IGUSA-TYPE FUNCTIONS ASSOCIATED TO FINITE FORMED SPACES AND THEIR FUNCTIONAL EQUATIONS

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Abstract. We study symmetries enjoyed by the polynomials enumerating non-degenerate flags in finite vector spaces, equipped with a non-degenerate alternating bilinear, hermitian or quadratic form. To this end we introduce Igusa-type rational functions encoding these polynomials and prove that they satisfy certain functional equations.

Some of our results are achieved by expressing the polynomials in question in terms of what we call parabolic length functions on Coxeter groups of type A. While our treatment of the orthogonal case exploits combinatorial properties of integer compositions and their refinements, we formulate a precise conjecture how in this situation, too, the polynomials may be described in terms of parabolic length functions.

1. Introduction

We fix a natural number $n \in \mathbb{N}$ and consider n-dimensional vector spaces V over a finite field F, equipped with

- a non-degenerate alternating bilinear form B (the 'symplectic case'),
- a hermitian form B (the 'unitary case') or
- a quadratic form f (the 'orthogonal case').

In the symplectic and unitary cases, we formally define $f : V \to F$ by $f(x) := B(x, x)$. In the orthogonal case, we let B denote the bilinear form obtained by polarising f: if $\operatorname{char} F \neq 2$, then B is non-degenerate symmetric, whereas, if $\operatorname{char} F = 2$, then B is alternating and possibly degenerate. The triple $\mathcal{V} := (V, B, f)$ will be called a *formed space*. We also introduce a parameter γ equal to 1 in the unitary case and equal to $1/2$ otherwise; with this convention $F = \mathbb{F}_{q^{2\gamma}}$ for a prime power q.

Recall that, by the classification of finite formed spaces (cf., e.g., [2, Section 3.3]), $\mathcal{V}$ decomposes as an orthogonal direct sum of a certain number of hyperbolic planes and an anisotropic space of dimension $d \in \{0, 1, 2\}$. In the orthogonal case we attach a sign $\varepsilon \in \{-1, 1\}$ to $\mathcal{V}$ if n is even, according to whether d equals 0 or 2. The six possibilities are given by the following table.

2000 *Mathematics Subject Classification.* 05E15; 15A63, 20F55.

Key words and phrases. Finite formed spaces, Coxeter groups, zeta functions, functional equations.

Draft version: March 24, 2006.

geometric type	n	d	ε	γ
symplectic	$2m$	0	–	1/2
unitary	$2m$	0	–	1
unitary	$2m+1$	1	–	1
orthogonal	$2m$	0	1	1/2
orthogonal	$2m+1$	1	–	1/2
orthogonal	$2m$	2	-1	1/2

In the current paper we study rational functions incorporating the numbers of F-rational points of the varieties of flags of non-degenerate subspaces in $\mathcal{V}$. More precisely, writing $[n-1]$ for $\{1,\dots,n-1\}$, by a *non-degenerate flag of type* $J=\{j_1,\dots,j_s\}_< \subseteq [n-1]$ we mean a family $\mathbf{U}_J=(U_j)_{j\in J}$ of non-degenerate subspaces of $\mathcal{V}$ with $U_{j_1}\subset\dots\subset U_{j_s}$ and $\dim U_j = j$ for each $j\in J$. Let

$$a^J_{\mathcal{V}}(q) := |\{\mathbf{U}_J \mid \mathbf{U}_J \text{ non-degenerate flag of type } J\}|.$$

Then $a^J_{\mathcal{V}}(q)$ is a monic polynomial in q (cf. the remarks at the end of this introduction regarding the orthogonal case), and we set

$$\alpha^J_{\mathcal{V}}(q^{-1}) := a^J_{\mathcal{V}}(q)/q^{\deg_q a^J_{\mathcal{V}}}.$$

We encode these numbers in rational functions as follows. Let $\mathbf{X}=(X_i)_i$ be a finite family of independent indeterminates. Fix a family of rational functions $\mathbf{F}=(F_J(\mathbf{X}))_{J\subseteq[n-1]}$ in $\mathbf{X}$ with the *inversion property* that

$$\text{(IP)} \qquad \text{for all } \; I\subseteq[n-1]: \; F_I(\mathbf{X}^{-1}) = (-1)^{|I|}\sum_{J\subseteq I} F_J(\mathbf{X}).$$

A simple and naturally occurring example (cf. [9, Part II]) of a family with this property is $\left(\prod_{j\in J}\frac{X_j}{1-X_j}\right)_{J\subseteq[n-1]}$. By defining

$$\mathrm{Ig}_{\mathcal{V}}(q^{-1},\mathbf{X}) := \mathrm{Ig}_{\mathcal{V},\mathbf{F}}(q^{-1},\mathbf{X}) := \sum_{J\subseteq[n-1]} \alpha^J_{\mathcal{V}}(q^{-1})F_J(\mathbf{X})$$

we associate to $\mathcal{V}$ and $\mathbf{F}$ a rational function. Some explicit examples of these *Igusa-type functions* may be found in the Appendix.

The first main result of this paper is

Theorem A. *For each n-dimensional, non-degenerate formed space $\mathcal{V}$ the associated Igusa-type function satisfies the functional equation*

$$\mathrm{Ig}_{\mathcal{V}}(q,\mathbf{X}^{-1}) = (-1)^a q^b\, \mathrm{Ig}_{\mathcal{V}}(q^{-1},\mathbf{X}),$$

where the integers a and b are given by the table below (with $m := \lfloor \frac{n}{2}\rfloor$).

geometric type	n	ε	a	b
symplectic	$2m$	–	$m-1$	$m(m-1)$
unitary	n	–	$\binom{n}{2}+n-1$	$\binom{n}{2}$
orthogonal	$2m$	1	$m+1$	m^2
orthogonal	$2m+1$	–	m	$m(m+1)$
orthogonal	$2m$	-1	m	m^2

The functions $\mathrm{Ig}_{\mathcal{V}}$ may be regarded as complementary to the rational functions built from the polynomials counting flags of *totally isotropic* subspaces in $\mathcal{V}$. These functions, which are thus related to the classical finite polar spaces associated to the formed spaces $\mathcal{V}$, are typical examples of the functions studied by Igusa in his work [9, Part II] on p-adic zeta functions associated to algebraic groups. After substitution of the variables X_i by suitable q-powers $q^{a_i - b_i s}$, where a_i, b_i are integers and s is a complex variable, such rational functions represent certain p-adic integrals. One of the main results in [9, Part II] establishes functional equations similar to the ones in Theorem A.

Igusa's functions are closely related to p-adic integrals associated to zeta functions of groups and algebras, where many instances of functional equations similar to the ones described in Theorem A occur. The analytic properties of Euler products of p-adic integrals of this type are also objects of intense study. We refer the reader to [3], [7], [4], [16] for more information on analytic properties of zeta functions of groups and their functional equations.

We note that in order to derive the functional equations in Theorem A it is essential to 'normalise' the polynomials $a^J_{\mathcal{V}}(q)$ by dividing them by their leading monomial. This is of course insubstantial in the setting of finite polar spaces, as the polynomials there enumerate the F-rational points of smooth projective varieties. Incidentally, one motivation for this normalisation comes from applications, where the functions $\mathrm{Ig}_{\mathcal{V}}$ give rise to certain rational *generating* functions; see the remarks following Conjecture C below. We also note that, by passing to the normalised polynomials $\alpha^J_{\mathcal{V}}(q^{-1})$, the assumption that $\mathcal{V}$ is non-degenerate means no loss of generality.

We now discuss the proof of Theorem A. In the symplectic and unitary case it follows from Witt's Extension Theorem that the respective isometry group acts transitively on the non-degenerate flags of a given type. A simple stabiliser computation reveals that the polynomials $\alpha^J_{\mathcal{V}}(q^{-1})$ may be expressed in terms of Gaussian polynomials (or q-binomial coefficients), which in turn admit a well-known description in terms of the *length function* on a Coxeter group of type $A_{\gamma n-1}$. The functional equation then follows with the same argument which Igusa has given in [9, Part II]. It rests on the fact that, in a Coxeter group, the effect of right-multiplication by the longest element on an element's length and descent set is well understood.

In the orthogonal case, however, things are more intricate. To begin with, the $a^J_{\mathcal{V}}(q)$ flags of type J come in up to $2^{|J|}$ isomorphism types and counting them together seems to be crucial for the occurrence of a functional equation. But of course the natural action of the respective orthogonal group on these flags is not transitive. The proof we give for this case of Theorem A is based on a combinatorial analysis of the polynomials $\alpha^J_{\mathcal{V}}(q^{-1})$ in terms of *integer compositions* and their refinements. Complementing this approach, we propose in Conjecture C an explicit formula which expresses these polynomials, too, in terms of Coxeter group data.

In the symplectic and unitary case, we prove a result which is slightly more general than Theorem A. Rather than counting flags which are non-degenerate with respect to a single non-degenerate sesquilinear form B, we study the numbers of flags which are non-degenerate with respect to a 'flag of forms'. Loosely speaking, a *flag of sesquilinear forms* $\boldsymbol{B}$ of type $I \subseteq [n-1]$ is a family of sesquilinear forms such that

- all but the first form are degenerate,
- each but the last form is defined on the radical of its successor,
- the last form is defined on the total space V and
- the non-zero radicals constitute a flag of type I in V.

'Non-degeneracy' is defined inductively; see Section 3 for details.

