Deforming the Lie superalgebras $D(2, 1; x)$

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Abstract. We describe the non-trivial deformations of the standard embedding of the Lie superalgebras $D(2, 1; x)$ into the derived contact superconformal algebra $K'(4)$, and realize $D(2, 1; x)$ as 4×4 matrices over a Weyl algebra.

1. Introduction

The Lie superalgebras $D(2, 1; x)$ have recently been studied by mathematicians and physicists from different points of view. In particular, they became important in the context of the AdS/CFT correspondence [2, 6, 13]. Recall that $D(2, 1; x)$, where $x \in \mathbb{C}\backslash\{0, -1\}$, is a one-parameter family of classical simple Lie superalgebras of dimension 17 [7]. The bosonic part of $D(2, 1; x)$ is $\mathfrak{sl}(2)\oplus\mathfrak{sl}(2)\oplus\mathfrak{sl}(2)$, and the action of $D(2, 1; x)_{\bar{0}}$ on $D(2, 1; x)_{\bar{1}}$ is the product of 2-dimensional representations. We will use another notation for these superalgebras (see [20]): $\Gamma(\sigma_1, \sigma_2, \sigma_3)$, where σ_i are nonzero complex numbers such that $\sigma_1 + \sigma_2 + \sigma_3 = 0$. Note that $\Gamma(\sigma_1, \sigma_2, \sigma_3) \cong D(2, 1; x)$, where $x = \sigma_1/\sigma_2$.

The family $D(2, 1; x)$ is closely connected with the derived contact superconformal algebra $K'(4)$ [3]. Recall that $K'(4)$ is spanned by 16 fields, one of which is a Virasoro field, and it is also known to physicists as the centerless big $N = 4$ superconformal algebra [8, 9]. It was shown in [21, 22] that the big $N = 4$ superconformal algebra contains $D(2, 1; x)$ as a subsuperalgebra.

In this work we consider the standard embedding of $\Gamma_\alpha = \Gamma(2, -1 - \alpha, \alpha - 1)$, where $\alpha \in \mathbb{C}$, into the Poisson superalgebra $P(4)$ of pseudodifferential symbols on the supercircle $S^{1|2}$ with two odd variables. Γ_α is naturally embedded into $K'(4) \subset P(4)$. We describe the infinitesimal deformations of this embedding, which are classified by $H^1(\Gamma_\alpha, K'(4))$. We prove that this cohomology space is one-dimensional and that the infinitesimal deformations are indeed the formal deformations of the embedding.

Integrability of infinitesimal deformations of embeddings of Lie algebras were studied by A. Nijenhuis and R. W. Richardson in [14, 19]. For the standard embeddings of $Vect(S^1)$ into the Poisson algebra on S^1 and into the Lie algebra of pseudodifferential symbols on S^1 they were studied in [15, 16]. Similar problems in the case of superalgebras $K(1)$ and $K(2)$ of contact vector fields on $S^{1|1}$ and $S^{1|2}$ were studied in [4, 1]. In our work we use the similar approach.

Note that in [17, 18] we constructed a different embedding of Γ_α into $P(4)$, where pseudodifferential symbols were essentially used. In this work we actually obtain an embedding of Γ_α into the Lie superalgebra of *differential operators* on $S^{1|2}$.

We also realize Γ_α as a Lie subsuperalgebra of 4×4 matrices over the Weyl algebra $\mathcal{W} = \sum_{i\geq 0} \mathbb{C}[t, t^{-1}]d^i$, where $d = \frac{\partial}{\partial t}$. This realization is different from the one given in [17, 18].

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2. Superalgebras $\Gamma(\sigma_1, \sigma_2, \sigma_3)$

Recall the definition of $\Gamma(\sigma_1, \sigma_2, \sigma_3)$ [20]. Let $\mathfrak{g} = \mathfrak{g}_{\bar{0}} \oplus \mathfrak{g}_{\bar{1}}$ be a Lie superalgebra, where $\mathfrak{g}_{\bar{0}} = sp(\psi_1) \oplus sp(\psi_2) \oplus sp(\psi_3)$ and $\mathfrak{g}_{\bar{1}} = V_1 \otimes V_2 \otimes V_3$, where V_i are 2-dimensional vector spaces, and ψ_i is a non-degenerate skew-symmetric form on V_i, $i = 1, 2, 3$. A representation of $\mathfrak{g}_{\bar{0}}$ on $\mathfrak{g}_{\bar{1}}$ is the tensor product of the standard representations of $sp(\psi_i)$ in V_i. Consider $sp(\psi_i)$ - invariant bilinear mapping

$$\mathcal{P}_i : V_i \times V_i \longrightarrow sp(\psi_i), \quad i = 1, 2, 3,$$

given by

$$\mathcal{P}_i(x_i, y_i)z_i = \psi_i(y_i, z_i)x_i - \psi_i(z_i, x_i)y_i$$

for all $x_i, y_i, z_i \in V_i$. Let $\mathcal{P}$ be a mapping

$$\mathcal{P} : \mathfrak{g}_{\bar{1}} \times \mathfrak{g}_{\bar{1}} \longrightarrow \mathfrak{g}_{\bar{0}}$$

given by

$$\begin{aligned}
&\mathcal{P}(x_1 \otimes x_2 \otimes x_3, y_1 \otimes y_2 \otimes y_3) = \\
&\sigma_1 \psi_2(x_2, y_2)\psi_3(x_3, y_3)\mathcal{P}_1(x_1, y_1) + \\
&\sigma_2 \psi_1(x_1, y_1)\psi_3(x_3, y_3)\mathcal{P}_2(x_2, y_2) + \\
&\sigma_3 \psi_1(x_1, y_1)\psi_2(x_2, y_2)\mathcal{P}_3(x_3, y_3)
\end{aligned}$$

for all $x_i, y_i \in V_i, i = 1, 2, 3$, where $\sigma_1, \sigma_2, \sigma_3$ are some complex numbers. The super Jacobi identity is satisfied if and only if $\sigma_1 + \sigma_2 + \sigma_3 = 0$. In this case $\mathfrak{g}$ is denoted by

