi^{p-1} \quad mod \quad \pi \end{aligned}$$

which easily concludes the proof because $a_{np}=a_n$. □

Proposition 4.11. *The form $\tau_\pi f_\pi^\sharp\in S_p^1$ in (4.10) belongs to pS_p^1. Moreover the form*

$$f_p^\sharp := \frac{2}{p}\tau_\pi f_\pi^\sharp\in S_p^1$$

is δ_π-overconvergent and satisfies the following congruence mod p

$$E(f_p^\sharp)\equiv\left(\sum_{(n,p)=1}\frac{a_n}{n}q^n\right)-\left(\sum_{n\geq 1}a_nq^{np}\right)\frac{\delta_p q}{q^p}$$

in the ring $R_p[[q]][\delta_p q]\hat{}$.

Proof. By Proposition 4.9, one gets

$$E(\tau_\pi f_\pi^\sharp)_{|\delta_p q=0}=-\frac{p(p-1)}{2}\sum_{(n,p)=1}\frac{a_n}{n}q^n.$$

The coefficient of $q^{p(n-1)}\delta_p q$ in $E(\tau_\pi f_\pi^\sharp)$ equals pa_n. Also, for $i\geq 2$, the coefficient of $q^{p(n-i)}(\delta_p q)^i$ in $E(\tau_\pi f_\pi^\sharp)$ equals

$$\frac{p-1}{2}\frac{p^i}{i!}a_n(n-1)(n-2)...(n-i+1).$$

In particular $E(\tau_\pi f_\pi^\sharp)$ is divisible by p in the ring $R[[q]][\delta_p q]\hat{}$. By the δ_p-Fourier expansion principle it follows that $\tau_\pi f_\pi^\sharp$ is divisible by p in S_p^1 which proves the first assertion of the Proposition. δ_π-overconvergence follows from Proposition 2.3. The rest of the Proposition then follows from the above coefficient computations. □

Remark 4.12. Let $\overline{f}=\sum a_m q^m\in k[[q]]$, $\overline{f^{(-1)}}:=\sum_{(n,p)=1}\frac{a_n}{n}q^n\in k[[q]]$ and let V be k-algebra endomorphism of $k[[q]]$ that sends q into q^p. Then the series in $k[[q]]$ obtained from the right hand side of the formula in Proposition 4.11 by reducing mod p equals

$$\overline{g}:=\overline{f^{(-1)}}-V(\overline{f})\frac{\delta_p q}{q^p}\in k[[q]][\delta_p q].$$

This series $\overline{g}$ is *Taylor δ_p-p-symmetric* in the sense of [9]. Also, recalling from [9] the operators denoted by "pU" and "$pT_0(p)$" acting on Taylor δ_p-p-symmetric series and using the fact that $T_{2,N}(p)\overline{f} = \overline{f}$ it is a trivial exercise (using the formulae in [9]) to check that "pU"$\overline{g} = \overline{g}$ and hence

$$\text{“}pT_0(p)\text{”}\overline{g} = \overline{g} + V(\overline{f^{(-1)}}).$$

In particular note that $\overline{g}$ is *not* an eigenvector of "$pT_0(p)$". On the other hand an action of the operators $T_0(n)$ (for level N) on $k[[q]][\delta_p q]$ was introduced in [9]; using the fact that $T_{2,N}(n)\overline{f} = a_n\overline{f}$ for $(n,p) = 1$ it follows (using the formulae in [9]) that $nT_0(n)\overline{g} = a_n\overline{g}$ for $(n,p) = 1$. So $\overline{g}$ *is* an eigenvector of all operators $nT_0(n)$ with eigenvalues a_n.

5. δ_π-OVERCONVERGENCE OF SOME BASIC δ_p-MODULAR FORMS

In this section we prove the δ_π-overconvergence of some of the basic δ_p-functions of the theory in [3, 5, 1, 6, 7, 11].