Now let $\boldsymbol{B}$ be a sesquilinear flag of forms of type $I \subseteq [n-1]$ on V. We denote by $a^J_{(V,\boldsymbol{B})}(q)$ the number of flags of type $J \subseteq [n-1]$ which are non-degenerate with respect to $\boldsymbol{B}$. In the symplectic case, both the type I of $\boldsymbol{B}$ and all the sets $J \subseteq [n-1]$ for which $a^J_{(V,\boldsymbol{B})}(q)$ is non-zero necessarily consist of even numbers. From the *normalised* polynomials

$$\alpha^J_{(V,\boldsymbol{B})}(q^{-1}) := a^J_{(V,\boldsymbol{B})}(q)/q^{\deg_q a^J_{(V,\boldsymbol{B})}}$$

and a family of rational functions $\mathbf{F} = (F_J(\mathbf{X}))_{J\subseteq[n-1]}$ with the inversion property (IP) we define, similarly as above, a rational function

$$\mathrm{Ig}_{(V,\boldsymbol{B})}(q^{-1},\mathbf{X}) := \mathrm{Ig}_{(V,\boldsymbol{B}),\mathbf{F}}(q^{-1},\mathbf{X}) = \sum_{J\subseteq[n-1]} \alpha^J_{(V,\boldsymbol{B})}(q^{-1})F_J(\mathbf{X}).$$

The second main result of this paper is

Theorem B. *For each n-dimensional vector space V, equipped with a flag of alternating bilinear (respectively hermitian) forms $\boldsymbol{B}$ of type $I = \{i_1,\dots,i_r\}_< \subseteq [n-1]$, the associated Igusa-type function satisfies the functional equation*

$$\mathrm{Ig}_{(V,\boldsymbol{B})}(q,\mathbf{X}^{-1}) = (-1)^a q^b \, \mathrm{Ig}_{(V,\widetilde{\boldsymbol{B}})}(q^{-1},\mathbf{X}),$$

where $\widetilde{\boldsymbol{B}}$ is a flag of forms of type $\widetilde{I} := \{n-i \mid i \in I\}$ and the integers a and b are given by the table below (with $m := \lfloor \frac{n}{2} \rfloor$).

geometric type	a	b
symplectic	$m-1$	$m(m-1) + ((i_2-i_1)i_1 + \cdots + (n-i_r)i_r)/2$
unitary	$n-1+b$	$\binom{n}{2} + (i_2-i_1)i_1 + \cdots + (n-i_r)i_r$

Note that, for $I = \varnothing$, Theorem B specialises to Theorem A in the symplectic and unitary case, respectively.

To prove Theorem B we show that the functions $\alpha^J_{(V,\boldsymbol{B})}(q^{-1})$ are polynomials which may be described in terms of a certain *statistic* on the Coxeter group W of type $A_{\gamma n-1}$. This statistic associates to an element $w \in W$ the sum of its ordinary length $l(w)$ with respect to the standard Coxeter generating set $S = \{s_1,\dots,s_{\gamma n-1}\}$ and its 'parabolic length' $l_{\mathrm{L}}^{(\gamma\widetilde{I})^c}(w)$. The *parabolic length* $l_{\mathrm{L}}^{(\gamma\widetilde{I})^c}(w)$ is the Coxeter length of the distinguished representative of shortest length in the left coset $wW_{(\gamma\widetilde{I})^c}$ of the standard parabolic subgroup $W_{(\gamma\widetilde{I})^c} = \langle s_i \in S \mid \gamma n - i \notin \gamma I\rangle$.

In fact, in Section 3 we show that Theorem B can be deduced from Theorem 1, a general result on rational functions defined in terms of linear combinations of parabolic length functions and characters on certain subgroups of finite Coxeter groups. Indeed, Theorem 1 extends to a slightly more general setting Igusa's key idea to deduce functional equations from features of the map induced by (right-) multiplication by the longest element.

Our initial interest in the Igusa-type functions $\mathrm{Ig}_{(V,\boldsymbol{B})}$ arose from our study of the zeta functions counting subgroups of higher Heisenberg groups. In [10] we introduce an equivalence relation, coarser than homothety, on the set of complete $\mathbb{Z}_p$-lattices in a non-degenerate symplectic p-adic vector space $\mathbb{Q}_p^{2m}$ such that equivalence classes of lattices are in one-to-one correspondence with the vertices of the affine Bruhat-Tits building for the symplectic group $\mathrm{Sp}_{2m}(\mathbb{Q}_p)$. For a flag of forms $\boldsymbol{B}$ of type $I = \{2i\}$, $i \in [m]$, and suitable choices of $\mathbf{F}$ the Igusa functions $\mathrm{Ig}_{(\mathbb{F}_q^{2m},\boldsymbol{B}),\mathbf{F}}$ may be regarded as generating functions, enumerating lattices in an equivalence class indexed by a special vertex of type i. We refer to [10] for details.

In the last part of the paper we formulate a precise conjecture describing the polynomials $\alpha_{\mathcal{V}}^J(q^{-1})$ in the orthogonal case. If it holds, the orthogonal case of Theorem A also follows from Theorem 1. Moreover, a proof of Conjecture C would constitute a first step towards extending Theorem B to the orthogonal case.

We introduce the subgroup $\mathcal{C}_n$ of 'chessboard elements' in the symmetric group $\mathcal{S}_n$ on n letters. A permutation is a *chessboard element* if the non-zero entries of its associated permutation matrix all fit either on the black or on the white squares of an $n \times n$-chessboard. In Section 5 we define linear characters χ_ε on $\mathcal{C}_n$ and a certain linear combination L of parabolic length functions on $\mathcal{S}_n$. By $D_{\mathrm{L}}(w)$ we denote the left-descent set of the permutation w (cf. Section 2).

Conjecture C. *For each n-dimensional, non-degenerate quadratic space $\mathcal{V}$ and each $J \subseteq [n-1]$,*

$$\alpha_{\mathcal{V}}^J(q^{-1}) = \sum_{\substack{w \in \mathcal{C}_n \\ D_{\mathrm{L}}(w) \subseteq J}} \chi_\varepsilon(w) q^{-L(w)}.$$

Remark (The orthogonal case in characteristic 2). As is well-known, quadratic forms are intimately related to symmetric bilinear forms. In fact, over a field of characteristic not equal to 2, the two notions lead to one and the same theory: a quadratic space $\mathcal{V} = (V, B, f)$ over a field F with $\mathrm{char}\, F \neq 2$ can equally well be regarded as a symmetric bilinear space and vice versa. Such a space $\mathcal{V}$ is said to be *non-degenerate* if the bilinear form B is non-degenerate, i.e. if the radical $\mathrm{Rad}(B) := \{x \in V \mid \forall y \in V : B(x, y) = 0\}$ is the zero subspace. In particular, enumerating non-degenerate flags in a quadratic space $\mathcal{V}$ is the same as counting non-degenerate flags in the symmetric bilinear space $\mathcal{V}$.

In characteristic 2, however, one has to distinguish more carefully between quadratic and symmetric bilinear forms. It is noteworthy that the analogous statement of Theorem A for symmetric bilinear spaces does not hold in characteristic 2: in the Appendix we display a 4-dimensional non-degenerate symmetric bilinear space whose associated 'Igusa-type' function does not satisfy a functional equation.

Now consider quadratic spaces $\mathcal{V} = (V, B, f)$ over a field F with $\mathrm{char}\, F = 2$. In this context B is alternating and carries less information than f. There are basically two notions of 'non-degeneracy', but unfortunately no standard terminology; cf. [2], [6], [11, Appendix 1], [12]. In this paper, we call $\mathcal{V}$ *non-defective* if the associated bilinear form B is non-degenerate, i.e. if the radical

$\mathrm{Rad}(B) := \{x \in V \mid \forall y \in V : B(x,y) = 0\}$ is the zero subspace. This can be thought of as a strong version of 'non-degeneracy'; in particular, every non-defective quadratic space is even-dimensional. But enumerating non-defective flags in a quadratic space over F is the same as counting non-degenerate flags in the induced alternating bilinear space, so we gain nothing new. We call a quadratic space $\mathcal{V}$ *non-degenerate* if the restriction of f to the radical $\mathrm{Rad}(B)$ is anisotropic, i.e. if for all $x \in \mathrm{Rad}(B)$ either $x = 0$ or $f(x) \neq 0$. This concept of 'non-degeneracy' is more flexible; in particular, there are non-degenerate quadratic spaces of any given dimension. Moreover, this turns out to be the right notion to formulate Theorem A. In fact, the polynomials $a^J_{\mathcal{V}}(q)$ counting non-degenerate flags of type J are the same in all characteristics; see Section 4.

The structure of the paper is as follows. In Section 2 we derive functional equations for rational functions defined in terms of parabolic length functions on Coxeter groups. Theorem 1, the main result of Section 2, is applied to prove Theorem B in Section 3. In Section 4 we prove the orthogonal case of Theorem A. In Section 5 we give a more precise statement of Conjecture C. Some explicit examples of Igusa-type functions can be found in the Appendix.

We use the following notation.

$\mathbb{N}$	the set $\{1,2,\dots\}$ of natural numbers
I_0	the set $I \cup \{0\}$ for $I \subseteq \mathbb{N}$
$[a,b]$	the interval $\{a, a+1, \dots, b\}$ for integers a, b
$[a]$	the interval $[1,a]$ for an integer a
$\{i_1,\dots,i_r\}_<$	the set $\{i_1,\dots,i_r\} \subseteq \mathbb{N}_0$ with $i_1 < \dots < i_r$
xI	the set $\{xi \mid i \in I\}$ for $I \subseteq \mathbb{N}$ and a rational number x
I^c	the set $[n-1] \setminus I$ for $I \subseteq [n-1]$, where n is clear from the context
$\widetilde{I}$	the set $\{n-i \mid i \in I\}$ for $I \subseteq [n-1]$, where n is clear from the context
$I - \mathbf{t}$	the set $\{i_1 - t_1, \dots, i_r - t_r\} \cap \mathbb{N}$ for $I = \{i_1,\dots,i_r\}_< \subseteq \mathbb{N}$ and $\mathbf{t} = (t_1,\dots,t_r) \in \mathbb{N}_0^r$
$I - j$	the set $\{i - j \mid i \in I\} \cap \mathbb{N}$ for $I \subseteq \mathbb{N}$, $j \in \mathbb{N}_0$
$\binom{a}{b}$	the ordinary binomial coefficient for $a, b \in \mathbb{N}_0$
$\binom{a}{b}_X$	the polynomial $\prod_{i=0}^{b-1}(1-X^{a-i})/(1-X^{b-i})$, where $a, b \in \mathbb{N}_0$ with $a \geq b$. Note: The *q-binomial coefficient* $\binom{a}{b}_q$ gives the number of subspaces of dimension b in $\mathbb{F}_q^a$.
$\binom{n}{J}_X$	the polynomial $\binom{n}{j_s}_X \binom{j_s}{j_{s-1}}_X \cdots \binom{j_2}{j_1}_X$, where $J = \{j_1,\dots,j_s\}_< \subseteq [n-1]_0$ for $n \in \mathbb{N}$. Note: $\binom{n}{J}_q$ gives the number of flags of type J in $\mathbb{F}_q^n$.
$\lfloor x \rfloor$	the greatest integer not exceeding the rational number x
$\mathcal{P}(S)$	the power set of a set S
$\mathcal{S}_n$	the symmetric group on n letters.