$\Gamma(\sigma_1, \sigma_2, \sigma_3)$. Superalgebras $\Gamma(\sigma_1, \sigma_2, \sigma_3)$ and $\Gamma(\sigma_1', \sigma_2', \sigma_3')$ are isomorphic if and only if there exists a nonzero element $k \in \mathbb{C}$ and a permutation π of the set $\{1, 2, 3\}$ such that

$$\sigma_i' = k \cdot \sigma_{\pi i} \text{ for } i = 1, 2, 3.$$

Superalgebras $\Gamma(\sigma_1, \sigma_2, \sigma_3)$ are simple if and only if $\sigma_1, \sigma_2, \sigma_3$ are all different from zero. Note that $\Gamma(\sigma_1, \sigma_2, \sigma_3) \cong D(2, 1; x)$ (see [7]) where $x = \sigma_1/\sigma_2$.

3. Embeddings into the Poisson superalgebra on $S^{1|2}$

The *Poisson algebra P of pseudodifferential symbols on the circle* is formed by the formal series

$$A(t, \tau) = \sum_{-\infty}^{n} a_i(t)\tau^i,$$

where $a_i(t) \in \mathbb{C}[t, t^{-1}]$, and the even variable τ corresponds to ∂_t, see [15]. The Poisson bracket is defined as follows:

$$\{A(t, \tau), B(t, \tau)\} = \partial_\tau A(t, \tau)\partial_t B(t, \tau) - \partial_t A(t, \tau)\partial_\tau B(t, \tau).$$

An associative algebra P_h, where $h \in (0, 1]$, is a deformation of P, see [16]. The multiplication in P_h is given as follows:

$$A(t, \tau) \circ_h B(t, \tau) = \sum_{n \geq 0} \frac{h^n}{n!}\partial_\tau^n A(t, \tau)\partial_t^n B(t, \tau).$$

The Lie algebra structure on the vector space P_h is given by

$$[A, B]_h = A \circ_h B - B \circ_h A,$$

so that

$$\lim_{h\to 0}\frac{1}{h}[A, B]_h = \{A, B\}.$$

Let $\Lambda(2N)$ be the Grassmann algebra in $2N$ variables $\xi_1, \ldots, \xi_N, \eta_1, \ldots, \eta_N$ with the parity $p(\xi_i) = p(\eta_i) = \bar{1}$. The *Poisson superalgebra of pseudodifferential symbols on $S^{1|N}$* is $P(2N) = P \otimes \Lambda(2N)$. The Poisson bracket is defined as follows:

$$\{A, B\} = \partial_\tau A\partial_t B - \partial_t A\partial_\tau B + (-1)^{p(A)+1}\sum_{i=1}^{N}(\partial_{\xi_i} A\partial_{\eta_i} B + \partial_{\eta_i} A\partial_{\xi_i} B).$$

Let $W(2N)$ be the Lie superalgebra of all superderivations of the associative superalgebra $\mathbb{C}[t, t^{-1}] \otimes \Lambda(2N)$. By definition,

$$K(2N) = \{D \in W(2N) \mid D\Omega = f\Omega \text{ for some } f \in \mathbb{C}[t, t^{-1}] \otimes \Lambda(2N)\},$$

where $\Omega = dt + \sum_{i=1}^{N} \xi_i d\eta_i + \eta_i d\xi_i$ is a differential 1-form, which is called a *contact form* [10]. Note that there exists an embedding

$$K(2N) \subset P(2N), \quad N \geq 0.$$

Consider a $\mathbb{Z}$-grading on the associative superalgebra $P(2N)$, defined by

$$\deg t = \deg \eta_i = \deg \tau = \deg \xi_i = 1 \text{ for } i = 1, \ldots, N.$$

With respect to the Poisson super bracket,

$$\{P_{(i)}(2N), P_{(j)}(2N)\} \subset P_{(i+j-2)}(2N).$$

Thus $P_{(2)}(2N)$ is a subsuperalgebra of $P(2N)$, and one can easily check that it is isomorphic to $K(2N)$. Note that this embedding of $K(2N)$ into $P(2N)$ is different from the embedding considered in [17, 18], which is based on another $\mathbb{Z}$-grading of $P(2N)$.

$K(2N)$ is simple if $N \neq 2$, and if $N = 2$, then the derived Lie superalgebra $K'(4) = [K(4), K(4)]$ is a simple ideal in $K(4)$ of codimension one, defined from the exact sequence

$$0 \to K'(4) \to K(4) \to \mathbb{C}t^{-1}\tau^{-1}\xi_1\xi_2\eta_1\eta_2 \to 0.$$

Proposition 3.1. For each $\alpha \in \mathbb{C}$ there exists an embedding

$$\rho_\alpha : \Gamma(2, -1-\alpha, \alpha-1) \to K'(4) \subset P(4).$$

$\Gamma_\alpha = \rho_\alpha(\Gamma(2, -1-\alpha, \alpha-1))$ is spanned by the following elements:

$$\begin{aligned}
&E_\alpha^1 = t^2, \quad F_\alpha^1 = \tau^2 - 2\alpha t^{-2}\xi_1\xi_2\eta_1\eta_2, \quad H_\alpha^1 = t\tau, \\
&E_\alpha^2 = \xi_1\xi_2, \quad F_\alpha^2 = \eta_1\eta_2, \quad H_\alpha^2 = \xi_1\eta_1 + \xi_2\eta_2, \\
&E_\alpha^3 = \xi_1\eta_2, \quad F_\alpha^3 = \xi_2\eta_1, \quad H_\alpha^3 = \xi_1\eta_1 - \xi_2\eta_2, \\
&T_\alpha^1 = t\eta_1, \quad T_\alpha^2 = t\eta_2, \quad T_\alpha^3 = t\xi_1, \quad T_\alpha^4 = t\xi_2, \\
&D_\alpha^1 = \tau\xi_1 + \alpha t^{-1}\xi_1\xi_2\eta_2, \quad D_\alpha^2 = \tau\xi_2 - \alpha t^{-1}\xi_1\xi_2\eta_1, \\
&D_\alpha^3 = \tau\eta_1 + \alpha t^{-1}\xi_2\eta_1\eta_2, \quad D_\alpha^4 = \tau\eta_2 - \alpha t^{-1}\xi_1\eta_1\eta_2.
\end{aligned} \tag{3.1}$$

Proof. Note that if $\alpha = 0$, then $\Gamma(2, -1, -1) \cong \mathfrak{spo}(2|4)$, and ρ_α is the standard embedding of $\mathfrak{spo}(2|4)$ into $P(4)$.