5.1. **Review of the δ_p-modular forms f_p^r [6].** We start by reviewing the construction of some basic δ_p-modular forms $f_p^r = f_{p,jet}^r \in M_p^1(-1-\phi^r)$, $r \geq 1$. These were introduced in [5, 6]. (There is a "crystalline definition" of these forms introduced in [5] for $r = 1$ and [1] for $r \geq 1$ in the case of level 1, and in [6] for arbitrary level; the equivalence of these definitions follows from [6], Proposition 8.86.) Below we follow [6], p. 263. The construction is as follows. We let $X \subset X_1(N)_{R_p}$ be an affine open set disjoint from $(cusps)$. Assume first that L is trivial on X and let x be a basis of L. Consider the universal elliptic curve $E \to X$ over R_p and view x as a relative 1-form on E/X. Cover E by affine open sets U_i. Then the natural projections $J_p^r(U_i) \to \hat{U}_i\hat{\otimes}_{S_p^0}S_p^r$ possess sections

$$s_{i,p} : \hat{U}_i\hat{\otimes}_{S_p^0}S_p^r \to J_p^r(U_i).$$

Let $N_p^r := Ker(J_p^r(E) \to \hat{E} \otimes_{S_p^0} S_p^r)$; it is a group object in the category of p-adic formal schemes over S_p^r. Then the differences $s_{i,p} - s_{j,p}$ define morphisms

$$s_{i,p} - s_{j,p} : \hat{U}_{ij}\hat{\otimes}_{S_p^0}S_p^r \to N_p^r$$

where the difference is taken in the group law of $J_p^r(E)/S_p^r$. On the other hand N_p^r identifies with the group $(\hat{\mathbb{A}}_{S_p^r}^r, [+])$ in (2.15) with coordinates given by the $\delta_p T, ..., \delta_p^r T$, where T is a parameter at the origin of E chosen such that $x \equiv dT$ mod T. Let L_p^r be the series in (2.16) attached to the formal group of E with respect to the same parameter T, viewed as a homomorphism $L_p^r : N_p^r = (\hat{\mathbb{A}}_{S_p^r}^r, [+]) \to \hat{\mathbb{G}}_{a,S_p^r}$. The compositions

$$L_p^r \circ (s_{i,p} - s_{j,p}) : \hat{U}_{ij}\hat{\otimes}_{S_p^0}S_p^r \to \hat{\mathbb{G}}_{a,S_p^r}$$

define a Cech cocycle of elements

$$\varphi_{ij}^r \in \mathcal{O}(\hat{U}_{ij}\hat{\otimes}_{S_p^0}S_p^r) \tag{5.1}$$

and hence a cohomology class φ^r in $H^1(E\hat{\otimes}_{S_p^0}S_p^r, \mathcal{O}) = H^1(E \otimes_{S_p^0} S_p^r, \mathcal{O})$. The expression

$$\langle \varphi^r, x \rangle x^{-1-\phi^r}, \tag{5.2}$$

where the brackets mean Serre duality, is a well defined element of $S_p^r \cdot x^{-1-\phi^r}$.

If L is not necessarily free on X we can make the above construction locally and the various expressions (5.2) glue together to give an element

$$f_p^r = f_{p,jet}^r \in M_p^r(-1-\phi^r). \tag{5.3}$$

5.2. δ_π**-overconvergence of** f_p^r**.**

Theorem 5.1. *Assume $v_p(\pi) \geq \frac{1}{p-1}$. Then the forms*

$$\frac{p}{\pi} f_p^r \in M_p^r(-1-\phi^r) \otimes_{R_p} R_\pi$$

belong to the image of the homomorphism

$$M_\pi^r(-1-\phi^r) \to M_p^r(-1-\phi^r) \otimes_{R_p} R_\pi.$$

In particular f_p^r are δ_π-overconvergent.