Throughout this paper $n \in \mathbb{N}$ and $m = \lfloor n/2 \rfloor$. We shall write

$$\begin{array}{lll} I = \{i_1, \dots, i_r\}_<, & J = \{j_1, \dots, j_s\}_< & \text{for subsets of } [n-1] \text{ or } [n], \text{ and} \\ G = \{g_1, \dots, g_k\}_<, & H & \text{for subsets of } [m]. \end{array}$$

2. Rational functions from Coxeter groups

In this section we prove functional equations for a family of rational functions associated to finite Coxeter systems. Theorem B will turn out to be a consequence of Theorem 1, the main result of the current Section.

Let (W, S) be a finite Coxeter system of rank $n-1$ with root system Δ. To ease notation we will frequently identify the set of Coxeter generators $S = \{s_1, \dots, s_{n-1}\}$ with the set of integers $[n-1]$. For each $I \subseteq S$ we denote by W_I the corresponding standard parabolic subgroup of W generated by the elements in I and by Δ_I the induced root system. We denote by l the length function on W with respect to S. The length of an element w may either be interpreted as the length of a shortest word in the elements of S representing the group element or as the number of positive roots that are sent to negative roots by w. The group W has a unique longest element w_0, whose length equals $|\Delta|/2$. It is well-known (cf. [8, Section 1.8]) that, for each $w \in W$,

$$l(w_0 w) + l(w) = l(w w_0) + l(w) = l(w_0).$$

The rational functions studied in this section are defined in terms of more general length functions. For each $I \subseteq S$ set

$$\begin{aligned} W_{\mathrm{L}}^I &:= \{w \in W \mid \forall s \in I : \ l(ws) > l(w)\}, \\ W_{\mathrm{R}}^I &:= \{w \in W \mid \forall s \in I : \ l(sw) > l(w)\}. \end{aligned}$$

We will need the following lemma ([13, Proposition 2.1.7]).

Lemma 1. *Let $I \subseteq S$. Then W_{L}^I (respectively W_{R}^I) is a left (respectively right) transversal to W_I in W, i.e. for every $w \in W$ there are unique elements*

$$u_{\mathrm{L}} \in W_{\mathrm{L}}^I, \ v_{\mathrm{L}} \in W_I \ \text{ and } \ u_{\mathrm{R}} \in W_{\mathrm{R}}^I, \ v_{\mathrm{R}} \in W_I$$

such that

$$w = u_{\mathrm{L}} v_{\mathrm{L}} = v_{\mathrm{R}} u_{\mathrm{R}}.$$

In particular, u_{L} is the unique element of shortest length in the left coset wW_I and u_{R} is the unique element of shortest length in the right coset $W_I w$. Moreover,

$$l(w) = l(u_{\mathrm{L}}) + l(v_{\mathrm{L}}) = l(v_{\mathrm{R}}) + l(u_{\mathrm{R}}).$$

The elements $u_{\mathrm{L}} \in wW_I$ and $u_{\mathrm{R}} \in W_I w$ are also characterised by the fact that they send positive roots of Δ_I to positive roots.

Definition 1 (Parabolic length)**.** For each $I \subseteq S$ and $w \in W$ we set

$$\begin{aligned} l_{\mathrm{L}}^I(w) &:= l(u_{\mathrm{L}}), \\ l_{\mathrm{R}}^I(w) &:= l(u_{\mathrm{R}}). \end{aligned}$$

We call l_{L}^I (respectively l_{R}^I) the *left* (respectively *right*) *parabolic length function* on W associated to I. We write $\mathbf{l}_L := (l_{\mathrm{L}}^I)_{I \subseteq S}$ and $\mathbf{l}_{\mathrm{R}} := (l_{\mathrm{R}}^I)_{I \subseteq S}$.

Note that for $I = \varnothing$ the corresponding parabolic length functions reduce to the ordinary Coxeter length function: $l_{\mathrm{L}}^{\varnothing} = l_{\mathrm{R}}^{\varnothing} = l$. Moreover, $l_{\mathrm{L}}^S = l_{\mathrm{R}}^S = 0$.

Lemma 2. *For each $I \subseteq S$ and $w \in W$ we have*

$$l_{\mathrm{L}}^{I}(w_0 w) + l_{\mathrm{L}}^{I}(w) = l_{\mathrm{L}}^{I}(w_0), \quad l_{\mathrm{L}}^{I}(w w_0) + l_{\mathrm{L}}^{I^{w_0}}(w) = l_{\mathrm{L}}^{I}(w_0), \tag{1}$$

$$l_{\mathrm{R}}^{I}(w_0 w) + l_{\mathrm{R}}^{I^{w_0}}(w) = l_{\mathrm{R}}^{I}(w_0), \quad l_{\mathrm{R}}^{I}(w w_0) + l_{\mathrm{R}}^{I}(w) = l_{\mathrm{R}}^{I}(w_0). \tag{2}$$

Proof. Let v_0 denote the longest element in W_I. Then

$$l_{\mathrm{L}}^{I}(w_0) = l(w_0) - l(v_0) = l(w_0) - |\Delta_I|/2.$$

Write $w = u_{\mathrm{L}} v_{\mathrm{L}}$ as in Lemma 1. We may then write

$$w_0 = u' v' v_{\mathrm{L}}^{-1} u_{\mathrm{L}}^{-1}$$

with $u' \in W$, $v' \in W_I$ such that $v' v_{\mathrm{L}}^{-1} = v_0$. It follows that

$$\begin{aligned} l(u') &= l(w_0) - l(v' v_{\mathrm{L}}^{-1} u_{\mathrm{L}}^{-1}) = l(w_0) - l(u_{\mathrm{L}} v_0) \\ &= l(w_0) - l(u_{\mathrm{L}}) - l(v_0) = (l(w_0) - l(v_0)) - l(u_{\mathrm{L}}) \\ &= l_{\mathrm{L}}^{I}(w_0) - l_{\mathrm{L}}^{I}(w). \end{aligned}$$

Clearly, $w_0 w W_I = u' W_I$. But u' sends positive roots of Δ_I to positive roots and is thus the unique coset representative of shortest length. Hence $l_{\mathrm{L}}^{I}(w_0 w) = l(u') = l_{\mathrm{L}}^{I}(w_0) - l_{\mathrm{L}}^{I}(w)$. This gives the first equation in (1).

For the second equation in (1), note that conjugation by w_0 yields $l_{\mathrm{L}}^{I}(w_0) = l_{\mathrm{L}}^{I^{w_0}}(w_0)$ and thus

$$l_{\mathrm{L}}^{I}(w w_0) = l_{\mathrm{L}}^{I^{w_0}}(w_0 w) = l_{\mathrm{L}}^{I^{w_0}}(w_0) - l_{\mathrm{L}}^{I^{w_0}}(w) = l_{\mathrm{L}}^{I}(w_0) - l_{\mathrm{L}}^{I^{w_0}}(w).$$

We omit the analogous proofs for the equations (2). □

Another important invariant of an element of a Coxeter group which we shall need is its (left) *descent set* $D_{\mathrm{L}}(w) := \{s \in S \mid l(sw) < l(w)\}$. Note that

$$D_{\mathrm{L}}(w w_0) = D_{\mathrm{L}}(w)^c := \{s \in S \mid s \notin D_{\mathrm{L}}(w)\}. \tag{3}$$

Elements of Coxeter groups of type A can be regarded as permutation matrices. It is noteworthy that both the descent sets and the values of the various parabolic length functions are easily read off from the associated matrices.

Lemma 3. *Let $(F_J(\mathbf{X}))_{J \subseteq S}$ be a family of rational functions with the inversion property* (IP)*. Then, for all $I \subseteq S$,*

$$\sum_{I \subseteq J \subseteq S} F_J(\mathbf{X}^{-1}) = (-1)^{|S|} \sum_{I^c \subseteq J \subseteq S} F_J(\mathbf{X})$$

Proof. This is an easy calculation. See [15, Lemma 7]. □

We now fix a family of rational functions $\mathbf{F} = (F_J(\mathbf{X}))_{J \subseteq S}$ with the inversion property (IP) and an independent indeterminate Y. We choose a family $\mathbf{b} = (b_I)_{I \subseteq S}$ of integers and define the statistics $\mathbf{b} \cdot \mathbf{l}_{\mathrm{L}}$ and $\mathbf{b} \cdot \mathbf{l}_{\mathrm{R}}$ on W by setting, for $w \in W$,

$$\mathbf{b} \cdot \mathbf{l}_{\mathrm{L}}(w) := \sum_{I \subseteq S} b_I l_{\mathrm{L}}^{I}(w), \qquad \mathbf{b} \cdot \mathbf{l}_{\mathrm{R}}(w) := \sum_{I \subseteq S} b_I l_{\mathrm{R}}^{I}(w).$$

Similarly, we write $\mathbf{b}^{w_0} \cdot \mathbf{l}_{\mathrm{L}}$ and $\mathbf{b}^{w_0} \cdot \mathbf{l}_{\mathrm{R}}$ to denote the statistics associating to w the elements $\sum_{I \subseteq S} b_I l_{\mathrm{L}}^{I^{w_0}}(w)$ and $\sum_{I \subseteq S} b_I l_{\mathrm{R}}^{I^{w_0}}(w)$, respectively. Let $W' \subseteq W$ be a subgroup with $w_0 \in W'$, and $\chi : W' \to \mathbb{C}^*$ a (linear) character of W'.