Let

$$V_1 = \mathrm{Span}(e_1, e_2), \quad V_2 = \mathrm{Span}(f_1, f_2), \quad V_3 = \mathrm{Span}(h_1, h_2),$$

and

$$\begin{aligned} \psi_1(e_1, e_2) &= -\psi_1(e_2, e_1) = 1, \\ \psi_2(f_1, f_2) &= -\psi_2(f_2, f_1) = 1, \\ \psi_3(h_1, h_2) &= -\psi_3(h_2, h_1) = 1. \end{aligned}$$

Explicitly an embedding ρ_α is given as follows:

$$\begin{aligned} &\rho_\alpha(\mathcal{P}_1(e_1, e_1)) = -E^1_\alpha, \quad \rho_\alpha(\mathcal{P}_1(e_2, e_2)) = -F^1_\alpha, \quad \rho_\alpha(\mathcal{P}_1(e_1, e_2)) = -H^1_\alpha, \\ &\rho_\alpha(\mathcal{P}_2(f_1, f_1)) = -2F^2_\alpha, \quad \rho_\alpha(\mathcal{P}_2(f_2, f_2)) = -2E^2_\alpha, \quad \rho_\alpha(\mathcal{P}_2(f_1, f_2)) = H^2_\alpha, \\ &\rho_\alpha(\mathcal{P}_3(h_1, h_1)) = -2F^3_\alpha, \quad \rho_\alpha(\mathcal{P}_3(h_2, h_2)) = 2E^3_\alpha, \quad \rho_\alpha(\mathcal{P}_3(h_1, h_2)) = H^3_\alpha, \\ &\rho_\alpha(e_1 \otimes f_1 \otimes h_1) = \sqrt{2}iT^1_\alpha, \quad \rho_\alpha(e_1 \otimes f_1 \otimes h_2) = \sqrt{2}iT^2_\alpha, \\ &\rho_\alpha(e_1 \otimes f_2 \otimes h_1) = -\sqrt{2}iT^4_\alpha, \quad \rho_\alpha(e_1 \otimes f_2 \otimes h_2) = \sqrt{2}iT^3_\alpha, \\ &\rho_\alpha(e_2 \otimes f_1 \otimes h_1) = \sqrt{2}iD^3_\alpha, \quad \rho_\alpha(e_2 \otimes f_1 \otimes h_2) = \sqrt{2}iD^4_\alpha, \\ &\rho_\alpha(e_2 \otimes f_2 \otimes h_1) = -\sqrt{2}iD^2_\alpha, \quad \rho_\alpha(e_2 \otimes f_2 \otimes h_2) = \sqrt{2}iD^1_\alpha. \end{aligned}$$

Thus $sp(\psi_i) \cong \mathrm{Span}(E^i_\alpha, H^i_\alpha, F^i_\alpha)$ for $i = 1, 2, 3$.

□

4. Deformations of embeddings

Let $\rho : \mathfrak{g} \to \mathfrak{h}$ be an embedding of Lie superalgebras, then $\mathfrak{h}$ is a $\mathfrak{g}$-module. A map $\rho + \beta\rho_1 : \mathfrak{g} \to \mathfrak{h}$, where $\rho_1 \in Z^1(\mathfrak{g}, \mathfrak{h})$ is a Lie superalgebra homomorphism up to quadratic terms in β. It is called an infinitesimal deformation. Infinitesimal deformations are classified by $H^1(\mathfrak{g}, \mathfrak{h})$, see [14, 19].

Describe obstructions to higher order prolongations of these infinitesimal deformations, see [15, 16]. Let

$$\tilde{\rho}_\beta = \rho + \sum_{k=1}^{\infty} \beta^k \rho_k : \mathfrak{g} \to \mathfrak{h},$$

where $\rho_k : \mathfrak{g} \to \mathfrak{h}$ are even linear maps, satisfy

$$\tilde{\rho}_\beta([X, Y]) = [\tilde{\rho}_\beta(X), \tilde{\rho}_\beta(Y)].$$

$\tilde{\rho}_\beta$ is called a formal deformation of ρ. Let $\varphi_\beta = \tilde{\rho}_\beta - \rho$. Then

$$[\varphi_\beta(X), \rho(Y)] + [\rho(X), \varphi_\beta(Y)] - \varphi_\beta([X, Y]) + \sum_{i,j>0} [\rho_i(X), \rho_j(Y)]\beta^{i+j} = 0. \quad (4.1)$$

The first three terms are $(d\varphi_\beta)(X, Y)$, where d stands for coboundary. For arbitrary linear maps $\varphi, \varphi' : \mathfrak{g} \to \mathfrak{h}$, define

$$\begin{aligned} &[[\varphi, \varphi']] : \mathfrak{g} \otimes \mathfrak{g} \to \mathfrak{h}, \\ &[[\varphi, \varphi']](X, Y) = [\varphi(X), \varphi'(Y)] + [\varphi'(X), \varphi(Y)]. \end{aligned} \quad (4.2)$$

The relation (4.1) is equivalent to

$$d\varphi_\beta + \frac{1}{2}[[\varphi_\beta, \varphi_\beta]] = 0.$$

Expanding this relation in power series in β, we have

$$d\rho_k + \frac{1}{2} \sum_{i+j=k} [[\rho_i, \rho_j]] = 0.$$

The first nontrivial relation is

$$d\rho_2 + \frac{1}{2}[[\rho_1, \rho_1]] = 0,$$

and it gives the first obstruction to integrability of an infinitesimal deformation. Note that (4.2) defines a bilinear map, called the cup-product:

$$H^1(\mathfrak{g}, \mathfrak{h}) \otimes H^1(\mathfrak{g}, \mathfrak{h}) \to H^2(\mathfrak{g}, \mathfrak{h}).$$

The obstructions to integrability of infinitesimal deformations lie in $H^2(\mathfrak{g}, \mathfrak{h})$. Thus we have to compute $H^1(\mathfrak{g}, \mathfrak{h})$ and the product classes in $H^2(\mathfrak{g}, \mathfrak{h})$.

