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Yaël Frégier*† Marco Zambon‡

Abstract

We consider the problem of deforming simultaneously a *pair* of given structures. We show that such deformations are governed by an L_∞-algebra, which we construct explicitly. Our machinery is based on Th. Voronov's derived bracket construction.

We consider algebraic and geometric applications including the deformations of morphisms of various kinds of algebras, of coisotropic submanifolds in Poisson manifolds, and of twisted Poisson structures.

Contents

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Introduction

Deformation theory was developed in the 50's by Kuranishi-Kodaira (complex structures in [25], [26], [27] and [30]) and by Gerstenhaber (associative algebras [16]). Nijenhuis-Richardson then gave an interpretation of deformations in terms of graded Lie algebras ([36] and [37]) which was later promoted by Deligne: *deformations of a given algebraic or geometric structure* Δ *are governed by a differential graded Lie algebra (DGLA) or, more generally, by an* L_∞*-algebra.*

For example Gerstenhaber in [16] introduced a graded Lie algebra $(L, [-,-])$ such that an associative algebra structure on a vector space V is given by $\Delta \in L_1$ such that $[\Delta, \Delta] = 0$. A *deformation* of Δ is an element $\Delta + \tilde{\Delta}$ such that $\tilde{\Delta} \in L_1$ and

$$0 = [\Delta + \tilde{\Delta}, \Delta + \tilde{\Delta}] = 2[\Delta, \tilde{\Delta}] + [\tilde{\Delta}, \tilde{\Delta}] = 2(d_\Delta \tilde{\Delta} + \frac{1}{2}[\tilde{\Delta}, \tilde{\Delta}]), \tag{1}$$

i.e. the DGLA $(L, d_\Delta, [\cdot, \cdot])$ governs deformations of the associative algebra Δ.

It is usually a hard task to show that the deformations of a given structure are governed by an L_∞-algebra, and even harder to construct explicitly the L_∞-algebra. When one succeeds in doing so, as a reward one gets the cohomology theory, analogues of Massey products and a natural equivalence relation on the space of deformations. Moreover, quasi-isomorphic L_∞-algebras govern equivalent deformation problems, a result with non-trivial applications to quantization (see [28]).

In this work we consider *simultaneous* deformations of *two* (interrelated) structures. A typical example is given by the simultaneous deformations of (Δ, Φ), where Δ denotes a pair of associative algebras and Φ is an algebra morphism between them. These deformations are characterized by a cubic equation (unlike eq. (1) which is quadratic) and are therefore governed by an L_∞-algebra with non trivial l_3-term.

Our main result, Thm. 3 in §1.3, constructs explicitly L_∞-algebras governing such simultaneous deformation problems.

Outline of the content of the paper. L_∞-*algebras*, introduced by Lada and Stasheff [32], consist of collections $\{l_i\}_{i\geq 1}$ of "multi-brackets" satisfying higher Jacobi identities. They can be built out of what we call *V-data* $(L, P, \mathfrak{a}, \Delta)$ via *derived bracket constructions* due to Th. Voronov [50] [51] (see Thm. 1 and 2).

Our main contribution is to determine L_∞-algebras governing *simultaneous* deformation problems (Thm. 3), by recognizing that they arise as in Voronov's Thm. 2. These results are collected in §1. We apply them to algebraic problems in §2 and §3, to geometric problems in §5, and we believe that the range of application of our tools is much broader than the examples we have examined (see §4).

We give algebraic applications to the study of simultaneous deformations of algebras and morphisms in the following categories: Lie, L_∞, Lie bi- (see §2) and associative algebras (see §3). Another application concerns Lie subalgebras of Lie algebras.

One could instead have used operadic methods, see for example [13], but our techniques have the advantage of not assuming any knowledge of operadic machinery and of easily delivering explicit formulae. Note that the graded Lie algebras appearing in our V-data can be seen as coderivations of certain coalgebras built from Koszul duality for operads (see §4).

The main novelty, concerning applications, is in geometry. In §5 we determine L_∞-algebras governing simultaneous deformations of: coisotropic submanifolds of Poisson manifolds; Dirac structures in Courant algebroids, with twisted Poisson structures as a special case. We also describe explicitly the equivalence relation on the spaces of twisted Poisson structures.

None of these examples, to our knowledge, falls under the scope of the operadic methods, and one should have in mind that in this geometric setting, no tool such as Koszul duality gives for free the graded Lie algebra L of the corresponding V-data.

In the appendix we give a proof of the fact (Prop. 2.17) that any L_∞-algebra structure on an arbitrary vector space can be recovered from Voronov's derived bracket construction, generalizing a well-known result valid for finite dimensional vector spaces (see for instance [50, Ex. 4.1]). We also provide background material on graded and formal geometry.

Deformation quantization of symmetries. One knows from [3] that the quantization of a mechanical system (Poisson manifold) can be understood as a deformation of the algebra of smooth functions "in the direction" of the Poisson structure, the first order term of the Taylor expansion of this deformation.

Kontsevich associates in [28] to any Poisson structure such a quantization: Poisson structures and their quantizations are Maurer-Cartan elements for suitable L_∞-algebras (Schouten and Gerstenhaber algebras, respectively), so it suffices to build a L_∞-morphism between these two L_∞-algebras (formality theorem). This morphism sends Maurer-Cartan elements to Maurer-Cartan elements, i.e. associates a quantization to any Poisson structure.

One of our first motivations was to apply this approach to symmetries. The notion of symmetry of a mechanical system $(C^\infty(M), \{-,-\})$ can be understood as a Lie algebra map $(\mathfrak{g}, [-,-]) \to (C^\infty(M), \{-,-\})$. This map can be extended, in the category of Poisson algebras, to $(S\mathfrak{g}, \{-,-\})$, the Poisson algebra of polynomial functions on $\mathfrak{g}^*$. Its graph can be regarded as a coisotropic submanifold of the Poisson manifold $\mathfrak{g}^* \times M$. Therefore, in §5.1 we construct an L_∞-algebra governing simultaneous deformations of Poisson tensors and their coisotropic submanifolds. This L_∞-algebra plays the role of the Schouten algebra in presence of symmetries. It generalizes the L_∞-algebras governing deformations of coisotropic submanifolds of Poisson manifolds considered by Oh and Park [38], and Cattaneo and Felder [7], since in their settings, the Poisson structure was kept fixed.

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1 L_∞-algebras via derived brackets and Maurer-Cartan elements

The purpose of this section is to establish Thm. 3, which produces the L_∞-algebras appearing in the rest of the article. Therefore, we first give some basic material about L_∞-algebras in §1.1, then we recall in §1.2 Voronov's constructions which will be the main tools used to establish in §1.3 our Theorem 3. We conclude justifying in §1.4 why no convergence issues arise in our machinery, and discussing equivalences in §1.5.

1.1 Background on L_∞-algebras

We start defining (differential) graded Lie algebras, which are special cases of L_∞-algebras.

Definition 1.1. A *graded Lie algebra* is a $\mathbb{Z}$-graded vector space $L = \bigoplus_{n\in\mathbb{Z}} L_n$ equipped with a degree-preserving bilinear bracket $[\cdot,\cdot]\colon L \otimes L \longrightarrow L$ which satisfies

1) graded antisymmetry: $[a, b] = -(-1)^{|a||b|}[b, a]$,

2) graded Leibniz rule: $[a, [b, c]] = [[a, b], c] + (-1)^{|a||b|}[b, [a, c]]$.

Here a, b, c are homogeneous elements of L and the degree $|x|$ of an homogeneous element $x \in L_n$ is by definition n.

Definition 1.2. A *differential graded Lie algebra* (DGLA for short) is a graded Lie algebra $(L, [\cdot,\cdot])$ equipped with a homological derivation $d\colon L \to L$ of degree 1. In other words:

1) $|da| = |a| + 1$ (d of degree 1),

2) $d[a, b] = [da, b] + (-1)^{|a|}[a, db]$ (derivation),

3) $d^2 = 0$ (homological).

In order to formulate the definition of an L_∞-algebra – a notion due to Lada and Stasheff [32] – let us give two notations. Given two elements v_1, v_2 in a graded vector space V, let us define the *Koszul sign* of the transposition $\tau_{1,2}$ of these two elements by

$$\epsilon(\tau_{1,2}, v_1, v_2) := (-1)^{|v_1||v_2|}.$$

We then extend multiplicatively this definition to an arbitrary permutation using a decomposition into transpositions. We will often abuse the notation $\epsilon(\sigma, v_1, \ldots, v_n)$ by writing $\epsilon(\sigma)$, and we define $\chi(\sigma) := \epsilon(\sigma)(-1)^\sigma$.

We will also need *unshuffles*: $\sigma \in S_n$ is called an $(i, n-i)$-unshuffle if it satisfies $\sigma(1) < \cdots < \sigma(i)$ and $\sigma(i+1) < \cdots < \sigma(n)$. The set of $(i, n-i)$-unshuffles is denoted by $S_{(i,n-i)}$. Following [31, Def. 2.1], we define

Definition 1.3. An L_∞-*algebra* is a $\mathbb{Z}$-graded vector space V equipped with a collection $(k \geq 1)$ of linear maps $l_k \colon \otimes^k V \longrightarrow V$ of degree $2-k$ satisfying, for every collection of homogeneous elements $v_1, \ldots, v_n \in V$:

1) graded antisymmetry: for every $\sigma \in S_n$

$$l_k(v_{\sigma(1)}, \ldots, v_{\sigma(n)}) = \chi(\sigma) l_k(v_1, \ldots, v_n),$$

2) relations: for all $n \geq 1$

$$\sum_{\substack{i+j=n+1\\ i,j\geq 1}} (-1)^{i(j-1)} \sum_{\sigma \in S_{(i,n-i)}} \chi(\sigma) l_j(l_i(v_{\sigma(1)}, \ldots, v_{\sigma(i)}), v_{\sigma(i+1)}, \ldots, v_{\sigma(n)}) = 0.$$

In a *curved* L_∞-*algebra* one additionally allows for an element $l_0 \in V_2$, one allows i and j to be zero in the relations 2), and one adds the relation corresponding to $n = 0$.

Notice that when all l_k vanish except for $k = 2$, we obtain graded Lie algebras.

In Def. 1.3 the multibrackets are graded antisymmetric and l_k has degree $2-k$, whereas in the next definition they are graded symmetric and all of degree 1.

Definition 1.4. An $L_\infty[1]$-*algebra* is a graded vector space W equipped with a collection $(k \geq 1)$ of linear maps $m_k \colon \otimes^k W \longrightarrow W$ of degree 1 satisfying, for every collection of homogeneous elements $v_1, \ldots, v_n \in W$:

1) graded symmetry: for every $\sigma \in S_n$

$$m_k(v_{\sigma(1)}, \ldots, v_{\sigma(n)}) = \epsilon(\sigma) m_k(v_1, \ldots, v_n),$$

2) relations: for all $n \geq 1$

$$\sum_{\substack{i+j=n+1\\ i,j\geq 1}} \sum_{\sigma \in S_{(i,n-i)}} \epsilon(\sigma) m_j(m_i(v_{\sigma(1)}, \ldots, v_{\sigma(i)}), v_{\sigma(i+1)}, \ldots, v_{\sigma(n)}) = 0.$$

In a *curved* $L_\infty[1]$-*algebra* one additionally allows for an element $m_0 \in W_1$ (which can be understood as a bracket with zero arguments), one allows i and j to be zero in the relations 2), and one adds the relation corresponding to $n = 0$.

Remark 1.5. There is a bijection between L_∞-algebra structures on a graded vector space V and $L_\infty[1]$-algebra structures on $V[1]$, the graded vector space defined by $(V[1])_i := V_{i+1}$ [50, Rem. 2.1]. The multibrackets are related by applying the décalage isomorphisms

$$(\otimes^n V)[n] \cong \otimes^n (V[1]), \quad v_1 \dots v_n \mapsto v_1 \dots v_n \cdot (-1)^{(n-1)|v_1| + \dots + 2|v_{n-2}| + |v_{n-1}|}, \tag{2}$$

where $|v_i|$ denotes the degree of $v_i \in V$. The bijection extends to the curved case.

From now on, for any $v \in V$, we denote by $v[1]$ the corresponding element in $V[1]$ (which has degree $|v| - 1$). Also, we denote the multibrackets in $L_\infty[1]$-algebras by $\{\cdots\}$, we denote by $d := m_1$ the unary bracket, and in the curved case we denote $\{\emptyset\} := m_0$ (the bracket with zero arguments).

Definition 1.6. Given an $L_\infty[1]$-algebra W, a *Maurer-Cartan element* is a degree 0 element Φ satisfying the Maurer-Cartan equation

$$\sum_{n=1}^{\infty} \frac{1}{n!} \{\underbrace{\Phi, \dots, \Phi}_{n \text{ times}}\} = 0. \tag{3}$$

One denotes by $MC(W)$ the set of its Maurer-Cartan elements.

If W is a curved $L_\infty[1]$-algebra, one defines Maurer-Cartan elements by adding $m_0 \in W_1$ to the left hand side of eq. (3) (i.e. by letting the sum in (3) start at $n = 0$).

There is an issue with the above definition: the l.h.s. of eq. (3) is generally an infinite sum. In this paper we solve this issue by considering *filtered* $L_\infty[1]$-algebras (see Def. 1.16), for which the above infinite sum automatically converges.

1.2 Th. Voronov's constructions of L_∞-algebras as derived brackets

Here we introduce V-data and recall how Voronov associates $L_\infty[1]$-algebras to a V-data.

Definition 1.7. A *V-data* consists of a quadruple $(L, \mathfrak{a}, P, \Delta)$ where

- L is a graded Lie algebra (we denote its bracket by $[\cdot, \cdot]$),
- $\mathfrak{a}$ an abelian Lie subalgebra,
- $P\colon L \to \mathfrak{a}$ a projection whose kernel is a Lie subalgebra of L,
- $\Delta \in Ker(P)_1$ an element such that $[\Delta, \Delta] = 0$.

When Δ is an arbitrary element of L_1 instead of $Ker(P)_1$, we refer to $(L, \mathfrak{a}, P, \Delta)$ as a *curved V-data*.

Theorem 1 ([50, Thm. 1, Cor. 1]). *Let $(L, \mathfrak{a}, P, \Delta)$ be a curved V-data. Then $\mathfrak{a}$ is a curved $L_\infty[1]$-algebra for the multibrackets $\{\emptyset\} := P\Delta$ and ($n \geq 1$)*

$$\{a_1, \dots, a_n\} = P[\dots[[\Delta, a_1], a_2], \dots, a_n]. \tag{4}$$

We obtain a $L_\infty[1]$-algebra exactly when $\Delta \in Ker(P)$.

When $\Delta \in Ker(P)$ there is actually a larger $L_\infty[1]$-algebra, which contains $\mathfrak{a}$ as in Thm. 1 as a $L_\infty[1]$-subalgebra.

Theorem 2 ([51, Thm. 2])**.** *Let $(L, \mathfrak{a}, P, \Delta)$ be a V-data, and denote $D := [\Delta, \cdot]\colon L \to L$. Then the space $L[1] \oplus \mathfrak{a}$ is a $L_\infty[1]$-algebra for the differential*

$$d(x[1], a) := (-(Dx)[1], P(x + Da)), \tag{5}$$

the binary bracket

$$\{x[1], y[1]\} = [x, y][1](-1)^{|x|} \in L[1],$$

and for $n \geq 1$:

$$\{x[1], a_1, \ldots, a_n\} = P[\ldots[x, a_1], \ldots, a_n] \in \mathfrak{a}, \tag{6}$$

$$\{a_1, \ldots, a_n\} = P[\ldots[Da_1, a_2], \ldots, a_n] \in \mathfrak{a}. \tag{7}$$

Here $x, y \in L$ and $a_1, \ldots, a_n \in \mathfrak{a}$. Up to permutation of the entries, all the remaining multibrackets vanish.

Notation 1.8. We will denote by

$$\mathfrak{a}^P_\Delta$$

and by

$$(L[1] \oplus \mathfrak{a})^P_\Delta$$

the $L_\infty[1]$-algebras produced by Thm. 1 and 2.

Given a curved V-data, assume that $\Phi \in \mathfrak{a}_0$ is such that $e^{[\cdot,\Phi]}$ is a well-defined (see Prop. 1.18 for a sufficient condition), giving an automorphisms of $(L, [\cdot,\cdot])$. Consider

$$P_\Phi := P \circ e^{[\cdot,\Phi]}\colon L \to \mathfrak{a}. \tag{8}$$

Notice that P_Φ is a projection since $e^{[\cdot,\Phi]}|_{\mathfrak{a}} = Id_{\mathfrak{a}}$ by the abelianity of $\mathfrak{a}$.

Remark 1.9. Let $(L, \mathfrak{a}, P, \Delta)$ be a curved V-data and $\Phi \in \mathfrak{a}_0$ as above. Then Φ is a Maurer-Cartan element of $\mathfrak{a}^P_\Delta$ iff

$$P_\Phi \Delta = 0, \tag{9}$$

or equivalently $\Delta \in ker(P_\Phi)$. This follows immediately from eq. (4) and will be used repeatedly in the proof of Thm. 3.

Remark 1.10. Let L' be a graded Lie subalgebra of L preserved by D (for example $L' = Ker(P)$). Then $L'[1] \oplus \mathfrak{a}$ is stable under the multibrackets of Thm. 2. We denote by $(L'[1] \oplus \mathfrak{a})^P_\Delta$ the induced $L_\infty[1]$-structure.

Remark 1.11. Voronov's [51, Thm. 2] is actually formulated for any degree 1 derivation D of L preserving $Ker(P)$ and satisfying $D \circ D = 0$. We restrict ourselves to inner derivations for the sake of simplicity, and since all the derivations that appear in our examples are of this kind.

A "semidirect product" $L_\infty[1]$-algebra similar to the one in Thm. 2 appeared in [2] [11].

1.3 Main theorem: an analogue of the Tangent complex within Voronov's theory.

It is well known [20, Prop.4.4] that one can twist an $L_\infty[1]$-algebra $\mathfrak{g}$ by one of its Maurer-Cartan elements Φ. One obtains a new $L_\infty[1]$-algebra $\mathfrak{g}_\Phi$, sometimes called the "tangent complex at Φ". Its n-th multibracket is

$$\{\dots\}_n^\Phi = \{\dots\}_n + \{\Phi, \dots\}_{n+1} + \frac{1}{2!}\{\Phi, \Phi, \dots\}_{n+2} + \dots$$

where $\{\dots\}_j$ denotes the j-th multibracket of $\mathfrak{g}$.

A property of the tangent complex $\mathfrak{g}_\Phi$ is that its Maurer-Cartan elements are in one to one correspondence with the deformations of Φ, i.e.

$$\Phi + \tilde{\Phi} \in MC(\mathfrak{g}) \quad \Leftrightarrow \quad \tilde{\Phi} \in MC(\mathfrak{g}_\Phi).$$

Our goal in this section, Thm. 3, is to have this property for simultaneous deformations. To this aim one needs to modify the notion of tangent complex in the setting of Voronov's theory.

We first reinterpret in Proposition 1.12 the tangent complex in terms of the twisted V-data $(L, \mathfrak{a}, P_\Phi, \Delta)$ and observe that "*twisting commutes with derived brackets*". We then establish in Thm. 3 that the construction given by Theorem 2 applied to the twisted V-data gives an $L_\infty[1]$-algebra whose Maurer-Cartan elements correspond to simultaneous deformations of Φ and Δ.

Proposition 1.12. *Let $(L, \mathfrak{a}, P, \Delta)$ be a filtered V-data and let $\Phi \in MC(\mathfrak{a}^P_\Delta)$. Then*

1) $(L, \mathfrak{a}, P_\Phi, \Delta)$ is also a V-data. Moreover $(\mathfrak{a}^P_\Delta)_\Phi = \mathfrak{a}^{P_\Phi}_\Delta$, i.e. "twisting commutes with derived brackets".

2) For any $\tilde{\Phi} \in \mathfrak{a}_0$:

$$\Phi + \tilde{\Phi} \in MC(\mathfrak{a}^P_\Delta) \quad \Leftrightarrow \quad \tilde{\Phi} \in MC(\mathfrak{a}^{P_\Phi}_\Delta).$$

The assumption *filtered* is there to ensure the convergences of the infinite sums appearing, and can be neglected on a first reading. We will address convergence issues in §1.4.

Proof. 1) P_Φ is well-defined by Prop. 1.18. $Ker(P_\Phi) = e^{[\cdot,-\Phi]}(Ker(P))$ is a Lie subalgebra of L since $e^{[\cdot,-\Phi]}$ is a Lie algebra automorphism of L and $ker(P)$ is a Lie subalgebra. Further $\Delta \in ker(P_\Phi)$ by Remark 1.9. Hence $(L, \mathfrak{a}, P_\Phi, \Delta)$ is a V-data, and by Thm. 1 we obtain the $L_\infty[1]$-algebra $\mathfrak{a}^{P_\Phi}_\Delta$.

The n-th multibracket $(n \geq 1)$ of $\mathfrak{a}^{P_\Phi}_\Delta$ is given by

$$\begin{aligned} &Pe^{[\cdot,\Phi]}[[[\Delta, \cdot], \dots], \cdot] \\ =&P[[[e^{[\cdot,\Phi]}\Delta, \cdot], \dots], \cdot] \\ =&\{\dots\}_n + \{\Phi, \dots\}_{n+1} + \frac{1}{2!}\{\Phi, \Phi, \dots\}_{n+2} + \dots \end{aligned}$$

which we recognize as the expression of $\{\dots\}_n^{\Phi}$. The first equality holds because $e^{[\cdot,\Phi]}$ is an automorphism of L and $e^{[\cdot,\Phi]}|_{\mathfrak{a}} = Id_{\mathfrak{a}}$.

2) We have

$$\begin{aligned}\tilde{\Phi} \in MC(\mathfrak{a}_{\Delta}^{P_\Phi}) &\Leftrightarrow P_\Phi(e^{[\cdot,\tilde{\Phi}]})\Delta = 0\\ &\Leftrightarrow P(e^{[\cdot,\Phi+\tilde{\Phi}]})\Delta = 0\\ &\Leftrightarrow \Phi + \tilde{\Phi} \in MC(\mathfrak{a}_{\Delta}^{P}).\end{aligned}$$

Here the first and last equivalences hold by Remark 1.9. In the second equivalence we used $e^{[\cdot,\Phi+\tilde{\Phi}]} = e^{[\cdot,\Phi]}e^{[\cdot,\tilde{\Phi}]}$, which holds since $\Phi, \tilde{\Phi}$ lie in the abelian subalgebra $\mathfrak{a}$. Notice that the sums appearing in the Maurer-Cartan equations converge both for $\mathfrak{a}_{\Delta}^{P}$ and $\mathfrak{a}_{\Delta}^{P_\Phi}$, by Prop. 1.18. □

Thanks to this result, one can see the classical tangent complex of $\mathfrak{a}_{\Delta}^{P}$ at Φ as the first derived bracket construction (Thm. 1) applied to the twisted V-data $(L, \mathfrak{a}, P_\Phi, \Delta)$. This suggests to consider, as a replacement of the notion of tangent complex at Φ, the result of the second derived bracket construction (Thm. 2) applied to the twisted V-data $(L, \mathfrak{a}, P_\Phi, \Delta)$. The main result of this paper is:

Theorem 3. *Let $(L, \mathfrak{a}, P, \Delta)$ be a filtered V-data and let $\Phi \in MC(\mathfrak{a}_{\Delta}^{P})$. Then for all $\tilde{\Delta} \in L_1$ and $\tilde{\Phi} \in \mathfrak{a}_0$:*

$$\begin{cases}[\Delta+\tilde{\Delta}, \Delta+\tilde{\Delta}] = 0\\ \Phi + \tilde{\Phi} \in MC(\mathfrak{a}_{\Delta+\tilde{\Delta}}^{P})\end{cases} \quad \Leftrightarrow (\tilde{\Delta}[1], \tilde{\Phi}) \in MC((L[1]\oplus\mathfrak{a})_{\Delta}^{P_\Phi}).$$

Moreover, $\mathfrak{a}_{\Delta+\tilde{\Delta}}^{P}$ is a curved $L_\infty[1]$-algebra. It is a $L_\infty[1]$-algebra exactly when $\tilde{\Delta} \in Ker(P)$.