Definition 2. With the given data we define the following rational functions:

$$\mathrm{IG}_{\mathrm{L}}^{W',\mathbf{b},\chi,\mathbf{F}}(Y,\mathbf{X}) := \sum_{w\in W'} \chi(w) Y^{\mathbf{b}\cdot\mathbf{l}_{\mathrm{L}}(w)} \sum_{D_{\mathrm{L}}(w)\subseteq J\subseteq S} F_J(\mathbf{X}),$$

$$\mathrm{IG}_{\mathrm{R}}^{W',\mathbf{b},\chi,\mathbf{F}}(Y,\mathbf{X}) := \sum_{w\in W'} \chi(w) Y^{\mathbf{b}\cdot\mathbf{l}_{\mathrm{R}}(w)} \sum_{D_{\mathrm{L}}(w)\subseteq J\subseteq S} F_J(\mathbf{X}).$$

The main result of the current section is

Theorem 1. *The following functional equations hold:*

$$\mathrm{IG}_{\mathrm{L}}^{W',\mathbf{b},\chi,\mathbf{F}}(Y^{-1},\mathbf{X}^{-1}) = (-1)^{|S|}\chi(w_0) Y^{-\mathbf{b}\cdot\mathbf{l}_{\mathrm{L}}(w_0)}\, \mathrm{IG}_{\mathrm{L}}^{W',\mathbf{b}^{w_0},\chi,\mathbf{F}}(Y,\mathbf{X}), \tag{4}$$

$$\mathrm{IG}_{\mathrm{R}}^{W',\mathbf{b},\chi,\mathbf{F}}(Y^{-1},\mathbf{X}^{-1}) = (-1)^{|S|}\chi(w_0) Y^{-\mathbf{b}\cdot\mathbf{l}_{\mathrm{R}}(w_0)}\, \mathrm{IG}_{\mathrm{R}}^{W',\mathbf{b},\chi,\mathbf{F}}(Y,\mathbf{X}). \tag{5}$$

Proof. The equations

$$\begin{aligned} \mathbf{b}^{w_0}\cdot\mathbf{l}_{\mathrm{L}}(ww_0) + \mathbf{b}\cdot\mathbf{l}_{\mathrm{L}}(w) &= \mathbf{b}\cdot\mathbf{l}_{\mathrm{L}}(w_0),\\ \mathbf{b}\cdot\mathbf{l}_{\mathrm{R}}(ww_0) + \mathbf{b}\cdot\mathbf{l}_{\mathrm{R}}(w) &= \mathbf{b}\cdot\mathbf{l}_{\mathrm{R}}(w_0) \end{aligned} \tag{6}$$

are immediate consequences of Lemma 2. Therefore, by (6), Lemma 3, and (3),

$$\begin{aligned} &\mathrm{IG}_{\mathrm{L}}^{W',\mathbf{b},\chi,\mathbf{F}}(Y^{-1},\mathbf{X}^{-1}) = \sum_{w\in W'} \chi(w) Y^{-\mathbf{b}\cdot\mathbf{l}_{\mathrm{L}}(w)} \sum_{D_{\mathrm{L}}(w)\subseteq J\subseteq S} F_J(\mathbf{X}^{-1})\\ &= (-1)^{|S|}\chi(w_0)^{-1} Y^{-\mathbf{b}\cdot\mathbf{l}_{\mathrm{L}}(w_0)} \sum_{w\in W'} \chi(ww_0) Y^{\mathbf{b}^{w_0}\cdot\mathbf{l}_{\mathrm{L}}(ww_0)} \sum_{D_{\mathrm{L}}(ww_0)\subseteq J\subseteq S} F_J(\mathbf{X})\\ &= (-1)^{|S|}\chi(w_0) Y^{-\mathbf{b}\cdot\mathbf{l}_{\mathrm{L}}(w_0)}\, \mathrm{IG}_{\mathrm{L}}^{W',\mathbf{b}^{w_0},\chi,\mathbf{F}}(Y,\mathbf{X}). \end{aligned}$$

The equation (5) is proved analogously. □

In this paper we shall see instances of both types of functional equations presented in Theorem 1. In Section 3 we demonstrate that Theorem B is a consequence of (4). Note that in the special case $\mathbf{F} = \left(\prod_{j\in J} \frac{X_j}{1-X_j}\right)_{J\subseteq[n-1]}$, replacing $\mathbf{b}$ by $\mathbf{b}^{w_0}$ in (4) simply amounts to inverting the order of the variables $X_1,\dots,X_{n-1}$. If Conjecture C holds, the orthogonal case of Theorem A follows from (5).

3. The symplectic and unitary case

In this section we study the polynomials enumerating flags which are non-degenerate with respect to a 'flag of sesquilinear forms'. Our aim is to proof Theorem B. Let V be an n-dimensional vector space over a field F. Let $I = \{i_1,\dots,i_r\}_< \subseteq [n-1]$, and set $i_0 := 0$, $i_{r+1} := n$.

Definition 3 (Flag of forms)**.** We say that V is equipped with a *flag of alternating bilinear* (respectively *hermitian*) *forms* $\boldsymbol{B} = (B_{i_1},\dots,B_{i_{r+1}})$ of type I if there is a filtration of subspaces

$$\{0\} =: R_{i_0} \subset R_{i_1} \subset \cdots \subset R_{i_r} \subset R_{i_{r+1}} := V$$

such that

(a) for all $i\in I$, $\dim R_i = i$;
(b) for all $\rho\in[r+1]$, B_{i_ρ} is an alternating bilinear (respectively hermitian) form $B_{i_\rho} : R_{i_\rho}\times R_{i_\rho}\to F$ with

$$\mathrm{Rad}(B_{i_\rho}) := \{x\in R_{i_\rho} \mid \forall y\in R_{i_\rho} :\ B_{i_\rho}(x,y) = 0\} = R_{i_{\rho-1}}.$$

We call the sequence $\mathbf{R} = (R_{i_1}, \dots, R_{i_r})$ the *flag of radicals* associated to the flag of forms $\boldsymbol{B}$.

Note that, given a flag of sesquilinear forms $\boldsymbol{B}$ of type I on V with flag of radicals $\mathbf{R}$ and $\rho \in [r+1]$, we have a flag of forms $(B_{i_1}, \dots, B_{i_\rho})$ of type $\{i_1, \dots, i_{\rho-1}\}$ on R_{i_ρ} with flag of radicals $(R_{i_1}, \dots, R_{i_{\rho-1}})$ and a flag of forms $(\overline{B_{i_{\rho+1}}}, \dots, \overline{B_{i_{r+1}}})$ of type $\{i_\varrho - i_\rho \mid \rho < \varrho \leq r\}$ on V/R_{i_ρ} with flag of radicals $(R_{i_{\rho+1}}/R_{i_\rho}, \dots, R_{i_r}/R_{i_\rho})$.

Definition 4 (Non-degeneracy)**.** Given a flag of sesquilinear forms $\boldsymbol{B}$ of type I on V with flag of radicals $\mathbf{R}$ as above, we say that a subspace $U \subseteq V$ is *non-degenerate* with respect to $\boldsymbol{B}$ if for each $\rho \in [r+1]$,

(a) $U \cap R_{i_\rho}$ is non-degenerate with respect to $(B_{i_1}, \dots, B_{i_\rho})$ and
(b) $(U + R_{i_\rho})/R_{i_\rho}$ is *non-degenerate* with respect to $(\overline{B_{i_{\rho+1}}}, \dots, \overline{B_{i_{r+1}}})$.

A flag $\mathbf{U}_J = (U_j)_{j \in J}$ of subspaces of V of type $J \subseteq [n-1]$, i.e. an ascending chain of subspaces with $\dim U_j = j$ for each $j \in J$, is said to be *non-degenerate* with respect to $\boldsymbol{B}$ if each of its constituents U_j is.

These definitions are illustrated by the following simple example.

Example. Suppose that n is even, $r = 1$ and $i_1 \in [n-1]$ is even. Then a flag of alternating bilinear forms $\boldsymbol{B} = (B_{i_1}, B_n)$ consists of a (degenerate) alternating bilinear form B_n on V with i_1-dimensional radical R_{i_1}, which in turn supports a non-degenerate form B_{i_1}. A subspace $U \subseteq V$ is non-degenerate with respect to $\boldsymbol{B} = (B_{i_1}, B_n)$ if $U \cap R_{i_1}$ is non-degenerate with respect to B_{i_1} and $(U + R_{i_1})/R_{i_1}$ is non-degenerate with respect to $\overline{B_n}$.

We shall now assume that V is an n-dimensional vector space over a finite field F, equipped with a flag of sesquilinear forms $\boldsymbol{B}$ of type I. As in the introduction we write $\gamma = 1/2$ in the symplectic case and $\gamma = 1$ in the unitary case so that $F = \mathbb{F}_{q^{2\gamma}}$ for some prime power q. Let $J \subseteq [n-1]$ and define $a^J_{(V,\boldsymbol{B})}(q)$ to be the number of flags of type J which are non-degenerate with respect to $\boldsymbol{B}$. We set

$$\alpha^J_{(V,\boldsymbol{B})}(q^{-1}) := a^J_{(V,\boldsymbol{B})}(q)/q^{\deg_q a^J_{(V,\boldsymbol{B})}}$$

and shall frequently write $a^J_{n,I}(q)$ for $a^J_{(V,\boldsymbol{B})}(q)$ and $\alpha^J_{n,I}(q^{-1})$ for $\alpha^J_{(V,\boldsymbol{B})}(q^{-1})$. Recall that, in the symplectic case, both the type I of a flag of forms and all the sets $J \subseteq [n-1]$ for which $a^J_{(V,\boldsymbol{B})}(q)$ is non-zero consist necessarily of even numbers.