Consider the embedding (3.1).

Theorem 4.1. $\dim H^1(\Gamma_\alpha, K'(4)) = 1$. The cohomology is spanned by the class of the 1-cocycle θ given as follows:

$$\begin{aligned} \theta(T^1_\alpha) &= \tau^{-1}\xi_2\eta_1\eta_2, & \theta(T^2_\alpha) &= -\tau^{-1}\xi_1\eta_1\eta_2, \\ \theta(T^3_\alpha) &= \tau^{-1}\xi_1\xi_2\eta_2, & \theta(T^4_\alpha) &= -\tau^{-1}\xi_1\xi_2\eta_1 \\ \theta(D^1_\alpha) &= t^{-1}\xi_1\xi_2\eta_2, & \theta(D^2_\alpha) &= -t^{-1}\xi_1\xi_2\eta_1, \\ \theta(D^3_\alpha) &= t^{-1}\xi_2\eta_1\eta_2, & \theta(D^4_\alpha) &= -t^{-1}\xi_1\eta_1\eta_2, \\ \theta(E^1_\alpha) &= 2\tau^{-2}\xi_1\xi_2\eta_1\eta_2, & \theta(F^1_\alpha) &= -2t^{-2}\xi_1\xi_2\eta_1\eta_2. \end{aligned} \quad (4.3)$$

The map $\tilde{\rho}_{\alpha,\beta} = \rho_\alpha + \beta\theta$ ($\beta \in \mathbb{C}$) is a formal deformation of the embedding (3.1).

Proof. Consider $\mathfrak{gl}(2) \cong \mathrm{Span}(\xi_i\eta_j \mid i,j=1,2) \subset \Gamma_\alpha$. The diagonal subalgebra of $\mathfrak{gl}(2)$ consists of $h = h_1\xi_1\eta_1 + h_2\xi_2\eta_2$, where $h_1, h_2 \in \mathbb{C}$. Let $\epsilon_i(h) = h_i, i = 1,2$. Obviously, $\mathrm{Span}(\xi_1,\xi_2)$ is the standard $\mathfrak{gl}(2)$-module, $\mathrm{Span}(\eta_1,\eta_2)$ is its dual, ξ_i and η_i have weight ϵ_i and $-\epsilon_i$. Note that $H^1(\Gamma_\alpha, K'(4))$ is a trivial $\mathfrak{gl}(2)$-module, since a Lie (super)algebra acts trivially on its cohomology [5]. Hence we have to compute only the 1-cocycles of weight zero. Note also that

$$H^1(\Gamma_\alpha, K'(4)) = \oplus_{n\in\mathbb{Z}} H^{1,n}(\Gamma_\alpha, K'(4)),$$

where the $\mathbb{Z}$-grading is given by the condition

$$\deg t = 1,\ \deg \tau = -1, \quad \deg \xi_i = \deg \eta_i = 0.$$

Let $c \in C^{1,n}(\Gamma_\alpha, K'(4))$ be a 1-cochain of weight zero. Note that if $c \neq 0$, then n is even: $n = 2m$, and c acts on the odd elements of Γ_α as follows:

$$\begin{aligned}
c(T^1_\alpha) &= g^m_1 t^{m+1}\tau^{-m}\eta_1 + s^m_1 t^m\tau^{-m-1}\xi_2\eta_1\eta_2, & c(D^1_\alpha) &= r^m_1 t^m\tau^{-m+1}\xi_1 + q^m_1 t^{m-1}\tau^{-m}\xi_1\xi_2\eta_2,\\
c(T^2_\alpha) &= g^m_2 t^{m+1}\tau^{-m}\eta_2 + s^m_2 t^m\tau^{-m-1}\xi_1\eta_1\eta_2, & c(D^2_\alpha) &= r^m_2 t^m\tau^{-m+1}\xi_2 + q^m_2 t^{m-1}\tau^{-m}\xi_1\xi_2\eta_1,\\
c(T^3_\alpha) &= g^m_3 t^{m+1}\tau^{-m}\xi_1 + s^m_3 t^m\tau^{-m-1}\xi_1\xi_2\eta_2, & c(D^3_\alpha) &= r^m_3 t^m\tau^{-m+1}\eta_1 + q^m_3 t^{m-1}\tau^{-m}\xi_2\eta_1\eta_2,\\
c(T^4_\alpha) &= g^m_4 t^{m+1}\tau^{-m}\xi_2 + s^m_4 t^m\tau^{-m-1}\xi_1\xi_2\eta_1, & c(D^4_\alpha) &= r^m_4 t^m\tau^{-m+1}\eta_2 + q^m_4 t^{m-1}\tau^{-m}\xi_1\eta_1\eta_2,
\end{aligned} \tag{4.4}$$

where $g^m_i, s^m_i, r^m_i, q^m_i \in \mathbb{C}$. Let

$$\begin{aligned}
c_0 &= t^{m+1}\tau^{-m+1}, & c_1 &= t^m\tau^{-m}\xi_1\eta_1,\\
c_2 &= t^m\tau^{-m}\xi_2\eta_2, & c_3 &= t^{m-1}\tau^{-m-1}\xi_1\xi_2\eta_1\eta_2.
\end{aligned}$$