Proof. The question is clearly local on X in the Zariski topology so we may assume that L is free on X with basis x and X has an étale coordinate $t : X \to \mathbb{A}^1$. We may also assume that each $U_i \to X$ factors though an étale map $t_i : U_i \to X \times \mathbb{A}^1$. Next we note that

$$\hat{U}_i \hat{\otimes}_{S_p^0} S_p^r \otimes_{R_p} R_\pi \simeq (\hat{U}_i \otimes_{R_p} R_\pi) \otimes_{S_\pi^0} (S_p^r \otimes_{R_p} R_\pi). \tag{5.4}$$

(This follows from the general fact that if S is a ring, S', C are S-algebras, A, B are C-algebras, and $A' = A \otimes_S S'$, $B' = B \otimes_S S'$, $C' = C \otimes_S S'$, then $A \otimes_C B \otimes_S S' \simeq A \otimes_C B' \simeq A \otimes_C C' \otimes_{C'} B' \simeq A' \otimes_{C'} B'$.) Consequently there is a canonical homomorphism from (5.4) to $\hat{U}_{i,R_\pi} \hat{\otimes}_{S_\pi^0} S_\pi^r$, where, as usual, $\hat{U}_{i,R_\pi} = \hat{U}_i \otimes_{R_p} R_\pi$. We claim that one can find sections $s_{i,p}$ and $s_{i,\pi}$ of the canonical projections making the following diagram commute:

$$\begin{array}{ccc} \hat{U}_i \hat{\otimes}_{S_p^0} S_p^r \otimes_{R_p} R_\pi & \stackrel{s_{i,p}}{\longrightarrow} & J_p^r(U_i) \otimes_{R_p} R_\pi \\ \downarrow & & \downarrow \\ \hat{U}_{i,R_\pi} \hat{\otimes}_{S_\pi^0} S_\pi^r & \stackrel{s_{i,\pi}}{\longrightarrow} & J_\pi^r(U_{i,R_\pi}) \end{array} \tag{5.5}$$

where the vertical morphisms are the canonical ones. Indeed consider the ring $B = \mathcal{O}(\hat{U}_{i,R_\pi})$ and the commutative diagram

$$\begin{array}{ccc} B[\delta_p t, ..., \delta_p^r t]\hat{} & \leftarrow & B[\delta_p t, ..., \delta_p^r t, \delta_p t_i, ..., \delta_p^r t_i]\hat{} \\ \uparrow & & \uparrow \\ B[\delta_\pi t, ..., \delta_\pi^r t]\hat{} & \leftarrow & B[\delta_\pi t, ..., \delta_\pi^r t, \delta_\pi t_i, ..., \delta_\pi^r t_i]\hat{} \end{array} \tag{5.6}$$

with horizontal arrows sending $\delta_p t_i, ..., \delta_p^r t_i$ and $\delta_\pi t_i, ..., \delta_\pi^r t_i$ into 0. Then the spaces in the diagram (5.5) are the formal spectra of the rings in the diagram (5.6) and we can take the horizontal arrows in the diagram (5.5) to be induced by the horizontal arrows in the diagram (5.6). The diagram (5.5) plus Proposition 2.16 then induces a commutative diagram

$$\begin{array}{ccccc} \hat{U}_{ij}\hat{\otimes}_{S_p^0}S_p^r\otimes_{R_p}R_\pi & \stackrel{s_{i,p}-s_{j,p}}{\longrightarrow} & N_p^r\otimes_{R_p}R_\pi & \stackrel{\frac{p}{\pi}L_p^r}{\longrightarrow} & \hat{\mathbb{G}}_{a,S_p^r\otimes_{R_p}R_\pi} \\ \downarrow & & \downarrow & & \downarrow \\ \hat{U}_{ij,R_\pi}\hat{\otimes}_{S_\pi^0}S_\pi^r & \stackrel{s_{i,\pi}-s_{j,\pi}}{\longrightarrow} & N_\pi^r & \stackrel{L_\pi^r}{\longrightarrow} & \hat{\mathbb{G}}_{a,S_\pi^r} \end{array} \tag{5.7}$$

where N_π^r is the kernel of the canonical projection $J_\pi^r(E_{R_\pi})\to\hat{E}_{R_\pi}\hat{\otimes}_{S_\pi^0}S_\pi^r$ and the vertical morphisms are the canonical ones. The diagram (5.7) shows that the cocycle $\frac{p}{\pi}\varphi_{ij}^r$ in (5.1) comes from a cocycle of elements in $\mathcal{O}(\hat{U}_{ij,R_\pi}\hat{\otimes}_{S_\pi^0}S_\pi^r)$. This immediately implies that the element $\frac{p}{\pi}\langle\varphi^r,x\rangle\in S_p^r\otimes_{R_p}R_\pi$ comes from an element in S_π^r and we are done. $\square$