Definition 5. Given a family $\mathbf{F} = (F_J(\mathbf{X}))_{J \subseteq [n-1]}$ of rational functions with the inversion property (IP) we define

$$\mathrm{Ig}_{(V,\boldsymbol{B})}(q^{-1}, \mathbf{X}) := \mathrm{Ig}_{(V,\boldsymbol{B}),\mathbf{F}}(q^{-1}, \mathbf{X}) = \sum_{J \subseteq [n-1]} \alpha^J_{(V,\boldsymbol{B})}(q^{-1}) F_J(\mathbf{X}).$$

Theorem B states that these Igusa-type functions satisfy a functional equation. In the remainder of the current section we show how this can be deduced from the first assertion of Theorem 1. Fix a family of rational functions $\mathbf{F} = (F_J(\mathbf{X}))_{J \subseteq [n-1]}$ with the inversion property (IP), and define

$$\gamma\mathbf{F} := (F_{\gamma^{-1}J'}(\mathbf{X}))_{J' \subseteq [\gamma n - 1]}.$$

Let $W' := W := \mathcal{S}_{\gamma n}$ be the full symmetric group on γn letters, let χ be the trivial character on W' and set, for each $J' \subseteq [\gamma n - 1]$,

$$b_{J'} := \delta(\text{‘}J' = \varnothing\text{’}) + \delta(\text{‘}J' = \{s_i \mid i \notin \gamma\widetilde{I}\}\text{’}),$$

where the Kronecker-delta $\delta(E) \in \{1,0\}$ reflects whether or not equation E holds.

In order to derive Theorem B from (4) it suffices to show that

$$\mathrm{Ig}_{(V,\boldsymbol{B}),\mathbf{F}}(q^{-1},\mathbf{X}) = \mathrm{IG}_{\mathrm{L}}^{W',\mathbf{b},\chi,\gamma\mathbf{F}}((-q)^{-1/\gamma},\mathbf{X})$$

for the given data W', $\mathbf{b}$, χ and $\gamma\mathbf{F}$. Clearly, it is enough to prove

Proposition 1. *Let $J \subseteq [n-1]$ such that $\gamma J \subseteq [\gamma n - 1]$. Then*

$$\alpha_{n,I}^{J}(q^{-1}) = a_{n,I}^{J}(q)/q^{2\gamma \deg_q \binom{n}{J}_q} = \sum_{\substack{w\in\mathcal{S}_{\gamma n}\\ D_{\mathrm{L}}(w)\subseteq\gamma J}} Y^{\lambda_{n,I}(w)}, \tag{7}$$

where $Y := (-q)^{-1/\gamma}$ and $\lambda_{n,I}(w) := l(w) + l_{\mathrm{L}}^{(\gamma\widetilde{I})^c}(w)$ for all $w \in W$.

Proof. We will first prove (7) in the case $I = \varnothing$. The proof consists of a simple index computation in the respective isometry group, i.e. in the symplectic group $\mathrm{Sp}_n(\mathbb{F}_q)$ or the unitary group $\mathrm{U}_n(\mathbb{F}_{q^2})$. The proof in the general case is then based on a recursive expression for the numbers $a_{n,I}^{J}(q)$.

So assume that $I = \varnothing$. Then $\boldsymbol{B} = (B)$ simply specifies a non-degenerate alternating bilinear (respectively hermitian) form on V. The respective isometry group $\mathrm{Sp}_n(\mathbb{F}_q)$ or $\mathrm{U}_n(\mathbb{F}_{q^2})$ acts transitively on the non-degenerate flags of type J, so it suffices to compute the stabiliser of any one of them. We construct a 'standard' non-degenerate flag $\mathbf{U}_J := (U_{j_1},\dots,U_{j_s})$ of type $J = \{j_1,\dots,j_s\}_<$ in the following way. In the symplectic case, choose a symplectic basis $E = (e_1, f_1, \dots, e_{n/2}, f_{n/2})$ for V (i.e. $B(e_i, f_j) = \delta_{ij}$, $B(e_i, e_j) = B(f_i, f_j) = 0$) and set $U_j := \langle e_1, f_1, \dots, e_{j/2}, f_{j/2}\rangle$ for $j \in J$. In the unitary case, choose a unitary basis $E = (e_1,\dots,e_n)$ for V (i.e. $B(e_i, e_j) = \delta_{ij}$) and set $U_j := \langle e_1,\dots,e_j\rangle$ for $j \in J$. It is not difficult to verify that an element of the respective isometry group of (V,B) stabilises $\mathbf{U}_J$ if and only if its matrix M_n with respect to the basis E is of block diagonal form

$$M_n = \begin{pmatrix} M_{j_1} & & & \\ & M_{j_2-j_1} & & \\ & & \ddots & \\ & & & M_{n-j_s} \end{pmatrix}$$

with $M_{j_\sigma - j_{\sigma-1}}$ in the respective smaller isometry group for all $\sigma \in [s+1]$, where $j_0 := 0, j_{s+1} := n$. Thus

$$a_{n,\varnothing}^{J}(q) = \begin{cases} |\mathrm{Sp}_n(\mathbb{F}_q)|/\prod_{\sigma\in[s+1]}|\mathrm{Sp}_{j_\sigma-j_{\sigma-1}}(\mathbb{F}_q)| & \text{in the symplectic case,}\\ |\mathrm{U}_n(\mathbb{F}_{q^2})|/\prod_{\sigma\in[s+1]}|\mathrm{U}_{j_\sigma-j_{\sigma-1}}(\mathbb{F}_{q^2})| & \text{in the unitary case.}\end{cases}$$

Employing the well-known formulae (cf. [1, p. 147], [6, Theorems 3.12 and 11.28])

$$|\,\mathrm{Sp}_n(\mathbb{F}_q)| = q^{\binom{n+1}{2}} \prod_{i\in[n/2]} (1 - q^{-2i}),$$
$$|\mathrm{U}_n(\mathbb{F}_{q^2})| = q^{n^2} \prod_{i\in[n]} (1 - (-q^{-1})^i)$$

and using the notation $Y = (-q)^{-1/\gamma}$ we obtain

$$\alpha_{n,\varnothing}^{J}(q^{-1}) = \frac{\prod_{i\in[\gamma n]}(1-Y^i)}{\prod_{\sigma\in[s+1]}\prod_{\iota\in[\gamma(j_\sigma-j_{\sigma-1})]}(1-Y^\iota)} = \binom{\gamma n}{\gamma J}_Y.$$

It is equally well-known (cf. [14, Example 2.2.5]) that Gaussian polynomials may be expressed in terms of Coxeter length functions on symmetric groups:

$$\binom{\gamma n}{\gamma J}_Y = \sum_{\substack{w \in \mathcal{S}_{\gamma n} \\ D_{\mathrm{L}}(w) \subseteq \gamma J}} Y^{l(w)}.$$

Equation (7) follows in the particular case $I = \varnothing$, as $l_{\mathrm{L}}^{(\gamma \widetilde{I})^c} = l_{\mathrm{L}}^{S} = 0$ and $\lambda_{n,I} = l$.

We now treat the general case $I = \{i_1, \dots, i_r\}_< \subseteq [n-1]$. To prove (7) we argue by induction on n. The base step $n = 0$ is trivial, so suppose that $n > 0$. We may further assume that $J = \{j_1, \dots, j_s\}_< \neq \varnothing$ and we define $j := j_1 = \min J$. Our first aim is to derive a recursive formula for $a_{n,I}^{J}(q)$, using the formula we obtained in the special case $I = \varnothing$. For this purpose we determine what are the possible first terms U_j of the flags $\mathbf{U}_J$ we intend to count. Then we consider in how many ways each such space U_j can be completed to yield a full flag $\mathbf{U}_J$.

Let T be the set of all r-tuples $\mathbf{t} = (t_1, \dots, t_r) \in ([j]_0)^r$ such that

$$\begin{aligned} &t_1 \leq \dots \leq t_r, \qquad \gamma\{t_1, \dots, t_r\} \subseteq [\gamma j]_0 \qquad \text{and} \\ &\forall \rho \in [r+1]: \; j - (n - i_\rho) - t_{\rho-1} \leq t_\rho - t_{\rho-1} \leq i_\rho - i_{\rho-1}, \end{aligned} \tag{8}$$

where $i_0 = t_0 = 0$ and $i_{r+1} := n, t_{r+1} := j$. These 'admissible' tuples encode the possible dimensions of the intersections $U_j \cap R_{i_\rho}$ of a j-dimensional subspace U_j of V, non-degenerate with respect to $\boldsymbol{B}$, with the members R_{i_ρ} of the flag of radicals associated to $\boldsymbol{B}$. Recalling that the underlying field F has cardinality $q^{2\gamma}$ and applying (7) for $I = \varnothing$, we note that for each $\mathbf{t} \in T$ there are precisely

$$\begin{aligned} A_{n,I}^{\mathbf{t}}(q) &= \prod_{\rho \in [r+1]} a_{(i_\rho - i_{\rho-1}),\varnothing}^{\{t_\rho - t_{\rho-1}\}}(q^{-1})\, q^{2\gamma(t_\rho - t_{\rho-1})(i_{\rho-1} - t_{\rho-1})} \\ &= \prod_{\rho \in [r+1]} q^{2\gamma(t_\rho - t_{\rho-1})(i_\rho - i_{\rho-1})} \prod_{\rho \in [r+1]} \sum_{\substack{w \in \mathcal{S}_{\gamma(i_\rho - i_{\rho-1})} \\ D_{\mathrm{L}}(w) \subseteq \gamma\{t_\rho - t_{\rho-1}\}}} Y^{l(w)} \end{aligned} \tag{9}$$

subspaces U_j, non-degenerate with respect to $\boldsymbol{B}$, such that $\dim(U_j \cap R_{i_\rho}) = t_\rho$ for all $\rho \in [r+1]$. Given such a subspace U_j, the number of non-degenerate flags $\mathbf{U}_J$ with first term U_j can be described inductively, using the notation $J - j = \{j_2 - j, \dots, j_s - j\}$ and $I - \mathbf{t} = \{i_1 - t_1, \dots, i_r - t_r\} \cap \mathbb{N}$; it equals

$$a_{n-j, I-\mathbf{t}}^{J-j}(q) = q^{2\gamma \deg_q \binom{n-j}{J-j}_q} \sum_{\substack{w \in \mathcal{S}_{\gamma(n-j)} \\ D_{\mathrm{L}}(w) \subseteq \gamma(J-j)}} Y^{\lambda_{n-j, I-\mathbf{t}}(w)}. \tag{10}$$