If $m \neq 0$, then the elements of weight zero in $C^{0,2m}(\Gamma_\alpha, K'(4))$ span the subspace $\mathrm{Span}(c_0, c_1, c_2, c_3)$. If $m = 0$, then the elements of weight zero in $C^{0,0}(\Gamma_\alpha, K'(4))$ span the subspace $\mathrm{Span}(c_0, c_1, c_2)$. Note that the coefficients g^m_i in (4.4) are as follows:

$$\begin{aligned}
&\text{if } c = dc_0, \text{ then } g^m_1 = g^m_2 = g^m_3 = g^m_4 = m-1,\\
&\text{if } c = dc_1, \text{ then } g^m_1 = -g^m_3 = 1, g^m_2 = g^m_4 = 0,\\
&\text{if } c = dc_2, \text{ then } g^m_1 = g^m_3 = 0, g^m_2 = -g^m_4 = 1.
\end{aligned}$$

Let $X, Y \in \Gamma_\alpha$. Note that

$$\begin{aligned}
dc(X,Y) &= \{X, c(Y)\} + \{Y, c(X)\} - c(\{X,Y\}), \quad &&\text{if } p(X) = p(Y) = \bar{1},\\
dc(X,Y) &= \{X, c(Y)\} - \{Y, c(X)\} - c(\{X,Y\}), \quad &&\text{if } p(X) = \bar{0},\ p(Y) = \bar{1},\\
dc(X,Y) &= \{X, c(Y)\} - \{Y, c(X)\} - c(\{X,Y\}), \quad &&\text{if } p(X) = p(Y) = \bar{0}.
\end{aligned}$$

Let $c \in Z^{1,2m}(\Gamma_\alpha, K'(4))$ be of weight zero. From the condition $dc(X,Y) = 0$, we have that

$$\begin{aligned} &\{T^1_\alpha, c(T^3_\alpha)\} + \{T^3_\alpha, c(T^1_\alpha)\} - c(E^1_\alpha) = 0, \\ &\{T^2_\alpha, c(T^4_\alpha)\} + \{T^4_\alpha, c(T^2_\alpha)\} - c(E^1_\alpha) = 0. \end{aligned} \tag{4.5}$$

It follows that

$$g^m_3 + g^m_1 = g^m_2 + g^m_4. \tag{4.6}$$

Case $m \neq 1$. One can change c by adding (or removing) coboundaries dc_i for $i = 0, 1, 2$, and thus assume that $g^m_i = 0$ for $i = 1, 2, 3$. Then from (4.6) $g^m_4 = 0$. Note that

$$\{T^1_\alpha, c(T^2_\alpha)\} + \{T^2_\alpha, c(T^1_\alpha)\} = 0, \tag{4.7}$$

hence $s^m_2 = -s^m_1$.

$$\{T^1_\alpha, c(T^4_\alpha)\} + \{T^4_\alpha, c(T^1_\alpha)\} = 0, \tag{4.8}$$

hence $s^m_4 = -s^m_1$.

$$\{T^2_\alpha, c(T^3_\alpha)\} + \{T^3_\alpha, c(T^2_\alpha)\} = 0, \tag{4.9}$$

hence $s^m_3 = -s^m_2 = s^m_1$.

Note that if $c = dc_3$, then in (4.4) $g^m_i = 0$ for $i = 1, \dots, 4$ and $s^m_1 = s^m_3 = -s^m_2 = -s^m_4 = 1$. Changing 1-cocycle c using the coboundary dc_3, we can assume in addition that $s^m_1 = 0$. Then $s^m_i = 0$ for $i = 2, 3, 4$. We have that

$$\{E^2_\alpha, c(T^1_\alpha)\} - \{T^1_\alpha, c(E^2_\alpha)\} + c(T^4_\alpha) = 0, \tag{4.10}$$

hence $c(E^2_\alpha) = 0$.

$$\{E^3_\alpha, c(T^1_\alpha)\} - \{T^1_\alpha, c(E^3_\alpha)\} - c(T^2_\alpha) = 0, \tag{4.11}$$

hence $c(E^3_\alpha) = 0$. Also

$$\{F^2_\alpha, c(T^3_\alpha)\} - \{T^3_\alpha, c(F^2_\alpha)\} + c(T^2_\alpha) = 0, \tag{4.12}$$

hence $c(F^2_\alpha) = 0$.

$$\{F^3_\alpha, c(T^3_\alpha)\} - \{T^3_\alpha, c(F^3_\alpha)\} - c(T^4_\alpha) = 0, \tag{4.13}$$

hence $c(F^3_\alpha) = 0$. Then

$$\{D^1_\alpha, c(T^4_\alpha)\} + \{T^4_\alpha, c(D^1_\alpha)\} = 0, \tag{4.14}$$

$$\{T^2_\alpha, c(D^1_\alpha)\} + \{D^1_\alpha, c(T^2_\alpha)\} = 0. \tag{4.15}$$

From (4.14) $(1 - m)r^m_1 + q^m_1 = 0$, and from (4.15) $(1 - m)r^m_1 - q^m_1 = 0$. Hence, $r^m_1 = q^m_1 = 0$.

$$\{T^1_\alpha, c(D^2_\alpha)\} + \{D^2_\alpha, c(T^1_\alpha)\} = 0, \tag{4.16}$$

$$\{D^2_\alpha, c(T^3_\alpha)\} + \{T^3_\alpha, c(D^2_\alpha)\} = 0. \quad (4.17)$$

From (4.16) $(1-m)r^m_2 + q^m_2 = 0$, and from (4.17) $(1-m)r^m_2 - q^m_2 = 0$. Hence, $r^m_2 = q^m_2 = 0$.

$$\{T^2_\alpha, c(D^3_\alpha)\} + \{D^3_\alpha, c(T^2_\alpha)\} = 0. \quad (4.18)$$

$$\{T^4_\alpha, c(D^3_\alpha)\} + \{D^3_\alpha, c(T^4_\alpha)\} = 0, \quad (4.19)$$

From (4.18) $(1-m)r^m_3 + q^m_3 = 0$ and from (4.19) $(m-1)r^m_3 + q^m_3 = 0$. Hence, $r^m_3 = q^m_3 = 0$.