Remark 5.2. Since $f_p^1\in M_p^1$ is δ_π-overconvergent it follows that its δ_p-Fourier expansion $E(f_p^1)\in R_p((q))[\delta_p q]\hat{}$ is also δ_π-overconvergent. But, as shown in [5], $E(f_p^1)$ equals the series Ψ_p in (2.14) and note that we knew already (cf. the remarks surounding Equation 2.14) that Ψ_p is δ_π-overconvergent. A similar remark holds for f_p^r, $r\geq 2$.

5.3. **δ_π-overconvergence of f_p^∂.** In this subsection we assume that $X\subset X_1(N)_{R_p}$ is an affine open set disjoint from (*cusps*) and (*ss*). There is a remarkable form $f_p^\partial\in M_p^1(\phi-1)$ playing a key role in the theory. This was introduced in [1] in the level 1 case; cf. [6], p. 269, for the arbitrary level case. The definition of f_p^∂ in [6], loc.cit. is crystalline but an alternative description of this form (up to a multiplicative factor in $R^\times$) can be given via [6], Proposition 8.64; here we shall follow this latter approach. Indeed one has a canonical R-derivation $\partial:\mathcal{O}(V)\to\mathcal{O}(V)$ defined by Katz [16] via the Gauss-Manin connection, generalizing the "Serre operator"; cf. [6], pp.254-255, for a review of this. (Here V is as in (4.1).) One can consider then the conjugate operator $\partial_1:M_p^1\to M_p^1=\mathcal{O}(J_p^1(V))$; cf. (2.17). One can also consider the *Ramanujan form* $P\in M_p^0(2)$; cf. [6], p. 255, for a review of this. Then one can define $f_p^\partial\in M_p^1$ by the formula

$$f_p^\partial:=\partial_1 f_p^1-p\phi(P)f_p^1\in M_p^1. \tag{5.8}$$

It turns out that actually f_p^∂ has weight $\phi-1$, i.e. $f_p^\partial\in M_p^1(\phi-1)$. (By the way, as shown in [1], f_p^∂ has δ_p-Fourier expansion $E(f_p^\partial)=1$.)

Theorem 5.1 plus Proposition 2.17 imply the following:

Theorem 5.3. *Assume $v_p(\pi)\geq\frac{1}{p-1}$. Then the element $\frac{p}{\pi}f_p^\partial\in M_p^1\otimes_{R_p}R_\pi$ belongs to the image of the map $M_\pi^1\to M_p^1\otimes_{R_p}R_\pi$. In particular f_p^∂ is δ_π-overconvergent.*

5.4. **Review of the δ_p-characters ψ_p of elliptic curves** [6]**.** We follow [6], pp. 194-197. Let A/R_p be an elliptic curve and fix a level $\Gamma_1(N)$ structure on A. (The construction below does not depend on this level structure.) If $Y_1(N)_{R_p}:=X_1(N)_{R_p}\backslash(cusps)$ we get an induced point $P_A:Spec\ R_p\to Y_1(N)_{R_p}$. Let $X\subset Y_1(N)_{R_p}$ be an affine open set "containing" the above point and such that the line bundle L on X is trivial with basis x. Let ω be the invertible 1-form on A defined by x. By the universality property of the p-jet spaces we get canonical morphisms $P_A^r:\mathcal{O}(J_p^r(X))\to R$ compatible with δ_p in the obvious sense. Then any δ_p-modular form $f\in M_p^r$ on X defines an element $f(A,\omega)\in R_p$ as follows:

we write $f = \tilde{f} \cdot x^w$ with $\tilde{f} \in \mathcal{O}(J^r_p(X))$ and one takes $f(A,\omega) \in R_p$ to be the image of $\tilde{f}$ in R_p via the above morphism P^r_A. In particular one can consider the δ_p-modular forms $f^1_p \in M^1_p(-1-\phi)$ and $f^2_p \in M^2_p(-1-\phi^2)$ in (5.3); we get elements $f^1_p(A,\omega), f^2_p(A,\omega) \in R_p$. We recall that $f^1_p(A,\omega) = 0$ if and only if A has a lift of Frobenius i.e. the p-power Frobenius of $A \otimes_{R_p} k$ lifts to a morphism of schemes $A \to A$ over $\mathbb{Z}$. Assume in what follows that A *does not have lift of Frobenius.* Then the quotient $\frac{f^2_p(A,\omega)}{f^1_p(A,\omega)}$, which is a priori an element of K_p, lies actually in R_p. On the other hand we may consider the cocycles (5.1). The images of these cocycles via the homomorphism $S^r_p = \mathcal{O}(J^r_p(X)) \to \mathcal{O}(J^r_p(X')) \stackrel{P^r_A}{\to} R_p$ yield cocycles

$$\varphi^r_{ij}(A) \in \mathcal{O}(\hat{U}_{ij,A})$$

where $U_{ij,A} = U_{ij} \cap A$. (Here we view A embedded into the universal elliptic curve E via the isomorphism $A \simeq E \times_{X,P_A} R_p$.) The cocycle

$$\varphi^2_{ij}(A) - \frac{f^2_p(A,\omega)}{f^1_p(A,\omega)}\varphi^1_{ij}(A) \in \mathcal{O}(\hat{U}_{ij,A})$$

turns out, by construction, to be a coboundary

$$\Gamma_i - \Gamma_j$$

with $\Gamma_i \in \mathcal{O}(\hat{U}_{i,A})$, $U_{i,A} = U_i \cap A$. Recall the series $L^r_p \in S^r_p[\delta_p T, ..., \delta^r_p T]\hat{}$; cf. (2.16). (Here T is an étale coordinate at the origin of E such that $x \equiv dT \bmod T$.) The images of L^r_p via $S^r_p \to R_p$ yield series $L^r_p(A) \in R_p[\delta_p T, ..., \delta^r_p T]\hat{}$. Take sections $s_{i,p} : \hat{U}_{i,A} \to J^2_p(U_{i,A})$ of the natural projections and let $N^2_{p,A}$ be the kernel of the projection $J^2_p(A) \to \hat{A}$. The maps

$$\tau_{i,p} : \hat{U}_{i,A} \hat{\times} N^2_{p,A} \to J^2_p(U_{i,A}), \tag{5.9}$$

given at the level of points by $(a,b) \mapsto s_{i,p}(a) + b$, are isomorphisms. Consider the functions

$$\begin{aligned} \psi_{i,p} &:= L^2_p(A) - \tfrac{f^2_p(A,\omega)}{f^1_p(A,\omega)} L^1_p(A) + \Gamma_i \\ &\in \mathcal{O}(\hat{U}_{i,A})[\delta_p T, \delta^2_p T]\hat{} \\ &= \mathcal{O}(\hat{U}_{i,A} \hat{\times} N^2_{p,A}). \end{aligned} \tag{5.10}$$

Then it turns out that the functions

$$\psi_{i,p} \circ \tau^{-1}_{i,p} \in \mathcal{O}(J^2_p(U_{i,A}))$$

glue together to give a function

$$\psi_p \in \mathcal{O}(J^2(A)). \tag{5.11}$$

This map turns out to be an homomorphism $J^2_p(A) \to \hat{\mathbb{G}}_a$ and was referred to in [6], Definition 7.24, as the *canonical* δ_p-character (of order 2) of A. (In loc. cit. ψ_p was denoted by ψ_{can}.)