Proof. By Prop. 1.12 we can apply Thm. 2 to obtain the $L_\infty[1]$-algebra $(L[1]\oplus\mathfrak{a})_{\Delta}^{P_\Phi}$, whose multibrackets we denote by $\{\dots\}$. We compute each summand appearing in the l.h.s of the Maurer-Cartan equation for $(\tilde{\Delta}[1], \tilde{\Phi})$ in $(L[1]\oplus\mathfrak{a})_{\Delta}^{P_\Phi}$, which reads

$$\sum_{n=1}^{\infty} \frac{1}{n!}\{(\tilde{\Delta}[1], \tilde{\Phi}), \dots, (\tilde{\Delta}[1], \tilde{\Phi})\}. \tag{10}$$

We have

$$\begin{aligned}\{(\tilde{\Delta}[1], \tilde{\Phi})\} &= (-[\Delta, \tilde{\Delta}][1],\ P_\Phi\tilde{\Delta} + P_\Phi[\Delta, \tilde{\Phi}]),\\ \{(\tilde{\Delta}[1], \tilde{\Phi}), (\tilde{\Delta}[1], \tilde{\Phi})\} &= (-[\tilde{\Delta}, \tilde{\Delta}][1],\ 2\cdot P_\Phi[\tilde{\Delta}, \tilde{\Phi}] + P_\Phi[[\Delta, \tilde{\Phi}], \tilde{\Phi}]),\\ \{\underbrace{(\tilde{\Delta}[1], \tilde{\Phi}), \dots, (\tilde{\Delta}[1], \tilde{\Phi})}_{n \text{ times}}\} &= (\ 0\ ,\ n\cdot P_\Phi[[[\tilde{\Delta}, \underbrace{\tilde{\Phi}], \dots], \tilde{\Phi}]}_{n-1 \text{ times}} + P_\Phi[[[[\Delta, \underbrace{\tilde{\Phi}], \tilde{\Phi}], \dots], \tilde{\Phi}]}_{n \text{ times}}).\end{aligned}$$

The last line refers to the n-th term for $n \geq 3$, and holds since the higher brackets with two or more entries in $L[1]\oplus\{0\}$ vanish.

The $L[1]$-component of (10) is just $-\frac{1}{2}[\Delta+\tilde{\Delta}, \Delta+\tilde{\Delta}][1]$. The $\mathfrak{a}$-component of (10) is

$$\begin{aligned}&P_\Phi\left(e^{[\cdot,\tilde{\Phi}]}\tilde{\Delta} + (e^{[\cdot,\tilde{\Phi}]} - 1)\Delta\right)\\ =&P_\Phi e^{[\cdot,\tilde{\Phi}]}(\Delta+\tilde{\Delta})\\ =&P e^{[\cdot,\Phi+\tilde{\Phi}]}(\Delta+\tilde{\Delta}),\end{aligned}$$

which by Remark 1.9 is the l.h.s. of the Maurer-Cartan equation in $\mathfrak{a}^P_{\Delta+\tilde{\Delta}}$ for $\Phi+\tilde{\Phi}$. Here in the first equation we used Remark 1.9. The last two statements follow from Thm. 1. Notice that the sums appearing in the Maurer-Cartan equations of $\mathfrak{a}^P_{\Delta+\tilde{\Delta}}$ and $(L[1]\oplus\mathfrak{a})^{P_\Phi}_{\Delta}$ both converge, by Prop. 1.18.

□

We obtain the following corollary about the space of curved $L_\infty[1]$-algebra structures arising as in Thm. 1 and Maurer-Cartan elements in there:

Corollary 1.13. *Let $L,\mathfrak{a},P$ such that $(L,\mathfrak{a},P,0)$ is a filtered V-data. The only non-vanishing multibrackets of $(L[1]\oplus\mathfrak{a})^P_0$, up to permutations of the entries, are*

$$\begin{aligned} d(x[1]) &= Px, \\ \{x[1],y[1]\} &= [x,y][1](-1)^{|x|}, \\ \{x[1],a_1,\dots,a_n\} &= P[\dots[x,a_1],\dots,a_n] \qquad \textit{for all } n\geq 1 \end{aligned}$$

where $x,y\in L[1]$ and $a_1,\dots,a_n\in\mathfrak{a}$.

Its Maurer-Cartan elements are characterized by: for all $\tilde{\Delta}\in L_1$ and $\tilde{\Phi}\in\mathfrak{a}_0$

$$\begin{cases} [\tilde{\Delta},\tilde{\Delta}]=0 \\ \tilde{\Phi} \textit{ is a MC element of } \mathfrak{a}^P_{\tilde{\Delta}} \end{cases} \Leftrightarrow (\tilde{\Delta}[1],\tilde{\Phi}) \textit{ is a MC element of } (L[1]\oplus\mathfrak{a})^P_0.$$

Proof. Applying Thm. 3 with $\Delta=0$ and $\Phi=0$, we obtain the $L_\infty[1]$-algebra $(L[1]\oplus\mathfrak{a})^P_0$, whose multibrackets are given by setting $D=0$ in Thm. 2 and are displayed above. □

1.4 Convergence issues

The left hand side of the Maurer-Cartan equation (3) is generally an infinite sum. In this subsection we review Getzler's notion of filtered L_∞-algebra [18], which guarantees that the above infinite sum converges. We show that simple assumptions on V-data ensure that the Maurer-Cartan equations of the (curved) $L_\infty[1]$-algebras we construct in Prop. 1.12 and Thm. 3 do converge.

Definition 1.14. Let V be a graded vector space. A *complete filtration* is a descending filtration by graded subspaces

$$V=\mathcal{F}^{-1}V\supset\mathcal{F}^0V\supset\mathcal{F}^1V\supset\dots$$

such that the canonical projection $V\to\varprojlim V/\mathcal{F}^nV$ is an isomorphism. Here

$$\varprojlim V/\mathcal{F}^nV{:=}\{\vec{x}\in\Pi_{n\geq-1}V/\mathcal{F}^nV\ :\ P_{i,j}(x_j)=x_i \text{ when } i<j\},$$

where $P_{i,j}\colon V/\mathcal{F}^jV\longrightarrow V/\mathcal{F}^iV$ is the canonical projection induced by the inclusion $\mathcal{F}^jV\subset\mathcal{F}^iV$.

Remark 1.15. If V can be written as a direct product of subspaces $V=\prod_{k\geq-1}V^k$, then $\{\mathcal{F}^nV\}_{n\geq-1}$ is a complete filtration of V, where $\mathcal{F}^nV:=\prod_{k\geq n}V^k$.

Definition 1.16. Let W be a curved $L_\infty[1]$-algebra. We say that W is *filtered*[1] if there exists a complete filtration on W such that all multibrackets $\{\dots\}$ have filtration degree -1.

Notice that for an element $\Phi \in W$ of filtration degree 1, we have $\{\Phi,\dots,\Phi\}_n \in \mathcal{F}^{n-1}W$ for all n, so the infinite sum

$$\sum_{n=0}^{\infty} \frac{1}{n!}\{\Phi,\dots,\Phi\}_n \tag{11}$$

converges in W by the completeness of the filtration. Indeed, setting $w_i := \sum_{n=0}^{i} \frac{1}{n!}\{\Phi,\dots,\Phi\}_n$ mod $\mathcal{F}^iW$ for all i defines an element $\overrightarrow{w} \in \Pi_{n\geq -1} W/\mathcal{F}^nW$ which turns out to belong to $\varprojlim V/\mathcal{F}^nV \cong W$.

We define *Maurer-Cartan elements* to be $\Phi \in W_0 \cap \mathcal{F}^1W$ for which the infinite sum (11) vanishes.

Definition 1.17. Let $(L, \mathfrak{a}, P, \Delta)$ be a curved V-data (Def. 1.7). We say that this curved V-data is *filtered* if there exists a complete filtration on L such that

a) The Lie bracket has filtration degree zero, i.e. $[\mathcal{F}^iL, \mathcal{F}^jL] \subset \mathcal{F}^{i+j}L$ for all $i, j \geq -1$,

b) $\mathfrak{a}_0 \subset \mathcal{F}^1L$,

c) the projection P has filtration degree zero, i.e. $P(\mathcal{F}^iL) \subset \mathcal{F}^iL$ for all $i \geq -1$.

Proposition 1.18. *Let $(L, \mathfrak{a}, P, \Delta)$ be a filtered, curved V-data. Then for every $\Phi \in MC(\mathfrak{a}^P_\Delta) \subset \mathfrak{a}_0$:*

1) the projection $P_\Phi := P \circ e^{[\cdot,\Phi]} \colon L \to \mathfrak{a}$ is well-defined and has filtration degree zero.

2) the curved $L_\infty[1]$-algebra $\mathfrak{a}^{P_\Phi}_\Delta$ given by Thm. 1 is filtered by $\mathcal{F}^n\mathfrak{a} := \mathcal{F}^nL \cap \mathfrak{a}$. Further, the sum (11) converges for any element of $\mathfrak{a}_0$.

3) if $\Delta \in ker(P)$: the $L_\infty[1]$-algebra $(L[1]\oplus\mathfrak{a})^{P_\Phi}_\Delta$ given by Thm. 2 is filtered by $\mathcal{F}^n(L[1]\oplus \mathfrak{a}) := (\mathcal{F}^nL)[1]\oplus\mathcal{F}^n\mathfrak{a}$. Further, the sum (11) converges for any element of $(L[1]\oplus\mathfrak{a})_0$.

Proof. 1) For every $x \in L$, say $x \in \mathcal{F}^iL$, by Def. 1.17 a)b) we have

$$[[\dots[x, \underbrace{\Phi],\dots], \Phi]}_{n \text{ times}} \in \mathcal{F}^{i+n}L.$$

Hence the completeness of the filtration on L implies that $e^{[\cdot,\Phi]}$ is a well-defined endomorphism of L. The above also shows that $e^{[\cdot,\Phi]}$ has filtration degree zero, and since P does by Def. 1.17 c), we conclude that the projection P_Φ has filtration degree zero.

2) We first check that $\{\mathcal{F}^n\mathfrak{a}\}_{n\geq-1}$ is a complete filtration of the vector space $\mathfrak{a}$.

The map $\mathfrak{a} \to \varprojlim \mathfrak{a}/\mathcal{F}^n\mathfrak{a}$ is surjective. Indeed, take an element of $\varprojlim \mathfrak{a}/\mathcal{F}^n\mathfrak{a}$, and consider its image under the canonical embedding $\varprojlim \mathfrak{a}/\mathcal{F}^n\mathfrak{a} \hookrightarrow \varprojlim W/\mathcal{F}^nW$. It is a sequence of

[1]Our definition differs from Getzler's, which requires that $W = \mathcal{F}^0W$ and that the multibrackets have filtration degree zero except for the zero-th bracket which has filtration degree one.

elements $\{a_i \bmod \mathcal{F}^i W\}_{i\geq -1}$ where $a_i \in \mathfrak{a}$. The surjectivity of $W \to \varprojlim W/\mathcal{F}^n W$ implies that there is an element $w \in W$ such that $a_i \bmod \mathcal{F}^i W = w \bmod \mathcal{F}^i W$ for all i, which implies $w \in \mathcal{F}^i W + \mathfrak{a}$ for all i and hence $w \in \cap_i(\mathcal{F}^i W + \mathfrak{a})$. Since $\cap_i(\mathcal{F}^i W) = \{0\}$ (by the injectivity of $W \to \varprojlim W/\mathcal{F}^n W$), this means that $w \in \mathfrak{a}$.

The map $\mathfrak{a} \to \varprojlim \mathfrak{a}/\mathcal{F}^n\mathfrak{a}$ is injective. Indeed, an element $a \in \mathfrak{a}$ is sent to 0 if and only if $a \in \cap_i(\mathcal{F}^i\mathfrak{a})$. But $\cap_i(\mathcal{F}^i\mathfrak{a}) \subset \cap_i(\mathcal{F}^i W)$, which is $\{0\}$ as seen above.

The multibracktets of $\mathfrak{a}^{P_\Phi}_\Delta$ is given by $P_\Phi[\dots[[\Delta, \bullet], \bullet], \dots, \bullet]$ (see eq. (4)). Using 1) and Def. 1.17 a), we see that this multibracket has filtration degree -1.

For the last statement, notice that $\mathfrak{a}_0 \subset \mathcal{F}^1\mathfrak{a}$ by Def. 1.17 b).

3) $\{(\mathcal{F}^n L)[1] \oplus \mathcal{F}^n\mathfrak{a}\}_{n\geq -1}$ is a complete filtration of the vector space $L[1] \oplus \mathfrak{a}$ because the two summands are complete filtrations of $L[1]$ and $\mathfrak{a}$ respectively (by assumption and by 2) respectively). The multibracktets of $(L[1] \oplus \mathfrak{a})^{P_\Phi}_\Delta$ are given in Thm. 2, and all have filtration degree -1 by 1) and Def. 1.17 a).

For the last statement, notice that the non-vanishing multibrackets of $(L[1]\oplus\mathfrak{a})^{P_\Phi}_\Delta$ accept at most two entries from $L[1]$, and use again $\mathfrak{a}_0 \subset \mathcal{F}^1\mathfrak{a}$. □

A common way to deal with convergence issues is to work formally (i.e. in terms of power series in a formal variable ε). We make precise how this goes in the present context.

Lemma 1.19. *Let $(L, \mathfrak{a}, P, \Delta)$ be a curved V-data. Then the conclusions of Prop. 1.18 hold in the formal setting, provided one replaces $\mathfrak{a}_0$ by $\mathfrak{a}_0 \otimes \varepsilon \cdot \mathbb{R}[[\varepsilon]]$.*

Proof. One checks easily that the following is a curved V-data:

- the graded Lie algebra $L \otimes \mathbb{R}[[\varepsilon]]$
- its abelian subalgebra $\mathfrak{a} \otimes \mathbb{R}[[\varepsilon]]$
- the natural projection $P_\varepsilon \colon L \otimes \mathbb{R}[[\varepsilon]] \to \mathfrak{a} \otimes \mathbb{R}[[\varepsilon]]$
- Δ,

where the the first three structures are defined by $\mathbb{R}[[\varepsilon]]$-linear extension. The natural complete filtration $\{\mathcal{F}^n\}_{n\geq 0}$ by $\mathcal{F}^n := L \otimes \varepsilon^n\mathbb{R}[[\varepsilon]]$ satisfies conditions a),c) of Def. 1.17. Hence one can apply Prop. 1.18 to the above listed curved V-data, taking care to replace $\mathfrak{a}_0$ by $\mathfrak{a}_0 \otimes \varepsilon \cdot \mathbb{R}[[\varepsilon]]$.

Notice that the curved $L_\infty[1]$-algebra induced by the above listed curved V-data is canonically isomorphic to $(\mathfrak{a}^P_\Delta) \otimes \mathbb{R}[[\varepsilon]]$. □

1.5 Equivalences of Maurer-Cartan elements

Let W be an $L_\infty[1]$-algebra. On $MC(W)$, the set of Maurer-Cartan elements, there is a canonical involutive (singular) distribution $\mathcal{D}$ which induces an equivalence relation on $MC(W)$ known as *gauge equivalence*. More precisely, each $z \in W_{-1}$ defines a vector field $\mathcal{Y}^z$ on W_0, whose value at $m \in W_0$ is[2]

[2] Th infinite sum (12) is guaranteed to converge if W is filtered and $W_{-1} \subset \mathcal{F}^1 W$, see §1.4. For both the examples we consider in this paper, this sum is actually finite.

$$\mathcal{Y}^z|_m := dz + \{z, m\} + \frac{1}{2!}\{z, m, m\} + \frac{1}{3!}\{z, m, m, m\} + \dots \tag{12}$$

This vector field is tangent to $MC(W)$. The distribution at the point $m \in MC(W)$ is defined as $\mathcal{D}|_m = \{\mathcal{Y}^z|_m : z \in W_{-1}\}$.

Remark 1.20. We give a justification of the above statements, see also [29, §3.4.2] [35, §2.5][15, §2.2]. Suppose W is finite-dimensional, so that the $L_\infty[1]$-algebra structure is encoded[3] by a degree 1, self-commuting vector field Q on W [50, Ex. 4.1]. We recall the following fact, that holds for any vector field X on W_0 and any element $m \in W_0$ (which defines a constant vector field m on W_0):

$$X|_m = (e^{[m,\cdot]}X)|_0. \tag{13}$$

Indeed, both sides equal $((\phi_{-1})_* X)|_0$, where ϕ denotes the time one flow of m (translation by m). Eq. (13) applied to $X = Q$ implies immediately that a point $m \in W_0$ is a zero of Q iff $-m$ satisfies the Maurer-Cartan equation (3).

View $z \in W_{-1}$ as a constant (degree -1) vector field on W. Then $[Q, z]$ is a degree zero vector field. As $\mathcal{L}_{[Q,z]}Q = [[Q, z], Q] = 0$, the flow of $[Q, z]$ preserves the set of zeros of Q, and hence $[Q, z]$ is tangent to this set. Eq. (13) applied to $X = [Q, Z]$ implies that $[Q, z]|_{W_0}$ is the pushforward by $-Id_{W_0}$ of $\mathcal{Y}^z$, therefore $\mathcal{Y}^z$ is tangent to $MC(W)$.

A computation shows that $\mathcal{D}$ can also be described in terms of all degree -1 vector fields: $\mathcal{D}|_m = \{[Q, Z]|_m : Z \in \chi_{-1}(W)\}$ for all $m \in MC(W)$. Since $[[Q, Z], [Q, Z']] = [Q, [[Q, Z], Z']]$ it follows that $\mathcal{D}$ is involutive.

Remark 1.21. When the differential d vanishes, the Jacobiator of the binary bracket $\{\cdot, \cdot\}$ is zero. Hence $\{\cdot, \cdot\}$ makes the vector space W_{-1} into an ordinary Lie algebra, and the assignment $W_{-1} \to \chi_0(W_0), z \mapsto (\mathcal{Y}^z)_{lin} := \{z, \cdot\} \in \chi_0(W_0)$ to the linear part of $\mathcal{Y}^z$ is a Lie algebra morphism.

Consider in particular the $L_\infty[1]$-subalgebra $ker(P)[1] \oplus \mathfrak{a}$ of the $L_\infty[1]$-algebra of Cor. 1.13. Notice that the differential vanishes, so Remark 1.21 applies. The vector field associated to a degree -1 element $z = (z_L[1], z_\mathfrak{a}) \in ker(P)[1] \oplus \mathfrak{a}$, evaluated at $m = (m_L[1], m_\mathfrak{a}) \in MC(ker(P)[1] \oplus \mathfrak{a})$, reads

$$\mathcal{Y}^z|_m = [z_L, m_L][1] + \sum_{n \geq 1} \frac{1}{n!} P[[z_L, \underbrace{m_\mathfrak{a}], \dots, m_\mathfrak{a}]}_{n \text{ times}} + \sum_{n \geq 1} \frac{1}{(n-1)!} P[[[m_L, z_\mathfrak{a}], \underbrace{m_\mathfrak{a}], \dots, m_\mathfrak{a}]}_{n-1 \text{ times}} \tag{14}$$

where the square bracket is the graded Lie algebra structure on L.

We will display explicitly the equivalence relations induced on morphisms between Lie algebras in §2.1 and on twisted Poisson structures in §5.3. In both cases it turns out that the equivalence classes coincide with the orbits of a group action.

[3] The multibrackets on W are recovered from Q applying Thm. 1 to the V-data ($L = \chi(W), \mathfrak{a} = \{\text{constant vector fields on } W\}, P(X) = X|_0, Q$).

2 Applications to Lie theory

In this section we apply the machinery developed in the previous section to instances in Lie theory. The results of §2.1 recover a theorem in [13].

We refer the reader to Appendix A.1 for the background material needed in §2.1- 2.3., and to Appendix A.2 for that needed in §2.4 - 2.5.

2.1 Lie algebra morphisms.

Let $(U, [\cdot,\cdot]_U)$ and $(V, [\cdot,\cdot]_V)$ be finite dimensional Lie algebras. We show that the deformations of Lie algebra morphisms $U \to V$ are ruled by a DGLA, recovering classical results of Nijenhuis and Richardson [37], and that more generally the simultaneous deformations of the Lie algebra structures and Lie algebra morphisms are ruled by a L_∞-algebra, recovering a theorem in [13] by the first author, Markl and Yau. The set-up of this subsection is a special case of the one of §2.5. We consider the simple instance of Lie algebras separately for the sake of concreteness and clarity of exposition.

We consider the graded manifold $(U \times V)[1]$, and encode the above data as vector fields on this graded manifold. See Appendix A.1 for some basic notions on graded manifolds and the notation; in particular $\chi(U[1])$ denotes the space of vector fields on $U[1]$, and $\iota\colon U \to \chi_{-1}(U[1])$ identifies elements of U with constant vector fields. We adopt the following conventions:

- The Lie bracket $[\cdot,\cdot]_U$ is encoded by the homological vector field $Q_U \in \chi_1(U[1])$ defined by $[[Q_U, \iota_X], \iota_Y] = \iota_{[X,Y]_U}$ for all $X, Y \in U$
- A linear map $\phi\colon U \to V$ is encoded by $\Phi \in \chi_0((U \times V)[1])$ defined by $[\Phi, \iota_X] = \iota_{\phi(X)}$ for all $X \in U$.

Remark 2.1. We give coordinate expressions for the vector fields Q_U, Q_V, Φ. Choose a basis of U, giving rise to coordinates $\{u_i\}$ on $U[1]$, and similarly choosing a basis of V get coordinates $\{v_\alpha\}$ on $V[1]$. Then

$$Q_U = -\frac{1}{2} c_{ij}^k u_i u_j \frac{\partial}{\partial u_k}, \qquad Q_V = -\frac{1}{2} d_{\alpha\beta}^\gamma v_\alpha v_\beta \frac{\partial}{\partial v_\gamma}, \qquad \Phi = -A_{l\eta} u_l \frac{\partial}{\partial v_\eta} \tag{15}$$

where c_{ij}^k and $d_{\alpha\beta}^\gamma$ are the structural constants of the Lie algebras U and V respectively and $A_{l\eta}$ is the matrix respresenting ϕ in the chosen basis.

The map $\phi\colon U \to V$ is a Lie algebra morphism exactly when

$$[Q_U, \Phi] + \frac{1}{2}[[Q_V, \Phi], \Phi] = 0, \tag{16}$$

see for example [37, p. 176].

Lemma 2.2. *The following quadruple forms a V-data:*

- *the graded Lie algebra* $L := \chi((U \times V)[1])$
- *its abelian subalgebra* $\mathfrak{a} := C(U[1]) \otimes V[1]$

- *the natural projection $P\colon L \to \mathfrak{a}$ with kernel*

$$ker(P) = \big(C(U[1]) \otimes C_{\geq 1}(V[1]) \otimes V[1]\big) \;\oplus\; \big(C(U[1] \times V[1]) \otimes U[1]\big)$$

- $\Delta := Q_U + Q_V$*,*

hence by Thm. 1 we obtain a $L_\infty[1]$-structure $\mathfrak{a}^P_\Delta$. For every linear map $\phi\colon U \to V$ we have: $\Phi \in \mathfrak{a}_0$ is a Maurer-Cartan element in $\mathfrak{a}^P_\Delta$ iff ϕ is a Lie algebra morphism.

Proof. $Ker(P)$ is a Lie subalgebra of L. This can be seen in coordinates, or noticing that the kernel consists exactly of vector fields on $(U \times V)[1]$ which are tangent to $(U \times \{0\})[1]$. Hence the above quadruple forms a V-data.

The $L_\infty[1]$-structure induced on $\mathfrak{a}$ by Thm. 1 is given by the multibrackets $P[[[Q_U + Q_V, \cdot], \cdots], \cdot]$. One computes easily in coordinates using (15) that $P[Q_V, \cdot]$, $[[Q_U, \cdot], \cdot]$ and $[[[Q_V, \cdot], \cdot], \cdot]$ vanish when applied to elements of $\mathfrak{a}$. Hence only the unary and binary brackets are non-zero, and they are given by

$$\begin{gathered}[Q_U, \cdot] \\ [[Q_V, \cdot], \cdot]\end{gathered}$$

respectively. Therefore the Maurer-Cartan equation of $\mathfrak{a}^P_\Delta$ is given by (16). □

Lemma 2.2 allows us to apply Prop. 1.12 and Thm. 3. Hence we deduce:

Corollary 2.3. *Let U, V finite dimensional Lie algebras and $\phi\colon U \to V$ a morphism. Let $(L, \mathfrak{a}, P, \Delta)$ as in Lemma 2.2.*

1) Let $\tilde{\phi}\colon U \to V$ be a linear map. Then

$$\phi + \tilde{\phi} \textit{ is a Lie algebra morphism} \quad \Leftrightarrow \quad \tilde{\Phi} \textit{ is a MC element of } \mathfrak{a}^{P_\Phi}_\Delta.$$

2) For all quadratic vector fields $\tilde{Q}_U$ on $U[1]$ and $\tilde{Q}_V$ on $V[1]$ and for all linear maps $\tilde{\phi}\colon U \to V$:

$$\begin{cases} Q_U + \tilde{Q}_U \textit{ and } Q_V + \tilde{Q}_V \textit{ define Lie algebra structures on } U \textit{ and } V \\ \phi + \tilde{\phi} \textit{ is a Lie algebra morphism between these new Lie algebra structures} \end{cases}$$
$$\Leftrightarrow ((\tilde{Q}_U + \tilde{Q}_V)[1], \tilde{\Phi}) \textit{ is a MC element of } (L[1] \oplus \mathfrak{a})^{P_\Phi}_\Delta.$$

Remark 2.4. We check that $(L, \mathfrak{a}, P, \Delta)$ is filtered V-data (Def. 1.17), as this is a hypothesis in Thm. 3. We have a direct sum decomposition $L = \oplus_{k \geq -1} L^k$ where $L^k := C_{k+1}(U[1]) \otimes C(V[1]) \otimes U[1] \,\oplus\, C_k(U[1]) \otimes C(V[1]) \otimes V[1]$. In other words, L^k is spanned by monomials in L whose total number of u's and $\frac{\partial}{\partial v}$'s, in coordinates, is exactly $k+1$. Then $\mathcal{F}^n L := \oplus_{k \geq n} L^k$ is a complete filtration of the vector space L. One checks easily that $(L, \mathfrak{a}, P, \Delta)$ is filtered V-data.