$$\{T^3_\alpha, c(D^4_\alpha)\} + \{D^4_\alpha, c(T^3_\alpha)\} = 0. \quad (4.20)$$

$$\{T^1_\alpha, c(D^4_\alpha)\} + \{D^4_\alpha, c(T^1_\alpha)\} = 0, \quad (4.21)$$

from (4.20) $(m-1)r^m_4 - q^m_4 = 0$ and from (4.21) $(m-1)r^m_4 + q^m_4 = 0$. Hence, $r^m_4 = q^m_4 = 0$. Hence the cocycle c is zero on the odd elements.
Case $m = 1$. Changing 1-cocycle c using coboundaries dc_i, we can assume that $g^1_1 = g^1_2 = 0$. Then from (4.6) $g^1_3 = g^1_4$. In addition, changing c by a multiple of dc_3, we can assume that $s^1_1 = 0$. Next

$$\{T^3_\alpha, c(T^4_\alpha)\} + \{T^4_\alpha, c(T^3_\alpha)\} = 0,$$

Hence $s^1_3 = -s^1_4$. From (4.7) $s^1_2 = -s^1_1 = 0$. From (4.8) $s^1_4 = g^1_4$. From (4.9) $s^1_3 = -g^1_3 = -g^1_4$. Note that from (4.5) we have that

$$c(E^1_\alpha) = g^1_4(-t^2\tau^{-2}\xi_1\eta_1 - t^2\tau^{-2}\xi_2\eta_2 + t^3\tau^{-1}).$$

Since the coefficient of ξ_1 in

$$\{E^1_\alpha, c(D^1_\alpha)\} - \{D^1_\alpha, c(E^1_\alpha)\} + 2c(T^3_\alpha) = 0$$

is $-2g^1_4$, then $g^1_4 = 0$. Thus $c(T^i_\alpha) = 0$ for $i = 1, 2, 3, 4$, and $c(E^1_\alpha) = 0$. From (4.10) $c(E^2_\alpha) = 0$. From (4.11) $c(E^3_\alpha) = 0$. From (4.12) $c(F^2_\alpha) = 0$. From (4.13) $c(F^3_\alpha) = 0$. From

$$\{D^1_\alpha, c(T^4_\alpha)\} + \{T^4_\alpha, c(D^1_\alpha)\} - (1+\alpha)c(E^2_\alpha) = 0,$$

$q^1_1 = 0$. From

$$\{T^1_\alpha, c(D^2_\alpha)\} + \{D^2_\alpha, c(T^1_\alpha)\} - (1-\alpha)c(F^3_\alpha) = 0,$$

$q^1_2 = 0$. From

$$\{T^2_\alpha, c(D^3_\alpha)\} + \{D^3_\alpha, c(T^2_\alpha)\} - \alpha c(F^2_\alpha) = 0,$$

$q^1_3 = 0$. From

$$\{T^3_\alpha, c(D^4_\alpha)\} + \{D^4_\alpha, c(T^3_\alpha)\} - (\alpha-1)c(E^3_\alpha) = 0,$$

$q^1_4 = 0$. From

$$\{D^1_\alpha, c(D^2_\alpha)\} + \{D^2_\alpha, c(D^1_\alpha)\} = 0,$$

$(1+\alpha)r_2^1 - (1+\alpha)r_1^1 = 0$. From

$$\{D_\alpha^3, c(D_\alpha^4)\} + \{D_\alpha^4, c(D_\alpha^3)\} = 0,$$

$(1+\alpha)r_4^1 - (1+\alpha)r_3^1 = 0$. From

$$\{D_\alpha^1, c(D_\alpha^4)\} + \{D_\alpha^4, c(D_\alpha^1)\} = 0,$$

$(1-\alpha)r_4^1 - (1-\alpha)r_1^1 = 0$. From

$$\{D_\alpha^2, c(D_\alpha^3)\} + \{D_\alpha^3, c(D_\alpha^2)\} = 0,$$

$(1-\alpha)r_3^1 - (1-\alpha)r_2^1 = 0$.
If $\alpha \neq \pm 1$, then $r_1^1 = r_2^1 = r_3^1 = r_4^1$. Then c is a multiple of dc_0.
Subcase $\alpha = 1$. In this case $r_1^1 = r_2^1$ and $r_3^1 = r_4^1$. One can change c by a multiple of dc_0 and assume that $r_1^1 = r_2^1 = 0$. From

$$\{D_\alpha^1, c(D_\alpha^3)\} + \{D_\alpha^3, c(D_\alpha^1)\} - c(F_\alpha^1) = 0, \tag{4.22}$$

$c(F_\alpha^1) = r_3^1(\xi_1\eta_1 + \xi_2\eta_2 + t\tau)$. From

$$\{F_\alpha^1, c(T_\alpha^1)\} - \{T_\alpha^1, c(F_\alpha^1)\} - 2c(D_\alpha^3) = 0, \tag{4.23}$$

$2r_3^1 t\eta_1 = 0$, hence $r_3^1 = r_4^1 = 0$. Hence the cocycle c is zero on the odd elements.
Subcase $\alpha = -1$. In this case $r_1^1 = r_4^1$ and $r_2^1 = r_3^1$. One can change c by a multiple of dc_0 and assume that $r_1^1 = r_4^1 = 0$. From (4.22) $c(F_\alpha^1) = r_3^1(\xi_1\eta_1 - \xi_2\eta_2 + t\tau)$. From (4.23) $2r_3^1 t\eta_1 = 0$, hence $r_3^1 = r_2^1 = 0$. Hence the cocycle c is zero on the odd elements.
Finally, from (4.5) $c(E_\alpha^1) = 0$, from (4.23) $c(F_\alpha^1) = 0$. From

$$\{E_\alpha^1, c(F_\alpha^1)\} - \{F_\alpha^1, c(E_\alpha^1)\} + 4c(H_\alpha^1) = 0$$