In case A has a lift of Frobenius a different (but similar, and in fact easier) construction leads to what in cf. [6], Definition 7.24 was referred to as the *canonical δ_p-character* (of order 1) of A. We will denote it again by

$$\psi_p \in \mathcal{O}(J^1_p(A)). \tag{5.12}$$

In [6], loc.cit. this δ_p-character was again denoted by ψ_{can}.

5.5. **δ_π-overconvergence of ψ_p.** Let A/R_p be an elliptic curve and let r be 1 or 2 according as A has a lift of Frobenius or not.

Theorem 5.4. *Assume $v_p(\pi) \geq \frac{1}{p-1}$. Then the function $\frac{p}{\pi}\psi_p$ belongs to the image of the map*

$$\mathcal{O}(J^r_\pi(A_{R_\pi})) \to \mathcal{O}(J^r_p(A)) \otimes_{R_p} R_\pi.$$

In particular ψ_p is δ_π-overconvergent.

Proof. We give the proof in case $r = 2$. The proof in case $r = 1$ is similar. It is enough to show that one can choose the data in our construction such that:

1) The functions $\frac{p}{\pi}\psi_{i,p}$ (where $\psi_{i,p}$ is as in (5.10)) belong to the image of

$$\mathcal{O}(\hat{U}_{i,A} \otimes_{R_p} R_\pi)[\delta_\pi T, \delta_\pi^2 T]\hat{} \to \mathcal{O}(\hat{U}_{i,A} \otimes_{R_p} R_\pi)[\delta_p T, \delta_p^2 T]\hat{};$$

2) There are commutative diagrams

$$\begin{array}{ccc} (\hat{U}_{i,A} \otimes_{R_p} R_\pi)\hat{\times}(N^2_{p,A} \otimes_{R_p} R_\pi) & \stackrel{\tau_{i,p}}{\to} & J^2_p(U_{i,A}) \otimes_{R_p} R_\pi \\ \downarrow & & \downarrow \\ (\hat{U}_{i,A} \otimes_{R_p} R_\pi)\hat{\times}N^2_{\pi,A} & \stackrel{\tau_{i,\pi}}{\to} & J^2_\pi(U_{i,A} \otimes_{R_p} R_\pi) \end{array}$$

for isomorphisms $\tau_{i,\pi}$.

Now 1) follows from the fact that $\frac{p}{\pi}L^r_p(A) \in R_\pi[\delta_\pi T, \delta_\pi^2 T]\hat{}$ (cf. Theorem 2.16), and $\Gamma_i \in \mathcal{O}(\hat{U}_{i,A})$. On the other hand 2) follows from the fact that one can choose the sections $s_{i,p}$ together with sections $s_{i,\pi}$ as in (5.5); then one can define the isomorphisms $\tau_{i,\pi}$ using $s_{i,\pi}$ in the obvious way. This ends the proof. □

5.6. **δ_π-overconvergence of $f^\sharp_p$ for f on $\Gamma_0(N)$.** We first recall the construction of the δ_p-modular forms $f^\sharp_p$ attached to newforms on $\Gamma_0(N)$ given in [7, 11]. As usual we let $N > 4$, $(N, p) = 1$. Fix, in what follows, a normalized newform $f = \sum_{n\geq 1} a_n q^n$ of weight 2 on $\Gamma_0(N)$ over $\mathbb{Q}$ and an elliptic curve A over $\mathbb{Q}$ of conductor N such that f and A correspond to each other in the sense of Theorem 3.1; recall that this means that there exists a morphism

$$\Phi : X_0(N) \to A \tag{5.13}$$

over $\mathbb{Q}$ such that the pull back to $X_0(N)$ of some 1-form on A over $\mathbb{Q}$ corresponds to f and $L(A, s) = \sum a_n n^{-s}$. Fix an embedding $\mathbb{Z}[1/N, \zeta_N] \subset R_p$. Let A_{R_p} be the Néron model of $A \otimes_{\mathbb{Q}} K_p$ over R_p (which is an elliptic curve) and let $X_1(N)_{R_p}$ be the (smooth) "canonical" model of $X_1(N)$ over R_p which has been considered before. By the Néron model property there is an induced morphism $\Phi_p : X_1(N)_{R_p} \to A_{R_p}$. Let $X \subset X_1(N)_{R_p}$ be any affine open set. Let r be 1 or 2 according as A_{R_p} has or has not a lift of Frobenius. (Note that we always have $r = 2$ if A has no complex multiplication.) The image of the canonical δ_p-character $\psi_p \in \mathcal{O}(J^r_p(A_{R_p}))$ in (5.11) (respectively (5.12)) via the map