An alternative way to check that there are no convergence issues for $e^{[\cdot, \Phi]}$ and the Maurer-Cartan equations appearing in Cor. 2.3 is to recall that $U \times V$ is finite dimensional and use a variant of Lemma 2.6 below.

In the rest of this subsection we make more explicit the structures of $\mathfrak{a}^{P_\Phi}_\Delta$ and $(L'[1] \oplus \mathfrak{a})^{P_\Phi}_\Delta$, where $L' \subset L$ is specified just after Lemma 2.5.

Given a morphism of Lie algebras $\phi\colon U \to V$, the associated Richardson-Nijenhius DGLA is given by $\oplus_i \wedge^i U^* \otimes V$, the differential being the Chevalley-Eilenberg differential of U with values in the module V (the module structure is given by $e \in U \mapsto [\phi(e), \cdot]_V$) and the bracket being the Lie bracket on V combined with the wedge product on $\wedge U^*$ (see [37, p. 175-6] or [12, §2.3]).

Lemma 2.5. *$\mathfrak{a}^{P_\Phi}_\Delta$ is the suspension of the Richardson-Nijenhius DGLA introduced in [37, §1].*

Proof. The n-ary bracket of $\mathfrak{a}^{P_\Phi}_\Delta$, evaluated on $a_1, \dots, a_n \in \mathfrak{a}$ is

$$P_\Phi[[[Q_U + Q_V, a_1], \cdots], a_n]$$

One computes easily in coordinates that only unary and binary brackets are non-zero, and they are given by

$$P[Q_U + [Q_V, \Phi], \cdot] = [Q_U + [Q_V, \Phi], \cdot] \tag{17}$$

$$P[[Q_V, \cdot], \cdot] = [[Q_V, \cdot], \cdot]. \tag{18}$$

respectively. The r.h.s. of (17) is exactly the Chevalley-Eilenberg differential of the Lie algebra U with values in the module V. The r.h.s. of (18) is given by the Lie bracket on V combined with the wedge product on $\wedge U^*$. Hence we obtain the Nijenhuis-Richardson DGLA. □

Up to this point we only looked at deformations of the morphism $\phi\colon U \to V$. Now we also deform the Lie algebra structures on the vector spaces U and V.

Define $L' := \chi(U[1]) \oplus \chi(V[1]) \subset L$. By Thm. 3 and Rem. 1.10 we obtain an $L_\infty[1]$-algebra $(L'[1] \oplus \mathfrak{a})^{P_\Phi}_\Delta$, governing the simultaneous deformations of the Lie algebra structures on U, V and of the morphisms.

Lemma 2.6. *$(L'[1] \oplus \mathfrak{a})^{P_\Phi}_\Delta$ has multibrackets of order up to dim$(V)+1$. Its Maurer-Cartan equation is cubic, given by eq. (21) below.*

Proof. We write down explicitly the multibrackets of $(L'[1] \oplus \mathfrak{a})^{P_\Phi}_\Delta$, as given in Thm. 2. We denote by $\tilde{Q}^i_U, \tilde{Q}^i_V$ and $\tilde{\Phi}^i$ general (homogeneous) elements of $\chi(U[1]), \chi(V[1])$ and $\mathfrak{a}$ respectively ($i = 1, 2, \dots$). The multibrackets involving only $\tilde{\Phi}$ are given exactly by (17) and (18) since $\mathfrak{a}^{P_\Phi}_\Delta$ is a L_∞-subalgebra of $(L'[1] \oplus \mathfrak{a})^{P_\Phi}_\Delta$. Explicitly, they are

$$d(\tilde{\Phi}) = [Q_U + [Q_V, \Phi], \tilde{\Phi}] \in \mathfrak{a}$$

and

$$\{\tilde{\Phi}^1, \tilde{\Phi}^2\} = [[Q_V, \tilde{\Phi}^1], \tilde{\Phi}^2] \in \mathfrak{a}.$$

Now we compute the multibrackets involving at least one of $\tilde{Q}_U[1]$ or $\tilde{Q}_V[1]$. For the differential we have

$$d(\tilde{Q}_U[1]) = -[Q_U + Q_V, \tilde{Q}_U][1] + P_\Phi(\tilde{Q}_U) = -[Q_U, \tilde{Q}_U][1] + [\tilde{Q}_U, \Phi] \in L[1] \oplus \mathfrak{a}$$

$$d(\tilde{Q}_V[1]) = -[Q_U + Q_V, \tilde{Q}_V][1] + P_\Phi(\tilde{Q}_V) = -[Q_V, \tilde{Q}_V][1] + \frac{1}{k!}[[\dots[\tilde{Q}_V, \underbrace{\Phi], \dots], \Phi}_{k}] \in L[1] \oplus \mathfrak{a}$$

where $k = |\tilde{Q}_V| + 1$. For the binary bracket we have

$$\begin{aligned}\{(\tilde{Q}_U^1 + \tilde{Q}_V^1)[1], (\tilde{Q}_U^2 + \tilde{Q}_V^2)[1]\} &= (-1)^{|\tilde{Q}_U^1 + \tilde{Q}_V^1|}([\tilde{Q}_U^1, \tilde{Q}_U^2] + [\tilde{Q}_V^1, \tilde{Q}_V^2])[1] \in L[1] \\ \{\tilde{Q}_U[1], \tilde{\Phi}\} &= P_\Phi[\tilde{Q}_U, \tilde{\Phi}] = [\tilde{Q}_U, \tilde{\Phi}] \in \mathfrak{a} \\ \{\tilde{Q}_V[1], \tilde{\Phi}\} &= P_\Phi[\tilde{Q}_V, \tilde{\Phi}] \in \mathfrak{a}.\end{aligned} \tag{19}$$

From (19) it follows that the only non-zero n-brackets with $n \geq 3$ are

$$\{\tilde{Q}_V[1], \tilde{\Phi}^1, \ldots, \tilde{\Phi}^n\} = P_\Phi[[\tilde{Q}_V, \tilde{\Phi}^1], \ldots, \tilde{\Phi}^n] \in \mathfrak{a}. \tag{20}$$

In coordinates it is clear that the operation $[\cdot, \tilde{\Phi}]$ sends $C(U[1]) \otimes C_i(V[1]) \otimes V[1]$ to $C(U[1]) \otimes C_{i-1}(V[1]) \otimes V[1]$. As $\tilde{Q}_V \in \chi(V[1]) \cong \sum_{i=1}^{dim(V)} C_i(V[1]) \otimes V[1]$, it is clear from eq. (20) that all n-brackets vanish for $n > dim(V) + 1$.

To write down the Maurer-Cartan elements , we can use eq. (3) and the formulae for the multibrackets derived above. Alternatively, by virtue of Cor. 2.3, we know that Maurer-Cartan elements $\tilde{Q} = \tilde{Q}_U[1] + \tilde{Q}_V[1] + \tilde{\Phi}$ are characterized by the equations $[Q_U + \tilde{Q}_U, Q_U + \tilde{Q}_U] = 0$, $[Q_V + \tilde{Q}_V, Q_V + \tilde{Q}_V] = 0$ and by the equation obtained replacing Q_U by $Q_U + \tilde{Q}_U$ (and similarly for Q_V,Φ) in eq. (16) . The latter equation reads

$$\begin{aligned}0 =& [\tilde{Q}_U, \Phi] + \frac{1}{2}[[\tilde{Q}_V, \Phi], \Phi] + [Q_U + [Q_V, \Phi], \tilde{\Phi}] \\ &+ [\tilde{Q}_U, \tilde{\Phi}] + [[\tilde{Q}_V, \tilde{\Phi}], \Phi] + \frac{1}{2}[[Q_V, \tilde{\Phi}], \tilde{\Phi}] \\ &+ \frac{1}{2}[[\tilde{Q}_V, \tilde{\Phi}], \tilde{\Phi}].\end{aligned} \tag{21}$$

□

2.1.1 Equivalences of Lie algebras morphisms

Consider the $L_\infty[1]$-algebra whose Maurer-Cartan elements are pairs of Lie algebra structures and morphisms between them, that is, the $L_\infty[1]$-algebra $\mathcal{L} := (L'[1] \oplus \mathfrak{a})^P_{\Delta=0}$ as in Cor. 1.13.

Elements of $\mathcal{L}_{-1}$ are of the form

$$z = (z_U[1], z_V[1], z_\mathfrak{a}) \in \ \chi_0(U[1])[1] \ \oplus \ \chi_0(V[1])[1] \ \oplus \ V[1].$$

Restricting the binary bracket $\{\cdot, \cdot\}_2$ to $\mathcal{L}_{-1}$ and using the identifications at the beginning of §2.1 we obtain the ordinary Lie algebra

$$End(U) \times (End(V) \ltimes V)$$

where $End(U)$ and $End(V)$ are endowed with the commutator bracket, V is abelian and $[A, f] = Af \in V$ for $A \in End(V)$ and $f \in V$.

Maurer-Cartan elements lie in $\mathcal{L}_0$, so they are of the form

$$m = (m_U[1], m_V[1], m_\mathfrak{a}) \in \ \chi_1(U[1])[1] \ \oplus \ \chi_1(V[1])[1] \ \oplus \ (U[1])^* \otimes V[1],$$

and as described at the beginning of §2.1 its components correspond respectively to a Lie bracket $[\cdot,\cdot]_{m_U}$ on U, a Lie bracket $[\cdot,\cdot]_{m_V}$ on V, and a Lie algebra morphism $\phi\colon U \to V$. By degree reasons eq. (14) reads simply

$$\begin{aligned}\mathcal{Y}^z|_m = &[z_U, m_U][1] \;\oplus\; [z_V, m_V][1] \;\oplus\; [z_U + z_V, m_\mathfrak{a}] + [[m_V, z_\mathfrak{a}], m_\mathfrak{a}] \\ \in &T_z\left(\chi_1(U[1])[1] \;\oplus\; \chi_1(V[1])[1] \;\oplus\; (U[1])^* \otimes V[1]\right).\end{aligned} \tag{22}$$

The assignment $z \mapsto \mathcal{Y}^z$ vector field is not a Lie algebra action: $z^1 = (0,0,z^1_\mathfrak{a})$ and $z^2 = (0,0,z^2_\mathfrak{a})$ commute, however the vector fields $\mathcal{Y}^{z^1}$ and $\mathcal{Y}^{z^2}$ do not commute. However restricting suitably we obtain an infinitesimal action, which integrates to the group action of symmetries given in [12, §3]:

Proposition 2.7. *The assignment $End(U) \times End(V) \to \chi(MC), z \mapsto \mathcal{Y}^z$ is a Lie algebra morphism. It integrates to the group action*

$$\begin{aligned}&(GL(U) \times GL(V)) \times MC \to MC \\ &((g,h)\ ,\ ([\cdot,\cdot]_{m_U}, [\cdot,\cdot]_{m_U}, \phi) \mapsto (g^*([\cdot,\cdot]_{m_U}), h^*([\cdot,\cdot]_{m_V}), h \circ \phi \circ g^{-1}).\end{aligned}$$

Here the Lie bracket $g^*([\cdot,\cdot]_{m_U})$ is defined as $g[g^{-1}\cdot, g^{-1}\cdot]_{m_U}$, and similarly for $h^*([\cdot,\cdot]_{m_V})$.

The equivalence classes induced by the singular distribution $\mathcal{D} := \{\mathcal{Y}^z : z \in \mathcal{L}_{-1}\}$ on MC agree with the orbits of the this action.

Proof. Notice that for $z \in End(U) \times End(V)$ the vector field $\mathcal{Y}^z$ is linear, hence $z \mapsto \mathcal{Y}^z$ is a Lie algebra morphism by Remark 1.21. We compute the integral curve of $\mathcal{Y}^z$ starting at $m = (m_U[1], m_V[1], m_\mathfrak{a}) \in MC$.

The first component of $\mathcal{Y}^z$ is $[z_U, \cdot][1]$. Its integral curve starting at $m_U[1]$ is $t \mapsto e^{t[z_U,\cdot]}m_U[1]$, since the latter forms a 1-parameter group and differentiates to $[z_V, \cdot]$ at time zero. The Lie bracket on U induced by $e^{[z_U,\cdot]}m_U[1]$ is $(exp(z_U))^*([\cdot,\cdot]_{m_U})$ where $exp(z_U)$ is the usual matrix exponential of $z_U \in \mathfrak{gl}(U)$ (this follows from the fact that $e^{[z_U,\cdot]}$ is an automorphism of $[\cdot,\cdot]$). The same argument applies to the second component of $\mathcal{Y}^z$.

For the third component, the integral curve of $[z_U + z_V, \cdot]$ starting at $m_\mathfrak{a}$ is $t \mapsto e^{t[z_U+z_V,\cdot]}m_\mathfrak{a}$. The element $e^{[z_U+z_V,\cdot]}m_\mathfrak{a} \in (U[1])^* \otimes V[1]$ corresponds to $exp(z_V) \circ \phi \circ exp(-z_U)\colon U \to V$. This shows that the group action in the statement of this proposition integrates the given Lie algebra action.

For the last statement we fix $m \in MC$ and show that $\mathcal{D}_m = \{\mathcal{Y}^z|_m : z = (z_U[1], z_V[1], 0)\}$. To this aim, just notice that $\mathcal{Y}^{(0,0,z_\mathfrak{a})}|_m = \mathcal{Y}^{(0,[m_V,z_\mathfrak{a}],0)}|_m$ for all $z_\mathfrak{a} \in V[1]$, as a consequence of $[m_V, m_V] = 0$. $\square$

2.2 Subalgebras of Lie algebras

Let $\mathfrak{g}$ be a finite dimensonal Lie algebra, $U \subset \mathfrak{g}$ a Lie subalgebra. We study deformations of the Lie algebra structure on $\mathfrak{g}$ and of the subspace U as a Lie subalgebra, similarly to Richardson [41].

At first, let $U \subset \mathfrak{g}$ be simply a subspace. We denote by $Q_\mathfrak{g} \in \chi(\mathfrak{g}[1])$ the homological vector field encoding the Lie algebra structure on $\mathfrak{g}$. Choose a subspace V in $\mathfrak{g}$ complementary to U. Given a linear map $\phi\colon U \to V$, we view it as an element $\Phi \in C_1(U[1]) \otimes \chi_{-1}(V[1]) \subset \chi_0(\mathfrak{g}[1])$ defined by $[\Phi, \iota_X] = \iota_{\phi(X)}$ for all $X \in U$.

Lemma 2.8. *The following quadruple forms a curved V-data:*

- *the graded Lie algebra* $L := \chi(\mathfrak{g}[1])$
- *its abelian subalgebra* $\mathfrak{a} := C(U[1]) \otimes V[1]$
- *the natural projection* $P\colon L \to \mathfrak{a}$ *with kernel*
$$ker(P) = \big(C(U[1]) \otimes C_{\geq 1}(V[1]) \otimes V[1]\big) \;\oplus\; \big(C(\mathfrak{g}[1]) \otimes U[1]\big)$$
- $\Delta := Q_{\mathfrak{g}}$,

hence by Thm. 1 we obtain a curved $L_\infty[1]$*-structure* $\mathfrak{a}^P_\Delta$.

$\Phi \in \mathfrak{a}_0$ *is a MC element in* $\mathfrak{a}^P_\Delta$ *iff* $graph(\phi)$ *is a Lie subalgebra of* $\mathfrak{g}$.

Further, the above quadruple forms a V-data iff U *is a Lie subalgebra of* $\mathfrak{g}$.

Proof. To show that the above quadruple forms a curved V-data proceed as in the proof of Lemma 2.2.

Rem. 1.9 says that Φ is a Maurer-Cartan element in $\mathfrak{a}^P_\Delta$ iff $e^{-[\Phi,\cdot]}Q_{\mathfrak{g}} \in ker(P)$. This condition is equivalent to asking that for all $X, Y \in U$:

$$\left[\left[e^{-[\Phi,\cdot]}Q_{\mathfrak{g}}, \iota_X\right], \iota_Y\right] \in U[1]$$

Using the fact that $e^{-[\Phi,\cdot]}$ is a Lie algebra automorphism of L (to pull it out of the brackets) and that $e^{[\Phi,\cdot]}\iota_X = \iota_X + [\Phi, \iota_X] = \iota_{X+\phi(X)}$, we see that the above is equivalent to

$$[X + \phi(X), Y + \phi(Y)] \in \{Z + \phi(Z) : Z \in U\} = graph(\phi),$$

i.e. to $graph(\phi)$ being a Lie subalgebra of $\mathfrak{g}$.

The last statement can be proven as follows: $Q_{\mathfrak{g}} \in ker(P)$ is equivalent to $[[Q_{\mathfrak{g}}, \iota_X], \iota_Y] \in U[1]$ for all $X, Y \in U$, which in turn means that U is a Lie subalgebra of $\mathfrak{g}$. (Alternatively, it follows from the above noticing that 0 is a Maurer-Cartan element of $\mathfrak{a}^P_\Delta$ iff $PQ_{\mathfrak{g}} = 0$.) ☐

Lemma 2.8 allow us to apply Thm. 3 with $\Phi = 0$.

We deduce:

Corollary 2.9. *Let* $\mathfrak{g}$ *be a Lie algebra,* $U \subset \mathfrak{g}$ *a Lie subalgebra. Choose a subspace* $V \subset \mathfrak{g}$ *complementary to* U*, and let* $(L, \mathfrak{a}, P, \Delta)$ *be the V-data as in Lemma 2.8.*

For all $\tilde{Q}_{\mathfrak{g}} \in L_1$ *and for all linear maps* $\tilde{\phi}\colon U \to V$*:*

$$\begin{cases} Q_{\mathfrak{g}} + \tilde{Q}_{\mathfrak{g}} \text{ defines a Lie algebra structure on } \mathfrak{g} \\ graph(\tilde{\phi}) \text{ is a Lie subalgebra of it} \end{cases}$$
$$\Leftrightarrow (\tilde{Q}_{\mathfrak{g}}[1], \tilde{\Phi}) \text{ is a MC element of } (L[1] \oplus \mathfrak{a})^P_\Delta.$$

Remark 2.10. The proof that $(L, \mathfrak{a}, P, \Delta)$ is a filtered V-data is given in Remark 2.4.

Remark 2.11. By Cor. 2.9, the Maurer-Cartan elements of $(L[1] \oplus \mathfrak{a})^P_\Delta$ are in bijection with deformations of the Lie algebra structure on $\mathfrak{g}$ and deformations of the subspace U as a Lie subalgebra.

Applying Cor. 2.3 to the Lie algebra U, to the Lie algebra $\mathfrak{g}$ and to the inclusion $i\colon U \hookrightarrow \mathfrak{g}$, we obtain an $L_\infty[1]$-algebra whose Maurer-Cartan elements are deformations of the Lie algebra structure on $\mathfrak{g}$ and deformations of i to linear maps $i + \tilde{i}\colon U \to \mathfrak{g}$ whose image is a Lie subalgebra of the new Lie algebra structure on $\mathfrak{g}$. Notice that the two Maurer-Cartan sets are quite different, as different maps $i + \tilde{i}$ can have the same image.

2.3 Lie bialgebra morphisms.

Let U and V be Lie bialgebras. We show that the simultaneous deformations of the Lie bialgebra structures and Lie bialgebra morphisms are ruled by some L_∞-algebra.

Definition 2.12. A finite dimensional vector space U is a *Lie bialgebra* if U is endowed with a Lie algebra structure, the dual U^* is endowed with a Lie algebra structure $[\cdot,\cdot]_{U^*}$, and the Chevalley-Eilenberg differential of U is a graded derivation of $[\cdot,\cdot]_{U^*}$ (extended to $\wedge U^*$).

A *morphism* between from a Lie bialgebra U to a Lie bialgebra V is a Lie algebra morphism $\phi\colon U \to V$ such that its dual $\phi^*\colon V^* \to U^*$ is also a Lie algebra morphism (see for instance [4]).

In order to rephrase the above definitions, we recall few notions from graded geometry. Let U be a vector space. The graded manifold $\mathcal{M} := T^*[2]U[1] = U[1]\times U^*[1]$ is symplectic, hence the space of functions is endowed with a degree -2 Poisson bracket[4]. Explicitly, the degree k functions are[5] $C_k(\mathcal{M}) = \wedge^k(U^* \times U)$. If we choose a basis for U, giving rise to degree 1 coordinates u_i on $U[1]$ and degree 1 coordinates on $U^*[1]$ which we denote by $\frac{\partial}{\partial u_i}$, the Poisson bracket is given by

$$\{u_i, u_j\} = 0, \quad \{\frac{\partial}{\partial u_i}, \frac{\partial}{\partial u_j}\} = 0, \quad \{u_i, \frac{\partial}{\partial u_j}\} = \delta_{ij} = \{\frac{\partial}{\partial u_j}, u_i\}.$$

Notice that $C(\mathcal{M})$ is not only graded but actually bigraded, by $C_{(i,j)}(\mathcal{M}) = \wedge^i U^* \otimes \wedge^j U$.

Since the Poisson bracket on $C(\mathcal{M})$ has degree -2, it follows that $C(\mathcal{M})[2]$ is a graded Lie algebra. There is a canonical (degree preserving) embedding

$$\chi(U[1]) \hookrightarrow C(\mathcal{M})[2],$$

whose image consists exactly of $\oplus_i C_{(i,1)}(\mathcal{M})$ (the fiber-wise linear functions on $\mathcal{M} = T^*[2]U[1]$). The embedding also preserves the brackets by [42, Lemma 3.3.1], i.e., it is an embedding of graded Lie algebras. Notice that there is a canonical symplectomorphism $T^*[2]U[1] = U[1] \times U^*[1] \cong U^*[1] \times U[1] = T^*[2]U^*[1]$, which provides a canonical embedding of graded Lie algebras $\chi(U^*[1]) \hookrightarrow C(\mathcal{M})[2]$.

We can now state, following [42, §3.1]: a Lie bialgebra structure on U is equivalent to an element $Q_U \in C_{(2,1)}(\mathcal{M})$ and an element $Q_{U^*} \in C_{(1,2)}(\mathcal{M})$ such that $Q_U + Q_{U^*}$ commutes with itself w.r.t. $\{\cdot,\cdot\}$, or equivalently so that $Q_U - Q_{U^*}$ self-commutes.

Further, if U and V are Lie bialgebras and given a linear map $\phi\colon U \to V$, consider the corresponding element $\Phi \in \chi((U \times V)[1]) \subset C(T^*[2](U \times V)[1])[2]$ as at the beginning of this section. Notice that the element of $C(T^*[2](U\times V)[1])[2]$ associated to ϕ^* is $-\Phi$. Using (16) we see that ϕ is a morphism of Lie bialgebras iff

$$\{Q_U, \Phi\} + \frac{1}{2}\{\{Q_V, \Phi\}, \Phi\} = 0, \tag{23}$$

$$\{Q_{V^*}, -\Phi\} + \frac{1}{2}\{\{Q_{U^*}, -\Phi\}, -\Phi\} = 0. \tag{24}$$

[4]This bracket is sometimes called "big bracket".

[5]Here we use $\wedge$ to denote the ordinary exterior power, and regard elements of U and U^* as having degree one.

Lemma 2.13. *Let (U, Q_U, Q_{U^*}) and (V, Q_V, Q_{V^*}) be finite dimensional Lie bialgebras. The following quadruple forms a V-data:*

- *the graded Lie algebra $L := C_{(\geq 1, \geq 1)}\big(T^*[2](U \times V)[1]\big)[2] = (\wedge^{\geq 1}(U^* \times V^*) \otimes \wedge^{\geq 1}(U \times V))[2]$*
- *its abelian subalgebra $\mathfrak{a} := (\wedge^{\geq 1} U^* \otimes \wedge^{\geq 1} V)[2]$*
- *the natural projection $P \colon L \to \mathfrak{a}$ with kernel*

$$ker(P) = \big(\wedge U^* \otimes \wedge^{\geq 1} V^* \otimes \wedge^{\geq 1}(U \times V)\big)[2] + \big(\wedge^{\geq 1}(U^* \times V^*) \otimes \wedge^{\geq 1} U \otimes \wedge V\big)[2]$$

- *$\Delta := Q_U + Q_{U^*} + Q_V - Q_{V^*}$,*

hence by Thm. 1 we obtain a $L_\infty[1]$-structure $\mathfrak{a}^P_\Delta$.