$c(H_\alpha^1) = 0$. From (4.10) $c(E_\alpha^2) = 0$, from (4.11) $c(E_\alpha^3) = 0$, from (4.12) $c(F_\alpha^2) = 0$, and from (4.13) $c(F_\alpha^3) = 0$. Hence $c(H_\alpha^2) = c(H_\alpha^3) = 0$, and c is the zero cocycle. This proves that if $m \neq 0$, then each 1-cocycle c of weight zero has the zero cohomology class, and the cohomology is spanned by the cocycle dc_3 where $m = 0$, because $c_3 = t^{-1}\tau^{-1}\xi_1\xi_2\eta_1\eta_2 \notin K'(4)$. The coefficients in (4.4) for this cocycle are $g_i = r_i = 0$, $s_1 = s_3 = -s_2 = -s_4 = 1$, and $q_1 = q_3 = -q_2 = -q_4 = 1$. Thus $dc_3 = \theta$ as it is given in (4.3).

According to the Richardson-Nijenhuis theory, one has to determine the cup product $[[\theta, \theta]]$ [15, 16]. It is easy to see that this cup product is identically zero (and not only in cohomology). Thus $\tilde{\rho}_{\alpha,\beta} = \rho_\alpha + \beta\theta$ is a formal deformation of the embedding ρ_α.

□

5. Matrices over a Weyl algebra

By definition, a Weyl algebra is

$$\mathcal{W} = \sum_{i\geq 0} \mathcal{A} d^i,$$

where $\mathcal{A}$ is an associative commutative algebra and $d : \mathcal{A} \to \mathcal{A}$ is a derivation of $\mathcal{A}$, with the relations

$$da = d(a) + ad, \quad a \in \mathcal{A},$$

see [11, 12]. Set

$$\mathcal{A} = \mathbb{C}[t, t^{-1}],\ d = \frac{\partial}{\partial t}.$$

Let $\mathrm{End}(\mathcal{W}^{2|2})$ be the Lie superalgebra of 4×4 matrices over $\mathcal{W}$.
Theorem 5.1. For each $\alpha \in \mathbb{C}$ there exists an embedding

$$\bar{\rho}_\alpha : \Gamma(2, -1-\alpha, \alpha-1) \to \mathrm{End}(\mathcal{W}^{2|2})$$

given as follows:

$$\bar{\rho}_\alpha(T^1_\alpha) = \left(\begin{array}{cc|cc} 0 & 0 & t & 0 \\ 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 \\ 0 & t & 0 & 0 \end{array}\right), \quad \bar{\rho}_\alpha(T^2_\alpha) = \left(\begin{array}{cc|cc} 0 & 0 & 0 & t \\ 0 & 0 & 0 & 0 \\ \hline 0 & -t & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right),$$

$$\bar{\rho}_\alpha(T^3_\alpha) = \left(\begin{array}{cc|cc} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & t \\ \hline t & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right), \quad \bar{\rho}_\alpha(T^4_\alpha) = \left(\begin{array}{cc|cc} 0 & 0 & 0 & 0 \\ 0 & 0 & -t & 0 \\ \hline 0 & 0 & 0 & 0 \\ t & 0 & 0 & 0 \end{array}\right),$$

$$\bar{\rho}_\alpha(D^1_\alpha) = \left(\begin{array}{cc|cc} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & d+\alpha t^{-1} \\ \hline d & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right), \quad \bar{\rho}_\alpha(D^2_\alpha) = \left(\begin{array}{cc|cc} 0 & 0 & 0 & 0 \\ 0 & 0 & -d-\alpha t^{-1} & 0 \\ \hline 0 & 0 & 0 & 0 \\ d & 0 & 0 & 0 \end{array}\right),$$

$$\bar{\rho}_\alpha(D^3_\alpha) = \left(\begin{array}{cc|cc} 0 & 0 & d+\alpha t^{-1} & 0 \\ 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 \\ 0 & d & 0 & 0 \end{array}\right), \quad \bar{\rho}_\alpha(D^4_\alpha) = \left(\begin{array}{cc|cc} 0 & 0 & 0 & d+\alpha t^{-1} \\ 0 & 0 & 0 & 0 \\ \hline 0 & -d & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right),$$

$$
\begin{aligned}
\bar\rho_\alpha(E^1_\alpha) &= t^2 1_{4|4}\\
\bar\rho_\alpha(F^1_\alpha) &= \left(\begin{array}{c|c} (d^2+\alpha t^{-1}d)1_{2|2} & 0 \\ \hline 0 & (d^2+\alpha d t^{-1})1_{2|2}\end{array}\right),\\
\bar\rho_\alpha(H^1_\alpha) &= (td+\frac{1+\alpha}{2})1_{4|4},
\end{aligned}
$$

$$
\bar\rho_\alpha(E^2_\alpha) = \left(\begin{array}{cc|cc} 0&0&0&0\\ 1&0&0&0\\ \hline 0&0&0&0\\ 0&0&0&0\end{array}\right), \bar\rho_\alpha(F^2_\alpha) = \left(\begin{array}{cc|cc} 0&-1&0&0\\ 0&0&0&0\\ \hline 0&0&0&0\\ 0&0&0&0\end{array}\right), \bar\rho_\alpha(H^2_\alpha) = \left(\begin{array}{cc|cc} -1&0&0&0\\ 0&1&0&0\\ \hline 0&0&0&0\\ 0&0&0&0\end{array}\right),
$$

$$
\bar\rho_\alpha(E^3_\alpha) = \left(\begin{array}{cc|cc} 0&0&0&0\\ 0&0&0&0\\ \hline 0&0&0&1\\ 0&0&0&0\end{array}\right), \bar\rho_\alpha(F^3_\alpha) = \left(\begin{array}{cc|cc} 0&0&0&0\\ 0&0&0&0\\ \hline 0&0&0&0\\ 0&0&1&0\end{array}\right), \bar\rho_\alpha(H^3_\alpha) = \left(\begin{array}{cc|cc} 0&0&0&0\\ 0&0&0&0\\ \hline 0&0&1&0\\ 0&0&0&-1\end{array}\right).
$$