For $\mathbf{t} \in T$, apply equations (9) and (10) together with the identities

$$\sum_{\rho \in [r+1]} (t_\rho - t_{\rho-1})(i_\rho - t_\rho) = j(n-j) - \sum_{\rho \in [r]} t_\rho(i_{\rho+1} - i_\rho - (t_{\rho+1} - t_\rho))$$

and

$$\binom{n}{J}_q = \binom{n}{j}_q \binom{n-j}{J-j}_q, \quad \deg_q \binom{n}{j}_q = j(n-j)$$

to obtain

$$\begin{aligned} \alpha^J_{n,I}(q^{-1}) &= a^J_{n,I}(q)/q^{2\gamma\deg_q\binom{n}{J}_q}\\ &= q^{-2\gamma\deg_q\binom{n}{J}_q}\sum_{\mathbf{t}\in T} A^{\mathbf{t}}_{n,I}(q)\; a^{J-j}_{n-j,I-\mathbf{t}}(q)\\ &= \sum_{\mathbf{t}\in T} Y^{2\left(\sum_{\rho\in[r]}\gamma t_\rho(\gamma(i_{\rho+1}-i_\rho)-\gamma(t_{\rho+1}-t_\rho))\right)} \qquad (11)\\ &\quad\cdot\Bigg(\sum_{\substack{w\in\mathcal{S}_{\gamma(n-j)}\\ D_{\mathrm{L}}(w)\subseteq\gamma(J-j)}} Y^{\lambda_{n-j,I-\mathbf{t}}(w)}\Bigg)\Bigg(\prod_{\rho\in[r+1]}\sum_{\substack{w\in\mathcal{S}_{\gamma(i_\rho-i_{\rho-1})}\\ D_{\mathrm{L}}(w)\subseteq\gamma\{t_\rho-t_{\rho-1}\}}} Y^{l(w)}\Bigg).\end{aligned}$$

We are looking to prove that the right hand side of equation (11) may be written as a sum over the elements in the symmetric group $\mathcal{S}_{\gamma n}$ whose left descent set is contained in γJ. In the following considerations we shall identify permutations $w\in\mathcal{S}_{\gamma n}$ (acting on $\{1,\dots,\gamma n\}$ from the right) with the corresponding $\gamma n\times\gamma n$-permutation matrices (acting on the set of standard row vectors by right-multiplication). Observe that for any element $w\in\mathcal{S}_{\gamma n}$ with $D_{\mathrm{L}}(w)\subseteq\gamma J$ the corresponding permutation matrix is ascending on the first $[\gamma j]$ rows. Define

$$\mathbf{t}=\mathbf{t}(w)=(t_1,\dots,t_r)$$

by

$$t_\rho := \gamma^{-1}|\{\varrho\in[\gamma j]\mid \varrho^w>\gamma(n-i_\rho)\}|\quad\text{for all }\rho\in[r],$$

and set $t_0:=0$, $t_{r+1}:=j$. Then $\mathbf{t}\in T$, as $n-i_\rho\geq j-t_\rho$ for all $\rho\in[r+1]$ and thus $\mathbf{t}$ satisfies (8). Applying suitable elementary column operations to w corresponding to left multiplication by elements of the parabolic subgroup $W_{(\gamma\widetilde{I})^c}$, it is easily seen that there are unique elements $u,v\in\mathcal{S}_{\gamma n}$ such that

(a) $w=uv$ and $l(w)=l(u)+l(v)$;
(b) for all $\rho\in[r+1]$:

$$\varrho\in[\gamma(j-t_\rho)+1,\gamma(j-t_{\rho-1})]\iff\varrho^u=\varrho+\gamma(n-i_\rho);$$

(c) $v\in W_{(\gamma\widetilde{I})^c}$, i.e. for all $\rho\in[r+1]$:

$$\varrho\in[\gamma(n-i_\rho)+1,\gamma(n-i_{\rho-1})]\iff\varrho^v\in[\gamma(n-i_\rho)+1,\gamma(n-i_{\rho-1})],$$

$$\text{and } D_{\mathrm{L}}(v)\subseteq\{\gamma(n-i_\rho)+\gamma(t_\rho-t_{\rho-1})\mid\rho\in[r+1]\}.$$

This is best seen in terms of permutation matrices. We write id_s for the $s\times s$-unit matrix. Then the permutation matrix u has the shape

$$\left(\begin{array}{c|c|c|c|c|c|c} \mathrm{id}_{\gamma(j-t_r)} & & \cdots & & & & \\ \hline & & & & & & \\ \hline & & \ddots & & & & \\ \hline & & & \mathrm{id}_{\gamma(t_2-t_1)} & & & \\ \hline & & & & & \mathrm{id}_{\gamma t_1} & \\ \hline & u_{r+1} & \cdots & & u_2 & & u_1 \end{array}\right), \qquad (12)$$

where u_ρ is an $\gamma(n-j) \times \gamma((i_\rho - i_{\rho-1}) - (t_\rho - t_{\rho-1}))$-matrix for $\rho \in [r+1]$. The permutation matrix v has the form

$$\left(\begin{array}{c|c|c|c} v_{r+1} & & & \\ \hline & \ddots & & \\ \hline & & v_2 & \\ \hline & & & v_1 \end{array}\right),$$

where v_ρ is an $\gamma(i_\rho - i_{\rho-1}) \times \gamma(i_\rho - i_{\rho-1})$-permutation matrix with at most one descent for $\rho \in [r+1]$. We may thus identify v with

$$(v_1, \dots, v_{r+1}) \in \mathcal{S}_{\gamma i_1} \times \mathcal{S}_{\gamma(i_2 - i_1)} \times \cdots \times \mathcal{S}_{\gamma(n-i_r)}$$

and have, by slight abuse of notation, for each $\rho \in [r+1]$,

$$D_{\mathrm{L}}(v_\rho) \subseteq \{\gamma(t_\rho - t_{\rho-1})\} \cap [\gamma(i_\rho - i_{\rho-1}) - 1]. \tag{13}$$

Remark. The above decomposition $w = uv$ is *not* the one from Lemma 1. It is important for our purpose that each v_ρ has at most one descent.

Note that, by deleting the first γj rows and respective columns in (12), the element u determines a unique $\gamma(n-j) \times \gamma(n-j)$-permutation matrix

$$u' := \begin{pmatrix} u_{r+1} & \cdots & u_2 & u_1 \end{pmatrix}$$

with descent set $D_{\mathrm{L}}(u') = D_{\mathrm{L}}(w) - \gamma j$.

As we indicated in Section 2, it is easy to determine the length of a permutation given by a permutation matrix: it is simply the number of entries 0 in the matrix which are not below or to the right of an entry 1. Thus

$$\begin{aligned} l(u) &= \sum_{\rho \in [r]} \gamma t_\rho (\gamma(i_{\rho+1} - i_\rho) - \gamma(t_{\rho+1} - t_\rho)) + l(u'), \\ l(v) &= \sum_{\rho \in [r+1]} l(v_\rho). \end{aligned}$$

Moreover, the parabolic length of w with respect to $(\gamma \widetilde{I})^c$ is determined by $\mathbf{t}$ and by the parabolic length of u' with respect to $(\gamma(\widetilde{I - \mathbf{t}}))^c$:

$$l_{\mathrm{L}}^{(\gamma \widetilde{I})^c}(w) = \sum_{\rho \in [r]} \gamma t_\rho (\gamma(i_{\rho+1} - i_\rho) - \gamma(t_{\rho+1} - t_\rho)) + l_{\mathrm{L}}^{(\gamma(\widetilde{I-\mathbf{t}}))^c}(u').$$

This gives

$$\begin{aligned} \lambda_{n,I}(w) &= (l(u) + l(v)) + l_{\mathrm{L}}^{(\gamma \widetilde{I})^c}(w) \\ &= 2 \sum_{\rho \in [r]} \gamma t_\rho (\gamma(i_{\rho+1} - i_\rho) - \gamma(t_{\rho+1} - t_\rho)) \\ &\quad + \lambda_{\gamma(n-j), I - \mathbf{t}}(u') + \sum_{\rho \in [r+1]} l(v_s). \end{aligned} \tag{14}$$

Conversely, any $\mathbf{t} \in T$ and any permutations $v_1, \dots, v_{r+1}, u'$ of the appropriate degrees such that (13) holds give rise to a permutation w satisfying

$$D_{\mathrm{L}}(w) \subseteq (D_{\mathrm{L}}(u') + \gamma j) \cup \{\gamma j\}.$$

Thus (14) shows that the right hand side of (11) is indeed equal to the right hand side of (7). □

4. The orthogonal case

Our aim in this section is to complete the proof of Theorem A. We consider non-degenerate quadratic spaces $\mathcal{V} = (V, B, f)$, where V is an n-dimensional vector space over the finite field $F = \mathbb{F}_q$, equipped with a quadratic form f, and B denotes the bilinear form obtained by polarising f. So for all $x, y \in V$,

$$B(x,y) = \begin{cases} f(x+y) + f(x) + f(y) & \text{if } \operatorname{char} F = 2, \\ \frac{1}{2}\left(f(x+y) - f(x) - f(y)\right) & \text{if } \operatorname{char} F \neq 2. \end{cases}$$

If $\operatorname{char} F \neq 2$, then B is non-degenerate symmetric and, as $f(x) = B(x,x)$ for all $x \in V$, the quadratic form f can easily be recovered from B. If $\operatorname{char} F = 2$, then B is alternating, possibly degenerate and carries less information than f.

Non-degenerate quadratic spaces over finite fields have been classified and can be described up to isomorphism as follows; cf. [1, p. 144], [2, Section 3.3]. If $\operatorname{char} F \neq 2$, then for any given dimension n there are two possible isomorphism types of non-degenerate quadratic spaces $\mathcal{V} = (V, B, f)$, namely

$$\begin{aligned} &\text{for } n \text{ odd:} && \mathcal{V} = \mathcal{H}_1 \perp \cdots \perp \mathcal{H}_m \perp \mathcal{A}_{1,1}, \\ & && \mathcal{V} = \mathcal{H}_1 \perp \cdots \perp \mathcal{H}_m \perp \mathcal{A}_{1,-1}, \\ &\text{for } n \text{ even:} && \mathcal{V} = \mathcal{H}_1 \perp \cdots \perp \mathcal{H}_{m-1} \perp \mathcal{H}_m, \\ & && \mathcal{V} = \mathcal{H}_1 \perp \cdots \perp \mathcal{H}_{m-1} \perp \mathcal{A}_2, \end{aligned}$$

where $m = \lfloor \frac{n}{2} \rfloor$, the $\mathcal{H}_i$ denote hyperbolic planes, $\mathcal{A}_{1,1}$ (respectively $\mathcal{A}_{1,-1}$) stands for an anisotropic line $\langle x \rangle$ with $f(x) \in (F^*)^2$ (respectively $f(x) \in F^* \setminus (F^*)^2$) and $\mathcal{A}_2$ is an anisotropic plane. For the purpose of counting non-degenerate flags in quadratic spaces $\mathcal{V}$ of given odd dimension, there is no significant difference between the two possible isomorphism types.