$\Phi \in \mathfrak{a}_0$ is a Maurer-Cartan element in $\mathfrak{a}^P_\Delta$ iff ϕ is a Lie bialgebra morphism.

Proof. Since $T^*[2](U \times V)[1]$ is endowed with a Poisson bracket of bidegree $(-1, -1)$, the shifted space of functions $C\big(T^*[2](U \times V)[1]\big)[2]$ is a graded Lie algebra and L is a graded Lie subalgebra. $Ker(P)$ is a Lie subalgebra of L, as can be checked in coordinates. Clearly Δ lies in $ker(P)$, and

$$\{\Delta, \Delta\} = \{Q_U + Q_{U^*}, Q_U + Q_{U^*}\} + \{Q_V - Q_{V^*}, Q_V - Q_{V^*}\} = 0.$$

Hence the above quadruple forms a V-data, and we can apply Thm. 1.

To compute the Maurer-Cartan elements of $\mathfrak{a}^P_\Delta$, take $\Phi \in \mathfrak{a}_0 = U^* \otimes V$. One computes easily in coordinates that

$$P\{\Delta, \Phi\} = \{Q_U - Q_{V^*}, \Phi\}$$
$$P\{\{\Delta, \Phi\}, \Phi\} = \{\{Q_V + Q_{U^*}, \Phi\}, \Phi\}$$

while all other terms of the Maurer-Cartan equation vanish. Separating the terms in $\wedge^2 U^* \otimes V$ from those in $U^* \otimes \wedge^2 V$ we conclude that Φ is a Maurer-Cartan element of $\mathfrak{a}^P_\Delta$ iff the equations (23) and (24) are satisfied, which in turn is equivalent to ϕ being a a Lie bialgebra morphism. □

Lemma 2.13 allows us to apply Prop. 1.12 and Thm. 3. Hence we deduce:

Corollary 2.14. *Let (U, Q_U, Q_{U^*}) and (V, Q_V, Q_{V^*}) be finite dimensional Lie bialgebras and $\phi \colon U \to V$ a Lie bialgebra morphism. Let $(L, \mathfrak{a}, P, \Delta)$ be as in Lemma 2.13.*

1) Let $\tilde{\phi} \colon U \to V$ a linear map. Then

$$\phi + \tilde{\phi} \text{ is a Lie bialgebra morphism} \quad \Leftrightarrow \quad \tilde{\Phi} \text{ is a MC element of } \mathfrak{a}^{P_\Phi}_\Delta.$$

2) For all $\tilde{Q}_U \in C_{(2,1)}(T^*[2]U[1])$ and $\tilde{Q}_{U^*} \in C_{(1,2)}(T^*[2]U[1])$, for all analogously defined $\tilde{Q}_V, \tilde{Q}_{V^*}$, and for all linear maps $\tilde{\phi} \colon U \to V$:

$$\begin{cases} (U, Q_U + \tilde{Q}_U, Q_{U^*} + \tilde{Q}_{U^*}) \text{ and } (V, Q_V + \tilde{Q}_V, Q_{V^*} + \tilde{Q}_{V^*}) \text{ are Lie bialgebras} \\ \phi + \tilde{\phi} \text{ is a Lie bialgebra morphism between these new Lie bialgebra structures} \end{cases}$$

$$\Leftrightarrow \ ((\tilde{Q}_U + \tilde{Q}_{U^*} + \tilde{Q}_V - \tilde{Q}_{V^*})[1], \tilde{\Phi}) \text{ is a MC element of } (L[1] \oplus \mathfrak{a})^{P_\Phi}_\Delta.$$

Remark 2.15. We check that the V-data appearing in Cor. 2.14 is filtered. We have a direct sum decomposition $L = \oplus_{k\geq -1} L^k$ where $L^k := L \cap \oplus_{q+r=k+1}\big(\wedge^q U^* \otimes \wedge V^* \otimes \wedge U \otimes \wedge^r V\big)[2]$. In other words, L^k is spanned by monomials in L whose total number of u's and $\frac{\partial}{\partial v}$'s, in coordinates, is exactly $k+1$. Then $\mathcal{F}^n L := \oplus_{k\geq n} L^k$ is a complete filtration of the vector space L. One checks easily that $(L, \mathfrak{a}, P, \Delta)$ is a filtered V-data.

Remark 2.16. It seems that there is no way to recover Cor. 2.14 simply applying the results of Cor. 2.3 twice (once to Lie algebra morphism $\phi\colon U \to V$ and once to the Lie algebra morphism $\phi^*\colon V^* \to U^*$), since the latter procedure would deform ϕ and ϕ^* to two Lie algebra morphisms $\alpha\colon U \to V$ and $\beta\colon V^* \to U^*$ which are not necessarily duals of each other.

2.4 Maurer-Cartan elements of L_∞-algebra structures

Fix a (possibly infinite dimensional) graded vector space W. We show that the space of pairs

$$(L_\infty[1]\text{-algebra structures on } W, \text{ Maurer-Cartan elements for this structure})$$

is governed by a Maurer-Cartan equation. We will ignore all convergence issues in this subsection; they are automatically dealt with if one works formally, see Lemma 1.19.

We refer to §A.2 for the background material on coderivations. In §A.3 we recall that $L_\infty[1]$-algebra structures on W are in bijection with degree 1 self-commuting coderivations Θ on $\overline{SW} := \oplus_{k=1}^{\infty} S^k W$, we show that there is an embedding $\alpha\colon W \hookrightarrow Coder(SW)$, and that there is a bracket-preserving embedding $\mathcal{J}\colon Coder(\overline{SW}) \hookrightarrow Coder(SW)$ whose image annihilates $1 \in SW$. In §A.3 we further prove that all $L_\infty[1]$-algebra structures are obtained by the derived bracket construction:

Proposition 2.17. *Let W be an L_∞[1]-algebra, and Θ the corresponding coderivation of $\overline{SW}$. The following quadruple forms a V-data:*

- *the graded Lie algebra $L := Coder(SW)$*
- *its abelian subalgebra $\mathfrak{a} := \{\alpha_w : w \in W\}$*
- *the projection $P\colon L \to \mathfrak{a}$, $\tau \mapsto \alpha_{\tau(1)}$*
- *$\Delta := \mathcal{J}\Theta$.*

The induced $L_\infty[1]$-structure on $\mathfrak{a}$ given by Thm. 1 is exactly the original L_∞[1]-structure on W, under the canonical identification $W \cong \mathfrak{a}, w \mapsto \alpha_w$.

We apply Cor. 1.13, choosing $\Theta = 0$ above and considering $Ker(P) \subset L$. we obtain

Corollary 2.18. *$\{\tau \in Coder(SW) : \tau(1) = 0\}[1] \oplus W$, endowed with the $L_\infty[1]$-algebra structure specified in Cor. 1.13, has the following property: for all $\tilde{\Theta} \in Coder(\overline{SW})_1$ and $\tilde{\Phi} \in W_0$:*

$$\begin{cases} \tilde{\Theta} \text{ defines an } L_\infty[1]\text{-algebra structure on } W \\ \tilde{\Phi} \text{ is a MC element of this } L_\infty[1]\text{-algebra structure on } W \end{cases}$$
$$\Leftrightarrow \ (\mathcal{J}\tilde{\Theta}[1], \tilde{\Phi}) \text{ is a MC element of } \{\tau \in Coder(SW) : \tau(1) = 0\}[1] \oplus W$$

2.5 L_∞-algebra morphisms

We consider deformations of a pair of arbitrary $L_\infty[1]$-algebras and of a $L_\infty[1]$-morphism between them. We show that deformations of the morphism with fixed $L_\infty[1]$-algebra structures are ruled by a $L_\infty[1]$-algebra (this follows also from Shoikhet [47, §3]), and then show that there is an $L_\infty[1]$-algebra governing arbitrary deformations.

In the next sections we will use the following notations. When E and F are two vector spaces, we will denote by $L(E,F)$ the set of linear maps from E to F and use $L(E) := L(E,F)$ when $E = F$.

Let U and V be two graded vector spaces. Denote $\overline{S(U \oplus V)} := \oplus_{k\geq 1} S^k(U \oplus V)$. Let

$$L := L\left(\overline{S(U \oplus V)}, U \oplus V\right) = \prod_{i\geq 1} \bigoplus_{q+r=i} L_U^{q,r} \oplus L_V^{q,r}, \tag{25}$$

where

$$L_U^{q,r} := \left\{\Pi_U \circ l \circ \Pi^{q,r} : l \in L(S^{q+r}(U \oplus V), U \oplus V)\right\}$$

for $\Pi^{q,r}\colon S^{q+r}(U \oplus V) \to S^q U \otimes S^r V$ and $\Pi_U \colon U \oplus V \to U$ the canonical projections. Consider the subspace

$$\mathfrak{a} := \prod_{q\geq 1} L_V^{q,0} \cong L(\overline{SU}, V).$$

Thanks to the decomposition (25) one has a projection $P\colon L \to \mathfrak{a}$. Notice that the vector space L has a natural $\mathbb{Z}$-grading: $L = \oplus_{n\in\mathbb{Z}} L_n$, where a map $l\colon \overline{S(U \oplus V)} \to U \oplus V$ lies in L_n if it raises the degree by n.

As remarked by Stasheff [48], L is a graded Lie algebra: the isomorphism of graded vector spaces $L \cong Coder(\overline{S(U \oplus V)})$ given in Proposition A.8 allows to define the Lie bracket on L, the *Gerstenhaber bracket*, as the pullback of the graded commutator of coderivations.

Proposition 2.19. *Let U and V be two graded vector spaces equipped with $L_\infty[1]$-algebra structures $\mu = (\mu_i)_{i\geq 1}$ and $\nu = (\nu_j)_{j\geq 1}$, where $\mu_i \in L_U^{i,0}$ and $\nu_j \in L_V^{0,j}$. The following quadruple (with the previous notations) forms a V-data:*

- *the graded Lie algebra L,*
- *its abelian subalgebra $\mathfrak{a}$,*
- *the projection $P\colon L \to \mathfrak{a}$,*
- $\Delta := \mu + \nu$.

Proof. The proof, which uses Lemma A.9, is analogous to the proof of Lemma 3.1 and is therefore left as an exercise to the reader. □

Proposition 2.20. $\Phi \in MC(\mathfrak{a}^P_\Delta) \Leftrightarrow \Phi$ *is a morphism of $L_\infty[1]$-algebras.*

Proof. Fix $\Phi \in \mathfrak{a}_0$. Our aim is to show that the condition for Φ to be a Maurer-Cartan element for the $L_\infty[1]$-algebra $\mathfrak{a}^P_\Delta$ (see Remark 1.9),

$$Pe^{[-,\Phi]}(\mu + \nu) = 0,$$

is equivalent to the condition for Φ to be a morphism of $L_\infty[1]$-algebras, i.e., for all $s \geq 1$ and $u_1, \dots, u_s \in U$:

$$\sum_{I \amalg J = [s]} \Phi_{|J|+1}(\mu_{|I|}(U_I) \cdot U_J) = \sum_{n=1}^{s} \frac{1}{n!} \sum_{I_1 \amalg \cdots \amalg I_n = [s]} \nu_n(\Phi_{|I_1|}(U_{I_1}) \cdot \dots \cdot \Phi_{|I_n|}(U_{I_n})), \tag{26}$$

where $[s] := \{1, \dots, s\}$, $\amalg$ means disjoint union and $U_I = u_{\alpha_1} \cdot \dots \cdot u_{\alpha_j}$ when $I = \{\alpha_1, \dots, \alpha_j\}$. Some of the I_i's in the expression $I_1 \amalg \cdots \amalg I_n = [s]$ can be empty. One will use the convention that $\Phi_{|\emptyset|}(U_\emptyset) = 0$ and $U_I \cdot U_\emptyset = U_I$. Here we decompose Φ as a sum of its homogeneous elements with respect to the polynomial degree, i.e. $\Phi = \sum \Phi_n$ where $\Phi_n \in L_V^{n,0}$.

It will be convenient to view the elements of L as coderivations, because in this case the Lie bracket is the graded commutator. The coderivation corresponding, by Proposition A.8, to Φ (resp. to μ, ν) will be denoted by $\bar{\Phi}$ (resp. $\bar{\mu}$, $\bar{\nu}$).

Φ is a Maurer-Cartan element of the $L_\infty[1]$-algebra $\mathfrak{a}_\Delta^P$ iff

$$P e^{[-,\bar{\Phi}]}(\bar{\mu} + \bar{\nu}) = 0.$$

But, with the notation $ad_\Phi := [-, \Phi]$, one has

$$e^{[-,\bar{\Phi}]} = \sum_{n \geq 0} \frac{1}{n!} {ad_{\bar{\Phi}}}^n,$$

and one can compute ${ad_{\bar{\Phi}}}^n(\bar{\mu})$ and ${ad_{\bar{\Phi}}}^n(\bar{\nu})$ with the expansion

$${ad_{\bar{\Phi}}}^n(\tau) = \sum_{k+l=n} (-1)^k \binom{n}{k} \bar{\Phi}^k \tau \bar{\Phi}^l.$$

Therefore everything boils down to compute terms of the form

$$\bar{\Phi}^k \tau \bar{\Phi}^l (u_1 \cdot \dots \cdot u_s).$$

The results of these computations for $\tau = \bar{\nu}$ and $\tau = \bar{\mu}$ with $n = k + l$ are claims 1 and 2 respectively, and give the two sides of the equation (26).

Claim 1. *The term*

$$pr_V(\bar{\Phi}^k \circ \bar{\nu} \circ \bar{\Phi}^l(U_{[s]}))$$

always vanishes except for $l = n$ for which one has

$$pr_V(\bar{\Phi}^0 \circ \bar{\nu} \circ \bar{\Phi}^n(U_{[s]})) = \sum_{I_1 \amalg \cdots \amalg I_n = [s]} \bar{\nu}_n(\Phi_{|I_1|}(U_{I_1}) \cdot \dots \cdot \Phi_{|I_n|}(U_{I_n})).$$

Claim 2. *The term*

$$pr_V(\bar{\Phi}^k \circ \bar{\mu} \circ \bar{\Phi}^l(U_{[s]}))$$

always vanishes, except for $k = n = 1$ for which one has

$$pr_V(\bar{\Phi}^1 \circ \bar{\mu}(U_{[s]})) = \sum_{I \amalg J = [s]} \Phi_{|J|+1}(\mu_{|I|}(U_I) \cdot U_J).$$

Combining the results of claims 1 and 2 finishes the proof of Proposition 2.20. □

We now state a lemma and use it to prove claims 1 and 2. All along we fix $s \geq 1$ and $u_1, \dots, u_s \in U$.

Lemma 2.21. *For all $t \geq 0$*

$$\bar{\Phi}^t(U_{[s]}) = \sum_{I_1 \amalg \cdots \amalg I_{t+1} = [s]} \Phi_{|I_1|}(U_{I_1}) \cdot \dots \cdot \Phi_{|I_t|}(U_{I_t}) \cdot U_{I_{t+1}}. \tag{27}$$

Proof. Apply formula (49) t times and remark that since Φ admits only elements in U, terms of the form $\Phi(\Phi(U_I) \cdot U_{I'})$ can not appear in the obtained expression. The case $t = 0$ is a convention. □

Proof of claim 1. We apply the formula (49) to $\bar{\nu}$ evaluated on the right hand side of the equation (27), with $t = l$ to get

$$\sum_{I_1 \amalg \cdots \amalg I_{l+2} = [s]} \sum_{J \amalg K = [l]} \nu_{|J|}(\Phi_{|I_{\alpha_1}|}(U_{I_{\alpha_1}}) \cdot \dots \cdot \Phi_{|I_{\alpha_j}|}(U_{I_{\alpha_j}}) \cdot U_{I_{l+1}}) \cdot \Phi_{|I_{\beta_1}|}(U_{I_{\beta_1}}) \cdot \dots \cdot \Phi_{|I_{\beta_k}|}(U_{I_{\beta_k}}) \cdot U_{I_{l+2}},$$

where $\{\alpha_1, \dots, \alpha_j\} = J$ and $\{\beta_1, \dots, \beta_k\} = K$.

Now, since ν admits only elements in U, the term $U_{I_{l+1}}$ must be absent in the previous expression, i.e. one has

$$\bar{\nu} \circ \bar{\Phi}^l(U_{[s]}) = \sum_{I_1 \amalg \cdots \amalg I_{l+1} = [s]} \sum_{J \amalg K = [l]} \nu_{|J|}(\Phi_{|I_{\alpha_1}|}(U_{I_{\alpha_1}}) \cdot \dots \cdot \Phi_{|I_{\alpha_j}|}(U_{I_{\alpha_j}})) \cdot \Phi_{|I_{\beta_1}|}(U_{I_{\beta_1}}) \cdot \dots \cdot \Phi_{|I_{\beta_k}|}(U_{I_{\beta_k}}) \cdot U_{I_{l+1}}.$$

We are interested in evaluating the expression $\bar{\Phi}^k \circ \bar{\nu} \circ \bar{\Phi}^l(U_{[s]})$, with $k + l = n$. By applying Lemma 2.21 with $t = k$ to the last expression, and by the fact that Φ admits only terms in U, one gets

$$\bar{\Phi}^k \circ \bar{\nu} \circ \bar{\Phi}^l(U_{[s]}) = \sum_{I_1 \amalg \cdots \amalg I_{n+1} = [s]} \sum_{J \amalg K = [n]} \nu_{|J|}(\Phi_{|I_{\alpha_1}|}(U_{I_{\alpha_1}}) \cdot \dots \cdot \Phi_{|I_{\alpha_j}|}(U_{I_{\alpha_j}})) \cdot \Phi_{|I_{\beta_1}|}(U_{I_{\beta_1}}) \cdot \dots \cdot \Phi_{|I_{\beta_k}|}(U_{I_{\beta_k}}) \cdot U_{I_{n+1}}.$$

Finally, if one considers the terms in the above formula which belong to V, one has

$$pr_V(\bar{\Phi}^k \circ \bar{\nu} \circ \bar{\Phi}^l(U_{[s]})) = \sum_{I_1 \amalg \cdots \amalg I_n = [s]} \nu_n(\Phi_{|I_1|}(U_{I_1}) \cdot \dots \cdot \Phi_{|I_n|}(U_{I_n})).$$

□

Proof of claim 2. We apply the formula (49) to $\bar{\mu}$ evaluated on the right hand side of the equation (27), with $t = l$ and remark that since μ admits only elements in U, terms of the form $\mu(\Phi(U_I) \cdot U_{I'})$ can not appear in the obtained expression. Therefore one has

$$\bar{\mu} \circ \bar{\Phi}^l(U_{[s]}) = \sum_{I_1 \amalg \cdots \amalg I_{l+2} = [s]} \Phi_{|I_1|}(U_{I_1}) \cdot \dots \cdot \Phi_{|I_l|}(U_{I_l}) \cdot \mu_{|I_{l+1}|}(U_{I_{l+1}}) \cdot U_{I_{l+2}}.$$

We now evaluate $\bar{\Phi}^k \circ \bar{\mu} \circ \bar{\Phi}^l(U_{[s]})$ by applying Lemma 2.21 to the previous expression, with $t = k$. Since Φ admits only elements in U, terms of the form $\Phi(\Phi(U_I) \cdot U_{I'})$ can not appear in the obtained expression. Hence one gets (remember that $n = k + l$)

$$\sum_{I_1 \amalg \cdots \amalg I_{n+2} = [s]} \Phi_{|I_1|}(U_{I_1}) \cdot \cdots \cdot \Phi_{|I_n|}(U_{I_n}) \cdot \mu_{|I_{n+1}|}(U_{I_{n+1}}) \cdot U_{I_{n+2}}$$
$$+ \sum_{I_1 \amalg \cdots \amalg I_{n+2} = [s]} \Phi_{|I_1|}(U_{I_1}) \cdot \cdots \cdot \Phi_{|I_n|+1}(U_{I_n} \cdot \mu_{|I_{n+1}|}(U_{I_n+1})) \cdot U_{I_{n+2}}.$$

In the previous expression, there are terms which belong to V only if n=k=1. In this case one has

$$pr_V(\bar{\Phi} \circ \bar{\mu}(U_{[s]})) = \sum_{I \amalg J = [s]} \Phi_{|J|+1}(\mu_{|I|}(U_I) \cdot U_J).$$

□

Prop. 2.19 and Prop. 2.20 allow us to apply Prop. 1.12 and Thm. 3 and deduce:

Corollary 2.22. *Let U, V be $L_\infty[1]$-algebras and $\Phi \in L(\overline{SU}, V)$ a $L_\infty[1]$-morphism from U to V and let $(L, \mathfrak{a}, P, \Delta)$ as in Prop. 2.19.*

1) Let $\tilde{\Phi} \in L_0(\overline{SU}, V) = \mathfrak{a}_0$. Then

$$\Phi + \tilde{\Phi} \textit{ is an } L_\infty[1]\textit{-morphism} \quad \Leftrightarrow \quad \tilde{\Phi} \in MC(\mathfrak{a}_\Delta^{P_\Phi}).$$

2) For all degree one coderivations $\tilde{Q}_U$ on $\overline{SU}$ and $\tilde{Q}_V$ on $\overline{SV}$ and for all $\tilde{\Phi} \in L_0(\overline{SU}, V)$:

$$\begin{cases} Q_U + \tilde{Q}_U \textit{ and } Q_V + \tilde{Q}_V \textit{ define } L_\infty[1]\textit{-algebra structures on } U, V \\ \Phi + \tilde{\Phi} \textit{ is a } L_\infty[1]\textit{-morphism between these } L_\infty[1]\textit{-algebra structures} \end{cases}$$
$$\Leftrightarrow \; ((\tilde{Q}_U + \tilde{Q}_V)[1], \tilde{\Phi}) \in MC((L[1] \oplus \mathfrak{a})_\Delta^{P_\Phi})$$

Remark 2.23. We have a direct product decomposition $L = \prod_{k \geq -1} L^k$ where $L^k := L_U^{k+1,\bullet} \oplus L_V^{k,\bullet}$. Here we use the short-hand notation $L_V^{k,\bullet} := \prod_{r \geq 0} L_V^{k,r}$. By Remark 1.15, $\mathcal{F}^n L := \prod_{k \geq n} L^k$ is then a complete filtration of the vector space L. One checks easily that $(L, \mathfrak{a}, P, \Delta)$ is filtered V-data (Def. 1.17).

3 Applications to associative algebras

In this section we treat the case of a morphism between two associative algebras. The cohomology theory governing simultaneous deformations of two associative algebras and a morphism between them has been defined in the context of cohomology of diagrams by M. Gerstenhaber and S.D. Schack in [17]. One of the problem remaining was the fact that the deformation equation could not be written as a Maurer-Cartan equation for a DGLA. The first author, Markl and Yau in [13] exhibited a L_∞-algebra which enabled to write this deformation equation as a Maurer-Cartan equation. This was based on operadic techniques. We show in this section how we can recover these results by means of our Thm. 3, which requires much less technology.

3.1 Morphisms of associative algebras

We will use some notations introduced in the previous section §2.5. Moreover, if E and F are two vector spaces, for any $n \geq 1$ and $I \amalg J = [n] := \{1, \ldots, n\}$, consider the notation

$$T^{I,J}(E, F) := \{x_1 \otimes \cdots \otimes x_n \in T^n(E \oplus F) : x_k \in E \text{ when } k \in I, x_k \in F \text{ otherwise } \}.$$

One has the decomposition

$$T^n(E \oplus F) = \bigoplus_{I \amalg J = [n]} T^{I,J}(E, F),$$

and therefore one can consider the projection $\Pi^{I,J}$ onto $T^{I,J}(E, F)$. One considers also the canonical projection Π_E (resp Π_F) from $E \oplus F$ onto E (resp F).

One will denote the set of n-linear maps from E to F by $L^n(E, F) := L(T^n E, F)$ and by $L^n(E) := L^n(E, E)$ when $E = F$. One has the decomposition:

$$L^n(E \oplus F) = \bigoplus_{I \amalg J = [n]} L_E^{I,J} \oplus L_F^{I,J}, \tag{28}$$

for $L_E^{I,J} := \{\Pi_E \circ l \circ \Pi^{I,J} : l \in L^n(E \oplus F)\}$. The decomposition (28) defines a projection

$$P \colon \prod\nolimits_{n \geq 1} L^n(E \oplus F) \to \oplus_{n \geq 1} L_F^{[n],\emptyset}.$$

Consider a morphism $\Phi \colon U \to V$ between two associative algebras (U, μ) and (V, ν), apply the above notations to $E := U[1]$ and $F := V[1]$, and consider μ and ν as elements of $L^2(U[1])$ and $L^2(V[1])$. As noticed by Stasheff in [48], the canonical identification $\prod_{n \geq 1} L^n(E \oplus F) \cong Coder(\overline{T(E \times F)})$ of Prop. A.8 makes $L := \prod_{n \geq 1} L^n(E \oplus F)$ into a graded Lie algebra.