Proof. For each $h \in (0,1]$ and each $\alpha \in \mathbb{C}$ there exists an embedding

$$\rho_{\alpha,h} : \Gamma(2,-1-\alpha,\alpha-1) \to P_h(4).$$

$\Gamma_{\alpha,h} = \rho_{\alpha,h}(\Gamma(2,-1-\alpha,\alpha-1))$ is spanned by the following elements:

$$
\begin{gathered}
E^1_{\alpha,h} = t^2\\
F^1_{\alpha,h} = \tau^2 - \alpha(2t^{-2}\xi_1\xi_2\eta_1\eta_2 + t^{-2}(\xi_1\eta_1+\xi_2\eta_2)h - t^{-1}\tau h),\\
H^1_{\alpha,h} = t\tau + \frac{\alpha+1}{2}h,\\
E^2_{\alpha,h} = \xi_1\xi_2, \quad F^2_{\alpha,h} = \eta_1\eta_2, \quad H^2_{\alpha,h} = \xi_1\eta_1+\xi_2\eta_2 - h,\\
E^3_{\alpha,h} = \xi_1\eta_2, \quad F^3_{\alpha,h} = \xi_2\eta_1, \quad H^3_{\alpha,h} = \xi_1\eta_1-\xi_2\eta_2,\\
T^1_{\alpha,h} = t\eta_1, \quad T^2_{\alpha,h} = t\eta_2,\\
T^3_{\alpha,h} = t\xi_1, \quad T^4_{\alpha,h} = t\xi_2,\\
D^1_{\alpha,h} = \tau\xi_1 + \alpha t^{-1}\xi_1\xi_2\eta_2, \quad D^2_{\alpha,h} = \tau\xi_2 - \alpha t^{-1}\xi_1\xi_2\eta_1,\\
D^3_{\alpha,h} = \tau\eta_1 + \alpha t^{-1}\eta_1\eta_2\xi_2, \quad D^4_{\alpha,h} = \tau\eta_2 - \alpha t^{-1}\eta_1\eta_2\xi_1,
\end{gathered}
$$

so that

$$\lim_{h\to 0}\Gamma_{\alpha,h} = \rho_\alpha(\Gamma_\alpha) \subset P(4).$$

Let $V = \mathbb{C}[t,t^{-1}] \otimes \Lambda(\xi_1,\xi_2)$. We fix $h=1$, and define a representation of $\Gamma(2,-1-\alpha,\alpha-1)$ in V according to the embedding $\rho_{\alpha,h=1}$. Namely, ξ_i is the operator of multiplication in $\Lambda(\xi_1,\xi_2)$, η_i is identified with ∂_{ξ_i}, and $1 \in P_{h=1}(4)$ acts by the identity operator. Consider the following basis in V:

$$v_m^0 = t^m, \quad v_m^1 = t^m\xi_1,$$
$$v_m^2 = t^m\xi_2, \quad v_m^3 = t^m\xi_1\xi_2 \text{ for all } m \in \mathbb{Z}.$$

Explicitly, the action of $\Gamma(2, -1-\alpha, \alpha-1)$ on V is given as follows

$$\begin{array}{l}
T_\alpha^1(v_m^3) = v_{m+1}^2, \quad T_\alpha^1(v_m^1) = v_{m+1}^0, \quad T_\alpha^2(v_m^3) = -v_{m+1}^1, \quad T_\alpha^2(v_m^2) = v_{m+1}^0,\\
T_\alpha^3(v_m^0) = v_{m+1}^1, \quad T_\alpha^3(v_m^2) = v_{m+1}^3, \quad T_\alpha^4(v_m^0) = v_{m+1}^2, \quad T_\alpha^4(v_m^1) = -v_{m+1}^3,\\
D_\alpha^1(v_m^0) = mv_{m-1}^1, \quad D_\alpha^1(v_m^2) = (m+\alpha)v_{m-1}^3, \quad D_\alpha^2(v_m^0) = v_{m-1}^2, \quad D_\alpha^2(v_m^1) = -(m+\alpha)v_{m-1}^3,\\
D_\alpha^3(v_m^3) = mv_{m-1}^2, \quad D_\alpha^3(v_m^1) = (m+\alpha)v_{m-1}^0, \quad D_\alpha^4(v_m^3) = -mv_{m-1}^1, \quad D_\alpha^4(v_m^2) = (m+\alpha)v_{m-1}^0,\\
E_\alpha^1(v_m^0) = v_{m+2}^0, \quad E_\alpha^1(v_m^3) = v_{m+2}^3, \quad E_\alpha^1(v_m^1) = v_{m+2}^1, \quad E_\alpha^1(v_m^2) = v_{m+2}^1,\\
F_\alpha^1(v_m^0) = m(m-1+\alpha)v_{m-2}^0, \quad F_\alpha^1(v_m^3) = m(m-1+\alpha)v_{m-2}^3,\\
F_\alpha^1(v_m^1) = (m+\alpha)(m-1)v_{m-2}^1, \quad F_\alpha^1(v_m^2) = (m+\alpha)(m-1)v_{m-2}^2,\\
H_\alpha^1(v_m^i) = (m + \dfrac{\alpha+1}{2})v_m^i, \quad i = 0,1,2,3,\\
E_\alpha^2(v_m^0) = v_m^3, \quad F_\alpha^2(v_m^3) = -v_m^0, \quad H_\alpha^2(v_m^0) = -v_m^0, \quad H_\alpha^2(v_m^3) = v_m^3,\\
E_\alpha^3(v_m^2) = v_m^1, \quad F_\alpha^3(v_m^1) = v_m^2, \quad H_\alpha^3(v_m^1) = v_m^1, \quad H_\alpha^3(v_m^2) = -v_m^2.
\end{array}$$

Thus we obtain the above-mentioned embedding $\bar{\rho}_\alpha$ of $\Gamma(2, -1-\alpha, \alpha-1)$ into $\mathrm{End}(\mathcal{W}^{2|2})$.

□

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