$$\mathcal{O}(J^r_p(A_{R_p})) \xrightarrow{\Phi_p^*} \mathcal{O}(J^r_p(X)) = S^r_p = M^r_p(0) \subset M^r_p$$

is denoted by $f^\sharp = f^\sharp_p$ and is a δ_p-modular form of weight 0; this form was introduced in [7] and played a key role in [11].

Putting together Theorem 5.4 and Remark 2.7 we get:

Theorem 5.5. *Assume $v_p(\pi) \geq \frac{1}{p-1}$. Then the function $\frac{p}{\pi} f_p^\sharp$ belongs to the image of the map*

$$\mathcal{O}(J^r_\pi(X_{R_\pi})) \to \mathcal{O}(J^r_p(X)) \otimes_{R_p} R_\pi.$$

In particular $f_p^\sharp$ is δ_π-overconvergent.

References

1. Barcau, M: Isogeny covariant differential modular forms and the space of elliptic curves up to isogeny, Compositio Math. 137, 237-273 (2003)
2. Breuil C., Conrad B., Diamond F., Taylor R.: On the modularity of elliptic curves over $\mathbb{Q}$: wild 3-adic exercises, J. Amer. Math. Soc. 14 (2001), 4, 843-939.
3. Buium, A.: Differential characters of Abelian varieties over $p-$adic fields, Invent. Math. 122, 309-340 (1995).
4. Buium, A.: Geometry of p-jets, Duke Math. J., 82, 2, (1996), 349-367.
5. Buium, A.: Differential modular forms, Crelle J., 520 (2000), 95-167.
6. Buium, A.: Arithmetic Differential Equations. Math. Surveys and Monographs 118, AMS (2005)
7. Buium A., Differential eigenforms, J. Number Theory 128 (2008), 979-1010.
8. Buium, A.: Differential characters on curves, Serge Lang Memorial Volume, to appear.
9. Buium, A., Saha, A., Hecke operators on differential modular forms mod p, preprint.
10. Buium A., Saha A., The ring of differential Fourier expansions, preprint.
11. Buium, A., Poonen, B.: Independence of points on elliptic curves arising from special points on modular and Shimura curves, II: local results, Compositio Math., 145 (2009), 566-602.
12. Deligne, P., Rappoport, M., Schemas de modules de courbes elliptiques, LNM 349, Springer 1973, pp. 143-316.
13. Diamond, F., and Im, J.: Modular forms and modular curves. In:Seminar on Fermat's Last Theorem, Conference Proceedings, Volume 17, Canadian Mathematical Society, pp. 39-134 (1995).
14. Gouvea, F., Arithmetic of p-adic modular forms, Lecture Notes in Math. 1304, Springer, 1985.
15. Gross, B. H., A tameness criterion for Galois representations associated to modular forms mod p, Duke Math. J., 61, 2, 445-517 (1990)
16. Katz, N.: $p-$adic properties of modular schemes and modular forms, LNM 350, Springer, Heidelberg (1973).
17. Knapp, A.: Elliptic Curves, Math. Notes, Princeton Univ. Press (1992)
18. Serre, J. P.: Formes modulaires et fonctions zéta $p-$adiques. In: LNM 350 (1973).
19. Silverman, J. H.: Arithmetic of Elliptic Curves. Springer, Heidelberg, New York (1985)
20. Silverman, J. H.: Advanced Topics in the Arithmetic of Elliptic Curves. Springer, Heidelberg, New York (1994)

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