Lemma 3.1. *The following quadruple forms a V-data:*

- *the graded Lie algebra* $L := \prod_{n \geq 0} L_n$ *with* $L_n := L^{n+1}((U \oplus V)[1])$ *with Gerstenhaber bracket* $[\cdot, \cdot]$
- *its abelian subalgebra* $\mathfrak{a} = \prod_{n \geq 0} \mathfrak{a}_n$ *with* $\mathfrak{a}_n := L_{V[1]}^{[n+1],\emptyset} \cong L(T^{n+1}U[1], V[1])$
- *the natural projection* $P \colon L \to \mathfrak{a}$ *given above*
- $\Delta := \mu + \nu$

hence by Thm. 1 we obtain a $L_\infty[1]$*-structure* $\mathfrak{a}_\Delta^P$*.*

$\Phi \in \mathfrak{a}_0$ *is a Maurer-Cartan element in* $\mathfrak{a}_\Delta^P$ *iff* Φ *is a morphism of associative algebras between* μ *and* ν*.*

Proof. To see that $\mathfrak{a}$ is an abelian graded Lie subalgebra of L, remark that elements of $\mathfrak{a}$ are maps which produce vectors in V and accept only terms in U. Therefore their composition is zero.

Next we show that $KerP$ is a graded Lie subalgebra of L. To this aim use the decomposition $KerP = A \oplus B$ where

$$A_n = \bigoplus_{I \amalg J = [n], |J|>0} L^{I,J}_{V[1]},$$
$$B_n = \bigoplus_{I \amalg J = [n]} L^{I,J}_{U[1]}.$$

Let $\alpha, \alpha' \in A, \beta \in B$ and $\gamma \in KerP$. One has $\alpha \circ \beta, \alpha \circ \alpha' \in A$ and $\beta \circ \gamma \in B$, showing that $KerP = A \oplus B$ is closed under the Gerstenhaber bracket. Further since $\nu \in A$ and $\mu \in B$, one has $\Delta \in KerP$.

Last we show that $[\Delta, \Delta] = 0$. Indeed,

$$[\Delta, \Delta] = [\mu, \mu] + [\nu, \nu] + 2[\mu, \nu].$$

Since μ and ν are associative algebras, by Prop. A.11, $[\mu, \mu]$ and $[\nu, \nu]$ vanish. Now, by definition of the bracket, $[\mu, \nu] = \mu \circ (\nu \otimes Id - Id \otimes \nu) - \nu \circ (\mu \otimes Id - Id \otimes \mu)$ but μ accepts only terms in V, whereas ν produces elements in U, hence the first summand of the right hand side vanishes. Similarly for the second summand. This concludes the proof that $(L, \mathfrak{a}, P, \Delta)$ forms a V-data.

Fix $\Phi \in \mathfrak{a}_0 = L(U[1], V[1])$. It will be convenient to view the elements of L as coderivations, because in this case the Lie bracket is the graded commutator. The coderivation corresponding to Φ (Proposition A.8) will be denoted by $\bar{\Phi}$. It is characterized by its only non vanishing corestriction, which is $\bar{\Phi}^1_1(u + v) = \Phi(u)$ where $u \in U$ and $v \in V$.

By Remark 1.9, Φ is a Maurer-Cartan element of the $L_\infty[1]$-algebra $\mathfrak{a}^P_\Delta$ iff

$$Pe^{[-,\bar{\Phi}]}(\mu + \nu) = 0. \tag{29}$$

Since

$$e^{[-,\bar{\Phi}]} = \sum_{n \geq 0} \frac{1}{n!} ad_{\bar{\Phi}}{}^n,$$

writing $ad_\Phi := [-, \Phi]$, we compute $ad_{\bar{\Phi}}{}^n(\mu)$ and $ad_{\bar{\Phi}}{}^n(\nu)$ with the expansion

$$ad_{\bar{\Phi}}{}^n(\tau) = \sum_{k=0}^{n} (-1)^k \bar{\Phi}^k \tau \bar{\Phi}^{n-k}.$$

Let us first remark that the commutator of a linear coderivation and a quadratic coderivation gives a quadratic coderivation. In particular $ad_{\bar{\Phi}}{}^n(\nu)$ and $ad_{\bar{\Phi}}{}^m(\mu)$ are quadratic coderivations and hence are only determined by their second Taylor coefficient, i.e. by their restriction to elements of $T^2(U \oplus V)$.

One observes that for elements x_1, x_2 in U (for elements in V, the expression would vanish),

$$\bar{\Phi}^2(x_1 \otimes x_2) = \bar{\Phi}(\Phi(x_1) \otimes x_2 + x_1 \otimes \Phi(x_2)) = 2\Phi(x_1) \otimes \Phi(x_2)$$

lies in $T^2(V)$. Therefore, Φ can not be applied anymore, meaning that $\bar{\Phi}^n(x_1 \otimes x_2) = 0$ for all $n > 2$. For the same reason, if τ has only quadratic Taylor coefficients, one has necessary

$\bar{\Phi}^n\tau_{|T^2(U\oplus V)} = 0$ for $n > 1$, and even $\bar{\Phi}\tau_{|T^2(U\oplus V)} = 0$ when the quadratic Taylor coefficients of τ have values in V. All these remark imply that the only non-vanishing $ad_{\bar{\Phi}}{}^n(\nu)_{|T^2(U\oplus V)}$ occurs for $n = 2$:

$$ad_{\bar{\Phi}}{}^2(\nu)(x_1 \otimes x_2) = 2\nu(\Phi(x_1) \otimes \Phi(x_2))$$

and the only non-vanishing $ad_{\bar{\Phi}}{}^n(\mu)_{|T^2(U\oplus V)}$ occurs for $n = 1$:

$$ad_{\bar{\Phi}}(\mu)(x_1 \otimes x_2) = -\Phi(\mu(x_1 \otimes x_2).$$

Since μ and ν commute, we obtain that the l.h.s. of eq. (29) is

$$Pe^{[\mu+\nu,\bar{\Phi}]}(x_1 \otimes x_2) = \nu(\Phi(x_1) \otimes \Phi(x_2)) - \Phi(\mu(x_1 \otimes x_2).$$

Hence $\bar{\Phi}$ satisfies eq. (29) iff Φ is a morphism of associative algebras. $\square$

To establish the connection with the problem of simultaneous deformations of morphisms and associative algebras, one considers the graded Lie subalgebra L' of L defined by

$$L'_i = L^{i+1}(U[1]) \;\oplus\; L^{i+1}(V[1]).$$

Thm. 3 and Remark 1.10 (which applies to L' since it contains Δ) imply:

Corollary 3.2. *Let (U,μ) and (V,ν) be associative algebras and $\Phi\colon U \to V$ a morphism of associative algebras. Let $(L,\mathfrak{a},P,\Delta)$ as in Lemma 3.1 and L' as above.*

For all $\tilde{\mu}+\tilde{\nu} \in L'_1$, and for all linear maps $\tilde{\Phi}\colon U \to V$:

$$\begin{cases} \mu+\tilde{\mu} \text{ and } \nu+\tilde{\nu} \text{ define associative algebra structures on } U \text{ and } V \\ \Phi+\tilde{\Phi} \text{ is an associative algebra morphism between these new associative algebra structures} \end{cases}$$

$\Leftrightarrow ((\tilde{\mu}+\tilde{\nu})[1],\tilde{\Phi}) \in MC((L'[1]\oplus\mathfrak{a})^{P_\Phi}_{\Delta})$.

Remark 3.3. Analogously to Remark 2.4, we have a direct product decomposition $L = \prod_{k\geq -1}\mathcal{L}^k$ where $\mathcal{L}^k := \bigoplus_{|I|=k+1} L^{I,\bullet}_{U[1]} \oplus \bigoplus_{|I|=k} \mathcal{L}^{I,\bullet}_{V[1]}$. Then $\mathcal{F}^n L := \prod_{k\geq n} L^k$ is a complete filtration of the vector space L by Remark 1.15. One checks easily that $(L,\mathfrak{a},P,\Delta)$ is filtered V-data (Def. 1.17).

We now write out explicitly the multi-brackets of $(L'[1]\oplus\mathfrak{a})^{P_\Phi}_{\Delta}$.

Let us denote by $\mathcal{P}_m[n]$ the set of ordered m-tuples of distinct points in $\{1,\dots,n+1\}$. For any $I \in \mathcal{P}_m[n]$ we will denote by $x_V \circ_I (a_1,\dots,a_n)$, the element obtained by plugging a_i into the I_i-th input of x_V, and by $x_V \circ_{I,\Phi} (a_1,\dots,a_n)$ the element obtained by further plugging Φ in the $n+1-m$ remaining inputs of x_V. Similarly, $a \circ_i \mu$ will mean the composition of a by μ at its $i$ith input. We will also use the notations

$$da = \nu(a\otimes\Phi) + \nu(\Phi\otimes a) - (-1)^n \sum_{i=1}^{n} a \circ_i \mu$$

and $d^\mu x_U = [\mu, x_U]$. With these notations, explicit formulas are given by:

Proposition 3.4. *Let (U,μ) and (V,ν) be associative algebras and $\Phi\colon U\to V$ a morphism of associative algebras, and adopt the notation of Corollary 3.2. The $L_\infty[1]$-multi-brackets of $(L'[1]\oplus\mathfrak{a})^{P_\Phi}_\Delta$ are given as follows:*

Given $(x[1],a)\in(L'[1]\oplus\mathfrak{a})_n$, i.e $x=(x_U,x_V)\in L^{n+1}(U[1])\oplus L^{n+1}(V[1])$ and $a\in L^n(U[1],V[1])$, one has

$$d(x[1],a)=(-d^\mu x_U-d^\nu x_V,-\Phi\circ x_U+x_V\circ\Phi^{\otimes^n}+da) \tag{30}$$

and

$$\{x,a\}=\sum_{i\in[n+1]}x_V\circ_{i,\Phi}a-(-1)^{|x||a|}\sum_j a\circ_j x_U. \tag{31}$$

If we moreover consider $a_1,\dots,a_m\in\mathfrak{a}$ (for $m\geq 2$), then one has

$$\{x,a_1,\dots,a_m\}=\sum_{I\in\mathcal{P}_m[n]}\epsilon(I)x_V\circ_{I,\Phi}(a_1,\dots,a_m) \tag{32}$$

$$\{a_1,a_2\}=\mu(a_1\otimes a_2). \tag{33}$$

Remark 3.5. In [13] formulas were given for an L_∞-algebra governing simultaneous deformations of associative algebras and their morphisms. Those formulas agree with the formulas of Prop. 3.4 modulo signs, which come from the fact that here we only give the $L_\infty[1]$-algebra multibrackets. If one wants to recover the original formulas of [13], one needs to desuspend this $L_\infty[1]$-algebra as indicated in Remark 1.5.

Proof. We first prove (30). It suffices to explicit the expression (5) of Theorem 2, therefore we will determine (a) $-D(x)[1]$ and (b) $P_\Phi(x+Da)$, where $D=[\Delta,\cdot]$.

(a) Since μ and x_V can not be composed, $[\mu,x_V]=0$ and hence $D(x_V)=[\nu,x_V]$. Since the similar result for x_U holds, one gets

$$-D(x)[1]=-d^\nu x_V-d^\mu x_U.$$

(b) Since a can not be composed on the right by ν and on the left by μ, one has $Da=\nu(a\otimes id)+\nu(id\otimes a)-(-1)^{|\mu||a|}\sum_{i=1}^n a\circ_i\mu$. In particular Da has only outputs in V, therefore Φ can only be right composed. Moreover it can only be right composed once since each of the summands of Da has at most one V input. Therefore $e^{[-,\Phi]}Da=Da+[Da,\Phi]$. After a look at the terms surviving the projection P, one gets

$$P_\Phi(Da)=\nu(a\otimes\Phi)+\nu(\Phi\otimes a)-(-1)^n\sum_{i=1}^n a\circ_i\mu.$$

One now remarks that in $e^{[-,\Phi]}x$, the only terms surviving the projection P are

$$P_\Phi x=-\Phi\circ x_U+x_V\circ\Phi^{\otimes^n},$$

therefore, identifying the terms in (5) gives (30).

We now prove (31). By definition of the Gerstenhaber bracket, one has

$$[x_U+x_V,a]=\sum_{i\in[n+1]}(x_U\circ_i a+x_V\circ_i a)-(-1)^{|x||a|}\sum_j(a\circ_j x_U+a\circ_j x_V).$$

But in this expression $x_U \circ_i a$ and $a \circ_i x_V$ vanish by incompatibility of the compositions. Now $P_\Phi(a \circ_j x_U) = a \circ_j x_U$ and $P_\Phi(x_V \circ_i a) = x_V \circ_{i,\Phi} a$, so one has proven (31), i.e.

$$\{x, a\} = \sum_{i \in [n+1]} x_V \circ_{i,\Phi} a - (-1)^{|x||a|} \sum_j a \circ_j x_U.$$

We now prove (32) for $m \geq 2$ by induction on m. Let us first start the induction by showing that

$$[[x, a_1], a_2] = \sum_{i,j} \epsilon(i,j) x_V \circ_I (a_1, a_2). \tag{34}$$

Let us remark that an element of L which has only U inputs and one V output can not be composed to the right or to the left by an element in $\mathfrak{a}$. This in particular applies to the element $[x_U, a_1]$, therefore one has $[[x_U, a_1], a_2] = 0$. Moreover, one has seen that

$$[x_V, a_1] = \sum_{i \in [n+1]} x_V \circ_i a_1.$$

But this term has one V output, therefore can not be left composed by a_2. This means, again by definition of the Gerstenhaber bracket, that one obtains eq. (34).

Let us now prove by induction that

$$[\dots [x, a_1], \dots, a_m] = \sum_{I \in \mathcal{P}_m[n]} \epsilon(I) x_V \circ_I (a_1, \dots, a_m).$$

We make the following observation (*Obs*): this element has a V output and therefore can not be composed to the left by an element in $\mathfrak{a}$. One has:

$$\begin{aligned}
[[\dots [x, a_1], \dots, a_m], a_{m+1}] &= [\sum_{I \in \mathcal{P}_m[n]} \epsilon(I) x_V \circ_I (a_1, \dots, a_m)\ ,\ a_{m+1}] \\
&\overset{Obs}{=} \sum_{I \in \mathcal{P}_m[n]} \epsilon(I) \sum_{i \in I^c} (x_V \circ_I (a_1, \dots, a_m)) \circ_i a_{m+1} \\
&= \sum_{I \in \mathcal{P}_{m+1}[n]} \epsilon(I) x_V \circ_I (a_1, \dots, a_{m+1}),
\end{aligned} \tag{35}$$

where in the first equality we used the induction step. It remains to apply the projection P_Φ. The above observation (*Obs*) applies in particular to the element Φ, therefore

$$e^{[-,\Phi]} x_V \circ_I (a_1, \dots, a_{m+1}) = \sum_{n \geq 0} \frac{1}{n!} x_V \circ_I (a_1, \dots, a_{m+1}) \circ \bar{\Phi}^n.$$

If one now compose this last equality with the projection P, one gets

$$P_\Phi(x_V \circ_I (a_1, \dots, a_{m+1})) = x_V \circ_{I,\Phi} (a_1, \dots, a_{m+1}).$$

Combining this last equality with (35) gives the result.

It remains to prove (33). But formula (7) is formally formula (6) with x replaced by Δ. Therefore one can compute the remaining brackets by replacing x by $\mu + \nu$ in formula (32). The only possibility is for $n = 2$, for which one gets (33). $\square$

4 Applications to algebras over Koszul operads

The objective of this short section is to indicate how the techniques used in §2.5 and §3 work for other types of algebras. The theory of Koszul duality for operads (see [21] or [34]), provides for a type of algebra $\mathcal{P}$, i.e. for an operad $\mathcal{P}$ (for example for the operad $\mathcal{A}s$, encoding the type of associative algebras), a cooperad $\mathcal{P}^{¡}$. In this setting, given a graded vector space U, one can define $(\mathcal{P}^{¡}(U), \delta)$, the cofree coalgebra of type $\mathcal{P}^{¡}$ co-generated by U. Since it is cofree, one has the identification, as vector spaces:

$$\mathcal{P}^{¡}(U^*) \otimes U \simeq Coder(\mathcal{P}^{¡}(U)).$$

By Remark A.5, $Coder(\mathcal{P}^{¡}(U))$ carries naturally the structure of a graded Lie algebra $[-,-]$, which can be pulled-back to $\mathcal{P}^{¡}(U^*) \otimes U$. An *algebra μ of type $\mathcal{P}_\infty$*, or *homotopy $\mathcal{P}$-algebra* on the vector space U can then be defined as an element $\mu \in \mathcal{P}^{¡}(U[1]^*) \otimes U[1]$ of internal degree 1 satisfying $[\mu,\mu]$=0. One can recover $\mathcal{P}$-algebras as the quadratic homogenous $\mathcal{P}_\infty$-algebras.

We are interested in deforming simultaneously two homotopy $\mathcal{P}$-algebras (U,μ) and (V,ν) and $\Phi\colon U \to V$ a morphism between them. The vector space $\mathcal{P}^{¡}(V)$ carries a polynomial grading, and one considers

$$L^i := \mathcal{P}^{¡^i}((U[1] \oplus V[1])^*) \otimes (U[1] \oplus V[1]).$$

The graded Lie algebra $L := \oplus_{i\geq 1} L^i$ admits an abelian subalgebra $\mathfrak{a} = \oplus_{i\geq 1}\mathfrak{a}^i$ with $\mathfrak{a}^i := \mathcal{P}^{¡^i}(U[1]^*) \otimes V[1]$. But one needs to work with the internal grading instead of the polynomial grading, and one will denote by $L := \oplus_{i\geq 1} L_i$ and $\mathfrak{a} = \oplus_{i\geq 1}\mathfrak{a}_i$ their decompositions in homogenous subspaces for the internal grading.

We believe that for any instance of Koszul operad $\mathcal{P}$, and any homotopy $\mathcal{P}$-algebras (U,μ) and (V,ν), the following Ansatz holds true.

Ansatz 4.1. *The following quadruple forms a filtered V-data (Def. 1.17):*

- *the graded Lie algebra* $L := \mathcal{P}^i((U \oplus V)[1]^*) \otimes (U \oplus V)[1]$ *with bracket* $[\cdot,\cdot]$
- *its abelian subalgebra* $\mathfrak{a} := \mathcal{P}^i(U[1]^*) \otimes V[1]$
- *the natural projection* $P\colon L \to \mathfrak{a}$
- $\Delta := \mu + \nu$.

Further, denoting by $\mathfrak{a}^P_\Delta$ the $L_\infty[1]$-algebra obtained by Thm. 1:
$\Phi \in \mathfrak{a}_0$ *lies in* $MC(\mathfrak{a}^P_\Delta)$ *iff* Φ *is a morphism of* $\mathcal{P}_\infty$*-algebras between* μ *and* ν*.*

Applying Thm. 3 we obtain:

Corollary 4.2. *Let $\mathcal{P}$ be a Koszul operad, (U,μ) and (V,ν) be $\mathcal{P}_\infty$-algebras and $\Phi\colon U \to V$ be a morphism of $\mathcal{P}_\infty$-algebras. Assume that Ansatz 4.1 holds true for the corresponding V-data $(L,\mathfrak{a},P,\Delta)$ and that Φ defines an element of L. Let $L' := \mathcal{P}^i(U[1]^*) \otimes U[1] \oplus \mathcal{P}^i(V[1]^*) \otimes V[1]$.*

Then for all $\tilde{\mu}+\tilde{\nu}$ in L'_1, and for all $\tilde{\Phi} \in \mathfrak{a}_0$:

$$\begin{cases} \mu+\tilde{\mu} \text{ and } \nu+\tilde{\nu} \text{ define } \mathcal{P}_\infty\text{-algebra structures on } U \text{ and } V \\ \Phi+\tilde{\Phi} \text{ is an } \mathcal{P}_\infty\text{-algebra morphism between these new } \mathcal{P}_\infty\text{-algebra structures} \end{cases}$$
$$\Leftrightarrow ((\tilde{\mu}+\tilde{\nu})[1], \tilde{\Phi}) \text{ is a MC element of } (L'[1]\oplus\mathfrak{a})^{P_\Phi}_\Delta.$$

Let us illustrate this in the case $\mathcal{P}=\mathcal{A}s$. It is well known that $\mathcal{A}s^{\text{i}}=\mathcal{A}s$ and that the free coassociative coalgebra on a vector space U is given by the tensor coalgebra, therefore

$$\mathcal{P}^{\text{i}}(U^*)\otimes U = T(U^*)\otimes U.$$

So in particular, Proposition 3.1 is nothing else than Ansatz 4.1 for $\mathcal{P}=\mathcal{A}s$, with U and V graded vector spaces concentrated in degree 0. In particular μ and ν must be associative algebras and not arbitrary $\mathcal{A}_\infty$-algebras.
Another illustration is given if we take $\mathcal{P}=\mathcal{L}ie$, the Lie operad. One has $\mathcal{L}ie^{\text{i}}=\mathcal{C}om$ (the cooperad of cocommutative coalgebras) and the free cocommutative coalgebra on a vector space U is given by the symmetric coalgebra, therefore

$$\mathcal{P}^{\text{i}}(U^*)\otimes U = S(U^*)\otimes U.$$

This fact enables to recognize Proposition 2.19 as Ansatz 4.1 in disguise.

5 Applications to Poisson geometry

In this section we apply the machinery developed in §1 to examples related to Poisson geometry.

5.1 Coisotropic submanifolds of Poisson manifolds

In this subsection we consider deformations of Poisson structures on a fix manifold M and deformations of coisotropic submanifolds. We build on work of Oh and Park [38], who realized that deformations of a coisotropic submanifold of a symplectic manifold are governed by a $L_\infty[1]$-algebra, and on work of Cattaneo and Felder [7] who associate an $L_\infty[1]$-algebra to any coisotropic submanifold of a Poisson manifold.

Our main reference for this deformation problem is [45, §3.2], which is based on [38] and [7]. Recall that a *Poisson structure* on M is a bivector field π on M such that $[\pi,\pi]=0$, where the bracket denotes the Schouten bracket, and that a submanifold $C\subset(M,\pi)$ is *coisotropic* if $\pi^\sharp TC^\circ \subset TC$, where $TC^\circ := \{\xi\in T^*M|_C : \xi|_{TC}=0\}$ and $\pi^\sharp\colon T^*M\to TM$ is the contraction with π [6].

Let M be a manifold. Let $C\subset M$ be a submanifold. Fix an embedding of the normal bundle $\nu C := TM|_C/TC$ into a tubular neighborhood of C in M, such that the embedding and its derivative are the identity on C. In the following we will identify νC with its image in M.

We say that a vector field on νC is *fiberwise polynomial* if it preserves the fiberwise polynomial functions on the vector bundle νC. Such a vector field X has *polynomial degree* n (denoted $|X|_{pol}=n$) if its action on fiberwise polynomial functions raises their degree (as polynomials) at most by n. Locally, choose local coordinates on C and linear coordinates along the fibers of νC, which we denote collectively by x and p respectively. Then the fiber-wise polynomial vector fields are exactly those which are sums of expressions $f_1(x)F_1(p)\frac{\partial}{\partial x}$ and $f_2(x)F_2(p)\frac{\partial}{\partial p}$ where $f_i \in C^\infty(C)$ and the F_i are polynomials. The polynomial degrees of the two vector fields exhibited here are $deg(F_1)$ and $deg(F_2) - 1$ respectively.

Consider $\chi^\bullet(\nu C)$, the space of multivector fields on the total space νC, and denote by $\chi^\bullet_{fp}(\nu C)$ the sums of products of fiberwise polynomial vector fields. $\chi^\bullet(\nu C))[1]$ is a graded Lie algebra when endowed with the Schouten bracket $[\cdot,\cdot]$, and $\chi^\bullet_{fp}(\nu C)[1]$ is a graded Lie subalgebra. The notion of polynomial degrees carries on to fiberwise polynomial multivector fields, by $|X_1 \wedge \cdots \wedge X_k|_{pol} = \sum_i |X_i|_{pol}$. The Schouten bracket preserves the polynomial degree (this is clear if we think of multivector fields as acting on tuples of functions).

Sections in $\Gamma(\wedge \nu C)$ can be regarded as elements of $\chi^\bullet_{fp}(\nu C)$ which are vertical (tangent to the fibers) and fiberwise constant. A *fiberwise polynomial Poisson* bivector field on νC is an element $\pi \in \chi^2_{fp}(\nu C)$ such that $[\pi,\pi] = 0$. Notice that the associated Poisson bracket raises the degree of fiberwise polynomial functions on νC by at most $|\pi|_{pol}$.

Remark 5.1. The condition that a Poisson structure be fiberwise polynomial is quite strong. For instance, if C is an arbitrary coisotropic submanifold of a symplectic manifold (M,ω), it does not seem possible to find an embedding of νC in M for which the bivector field ω^{-1} is fiberwise polynomial. This seems to fail even if one works locally (the coordinate expression for ω in [38, eq. (6.8)], once inverted, is fiberwise analytic but not fiberwise polynomial). We expect to be able to extend the results of this subsection to fiberwise analytic Poisson structures.