We now discuss the case $\operatorname{char} F = 2$. Then the above list still provides all isomorphism types of non-degenerate quadratic spaces, but becomes one term shorter: as every element of F is a square, in any given odd dimension there is (up to isomorphism) just one non-degenerate quadratic space. In any given even dimension there are still two isomorphism types. Note also that non-degenerate quadratic spaces of odd dimension are defective with 1-dimensional radical, whereas non-degenerate quadratic spaces of even dimension are non-defective.

Returning to the task of proving Theorem A, we recall from the introduction that, in the even-dimensional case, we attach a sign $\varepsilon = 1$ or $\varepsilon = -1$ to $\mathcal{V}$ according to whether the anisotropic kernel of $\mathcal{V}$ is 0- or 2-dimensional. More suggestively, we write $a^J_{2m+1}(q) := a^J_{\mathcal{V}}(q)$ if $n = 2m+1$ is odd and, similarly, $a^J_{2m,\varepsilon}(q) := a^J_{\mathcal{V}}(q)$ if $n = 2m$ is even and $\mathcal{V}$ of type ε. We are interested in the polynomials

$$\begin{aligned} \alpha^J_{2m+1}(q^{-1}) &:= a^J_{2m+1}(q)/q^{\deg_q a^J_{2m+1}}, \\ \alpha^J_{2m,\varepsilon}(q^{-1}) &:= a^J_{2m,\varepsilon}(q)/q^{\deg_q a^J_{2m,\varepsilon}}. \end{aligned}$$

Definition 6. Given a family $\mathbf{F} = (F_J(\mathbf{X}))_{J\subseteq[n-1]}$ of rational functions with the inversion property (IP) we define respectively

$$\mathrm{Ig}_{2m+1}(q^{-1},\mathbf{X}) := \mathrm{Ig}_{2m+1,\mathbf{F}}(q^{-1},\mathbf{X}) := \sum_{J\subseteq[n-1]} \alpha^J_{2m+1}(q^{-1})F_J(\mathbf{X}),$$
$$\mathrm{Ig}_{2m,\varepsilon}(q^{-1},\mathbf{X}) := \mathrm{Ig}_{2m,\varepsilon,\mathbf{F}}(q^{-1},\mathbf{X}) := \sum_{J\subseteq[n-1]} \alpha^J_{2m,\varepsilon}(q^{-1})F_J(\mathbf{X}).$$

To streamline notation, we will sometimes add in the odd-dimensional case a superfluous ε to expressions like $a^J_n(q)$, $\alpha^J_n(q^{-1})$ or $\mathrm{Ig}_n(q^{-1},\mathbf{X})$, thus writing e.g. $a^J_{n,\varepsilon}(q)$, $\alpha^J_{n,\varepsilon}(q^{-1})$ or $\mathrm{Ig}_{n,\varepsilon}(q^{-1},\mathbf{X})$, irrespective of the parity of n.

We now fix a family of rational functions $\mathbf{F} = (F_J(\mathbf{X}))_{J\subseteq[n-1]}$ with the inversion property (IP). The assertion of Theorem A in the orthogonal case then takes the following form.

Theorem 2. *The Igusa-type functions satisfy functional equations*

$$\mathrm{Ig}_{2m+1}(q,\mathbf{X}^{-1}) = (-1)^m q^{m^2+m}\,\mathrm{Ig}_{2m+1}(q^{-1},\mathbf{X}),$$
$$\mathrm{Ig}_{2m,\varepsilon}(q,\mathbf{X}^{-1}) = -\varepsilon(-1)^m q^{m^2}\,\mathrm{Ig}_{2m,\varepsilon}(q^{-1},\mathbf{X}).$$

We first give an outline of the proof of Theorem 2, deferring precise definitions for a moment. In Proposition 3 we derive explicit formulae for the polynomials $\alpha^J_{n,\varepsilon}(q^{-1})$ from the well-known formulae for the orders of the orthogonal groups. A key observation is that the map $J \mapsto \alpha^J_{n,\varepsilon}(q^{-1})$ factors over a 'bisecting' map $\varphi : \mathcal{P}([n]) \to \mathcal{P}([m])$. We are thus led to define, for $G \subseteq [m]$, $I \in \varphi^{-1}(G)$,

$$\alpha^{\uparrow G}_{n,\varepsilon}(q^{-1}) := \alpha^I_{n,\varepsilon}(q^{-1})$$

and

$$F_{\varphi^{-1}(G)}(\mathbf{X}) := \sum_{I\in\varphi^{-1}(G)} F_I(\mathbf{X})$$

so that

$$\mathrm{Ig}_{n,\varepsilon}(q^{-1},\mathbf{X}) = \sum_{G\subseteq[m]} \alpha^{\uparrow G}_{n,\varepsilon}(q^{-1})F_{\varphi^{-1}(G)}(\mathbf{X}).$$

As we shall see, any subset $G \subseteq [m]$ induces in a natural way a composition $C(G) := C(G,m)$ of a non-negative integer $N(G) \le m$. For $G, H \subseteq [m]$, we denote by $\|G\|$ the number of parts of $C(G)$ and by $c_{G,H}$ the number of ways the composition $C(H)$ refines a truncation of the composition $C(G)$. We then prove the following 'inversion equations'.

Proposition 2. (i) *For each $H \subseteq [m]$,*

$$F_{\varphi^{-1}(H)}(\mathbf{X}^{-1}) = (-1)^{n-1+\|H\|} \sum_{G\subseteq[m]} c_{G,H} F_{\varphi^{-1}(G)}(\mathbf{X}).$$

(ii) *For each $G \subseteq [m]$,*

$$\alpha^{\uparrow G}_{2m+1}(q) = (-1)^m q^{m^2+m} \sum_{H\subseteq[m]} (-1)^{\|H\|} c_{G,H}\, \alpha^{\uparrow H}_{2m+1}(q^{-1}),$$
$$\alpha^{\uparrow G}_{2m,\varepsilon}(q) = \varepsilon(-1)^m q^{m^2} \sum_{H\subseteq[m]} (-1)^{\|H\|} c_{G,H}\, \alpha^{\uparrow H}_{2m,\varepsilon}(q^{-1}).$$

Theorem 2 is an immediate consequence of Proposition 2: in the odd-dimensional case,

$$\begin{aligned}
\mathrm{Ig}_{2m+1}(q,\mathbf{X}^{-1}) &= \sum_{G\subseteq[m]} \alpha_{2m+1}^{\uparrow G}(q) F_{\varphi^{-1}(G)}(\mathbf{X}^{-1}) \\
&= (-1)^m q^{m^2+m} \sum_{G,H\subseteq[m]} (-1)^{\|H\|} c_{G,H}\, \alpha_{2m+1}^{\uparrow H}(q^{-1}) F_{\varphi^{-1}(G)}(\mathbf{X}^{-1}) \\
&= (-1)^m q^{m^2+m} \sum_{H\subseteq[m]} \alpha_{2m+1}^{\uparrow H}(q^{-1}) F_{\varphi^{-1}(H)}(\mathbf{X}) \\
&= (-1)^m q^{m^2+m}\, \mathrm{Ig}_{2m+1}(q^{-1},\mathbf{X}).
\end{aligned}$$

The functional equation for $\mathrm{Ig}_{2m,\varepsilon}(q^{-1},\mathbf{X})$ follows in a similar way. In the remainder of this section we give precise definitions of the above concepts, and we supply a proof of Proposition 2.

Definition 7 (Integer compositions)**.** By a *composition* C of a non-negative integer N into ρ parts we mean a tuple $(x_1,\dots,x_\rho)\in\mathbb{N}^\rho$ such that $N = x_1+\dots+x_\rho$.

Given $I=\{i_1,\dots,i_r\}_< \subseteq [n]$, we define

$$N(I,n) := \max([n]_0\setminus I) \quad\text{and}\quad \rho := \max\{\varrho\in[r+1]_0 \mid i_{\varrho-1} < N(I,n)\},$$

where $i_{-1} := -1$, $i_0 := 0$. Then I induces a composition $C(I,n)$ of $N(I,n)$ into $\|I\|_n := \rho$ parts, namely

$$C(I,n) := (i_1, i_2-i_1,\dots,i_{\rho-1}-i_{\rho-2}, N(I,n)-i_{\rho-1}).$$

Note that, if $I\subseteq[n-1]$, then $N(I,n)=n$ and $\rho=r+1$. The map $I\mapsto C(I,n)$ induces a bijection from $\mathcal{P}([n-1])$ onto the set of all compositions of n.

We define the *bisecting map*

$$\varphi:\mathcal{P}([n])\to\mathcal{P}([m])$$

as follows: for $I\subseteq[n]$ with $C(I,n)=(x_1,\dots,x_\rho)$ set

$$\begin{aligned}
\mathrm{cut}(I) &:= \left\lfloor\frac{x_1}{2}\right\rfloor + \left\lfloor\frac{x_2}{2}\right\rfloor + \dots + \left\lfloor\frac{x_\rho}{2}\right\rfloor, \\
\varphi_0(I) &:= \left\{ \left\lfloor\frac{x_1}{2}\right\rfloor + \left\lfloor\frac{x_2}{2}\right\rfloor + \dots + \left\lfloor\frac{x_\varrho}{2}\right\rfloor \mid \varrho\in[\rho]\right\} \setminus \{0,\mathrm{cut}(I)\}
\end{aligned}$$

and

$$\varphi(I) := \varphi_0(I)\cup[\mathrm{cut}(I)+1,m].$$

Note that $N(\varphi(I),m)=\mathrm{cut}(I)$ and $\|\varphi(I)\|_m\le\|I\|_n$. Moreover, φ maps $\mathcal{P}([n-1])$ surjectively onto $\mathcal{P}([m])$. For subsets $G\subseteq[m]$ we agree to write $N(G):=N(G,m)$ and $\|G\|:=\|G\|_m$.