Lemma 5.2. *Let π be a fiberwise polynomial Poisson structure on νC. The following quadruple forms a curved V-data:*

- *the graded Lie algebra $L := \chi^\bullet_{fp}(\nu C)[1]$*
- *its abelian subalgebra $\mathfrak{a} := \Gamma(\wedge \nu C)[1]$*
- *the natural projection $P\colon L \to \mathfrak{a}$ given by restriction to C and projection along $\wedge T(\nu C)|_C \to \wedge \nu C$*
- *$\Delta := \pi$,*

hence by Thm. 1 we obtain a curved $L_\infty[1]$-structure $\mathfrak{a}^P_\Delta$.

Its Maurer-Cartan equation reads

$$P \sum_{n=0}^{|\pi|_{pol}+2} \frac{1}{n!}[[\ldots[\pi, \underbrace{\Phi], \ldots], \Phi]}_{n \ times} = 0, \tag{36}$$

where $\Phi \in \Gamma(\nu C)[1]$ is seen as a vertical vector field. $\Phi \in \Gamma(\nu C)[1]$ is a Maurer-Cartan element in $\mathfrak{a}^P_\Delta$ iff $graph(-\Phi)$ is a coisotropic submanifold of $(\nu C, \pi)$.

Further, the above quadruple forms a V-data iff C is a coisotropic submanifold of $(\nu C, \pi)$.

Proof. The fact that the above quadruple forms a curved V-data is essentially the content of [7, §2.6]. For a more detailed proof we refer to [45, Lemma 3.3 in §3.3], use that $\chi^\bullet_{fp}(\nu C)$

is a graded Lie subalgebra of $\chi^\bullet(\nu C)$, and notice the $[\pi,\pi]=0$ by definition of Poisson structure.

To prove eq. (36) we argue as follows. Elements $a_i \in \mathfrak{a}_0 = \Gamma(\nu C)[1]$, seen as vertical vector fields on νC, have polynomial degree -1 (in coordinates they read $f(x)\frac{\partial}{\partial p}$). Since the Schouten bracket preserves the polynomial degree, $[[\dots[\pi,a_1],\dots],a_n]$ has polynomial degree $|\pi|_{pol}-n$. Since the polynomial degree of a non-vanishing bivector field is ≥ -2, we conclude that the above iterated brackets vanish for $n > |\pi|_{pol}+2$.

The equivalence[6] between $\Phi \in \Gamma(\nu C)[1]$ being a Maurer-Cartan element and $graph(-\Phi)$ being a coisotropic submanifold of $(\nu C,\pi)$ is proven in a separate note [14]. The idea is that the bivector field $e^{[\cdot,\Phi]}\pi$ is the pushforward of π by the flow of the vector field Φ, that this flow maps $graph(-\Phi)$ to C, and to interpret eq. (36) as saying that C is coisotropic w.r.t. $e^{[\cdot,\Phi]}\pi$. Notice that the curved $L_\infty[1]$-structure on $\Gamma(\wedge\nu C)[1]$ depends only on the jets in fiber-directions of π along C; this is clear by [7, Prop. 2.1] or eq. (36) above.

For the last statement, use Thm. 1 and notice that C is coisotropic iff we can write $\pi = \sum_j X_j \wedge Y_j$ with X_j tangent to C, i.e. iff $\pi \in ker(P)$. □

Hence we can apply Cor. 1.13 (choosing $\pi = 0$ above):

Corollary 5.3. *Let C be a submanifold of a manifold, and consider a tubular neighborhood νC. For all $\tilde\pi \in \chi^2_{fp}(\nu C)$ and $\tilde\Phi \in \Gamma(\nu C)$:*

$$\begin{cases} \tilde\pi \textit{ is a Poisson structure} \\ graph(-\tilde\Phi) \textit{ is a coisotropic submanifold of } (\nu C,\tilde\pi) \end{cases}$$
$$\Leftrightarrow (\tilde\pi[2],\tilde\Phi[1]) \textit{ is a MC element of the } L_\infty\textit{[1]-algebra } \chi^\bullet_{fp}(\nu C)[2] \oplus \Gamma(\wedge\nu C)[1].$$

The above $L_\infty[1]$-algebra structure is given by the multibrackets[7] (all other vanish)

$$\begin{aligned} d(X[1]) &= PX, \\ \{X[1],Y[1]\} &= [X,Y][1](-1)^{|X|}, \\ \{X[1],a_1,\dots,a_n\} &= P[\dots[X,a_1],\dots,a_n] \qquad \textit{for all } n\geq 1 \end{aligned}$$

where $X,Y \in \chi^\bullet_{fp}(\nu C)[1]$, $a_1,\dots,a_n \in \Gamma(\wedge\nu C)[1]$, and $[\cdot,\cdot]$ denotes the Schouten bracket on $\chi^\bullet_{fp}(\nu C)[1]$.

5.2 Dirac structures and Courant algebroids

In this subsection we consider a Courant algebroid structure on a fixed vector bundle and a Dirac subbundle A. We study deformations of the Courant algebroid structure (with the constraint that A remains Dirac for the new Courant algebroid), and of the Dirac subbundle A. Deformations of Dirac subbundles within a fixed Courant algebroid were studied by Liu, Weinstein, Xu [33] and by Bursztyn, Crainic, Ševera [5]. We will make use of facts from [5, §3] and Roytenberg's [44, §3] [42, §3][43]. We refer to §A.1 and to [45, §1.4] or [8] for some basic facts on graded geometry.

[6] See [45, Ex. 3.2 in §4.3] for an example where π is not fiberwise polynomial and the correspondence fails.

[7] The formulas for the multibrackets show that the Maurer-Cartan equation for $(\tilde\pi[2],\tilde\Phi[1])$ has at most $|\tilde\pi|_{pol}+2$ terms, by the same argument as in Lemma 5.2.

Recall that a *Courant algebroid* consists of a vector bundle $E \to M$ with a non-degenerate symmetric pairing on the fibers, a bilinear operation $[\![\cdot,\cdot]\!]$ on $\Gamma(E)$, and a bundle map $\rho\colon E \to TM$ satisfying compatibility conditions, see for instance [43, Def. 4.2]. An example is $TM \oplus T^*M$ with the natural pairing, $[\![X+\xi, Y+\eta]\!] := [X,Y] + \mathcal{L}_X\eta - \iota_Y d\xi$, and $\rho(X+\xi) = X$ (this is sometimes called the standard Courant algebroid). A *Dirac structure* is a subbundle $L \subset E$ such that L equals its orthogonal w.r.t. the pairing, and so that $\Gamma(L)$ is closed under $[\![\cdot,\cdot]\!]$, see [9]. Examples of Dirac structures for the standard Courant algebroid are provided by graphs of closed 2-forms and Poisson bivector fields.

Fix a Courant algebroid $E \to M$, a Dirac structure A, and a complementary isotropic subbundle K (not necessarily involutive), so $E = A \oplus K$ as a vector bundle. Identify $K \cong A^*$ via the pairing on the fibers of E. Consider the map

$$\Gamma(\wedge^2 A^*) \to \Gamma(A)\ ,\ \eta_1 \wedge \eta_2 \mapsto pr_A([\![(0,\eta_1),(0,\eta_2)]\!],$$

and view it as an element $\psi \in \Gamma(\wedge^3 A)$. Denote by d_A the degree 1 derivation of $\Gamma(\wedge A^*)$ given by the Lie algebroid structure on A (the latter is given by $[\cdot,\cdot]_A := [\![\cdot,\cdot]\!]|_A$ and anchor $\rho|_A\colon A \to TM$). Similarly denote by d_{A^*} the degree 1 derivation[8] of $\Gamma(\wedge A)$ given by the bracket $[\eta_1,\eta_2]_{A^*} := pr_{A^*}([\![(0,\eta_1),(0,\eta_2)]\!]$ on $\Gamma(A^*)$ and the bundle map $\rho|_{A^*}\colon A^* \to TM$. The data given ψ, by the Lie algebroid $(A, [\cdot,\cdot]_A, \rho|_A)$, and by $(A^*, [\cdot,\cdot]_{A^*}, \rho|_{A^*})$ form a *Lie quasi-bialgebroid* ([5, §3], see also [42, §3.8]). From them one can reconstruct the Courant algebroid structure on E: the bilinear operation is recovered as

$$[\![(a_1,\eta_1),(a_2,\eta_2)]\!] = ([a_1,a_2]_A + \mathcal{L}_{\eta_1}a_2 - \iota_{\eta_2}d_{A^*}a_1 + \psi(\eta_1,\eta_2,\cdot)\ ,\ [\eta_1,\eta_2]_{A^*} + \mathcal{L}_{a_1}\eta_2 - \iota_{a_2}d_A\eta_1) \tag{37}$$

and the anchor as $\rho_A + \rho_{A^*}\colon A \oplus A^* \to TM$ [5, §3].

Recall that Courant algebroids are in bijective correspondence with degree 2 symplectic graded manifolds $\mathcal{M}$ together with a degree 3 function $\Delta \in C(\mathcal{M})$ satisfying $\{\Delta,\Delta\} = 0$ [43, Thm. 4.5]. (Here $\{\cdot,\cdot\}$ denotes the degree -2 Poisson bracket on $C(\mathcal{M})$ induced by the symplectic structure). The Courant algebroid E corresponds to

$$\big(\mathcal{M} := T^*[2]A[1]\ ,\ \Delta = h_{d_A} + F^*(h_{d_{A^*}}) - \psi\big)$$

with the canonical symplectic structure, by [42, Thm. 3.8.2]. Here we view $\psi \in \Gamma(\wedge^3 A)$ as an element of $C_3(\mathcal{M})$. Further $h_{d_A} \in C_3(\mathcal{M})$ is the fiber-wise linear function induced by d_A, the function $h_{d_{A^*}} \in C_3(T^*[2]A^*[1])$ is defined similarly, and $F\colon T^*[2]A[1] \to T^*[2]A^*[1]$ is the canonical symplectomorphism known as *Legendre transformation* [42, §3.4]. We denote by π the contangent projection $\mathcal{M} \to A[1]$.

Lemma 5.4. *Fix a Courant algebroid $E \to M$, a Dirac structure A, and a complementary isotropic subbundle K. The following quadruple forms a V-data:*

- *the graded Lie algebra $L := C(\mathcal{M})[2]$ with Lie bracket*[9] $\{\cdot,\cdot\}$
- *its abelian subalgebra $\mathfrak{a} := \pi^*(C(A[1]))[2] \cong \Gamma(\wedge A^*)[2]$*
- *the natural projection $P\colon L \to \mathfrak{a}$ given by evaluation on the base $A[1]$*

[8] d_A squares to zero because A is a Lie algebroid, but d_{A^*} generally does not.

[9] $\{\cdot,\cdot\}$, as a bracket on L, has degree zero. Hence $(L,\{\cdot,\cdot\})$ is a graded Lie algebra.

- $\Delta = h_{d_A} + F^*(h_{d_{A^*}}) - \psi$,

hence by Thm. 1 we obtain a $L_\infty[1]$*-structure* $\mathfrak{a}^P_\Delta$*. For every* $\Phi \in \Gamma(\wedge^2 A^*)$ *we have:* $\Phi[2]$ *is a MC element of* $\mathfrak{a}^P_\Delta$ *iff*

$$graph(-\Phi) := \{(X - \iota_X \Phi) : X \in A\} \subset A \oplus A^* = E$$

is a Dirac structure.

Proof. Since $\{\cdot, \cdot\}$ is the canonical Poisson bracket on the cotangent bundle, the cotangent fibers and the base $A[1]$ are Lagrangian submanifolds. Hence $\mathfrak{a}$ is an abelian Lie subalgebra of L and $ker(P)$, which consists of function on $T^*[2]A[1]$ vanishing on the base, is a Lie subalgebra. We have $\{\Delta, \Delta\} = 0$ since Δ induces a Courant algebroid structure on $A \oplus A^*$. Finally, $\Delta \in ker(P)$ follows from the expression in coordinates for h_{d_A}, $F^*(h_{d_{A^*}})$ and ψ [42, eq. 3.11, eq. 3.15]. Hence the the above quadruple is a V-data, and by Thm. 1 we obtain an $L_\infty[1]$-algebra structure $\mathfrak{a}^P_\Delta$.

We compute the Maurer-Cartan equation of $\mathfrak{a}^P_\Delta$. Let $\Phi \in \mathfrak{a}_0 = \Gamma(\wedge^2 A^*)[2]$. From the expression in coordinates for $F^*(h_{d_{A^*}})$ it follows that $\{F^*(h_{d_{A^*}}), \Phi\}$ and $\{-\psi, \Phi\}$ vanish on the base $A[1]$. So

$$P\{\Delta, \Phi\} = \{h_{d_A}, \Phi\} = d_A \Phi \in \Gamma(\wedge^3 A^*)$$

where we used [42, Lemma 3.3.1 1)]. Further $\{\{h_{d_A}, \Phi\}, \Phi\} = 0$ since both $\{h_{d_A}, \Phi\}$ and Φ lie in the abelian Lie subalgebra $\pi^*(C(A[1]))$, and in coordinates it is clear that $\{\{-\psi, \Phi\}, \Phi\}$ vanishes on the base $A[1]$. So

$$P\{\{\Delta, \Phi\}, \Phi\} = \{\{F^*(h_{d_{A^*}}), \Phi\}, \Phi\} = -[\Phi, \Phi]_{A^*}$$

where we used [42, Lemma 3.6.2]. Further,

$$P\{\{\{\Delta, \Phi\}, \Phi\}, \Phi\} = \{\{\{-\psi, \Phi\}, \Phi\}, \Phi\} = -(\Phi^\sharp \wedge \Phi^\sharp \wedge \Phi^\sharp)\psi \in \Gamma(\wedge^3 A^*),$$

where $\Phi^\sharp \colon A \to A^*, v \mapsto \iota_v \Phi$ is the contraction in the first component. All the other terms of the Maurer-Cartan equation vanish. Hence we conclude that the Maurer-Cartan equation is

$$d_A \Phi - \frac{1}{2}[\Phi, \Phi]_{A^*} - \wedge^3 \tilde{\Phi}(\psi) = 0 \tag{38}$$

where $\wedge^3 \tilde{\Phi}$ is defined as in §5.3. This equation is equivalent to $graph(-\Phi)$ being a Dirac structure by [5, Prop. 3.5]. $\square$

Remark 5.5. Given a vector bundle $E \to M$ with a non-degenerate symmetric pairing on the fibers and a direct sum decomposition into maximal isotropic subbundles $E = A \oplus K$, [42, Thm. 3.8.2] shows: the Courant algebroid structures on E for which A is a Dirac subbundle are given exactly by self-commuting degree 3 functions on $\mathcal{M} := T^*[2]A[1]$ which vanish on the base $A[1]$.

Corollary 5.6. *Fix a Courant algebroid* $E \to M$*, a Dirac structure* A*, and a complementary isotropic subbundle* K*. Let* $(L, \mathfrak{a}, P, \Delta)$ *as in Lemma 5.4 For all* $\tilde{\Delta} \in C(\mathcal{M})_3$ *with* $\tilde{\Delta}[2] \in Ker(P)$ *and* $\tilde{\Phi} \in \Gamma(\wedge^2 A^*)$*:*

$$\begin{cases} \Delta + \tilde{\Delta} \textit{ defines a new Courant algebroid} \\ \textit{structure on the vector bundle } E, \\ graph(-\tilde{\Phi}) \textit{ is a Dirac structure there} \end{cases} \Leftrightarrow \quad (\tilde{\Delta}[3], \tilde{\Phi}[2]) \textit{ is a MC element of } (ker(P)[1] \oplus \mathfrak{a})^P_\Delta.$$

Proof. Apply Thm. 3 with $\Phi = 0$ and use Remark 5.5 to ensure that A is a Dirac subbundle for the new Courant algebroid structures. Notice that $ker(P)[1] \oplus \mathfrak{a}$ is a $L_\infty[1]$-subalgebra of $(L[1] \oplus \mathfrak{a})^P_\Delta$, by Remark 1.10. $\square$

Remark 5.7. The new Courant algebroid structures that appear in Cor. 5.6 are exactly those for which A is a Dirac subbundle, by Remark 5.5.

Remark 5.8. We check that the V-data $(L, \mathfrak{a}, P, \Delta)$ is filtered (Def. 1.17). $T^*[2]A^*[1]$ is a vector bundle over $A^*[1]$, so we can denote by $C^k(T^*[2]A^*[1])$ the functions which are polynomials of degree k on each fiber. Using the Legendre transformation F to identify $\mathcal{M} = T^*[2]A[1]$ with $T^*[2]A^*[1]$ we obtain a direct product decomposition $L = \prod_{k \geq -1} L^k$ where $L^k := C^{k+1}(T^*[2]A^*[1])$. Notice that an element of $\pi^*(C_{k+1}(A[1]))[2] \cong \Gamma(\wedge^{k+1} A^*)[2]$ lies in L^k. By Remark 1.15, $\mathcal{F}^n L := \prod_{k \geq n} L^k$ is a complete filtration of the vector space L. One checks easily that $(L, \mathfrak{a}, P, \Delta)$ is a filtered V-data.

5.3 Twisted Poisson structures

In this subsection we present a special case of the situation studied in §5.2. We apply Cor. 5.6 to the standard Courant algebroid over a manifold M and $A = T^*M$. We obtain a $L_\infty[1]$-algebra whose Maurer-Cartan elements consist of closed 3-forms and twisted Poisson structures [46], recovering the $L_\infty[1]$-algebra recently displayed by Getzler [19]. Twisted Poisson structures appeared in relation to deformations also in [39, §3].

We will need the following notation: for $\pi \in \wedge^a TM$ and $a \geq 1$ we define

$$\pi^\sharp \colon T^*M \to \wedge^{a-1} TM \quad , \quad \xi \to \iota_\xi \pi,$$

and we define $\pi^\sharp \equiv 0$ if $a = 0$. We also need an extension of the above to several multivectors: for $\pi_1 \in \wedge^{a_1} TM, \dots, \pi_n \in \wedge^{a_n} TM$ $(n \geq 1, a_i \geq 0)$, we define

$$\begin{aligned} \pi_1^\sharp \wedge \dots \wedge \pi_n^\sharp \quad : \quad & \wedge^n T^*M \to \wedge^{a_1 + \dots + a_n - n} TM, \\ & \xi_1 \wedge \dots \wedge \xi_n \mapsto \sum_{\sigma \in S_n} (-1)^\sigma \pi_1^\sharp(\xi_{\sigma(1)}) \wedge \dots \wedge \pi_n^\sharp(\xi_{\sigma(n)}) \end{aligned}$$

where $\xi_i \in T^*M$ and $(-1)^\sigma$ is the sign of the permutation σ.

Recall that, given a bivector field π and a closed 3-form H, one says that π is a *H-twisted Poisson structure* [46, eq. (1)] iff

$$[\pi, \pi]_{Sch} = 2 \wedge^3 \tilde{\pi}(H),$$

where $\wedge^3 \tilde{\pi} = \frac{1}{6}(\pi^\sharp \wedge \pi^\sharp \wedge \pi^\sharp)$.

Corollary 5.9. *Let M be a manifold. There is an $L_\infty[1]$-algebra structure on*

$$\mathfrak{L} := \Omega^{\bullet \geq 1}(M)[3] \oplus \chi^\bullet(M)[2]$$

whose only non-vanishing multibrackets are

a) minus the de Rham differential on differential forms,

b) $\{\pi_1, \pi_2\} = [\pi_1, \pi_2]_{Sch}(-1)^{a_1+1}$, where $\pi_i \in \chi^{a_i}(M)$ and $[\cdot, \cdot]_{Sch}$ denotes the Schouten bracket,

c) $\{H, \pi_1, \ldots, \pi_n\} = (-1)^{\sum_{i=1}^n a_i(n-i)}(\pi_1^\sharp \wedge \cdots \wedge \pi_n^\sharp)H$
for all $n \geq 1$*, where* $H \in \Omega^n(M)$ *and* $\pi_1 \in \chi^{a_1}(M), \ldots, \pi_n \in \chi^{a_n}(M)$.

Its Maurer-Cartan elements are exactly pairs $(H[3], \pi[2])$ *where* $H \in \Omega^3(M)$ *and* $\pi \in \chi^2(M)$ *are such that* $dH = 0$ *and* π *is a* H*-twisted Poisson structure.*

Remark 5.10. The graded vector space $\mathfrak{L} = \Omega^{\bullet \geq 1}(M)[3] \oplus \chi^\bullet(M)[2]$ is concentrated in degrees $\{-2, \ldots, dim(M) - 2\}$, and its degree i component is $\Omega^{i+3}(M) \oplus \chi^{i+2}(M)$.

Proof. We apply Cor. 5.6 to the standard Courant algebroid $TM \oplus T^*M$ (defined at the beginning of §5.2), to $A = T^*M$ and $K = TM$. Notice that it corresponds to the Lie bialgebroid (A, K), where A has the zero structure and $K = TM$ has its canonical Lie algebroid structure.

We use the following notation for the canonical local coordinates on $\mathcal{M} := T^*[2]T^*[1]M$: we denote by x_j arbitrary local coordinates on M, by p_j the canonical coordinates on the fibers of $T^*[1]M$ (so the degrees are $|x_j| = 0, |p_j| = 1$, for $j = 1, \ldots, dim(M)$). By P_j, v_j we denote the conjugate coordinates on the fibres of $\mathcal{M} \to T^*[1]M$, with degrees $|P_j| = 2, |v_j| = 1$. One has $\boldsymbol{\{}P_j, x_k\boldsymbol{\}} = \delta_{jk}$ and $\boldsymbol{\{}p_j, v_k\boldsymbol{\}} = \delta_{jk}$. The element of $C_3(\mathcal{M})$ corresponding to the standard Courant algebroid is $\mathcal{S} := \sum_i P_i v_i$.

The quadruple appearing in Lemma 5.4 reads

- $L := C(T^*[2]T^*[1]M)[2]$, whose Lie bracket we denote by $\boldsymbol{\{}\cdot, \cdot\boldsymbol{\}}$
- $\mathfrak{a} := C(T^*[1]M))[2] \cong \chi^\bullet(M)[2]$
- the natural projection $P \colon L \to \mathfrak{a}$ given by evaluation on the base $T^*[1]M$, i.e. setting $P_j = 0, v_j = 0$ for all j
- $\Delta = \sum_i P_i v_i$.

The multibrackets of the $L_\infty[1]$-algebra $(L[1] \oplus \mathfrak{a})^P_\Delta$ are given in Thm. 2. Notice that using the Legendre transformation F we have

$$\Omega(M)[2] = C(T[1]M)[2] \subset C(T^*[2]T[1]M)[2] \cong L,$$

and $\Omega^{\bullet \geq 1}(M)[2] \subset ker(P)$ is a Lie subalgebra preserved by $\boldsymbol{\{}\Delta, \cdot\boldsymbol{\}}$. So by Remark 1.10 it follows that $\mathfrak{L} = \Omega^{\bullet \geq 1}(M)[3] \oplus \chi^\bullet(M)[2]$ is a $L_\infty[1]$-subalgebra of $(L[1] \oplus \mathfrak{a})^P_\Delta$. We justify why the restriction of the multibrackets to $\mathfrak{L}$ is the one described in the statement of this corollary. a) follows from eq. (5) and

$$\boldsymbol{\{}\sum_i P_i v_i, F(x) v_{\epsilon(1)} \ldots v_{\epsilon(k)}\boldsymbol{\}} = \sum_i \frac{\partial F}{\partial x_i} v_i v_{\epsilon(1)} \ldots v_{\epsilon(k)},$$

where $\epsilon(i) = 1, \ldots, dim(M)$. b) follows from eq. (7) and [42, Lemma 3.6.2]. c) follow from eq. (6) and a lengthy but straightforward computation in coordinates.

For the statement on Maurer-Cartan elements we proceed as follows. Given $H \in \Omega^3(M)$, the degree 3 function $\sum_i P_i v_i + H$ on $\mathcal{M}$ defines a Courant algebroid structure (i.e., is self-commuting) iff H is closed, and in this case it induces the $(-H)$-twisted[10] Courant

[10] Recall that the K-twisted Courant algebroid is $TM \oplus T^*M$ with bilinear operation $[\![X + \xi, Y + \eta]\!]_K := [X, Y] + \mathcal{L}_X \eta - \iota_Y d\xi + \iota_Y \iota_X K$.

algebroid $(TM \oplus T^*M)_{-H}$ [43, §4][52, §8] . Hence, by Cor. 5.6, $(H[3], \pi[2])$ is a Maurer-Cartan element of $\mathfrak{L}$ iff H is closed and $graph(-\pi)$ is a Dirac structure in $(TM \oplus T^*M)_{-H}$. The latter condition is equivalent to $-\pi$ being a $(-H)$-twisted Poisson structure [46, §3], that is, to π being a H-twisted Poisson structure. ☐

5.3.1 Equivalences of twisted Poisson structures

Consider the $L_\infty[1]$-algebra $\mathfrak{L}$ of Cor. 5.9. Its degree -1 component is $\mathfrak{L}_{-1} = \Omega^2(M) \oplus \chi(M)$, and the binary bracket there reduces to the Lie bracket of vector fields on $\chi(M)$, making $\mathfrak{L}_{-1}$ into a Lie algebra. Fix $(B, X) \in \mathfrak{L}_{-1} = \Omega^2(M) \oplus \chi(M)$. It defines a vector field $\mathcal{Y}^{(B,X)}$ on $\mathfrak{L}_0 = \Omega^3(M) \oplus \chi^2(M)$. By eq. (12) and Cor. 5.9 at the point (H, π) the vector field reads

$$\mathcal{Y}^{(B,X)}|_{(H,\pi)} = \big(-dB \;,\; [X, \pi] + \wedge^2\tilde{\pi}(B - \iota_X H)\big) \tag{39}$$

where $\wedge^2\tilde{\pi} := \frac{1}{2}(\pi^\sharp \wedge \pi^\sharp)$.

For any diffeomorphism ϕ of M, we consider the vector bundle automorphism

$$TM \oplus T^*M,\; Y + \eta \mapsto \phi_* Y + (\phi^{-1})^*\eta,$$

which by abuse of notation we denote by ϕ_*. For any $B \in \Omega^2(M)$, we consider

$$e^B \colon TM \oplus T^*M,\; Y + \eta \mapsto Y + (\eta + \iota_Y B).$$

Recall that the vector bundle $TM \oplus T^*M$ is endowed with a canonical pairing on the fibers given by $\langle X_1 + \xi_1, X_2 + \xi_2\rangle = \frac{1}{2}(\iota_{X_1}\xi_2 + \iota_{X_2}\xi_1)$.

Remark 5.11. The group of vector bundle automorphisms of $TM \oplus T^*M$ preserving the canonical pairing and preserving[11] the canonical projection $TM \oplus T^*M \to TM$ is given exactly by $\{\phi_* e^B : \phi \in \mathrm{Diff}(M), B \in \Omega^2(M)\}$. This follows by the same argument as for [23, Prop. 2.5]. Further notice that $e^B \phi_* = \phi_* e^{\phi^* B}$.

Abusing notation, for any bivector field π such that $1 + B^\flat \pi^\sharp \colon T^*M \to T^*M$ is invertible, we denote by $e^B\pi$ the unique bivector field whose graph is $e^B(graph(\pi))$. (Here $B^\flat$ is the contraction in the first component of B.) In order to compute the flow of $\mathcal{Y}^{(B,X)}$ we need a lemma:

Lemma 5.12. *Let X be a vector field on a manifold M with flow ϕ^t defined for $t \in I \subset \mathbb{R}$, let $\{C_t\}_{t\in I}$ be a smooth family of 2-forms and let π be a bivector field. Denote $\pi_t := (\phi_t)_*(e^{C_t}\pi)$. Then*

$$\frac{d}{dt}\pi_t = [X, \pi_t] + \wedge^2\tilde{\pi}_t\left((\phi_{-t})^*(\frac{d}{dt}C_t)\right). \tag{40}$$

Proof. We have

$$\frac{d}{dt}(e^{C_t}\pi) = \wedge^2\widetilde{(e^{C_t}\pi)}(\frac{d}{dt}C_t). \tag{41}$$

[11] In the sense that the projection $TM \oplus T^*M \to TM$ is equivariant w.r.t. the vector bundle automorphism and the derivative of its base map.

This follows from $(e^{C_t}\pi)^\sharp = \pi^\sharp(1+C_t^\flat\pi^\sharp)^{-1}$ [46, §4], and from $\frac{d}{dt}(e^{C_t}\pi)^\sharp = -(e^{C_t}\pi)^\sharp(\frac{d}{dt}C_t)^\flat(e^{C_t}\pi)^\sharp$. Using eq. (41) in the first equality we obtain

$$\begin{aligned}\frac{d}{dt}\pi_t &= (\phi_t)_*\left(\frac{d}{dt}(e^{C_t}\pi) + [X, e^{C_t}\pi]\right)\\ &= (\phi_t)_*\left(\wedge^2\widetilde{(e^{C_t}\pi)}(\frac{d}{dt}C_t)\right) + (\phi_t)_*[X, e^{C_t}\pi]\end{aligned}$$

which equals the r.h.s. of eq. (40). $\square$

Proposition 5.13. *Let $(B,X) \in \Omega^2(M) \oplus \chi(M)$. The integral curve of $\mathcal{Y}^{(B,X)}$ starting at the point $(H,\pi) \in \mathfrak{L}_0$ reads*

$$t \mapsto (H - tdB, (\phi_t)_* e^{C_t^H}\pi) \tag{42}$$

where ϕ denotes the flow of X and

$$C_t^H := D_t + \int_0^t (\phi_s^*)(B - \iota_X H)ds$$

for D_t the unique solution with $D_0 = 0$ of

$$\frac{d}{dt}D_t = t(\phi_t^*)\iota_X dB.$$

(The above curve is defined as long as ϕ_t is defined and $1 + (C_t^H)^\flat\pi^\sharp$ is invertible.)

Proof. Fix $(H,\pi) \in \mathfrak{L}_0$ and consider the curve defined in eq. (42). The curve is tangent to the vector field $\mathcal{Y}^{(B,X)}$ at all times t, by virtue of Lemma 5.12 and since

$$(\phi_{-t})^*(\frac{d}{dt}C_t^H) = (\phi_{-t})^*[t(\phi_t^*)\iota_X dB + (\phi_t^*)(B - \iota_X H)] = B - \iota_X(H - tdB).$$

Since at time $t = 0$ the curve is located at the point (H,π), we are done. $\square$

Remark 5.14. Let $(B,X) \in \Omega^2(M) \oplus \chi(M)$ where B is closed, and let $(H,\pi) \in \mathfrak{L}_0$. Then $D_t = 0$, and consequently $(\phi_t)_* e^{C_t^H}$ is a one parameter *group* of orthogonal vector bundle automorphisms of $TM \oplus T^*M$ (see [23, Prop. 2.6]). Hence the second component of integral curve of $\mathcal{Y}^{(B,X)}$ starting at (H,π) is the image of (the graph of) π under a one parameter group of orthogonal vector bundle automorphisms of $TM \oplus T^*M$.

Consider the group $\Omega^2(M) \rtimes \mathrm{Diff}(M)$, with multiplication

$$(B_1,\phi_1)\cdot(B_2,\phi_2) = (B_1 + (\phi_1^{-1})^*B_2\ ,\ \phi_1 \circ \phi_2).$$

We consider two natural left actions of $\Omega^2(M) \rtimes \mathrm{Diff}(M)$:

- on $TM \oplus T^*M$ by $(B,\phi) \mapsto e^B\phi_*$ (preserving the canonical pairing)
- on $\Omega^3_{closed}(M)$ by $(B,\phi)\cdot H = (\phi^{-1})^*H - dB$.

Remark 5.15. The partial[12] action of $\Omega^2(M) \rtimes \mathrm{Diff}(M)$ on $\chi^2(M) \oplus \Omega^3(M)$ by

$$(B,\phi)\cdot(H,\pi) = ((\phi^{-1})^*(H) - dB\ ,\ e^B\phi_*\pi)$$

preserves

$$MC(\mathfrak{L}) = \{(H,\pi) \in \Omega^3_{closed}(M) \oplus \chi^2(M) : \pi \text{ is a } H\text{-twisted Poisson structure}\}.$$

This follows from Prop. 5.16 below, but can also easily be checked directly as follows. For every $H \in H^3_{closed}(M)$, the automorphism $e^B\phi_*$ maps the H-twisted Courant bracket into the $(\phi^{-1})^*(H) - dB$-twisted Courant bracket [23, §2.2]. Now use that π is a H-twisted Poisson structure iff $graph(\pi)$ is involutive w.r.t. the H-twisted Courant bracket.

Proposition 5.16. *The leaves of the involutive singular distribution*

$$span\{\mathcal{Y}^{(B,X)} : (B,X) \in \mathfrak{L}_{-1} = \Omega^2(M) \oplus \chi(M)\} \tag{43}$$

on $MC(\mathfrak{L})$ coincide with the orbits of the partial action of $\Omega^2(M) \rtimes$ Diff(M) on $MC(\mathfrak{L})$.

Proof. It suffices to show that (43) coincides with the singular distribution given by the infinitesimal action associated to the group action of $\Omega^2(M) \rtimes \mathrm{Diff}(M)$. Notice that the Lie algebra of this group is $\Omega^2(M) \oplus \chi(M)$, so take an element $(B,X) \in \Omega^2(M) \oplus \chi(M)$. We compute the corresponding generator of the action $\mathcal{Z}^{(B,X)}$ at a point $(H,\pi) \in MC(\mathfrak{L})$: we have

$$\mathcal{Z}^{(B,X)}|_{(H,\pi)} := \frac{d}{dt}|_{t=0}(tB,\phi_t)\cdot(H,\pi) = \big(-d(\iota_X H + B)\ ,\ [X,\pi] + \wedge^2\tilde{\pi}(B)\big) \tag{44}$$

where ϕ_t is the flow of X and using Lemma 5.12 to compute $\frac{d}{dt}|_{t=0}(\phi_t)_* e^{(\phi_t)^*(tB)}(\pi)$. Comparing this with eq. (39) we see that

$$\mathcal{Z}^{(B-\iota_X H, X)}|_{(H,\pi)} = \mathcal{Y}^{(B,X)}|_{(H,\pi)}.$$

This shows that the two singular distributions agree at the point (H,π), and repeating at every point of $MC(\mathfrak{L})$ we conclude that the two singular distributions agree on $MC(\mathfrak{L})$. $\square$

5.4 Generalized complex structures and Courant algebroids

In this subsection we consider deformations of Courant algebroid structures on a fixed vector bundle and of their generalized complex structures. Deformations of generalized complex structures within a fixed Courant algebroid were studied by Gualtieri in [22, §5].

Fix a Courant algebroid $E \to M$ and a *generalized complex structure* J, i.e. a vector bundle map $J\colon E \to E$ with $J^2 = -Id$, preserving the fiberwise pairing and satisfying an integrability condition [24][22, Def. 4.18]. J can be equivalently encoded by a complex Dirac structure $A \subset E \otimes \mathbb{C}$ transverse to the complex conjugate $\bar{A}$. The correspondence is as follows: given J, define A to be the $+i$-eigenbundle of the complexification of J. Given A, consider the complex endomorphism of $E\otimes\mathbb{C}$ with $+i$-eigenbundle A and $-i$-eigenbundle $\bar{A}$, and define J to be the restriction to E.

[12]The action is defined whenever $1 + B^\flat(\phi_*\pi)^\sharp$ is invertible.

Hence we are in the situation of §5.2, except that we consider *complex* Dirac structures in the complexification $E \otimes \mathbb{C}$ of a (real) Courant algebroid. Notice that E does not have a preferred splitting into Dirac subbundles. On the other hand, $E \otimes \mathbb{C}$ is a complex Courant algebroid with a splitting $E \otimes \mathbb{C} = A \oplus \bar{A}$ into complex Dirac subbundles. The construction of [43, Thm. 4.5] leads to a *complex* graded manifold[13] with a degree 2 symplectic structure $\{\cdot,\cdot\}$, namely $\mathcal{N} = T^*[2]A[1]$. We denote its "global functions", a graded commutative algebra over $\mathbb{C}$, by $C_{\mathbb{C}}(\mathcal{N})$.

Lemma 5.17. *Fix a Courant algebroid $E \to M$ and a generalized complex structure J, encoded by a complex Dirac structure A transverse to $\bar{A}$. The following quadruple forms a V-data:*

- *the complex graded Lie algebra $L := C_{\mathbb{C}}(\mathcal{N})[2]$ with Lie bracket $\{\cdot,\cdot\}$*
- *its complex abelian subalgebra $\mathfrak{a} := \pi^*(C_{\mathbb{C}}(A[1]))[2] \cong \Gamma(\wedge A^*)[2]$*
- *the natural projection $P\colon L \to \mathfrak{a}$ given by evaluation on the base $A[1]$*
- *$\Delta := h_{d_A} + F^*(h_{d_{A^*}})$, where h_{d_A}, F and $h_{d_{A^*}}$ are defined analogously to §5.2,*

hence by Thm. 1 we obtain a complex[14] *$L_\infty[1]$-structure $\mathfrak{a}^P_\Delta$.*

For all $\Phi \in \Gamma(\wedge^2 A^*)$ we have: $\Phi[2]$ is a Maurer-Cartan element in $\mathfrak{a}^P_\Delta$ iff

$$graph(-\Phi) := \{(X - \iota_X \Phi) : X \in A\} \subset A \oplus \bar{A} = E \otimes \mathbb{C}$$

is a complex Dirac structure in $E \otimes \mathbb{C}$.

Proof. Exactly as the proof of Lemma 5.4, but working over $\mathbb{C}$ and taking $K := \bar{A}$. □

As earlier, let $\mathcal{M}$ be the (real) degree 2 symplectic manifold with self-commuting function Δ corresponding to the Courant algebroid E. We have $C_{\mathbb{C}}(\mathcal{N}) = C(\mathcal{M}) \otimes \mathbb{C}$. Since Δ defines a complex Courant algebroid structure on $E \otimes \mathbb{C}$ which is the complexification of a (real) Courant algebroid structure on E, it follows that $\Delta \in C(\mathcal{M}) \subset C_{\mathbb{C}}(\mathcal{N})$. We are interested only in complex Courant algebroid structures on $E \otimes \mathbb{C}$ which are complexifications of Courant algebroid structures on E, so we deform Δ only within $C(\mathcal{M})$.

Corollary 5.18. *Fix a Courant algebroid $E \to M$ and a generalized complex structure J, encoded by a complex Dirac structure A. Let $\mathcal{M}$, $\mathcal{N}$, and the V-data $(L, \mathfrak{a}, P, \Delta)$ be as above. Then there exists a (real) $L_\infty[1]$-algebra structure on $(ker(P) \cap C(\mathcal{M}))[1] \oplus \mathfrak{a}$ with the property that for all $\tilde{\Delta} \in C(\mathcal{M})_3$ with $\tilde{\Delta}[2] \in Ker(P)$ and small enough $\tilde{\Phi} \in \Gamma(\wedge^2 A^*)$:*

$$\begin{cases} \Delta + \tilde{\Delta} \text{ defines a Courant algebroid structure on } E \\ graph(-\tilde{\Phi}) \text{ is the } +i\text{-eigenbundle of a generalized complex structure there} \end{cases}$$

$$\Leftrightarrow (\tilde{\Delta}[3], \tilde{\Phi}[2]) \text{ is a MC element of } (ker(P) \cap C(\mathcal{M}))[1] \oplus \mathfrak{a}.$$

Proof. Apply Thm. 3 (which holds over $\mathbb{C}$ as well) with $\Phi = 0$ to obtain the complex $L_\infty[1]$-structure $(ker(P)[1] \oplus \mathfrak{a})^P_\Delta$. View the latter as a real $L_\infty[1]$-structure. Since $\Delta \in C(\mathcal{M})$, it follows that $(ker(P) \cap C(\mathcal{M}))[1] \oplus \mathfrak{a}$ is a $L_\infty[1]$-subalgebra. □

Remark 5.19. To see that the above V-data is filtered, proceed exactly as in Remark 5.8.

[13] It is given by a sheaf of graded commutative algebras over $\mathbb{C}$ satisfying the usual locally triviality condition.

[14] Hence the underlying graded vector space is complex and the multibrackets are $\mathbb{C}$-linear.

A Appendix

This appendix collects some background material on graded and formal geometry needed in the main text. Further, it presents the proof of Prop. 2.17.

Recall that a *graded vector space* is just a vector space W with a direct sum decomposition into subspaces $W = \oplus_{i\in\mathbb{Z}} W_i$. We refer to elements of W_i as "elements of degree i" and $|x|$ denotes the degree of x. The dual of W is naturally a graded vector space with $(W^*)_i = (W_{-i})^*$. For any integer k, $W[k]$ denotes the graded vector space with $(W[k])_i = W_{i+k}$. The set

$$L(E,E') := \{\text{linear maps from } E \text{ to } E'\}$$

is a graded vector space, with grading inherited from those of E and E': an element $\phi \in L(E,E')$ is said to be of degree k if it raises degrees by k, i.e. if $|\phi(x)| = |x| + k$ for all homogeneous $x \in E$. One denotes by $L(E,E')_k$ the set of linear maps of degree k, and $L(E,E') = \oplus_{k\in\mathbb{Z}} L(E,E')_k$. One easily checks that

$$L(E) := L(E,E)$$

is a graded Lie algebra when endowed with the graded commutator

$$[\phi,\psi] := \phi\circ\psi - (-1)^{|\phi||\psi|}\psi\circ\phi.$$

A.1 A primer on graded geometry: graded spaces and homological vector fields

We recall the notions of graded geometry needed in §2.1 - 2.3. An extension of these notions is used in §5.2 and §5.4. See [45, §1.4] or [8] for more details.

Let W be a $\mathbb{Z}$-graded vector space. We introduce the symmetric algebra of W and its derivations.

- Let $TW := \mathbb{R}\oplus W \oplus W^{\otimes 2}\oplus\dots$ be the tensor algebra of W. It is a graded algebra, i.e., it is a graded vector space endowed with an associative morphism $TW\otimes TW \to TW$. Let SW be the quotient of TW by the ideal generated by $x\otimes y - (-1)^{|x||y|} y\otimes x$, where x and y range over homogeneous elements of W. SW is a graded commutative algebra (see [45, §2, Def. 4.1], called the *graded symmetric algebra* of W.

- For any integer k, $Der(SW)_k$ denotes the space of degree k *derivations* of SW, i.e. $Q \in L(SW)_k$ which satisfy

  $$Q(x\cdot y) = Q(x)\cdot y + (-1)^{k|x|} x\cdot Q(y).$$

  $Der(SW) := \oplus_{k\in\mathbb{Z}} Der(SW)_k$ is closed under the graded commutator of linear endomorphisms, i.e. $Der(SW)$ is a Lie subalgebra of $(L(SW),[-,-])$.

Now let U be an n-dimensional, real vector space. Then $U[1]$ (resp. $(U[1])^*$) is a graded vector space concentrated in degree -1 (resp. 1). Exactly as ordinary vector spaces are instances of smooth manifolds, graded vector spaces are instances of graded manifolds. We do not give the definition of graded manifold here (see [8, §2.1]). Rather, we describe explicitly the two algebraic structures associated to the graded manifold $U[1]$ that will be used in this article:

- The space of "*functions* on $U[1]$"

$$C(U[1]) := S((U[1])^*).$$

  It is a graded commutative algebra concentrated in degrees $0, \dots, n$. It is isomorphic to the ordinary exterior algebra $\wedge U^*$ of U^* (graded so that elements of $\wedge^k U^*$ have degree k).

- The space of "*vector fields* on $U[1]$"

$$\chi(U[1]) := Der(C(U[1])).$$

  It is a graded Lie algebra, concentrated in degrees ≥ -1. As a graded vector space it is just $S((U[1])^*) \otimes U[1]$.

We give "coordinate expressions" for the above functions and vector fields. Notice that there is a canonical identification $\iota \colon U \to \chi_{-1}(U[1])$. An element $X \in U$ is identified with the vector field ι_X that satisfies $\iota_X(u) = \langle X, u \rangle$ for all $u \in (U[1])^* = C_1(U[1])$, where the pointy brackets denote the pairing of a vector space with its dual. (It is enough to specify how ι_X acts on $(U[1])^*$, since the latter generates the graded commutative algebra $C(U[1])$.)

- Choose a basis $X_1, \dots, X_n$ of U. The dual basis, viewed as a basis of $(U^*)[-1] = (U[1])^*$, will be denoted by

$$u_1, \dots, u_n.$$

  We refer to the u_i as *coordinates on* $U[1]$. Notice that $|u_i| = 1$. The graded commutative algebra $C(U[1])$ is generated by the u_i, and a generic degree k element of $C(U[1])$ is a degree k polynomial expression in the u_i.

- $X_1, \dots, X_n$, under the identification of U with $\chi_{-1}(U[1])$, becomes a basis of $\chi_{-1}(U[1])$ which we denote by

$$\frac{\partial}{\partial u_1}, \dots, \frac{\partial}{\partial u_n}.$$

  Notice that $|\frac{\partial}{\partial u_i}| = -1$. We have $\frac{\partial}{\partial u_i}(u_j) = \delta_{ij}$. A general degree k element of $\chi(U[1])$ is of the form $\sum_{i=1}^n P_i \frac{\partial}{\partial u_i}$, where P_i is a degree $k+1$ polynomial expression on the u_j's.

Finally, by *homological vector field* on $U[1]$ we mean a degree 1 element $Q \in \chi(U[1])$ with the property that $[Q, Q] = 0$. Notice that a homological vector field is necessarily of the form $\sum c_{ij}^k u_i u_j \frac{\partial}{\partial u_k}$ for some constants c_{ij}^k.

A.2 A primer on formal geometry: coalgebras and homological coderivations

The notion of formal geometry is used in §2.4-2.5 and §3, and is dual to the notion of graded geometry. It is of use when one has to deal with infinite dimensional algebras. In this section we introduce the main objects of interest, homological coderivations. They are compact ways to handle algebras, or algebras up to homotopy: the brackets of these algebras are given by the Taylor coefficients of the corresponding coderivation. References for proofs can be found for example in [1], [10] or the appendix of [40].

Definition A.1. A *coalgebra* structure on a (possibly graded) vector space W consists of a (degree 0) linear map $\Delta : W \to W \otimes W$, called *coproduct*, satisfying the coassociativity condition

$$(\Delta \otimes Id) \circ \Delta = (Id \otimes \Delta) \circ \Delta.$$

The only examples which will be of use here are:

Example A.2. If V is a (graded) vector space over the field $\mathbb{K}$, let us consider $TV = \oplus_{k=0}^{\infty} V^{\otimes^k}$ and $SV = \oplus_{k=0}^{\infty} S^k V$. They are coalgebras for the (degree-preserving) coproducts given respectively by

$$\Delta(x_1 \otimes \cdots \otimes x_n) := \sum_{i=0}^{n} (x_1 \otimes \cdots \otimes x_{i-1}) \bigotimes (x_i \otimes \cdots \otimes x_n) \tag{45}$$

and

$$\Delta(x_1 \ldots x_n) := \sum_{i=0}^{n} \sum_{\sigma \in Sh_{(i,n-i)}} \epsilon(\sigma; x_1, \ldots, x_n) \cdot (x_{\sigma(1)} \ldots x_{\sigma(i)}) \bigotimes (x_{\sigma(i+1)} \ldots x_{\sigma(n)}). \tag{46}$$

We used the notation $\epsilon(\sigma; x_1, \ldots, x_n)$, the Koszul sign given by the permutation σ of the elements x_i and the convention that $x_1 \otimes \cdots \otimes x_n = x_{\sigma(1)} \ldots x_{\sigma(n)} = 1_{\mathbb{K}}$ when $n = 1$. In particular, $\Delta(1_{\mathbb{K}}) = 1_{\mathbb{K}} \otimes 1_{\mathbb{K}}$.

Most people rather work with the *reduced* tensor/symmetric coalgebras:

Example A.3. One defines $\overline{TV} = \oplus_{k=1}^{\infty} V^{\otimes^k}$ and $\overline{SV} = \oplus_{k=1}^{\infty} S^k V$. They are coalgebras for the coproducts (both denoted by $\bar{\Delta}$) given by replacing the element $1_{\mathbb{K}} \in V^{\otimes^0} = S^0 V = \mathbb{K}$ by 0 in eq. (45) and (46). In other words:

$$\Delta(x) = \bar{\Delta}(x) + 1_{\mathbb{K}} \otimes x + x \otimes 1_{\mathbb{K}}.$$

Definition A.4. A *coderivation* of a coalgebra (W, Δ) consists of a linear endomorphism Q of W satisfying the following (co) Leibniz condition:

$$(Q \otimes Id + Id \otimes Q) \circ \Delta = \Delta \circ Q. \tag{47}$$

One denotes by $Coder(W)$ the set of coderivations of (W, Δ). It is a graded Lie subalgebra of $(L(W), [-,-])$.

Remark A.5. If both Q and Q' are odd, then $[Q, Q'] = Q \circ Q' + Q' \circ Q$. This means that if Q is odd, then $Q \circ Q$ is a coderivation.

Definition A.6. A *homological* coderivation consists in a degree one coderivation Q satisfying

$$Q \circ Q = 0. \tag{48}$$

From now on we work with non-negatively graded coalgebras, i.e. such that $W = \oplus_{i \geq 0} W_i$. Let Q be a linear endomorphism of W. As a linear map, it is uniquely defined by its restrictions to the subspaces W_k: if one denotes $Q_k := Q|_{W_k}$, one has $Q = \prod_{k=0}^{\infty} Q_k$. Let us consider the natural projection $\Pi_{W_l} : W \to W_l$, for every l. One denotes $Q^l := \Pi_{W_l} \circ Q$. Clearly:

$$Q = \prod\nolimits_{k,l=0}^{\infty} Q_k^l.$$

Definition A.7. The collection $Q^1 := \{Q^1_0, \dots, Q^1_i, \dots\}$ is called the set of *Taylor coefficients* of Q. The coderivation Q is said to be *quadratic* if its only non zero Taylor coefficient is Q^1_2.

Coderivations are most of the time encountered through their Taylor coefficients. Proposition A.8 shows how to reconstruct a coderivation from its Taylor coefficients, and formula (50) expresses the condition of being homological in these terms:

Proposition A.8. *A coderivation Q of $\overline{TV}$ (resp. $\overline{SV}$) is uniquely determined by the collection $\{Q^1_1, \dots, Q^1_i, \dots\}$ of its Taylor coefficients by the formula*

$$Q^i_n = \sum_{s=1}^{i} Id^{s-1} \otimes Q^1_{n-i+1} \otimes Id^{i-s},$$

resp.

$$Q = m_{\overline{SV}} \circ (Q^1 \otimes Id) \circ \bar{\Delta}, \tag{49}$$

where $m_{\overline{SV}}$ denotes the multiplication of $\overline{SV}$.

Prop. A.8, whose proof can be found in [1], [10] or the appendix of [40], enables to reformulate the condition (48) for a coderivation to be homological:

Lemma A.9. *A coderivation Q of $\overline{TV}$ is homological if and only if its Taylor coefficients satisfy the set of equations ($n \geq 1$)*

$$\sum_{i=1}^{n} \sum_{s=1}^{i} Q^1_i \circ (Id^{s-1} \otimes Q^1_{n-i+1} \otimes Id^{i-s}) = 0. \tag{50}$$

In particular, a quadratic homological coderivation of $\overline{TV}$ is equivalent to the equation

$$Q^1_2 \circ (Q^1_2 \otimes Id) + Q^1_2 \circ (Id \otimes Q^1_2) = 0. \tag{51}$$

In the same way, a coderivation Q of $\overline{SV}$ is homological if and only if its Taylor coefficients form a $L_\infty[1]$-algebra on V (see Def. 1.4).

Proof. Let Q be a homological coderivation of $\overline{TV}$. By Remark A.5, $Q \circ Q$ is a coderivation, and we can apply Proposition A.8. Therefore we will get a series of equations, namely, the annihilation of all its Taylor coefficients $(Q \circ Q)^1_i$. But one has, by use of Proposition A.8, the following expression for the these coefficients:

$$\begin{aligned} (Q \circ Q)^1_n \quad &= \quad \sum_{i=1}^{n} Q^1_i \circ Q^i_n \\ &\overset{Prop\ A.8}{=} \quad \sum_{i=1}^{n} Q^1_i \circ (\sum_{s=1}^{i} Id^{s-1} \otimes Q^1_{n-i+1} \otimes Id^{i-s}). \end{aligned}$$

The proof of the statement for $\overline{SV}$ goes along the same lines and can be found in [1]. $\square$

Lemma A.9 is important, since it is the link between homological coderivations and algebras. But to have this link explicit, one still needs to "desuspend" the relation. First, let us recall the shift operator $[1]\colon V \to V[1]$, which maps an element of $v \in V_i$ to itself seen as an element of $(V[1])_{i-1}$. (In other words, [1] shifts the degree of an element by 1.) Sometimes we write $v[1]$ for $[1]v$.

Definition A.10. Let Q be a coderivation of $\overline{T(V[1])}$. We define the *desuspension operator* d by

$$dQ_n^1 := [1]Q_n^1[-1]^{\otimes^n} \colon \overline{TV} \to V, \tag{52}$$

and similarly for $\overline{S(V[1])}$.

This desuspension operator constitutes the link between homological coderivations and homotopy algebras. This is the content of the following proposition, whose proof can be found in [1] and [10].

Proposition A.11. *The operator d defined by equation (52) gives a bijection between the sets of quadratic homological coderivations of $\overline{T(V[1])}$ and of associative algebra structures on V.*

Similarly, it also gives a bijection between the sets of homological coderivations of $\overline{S(V[1])}$ and of L_∞-algebra structures on V. The latter restricts to a bijection between the quadratic homological coderivations of $\overline{S(V[1])}$ and the graded Lie algebras structures on V.

This last result suggests the definition of an A_∞-algebra, introduced by Stasheff in [49].

Definition A.12. An *associative algebra up to homotopy* (or *A_∞-algebra*) is a graded vector space V equipped with a collection of maps $\{m_1, \dots, m_l, \dots\}$, obtained by desuspension of the Taylor coefficients of a homological coderivation Q of $\overline{T(V[1])}$.

A.3 The proof of Prop. 2.17: infinite dimensional L_∞-algebras via derived brackets

It is well-known that an L_∞-algebra structure on a finite dimensional graded vector space V is equivalent to a homological vector field on $V[1]$. The L_∞-multibrackets can be recovered with a derived bracket construction [50, Ex. 4.1]. If V is an infinite dimensional, the above procedure does not apply (it involves considering the dual of $V[1]$). Instead, as stated in Lemma A.9, a L_∞-structure on V can be encoded by a suitable *coderivation* on a *reduced* symmetric coalgebra. In this section we show that the L_∞-structure can also be recovered from a coderivation on a (unreduced) symmetric coalgebra by a derived bracket construction, proving Prop. 2.17.

Let W be a (possibly infinite dimensional) graded vector space. We will apply Voronov's derived bracket construction (Thm. 1) to the graded Lie algebra $(Coder(SW), [-,-])$ of Def. A.4. Let us introduce the abelian subalgebra $\mathfrak{a}$ which we will need.

Lemma A.13. *For every homogeneous $w \in W$,*

$$\begin{aligned} \alpha_w \colon SW &\to SW, \\ x_1 \dots x_n &\mapsto w \cdot x_1 \dots x_n \end{aligned}$$

is a coderivation of SW of degree $|w|$. Further, $\mathfrak{a} := \{\alpha_w : w \in W\}$ is an abelian Lie subalgebra of $Coder(SW)$.

Proof. We show that α_w is a coderivation, i.e., that is satisfies eq. (A.4). With the notations $x_\emptyset := 1$ and $x_I := x_{i_1} \cdot \dots \cdot x_{i_n}$ for $I = \{i_1, \dots, i_n\}$, one can abbreviate

$$\Delta x_I = \sum_{I' \amalg I'' = I} \pm x_{I'} \otimes x_{I''},$$

where $\pm$ are the signs which appear in formula (46). On the one hand one has

$$(\alpha_w \otimes Id + Id \otimes \alpha_w)\Delta(x_I) = \sum_{I' \amalg I'' = I} \pm w \cdot x_{I'} \otimes x_{I''} + (-1)^{|w||x_{I'}|} \pm x_{I'} \otimes w \cdot x_{I''}.$$

On the other hand, if one denotes $w \cdot x_I = x_{\{\star \amalg I\}}$, one gets

$$\Delta(\alpha_w(x_I)) = \sum_{J' \amalg J'' = \star \amalg I} \pm x_{J'} \otimes x_{J''} = \sum_{I' \amalg I'' = I} \pm w \cdot x_{I'} \otimes x_{I''} + (-1)^{|w||x_{I'}|} \pm x_{I'} \otimes w \cdot x_{I''}.$$

Hence α_w is a coderivation.

To show that $\mathfrak{a}$ is abelian we compute for all homogeneous $v, w \in W$ and $x \in SW$ that $[\alpha_v, \alpha_w]x = v \cdot w \cdot x - (-1)^{|v||w|} w \cdot v \cdot x = 0$. □

Lemma A.14. *For every $\tau \in Coder(SW)$ one has $\tau(1) \in W$.*

Proof. For any element $w \in SW$ the following holds: $\Delta w = w \otimes 1 + 1 \otimes w$ iff $w \in W$.

Applying eq. (47) to $1 \in SW$ we see that $\tau(1)$ satisfies the above relation, so it must lie in W. □

There is an embedding of the coderivations on $\overline{SW}$ into those on SW:

Lemma A.15. *Consider the map*

$$\mathcal{J} \colon Coder(\overline{SW}) \to Coder(SW)$$

defined by $(\mathcal{J}\Theta)(1) = 0$ and, for all $n \geq 1$, by

$$(\mathcal{J}\Theta)(w_1 \dots w_n) = \Theta(w_1 \dots w_n).$$

$\mathcal{J}$ is well defined, injective, and bracket-preserving.

Proof. We check that $\mathcal{J}\Theta$ lies in $Coder(SW)$. The relation (47) is trivially satisfied on the element 1. Let now x be an element of $\overline{SW}$. We have

$$\begin{aligned}(\mathcal{J}\Theta \otimes Id + Id \otimes \mathcal{J}\Theta)(\Delta(x)) &= \Theta(x) \otimes 1 + 1 \otimes \Theta(x) + (\Theta \otimes Id + Id \otimes \Theta)(\bar{\Delta}(x)) \\ &= \Theta(x) \otimes 1 + 1 \otimes \Theta(x) + \bar{\Delta}(\Theta(x)) \\ &= \Delta(\mathcal{J}\Theta(x)),\end{aligned}$$

where in the first equality we used $(\mathcal{J}\Theta)(1) = 0$ and in the second that $\Theta \in Coder(\overline{SW})$. Hence $\mathcal{J}\Theta$ is a coderivation.

$\mathcal{J}$ is bracket-preserving since, for any $\Theta_i \in Coder(\overline{SW})$, the graded commutator $[\mathcal{J}\Theta_1, \mathcal{J}\Theta_2]$ vanishes on $1 \in SW$ and agrees with $[\Theta_1, \Theta_2]$ on $\oplus_{k=1}^{\infty} S^k W$. □

Now we are ready to prove Prop. 2.17, which recovers $L_\infty[1]$-algebra structures on W via derived brackets. We repeat the proposition for the reader's convenience:

Proposition. *Let W be an L_∞[1]-algebra, and Θ the corresponding coderivation of $\overline{SW}$ given by Lem. A.9. The following quadruple forms a V-data:*

- *the graded Lie algebra $L := Coder(SW)$*
- *its abelian subalgebra $\mathfrak{a} := \{\alpha_w : w \in W\}$*
- *the projection $P\colon L \to \mathfrak{a}$, $\tau \mapsto \alpha_{\tau(1)}$*
- *$\Delta := \mathcal{J}\Theta$.*

The induced $L_\infty[1]$-structure on $\mathfrak{a}$ given by Thm. 1 is exactly the original L_∞[1]-structure on W, under the canonical identification $W \cong \mathfrak{a}, w \mapsto \alpha_w$.

Proof. $\mathfrak{a}$ is an abelian Lie subalgebra of $Coder(SW)$ by Lemma A.13. The map P is well-defined by Lemma A.14, and is clearly a projection (that is, $P^2 = P$). Its kernel $ker(P)$ agrees with the subspace of coderivations vanishing on $1 \in SW$. Hence $ker(P)$ is a Lie subalgebra of $Coder(SW)$ and it contains $\mathcal{J}\Theta$. Further $[\mathcal{J}\Theta, \mathcal{J}\Theta] = 0$ by Lemma A.15.

We conclude that $(L, \mathfrak{a}, P, \Delta)$ is a V-data and the assumptions of Thm. 1 are satisfied.

To compute the induced multibrackets on $\mathfrak{a}$, notice that for every $n \geq 1$

$$[\cdots[\mathcal{J}\Theta, \alpha_{w_1}], \cdots, \alpha_{w_n}](1) = \mathcal{J}\Theta \circ \alpha_{w_1} \circ \cdots \circ \alpha_{w_n}(1) + \sum_{i=1}^{n} \alpha_{v_i} M_i \tag{53}$$

for certain elements $M_i \in \oplus_{k=1}^{\infty} S^k W$. In particular the sum on the r.h.s. lies in $\oplus_{k=2}^{\infty} S^k W$. Hence

$$\begin{aligned}
[\cdots[\mathcal{J}\Theta, \alpha_{w_1}], \cdots, \alpha_{w_n}](1) &= pr_W\big([\cdots[\mathcal{J}\Theta, \alpha_{w_1}], \cdots, \alpha_{w_n}](1)\big) \\
&= pr_W\big(\mathcal{J}\Theta \circ \alpha_{w_1} \circ \cdots \circ \alpha_{w_n}(1)\big) \\
&= pr_W\big(\mathcal{J}\Theta(w_1 \cdots w_n)\big) \\
&= \{w_1, \cdots, w_n\}
\end{aligned}$$

where in the first equality we used Lemma A.14, in the second we used eq. (53), and in the fourth Lemma A.9. Hence

$$P[\cdots[\mathcal{J}\Theta, \alpha_{w_1}], \cdots, \alpha_{w_n}] = \alpha_{[\cdots[\mathcal{J}\Theta, \alpha_{w_1}], \cdots, \alpha_{w_n}](1)} = \alpha_{\{w_1, \cdots, w_n\}}.$$

□

A.3.1 Application: L_∞ algebras associated to A_∞ algebras

It is well known ([32] or [34, Prop. 13.2.16]) that, in the same way that one can associate a Lie algebra to an associative algebra (by taking the commutator), one can associate a L_∞-algebra to an A_∞-algebra. In this subsection, which is not used in the rest of the paper, we show that it is indeed possible to understand this in terms of derived brackets.

Let W be a (possibly infinite dimensional) graded vector space and $w \in W$. Let us define the map

$$\alpha_w \colon TW \longmapsto TW, \quad \alpha_w(w_1 \otimes \cdots \otimes w_n) = \sum_{i=0}^{n} w_1 \otimes \cdots \otimes w_i \otimes w \otimes w_{i+1} \otimes \cdots \otimes w_n.$$

In the following we use the notations of the previous section, modulo the replacement of the symmetric product by the tensor product.

Proposition A.16. *Let $W[-1]$ be an A_∞-algebra, and Θ the corresponding coderivation of $\overline{TW}$ given by Def. A.12. The following quadruple forms a V-data:*

- *the graded Lie algebra $L := Coder(TW)$*
- *its abelian subalgebra $\mathfrak{a} := \{\alpha_w : w \in W\}$*
- *the projection $P \colon L \to \mathfrak{a}$, $\tau \mapsto \alpha_{\tau(1)}$*
- *$\Delta := \mathcal{J}\Theta$.*

The induced $L_\infty[1]$-structure on $\mathfrak{a}$ given by Thm. 1 is exactly the L_∞[1]-structure on W obtained by symmetrization of the $A_\infty[1]$-algebra structure on W.

Proof. One can easily check by mimicking §A.3 that the map α_w is a coderivation of TW, $\mathfrak{a}$ is an abelian subalgebra of L, and $Ker(P)$ is a subalgebra of L. To show that $(L, \mathfrak{a}, P, \Delta)$ forms a V-data, it remains to show that $[\Delta, \Delta] = 0$. But by definition A.12, an A_∞-algebra on $W := V[-1]$ is equivalent to a Maurer-Cartan element of $Coder(\overline{TW})$, i.e. a coderivation Θ of degree 1 such that $[\Theta, \Theta] = 0$. Now use the fact that the map $\mathcal{J}$ is bracket preserving.

Therefore the derived bracket construction of Thm. 1 can be applied to the V-data $(L, \mathfrak{a}, P, \Delta)$ above, associating a $L_\infty[1]$-algebra to the given A_∞-algebra.

It remains to check that the obtained $L_\infty[1]$-structure on W can alternatively be obtained by symmetrization: The computation following eq. (53) gives in particular

$$pr_W\big(\mathcal{J}\Theta \circ \alpha_{w_1} \circ \cdots \circ \alpha_{w_n}(1)\big) = \{w_1, \cdots, w_n\}. \tag{54}$$

One remarks (proof by induction) that $\alpha_{w_1} \circ \cdots \circ \alpha_{w_n}(1) = \sum_{\sigma \in S_n} w_{\sigma(1)} \otimes \cdots \otimes w_{\sigma(n)}$. So (54) rewrites as

$$\{w_1, \cdots, w_n\} = \sum_{\sigma \in S_n} \Theta_n^1(w_{\sigma(1)} \otimes \cdots \otimes w_{\sigma(n)}),$$

i.e. the n-th bracket of the $L_\infty[1]$-structure on $\mathfrak{a}$ given by Thm. 1 is obtained by symmetrization of the n-th bracket of the original $A_\infty[1]$-structure. □

References

[1] D. Arnal, D. Manchon, and M. Masmoudi. Choix des signes pour la formalité de M. Kontsevich. *Pacific J. Math.*, 203(1):23–66, 2002.

[2] G. Barnich, R. Fulp, T. Lada, and J. Stasheff. The sh Lie structure of Poisson brackets in field theory. *Comm. Math. Phys.*, 191(3):585–601, 1998.

[3] F. Bayen, M. Flato, C. Fronsdal, A. Lichnerowicz, and D. Sternheimer. Quantum mechanics as a deformation of classical mechanics. *Lett. Math. Phys.*, 1:521–530, 1977.

[4] M. Benayed. Lie bialgebras real cohomology [Lie bialgebras and cohomology]. *J. Lie Theory*, 7(2):287–292, 1997.

[5] H. Bursztyn, M. Crainic, and P. Ševera. Quasi-Poisson structures as Dirac structures. In *Travaux mathématiques. Fasc. XVI*, Trav. Math., XVI, pages 41–52. Univ. Luxemb., Luxembourg, 2005.

[6] A. Cannas da Silva and A. Weinstein. *Geometric models for noncommutative algebras*, volume 10 of *Berkeley Mathematics Lecture Notes*. American Mathematical Society, Providence, RI, 1999.

[7] A. S. Cattaneo and G. Felder. Relative formality theorem and quantisation of coisotropic submanifolds. *Adv. in Math.*, 208:521–548, 2007.

[8] A. S. Cattaneo and F. Schätz. Introduction to supergeometry. *Rev. Math. Phys.*, 23(6):669–690, 2011.

[9] T. Courant. Dirac manifolds. *Trans. Amer. Math. Soc.*, 319(2):631–661, 1990.

[10] M. Doubek, M. Markl, and P. Zima. Deformation theory (lecture notes). *Archivum mathematicum 43(5), 2007, 333-371*, 05 2007, 0705.3719.

[11] D. Fiorenza and M. Manetti. L_∞ structures on mapping cones. *Algebra Number Theory*, 1(3):301–330, 2007.

[12] Y. Frégier. A new cohomology theory associated to deformations of Lie algebra morphisms. *Lett. Math. Phys.*, 70(2):97–107, 2004.

[13] Y. Frégier, M. Markl, and D. Yau. The L_∞-deformation complex of diagrams of algebras. *New York J. Math.*, 15:353–392, 2009.

[14] Y. Fregier and M. Zambon. Actual deformations of coisotropic submanifolds. In preparation.

[15] K. Fukaya. Deformation theory, homological algebra and mirror symmetry. In *Geometry and physics of branes (Como, 2001)*, Ser. High Energy Phys. Cosmol. Gravit., pages 121–209. IOP, Bristol, 2003. `www.mat.uniroma1.it/people/manetti/GeoSup0708/fukaya.pdf`.

[16] M. Gerstenhaber. On the deformation of rings and algebras. *Ann. of Math.*, 79:55–70, 1964.

[17] M. Gerstenhaber and S. D. Schack. On the deformation of algebra morphisms and diagrams. *Trans. Amer. Math. Soc.*, (279):1–50, 1983.

[18] E. Getzler. Homotopy theory for L_∞-algebras. Course given at the workshop “Higher Structure in Topology and Geometry IV”, Göttingen, June 2-4, 2010. Partial notes available at `http://nlab.mathforge.org/nlab/show/descent+for+L-infinity+algebras`.

[19] E. Getzler. Talk at the conference *Poisson 2010*, Rio de Janeiro, August 2010.

[20] E. Getzler. Lie theory for nilpotent L_∞-algebras. *Annals of Mathematics*, (170):271–301, 2009.

[21] V. Ginzburg and M. Kapranov. Koszul duality for operads. *Duke Math. J.*, 76(1):203–272, 1994.

[22] M. Gualtieri. Generalized complex geometry, `arXiv:math.DG/0401221`.

[23] M. Gualtieri. Generalized complex geometry. *Ann. of Math. (2)*, 174(1):75–123, 2011.

[24] N. Hitchin. Generalized Calabi-Yau manifolds. *Q. J. Math.*, 54(3):281–308, 2003.

[25] K. Kodaira and D. Spencer. On deformations of complex-analytic structures i. *Ann.of Math.*, 67(2):328–401, 1958.

[26] K. Kodaira and D. Spencer. On deformations of complex-analytic structures ii. *Ann.of Math.*, 67(3):403–466, 1958.

[27] K. Kodaira and D. Spencer. On deformations of complex-analytic structures iii. *Ann.of Math.*, 71(1):43–76, 1960.

[28] M. Kontsevich. Deformation quantization of Poisson manifolds. *Lett. Math. Phys.*, 66(3):157–216, 2003.

[29] M. Kontsevich and Y. Soibelman. Deformation Theory. I. Draft. `http://www.math.ksu.edu/ soibel/Book-vol1.ps`.

[30] M. Kuranishi. New proof for the existence of locally complete families of complex structures. In *Proc. Conf. Complex Analysis (Minneapolis, 1964)*. Springer, 1965.

[31] T. Lada and M. Markl. Strongly homotopy Lie algebras. *Comm. Algebra*, 23(6):2147–2161, 1995.

[32] T. Lada and J. Stasheff. Introduction to sh lie algebras for physicists. *Internat. J. Theoret. Phys.*, 7(32):1087–1103, 1993.

[33] Z.-J. Liu, A. Weinstein, and P. Xu. Manin triples for Lie bialgebroids. *J. Differential Geom.*, 45(3):547–574, 1997.

[34] J. L. Loday and B. Vallette. Algebraic operads. *available at* *http://math.unice.fr/∼brunov/*.

[35] S. A. Merkulov. An L_∞-algebra of an unobstructed deformation functor. *Internat. Math. Res. Notices*, (3):147–164, 2000.

[36] A. Nijenhuis and R. W. Richardson, Jr. Cohomology and deformations in graded Lie algebras. *Bull. Amer. Math. Soc.*, 72:1–29, 1966.

[37] A. Nijenhuis and R. W. Richardson, Jr. Deformations of homomorphisms of Lie groups and Lie algebras. *Bull. Amer. Math. Soc.*, 73:175–179, 1967.

[38] Y.-G. Oh and J.-S. Park. Deformations of coisotropic submanifolds and strong homotopy Lie algebroids. *Invent. Math.*, 161(2):287–360, 2005.

[39] J.-S. Park. Topological open p-branes. In *Symplectic geometry and mirror symmetry (Seoul, 2000)*, pages 311–384. World Sci. Publ., River Edge, NJ, 2001.

[40] D. Quillen. Rational homotopy theory. *Ann. of Math. (2)*, 90:205–295, 1969.

[41] R. W. Richardson, Jr. Deformations of subalgebras of Lie algebras. *J. Differential Geometry*, 3:289–308, 1969.

[42] D. Roytenberg. Courant algebroids, derived brackets and even symplectic supermanifolds. Ph.D. Thesis. Arxiv `math/9910078`.

[43] D. Roytenberg. On the structure of graded symplectic supermanifolds and Courant algebroids. In *Quantization, Poisson brackets and beyond (Manchester, 2001)*, volume 315 of *Contemp. Math.*, pages 169–185. Amer. Math. Soc., Providence, RI, 2002.

[44] D. Roytenberg. Quasi-Lie bialgebroids and twisted Poisson manifolds. *Lett. Math. Phys.*, 61(2):123–137, 2002.

[45] F. Schätz. Coisotropic Submanifolds and the BFV-Complex, PhD Thesis (University of Zürich), 2009, `http://www.math.ist.utl.pt/~fschaetz/`.

[46] P. Ševera and A. Weinstein. Poisson geometry with a 3-form background. *Progr. Theoret. Phys. Suppl.*, (144):145–154, 2001. Noncommutative geometry and string theory (Yokohama, 2001).

[47] B. Shoikhet. An explicit construction of the Quillen homotopical category of dg Lie algebras, arXiv 0706.1333v1.

[48] J. Stasheff. The intrinsic bracket on the deformation complex of an associative algebra. *Journal of Pure and Applied Algebra*, 89(1-2):231 – 235, 1993.

[49] J. D. Stasheff. Homotopy associativity of H-spaces. I, II. *Trans. Amer. Math. Soc. 108 (1963), 275-292; ibid.*, 108:293–312, 1963.

[50] T. Voronov. Higher derived brackets and homotopy algebras. *J. Pure Appl. Algebra*, 202(1-3):133–153, 2005.

[51] T. T. Voronov. Higher derived brackets for arbitrary derivations. In *Travaux mathématiques. Fasc. XVI*, Trav. Math., XVI, pages 163–186. Univ. Luxemb., Luxembourg, 2005.

[52] M. Zambon. L_∞-algebras and higher analogues of Dirac structures and Courant algebroids. *J. Symplectic Geometry*, 10(6), 2012.