We now give explicit formulae for the polynomials $\alpha_{n,\varepsilon}^J(q^{-1})$.

Proposition 3. *Let $J\subseteq[n-1]$, $H:=\varphi(J)\subseteq[m]$, and put $Y:=q^{-2}$.*

(i) *For $n = 2m+1$ odd,*

$$\alpha^J_{2m+1}(q^{-1}) = \binom{N(H)}{\varphi_0(J)}_Y \prod_{i=N(H)+1}^{m} (1-Y^i)$$
$$= \binom{m}{H \cup \{N(H)\}}_Y (1-Y)^{m-N(H)}.$$

(ii) *For $n = 2m$ even,*

$$\alpha^J_{2m,\varepsilon}(q^{-1}) = \binom{m}{J/2}_Y = \binom{m}{H}_Y \qquad \text{if } J \subseteq 2\mathbb{N},$$
$$\alpha^J_{2m,\varepsilon}(q^{-1}) = \binom{N(H)}{\varphi_0(J)}_Y \frac{\prod_{i=N(H)+1}^{m}(1-Y^i)}{1+\varepsilon q^{-m}}$$
$$= \binom{m}{H \cup \{N(H)\}}_Y \frac{(1-Y)^{m-N(H)}}{1+\varepsilon q^{-m}} \qquad \text{otherwise.}$$

Proof. First we are going to prove the assertions in odd characteristic, where the discriminant helps to distinguish isomorphism types of quadratic spaces and where we can freely apply Witt's Extension and Cancellation Theorem. Afterwards we explain why the formulae also remain true in characteristic 2.

So first suppose that the underlying field $F = \mathbb{F}_q$ has odd characteristic. Recall the formulae for the orders of the respective orthogonal groups

$$|\mathrm{O}_{2m+1}(\mathbb{F}_q)| = 2q^{m^2} \prod_{i \in [m]} (q^{2i}-1) =: p_{2m+1}(q) =: p_{2m+1},$$
$$|\mathrm{O}^{\varepsilon}_{2m}(\mathbb{F}_q)| = 2q^{m^2-m}(q^m-\varepsilon) \prod_{i \in [m-1]} (q^{2i}-1) =: p_{2m,\varepsilon}(q) =: p_{2m,\varepsilon}$$

(cf. [1, p. 147], [6, Theorem 9.11]), and put

$$p^\sharp_n := p^\sharp_n(q) := \begin{cases} p_{2m+1} & \text{if } n = 2m+1 \text{ odd,} \\ (q^m+\varepsilon)p_{2m,\varepsilon} & \text{if } n = 2m \text{ even.} \end{cases}$$

Let $J = \{j_1, \ldots, j_s\}_< \subseteq [n-1]$, and put $j_0 := 0$, $j_{s+1} := n$. Counting non-degenerate flags $\mathbf{U}_J = (U_j)_{j \in J}$ of type J in $\mathcal{V}$ is equivalent to counting (ordered) orthogonal decompositions

$$\mathcal{V} = \mathcal{W}_1 \perp \cdots \perp \mathcal{W}_{s+1} \tag{15}$$

with $\dim \mathcal{W}_\sigma = k_\sigma := j_\sigma - j_{\sigma-1}$ for all $\sigma \in [s+1]$. The isomorphism type of such an orthogonal decomposition is determined by the discriminants $\operatorname{disc} \mathcal{W}_\sigma \in \mathbb{F}_q^*/(\mathbb{F}_q^*)^2 \cong \{1,-1\}$ of the non-degenerate spaces $\mathcal{W}_\sigma$, $\sigma \in [s+1]$.

Let $\eta \in \{1,-1\}$, according to whether -1 is a square in $\mathbb{F}_q$ or not. At this point it is advantageous to assign, also to an odd-dimensional non-degenerate quadratic space $\mathcal{W}$ a sign $\varepsilon(\mathcal{W}) \in \{1,-1\}$, namely the discriminant of the (one-dimensional) anisotropic kernel of $\mathcal{W}$. We then have $\operatorname{disc} \mathcal{W} = \varepsilon(\mathcal{W})\eta^{\lfloor \dim \mathcal{W}/2 \rfloor}$ for any non-degenerate quadratic space $\mathcal{W}$, irrespective of the parity of $\dim \mathcal{W}$.

Thus the isomorphism type of an orthogonal decomposition of the form (15) can be encoded in a tuple $\boldsymbol{\varepsilon} = (\varepsilon_1, \ldots, \varepsilon_{s+1}) \in \{1,-1\}^{s+1}$ such that $\mathcal{W}_\sigma$ is of type ε_σ for all $\sigma \in [s+1]$. Moreover, the tuples $\boldsymbol{\varepsilon}$ which arise in this way are precisely the elements of $E := E(\mathcal{V}) := \{\boldsymbol{\varepsilon} \mid \varepsilon_1 \cdots \varepsilon_{s+1} = \eta^{m-N(\varphi(J))}\varepsilon\}$, and

Witt's Extension and Cancellation Theorem implies that the number of ordered orthogonal decompositions of isomorphism type $\boldsymbol{\varepsilon} \in E$ equals

$$\frac{|\mathrm{O}_n^{\varepsilon}(\mathbb{F}_q)|}{|\prod_{\sigma=1}^{s+1} \mathrm{O}_{k_\sigma}^{\varepsilon_\sigma}(\mathbb{F}_q)|} = \frac{p_{n,\varepsilon}}{\prod_{\sigma=1}^{s+1} p_{k_\sigma,\varepsilon_\sigma}};$$

cf. [1, p. 147f]. Setting

$$\mathcal{E}(J) := \{\sigma \in [s+1] \mid k_\sigma \equiv 0 \mod 2\}$$

we thus obtain

$$a_{n,\varepsilon}^J(q) = p_{n,\varepsilon} \sum_{\boldsymbol{\varepsilon} \in E} \left(\prod_{\sigma=1}^{s+1} p_{k_\sigma,\varepsilon_\sigma} \right)^{-1} = \frac{p_{n,\varepsilon}}{\prod_{\sigma=1}^{s+1} p_{k_\sigma}^{\sharp}} \sum_{\boldsymbol{\varepsilon} \in E} \prod_{\sigma \in \mathcal{E}(J)} (q^{k_\sigma/2} + \varepsilon_\sigma).$$

Note that

$$\begin{aligned} \sum_{\boldsymbol{\varepsilon} \in E} \prod_{\sigma \in \mathcal{E}(J)} (q^{k_\sigma/2} + \varepsilon_\sigma) &= 2^s \prod_{\sigma \in \mathcal{E}(J)} q^{k_\sigma/2} && \text{if } \mathcal{E}(J) \neq [s+1], \\ \sum_{\boldsymbol{\varepsilon} \in E} \prod_{\sigma \in \mathcal{E}(J)} (q^{k_\sigma/2} + \varepsilon_\sigma) &= 2^s (q^{n/2} + \varepsilon) && \text{if } \mathcal{E}(J) = [s+1]. \end{aligned}$$

From this the claim follows for $\operatorname{char} F \neq 2$.

Before turning our attention to the case $\operatorname{char} F = 2$, we record a set of formulae for later use. Let $j \in [n-1]$ and $\delta \in \{1,-1\}$. If $j = 2h+1$ is odd, let $a_{\mathcal{V}}^j(q)$ denote the number of non-degenerate j-dimensional subspaces in $\mathcal{V}$. If $j = 2h$ is even, let $a_{\mathcal{V}}^{j,\delta}(q)$ denote the number of non-degenerate j-dimensional subspaces of type δ in $\mathcal{V}$. According to whether $\mathcal{V}$ is odd- or even-dimensional, we also write $a_{2m+1}^{2h+1}(q)$, $a_{2m,\varepsilon}^{2h+1}(q)$ in the former and $a_{2m+1}^{2h,\delta}(q)$, $a_{2m,\varepsilon}^{2h,\delta}(q)$ in the latter case. Our calculations above, based on Witt's Extension and Cancellation Theorem, show in particular that, if $\operatorname{char} F \neq 2$,

$$\begin{aligned} a_{2m+1}^{2h+1}(q) &= \frac{|\mathrm{O}_{2m+1}(\mathbb{F}_q)|}{|\mathrm{O}_{2h+1}(\mathbb{F}_q)||\mathrm{O}_{2m-2h}^{+}(\mathbb{F}_q)|} + \frac{|\mathrm{O}_{2m+1}(\mathbb{F}_q)|}{|\mathrm{O}_{2h+1}(\mathbb{F}_q)||\mathrm{O}_{2m-2h}^{-}(\mathbb{F}_q)|} \\ &= \frac{2\, q^{m-h}\, p_{2m+1}}{p_{2h+1}\, p_{2m-2h}^{\sharp}}, \end{aligned} \tag{16}$$

$$a_{2m,\varepsilon}^{2h+1}(q) = 2 \cdot \frac{|\mathrm{O}_{2m}^{\varepsilon}(\mathbb{F}_q)|}{|\mathrm{O}_{2h+1}(\mathbb{F}_q)||\mathrm{O}_{2m-2h-1}(\mathbb{F}_q)|} = \frac{2\, p_{2m,\varepsilon}}{p_{2h+1}\, p_{2m-2h-1}}, \tag{17}$$

$$a_{2m+1}^{2h,\delta}(q) = \frac{|\mathrm{O}_{2m+1}(\mathbb{F}_q)|}{|\mathrm{O}_{2h}^{\delta}(\mathbb{F}_q)||\mathrm{O}_{2m-2h+1}(\mathbb{F}_q)|} = \frac{p_{2m+1}}{p_{2h,\delta}\, p_{2m-2h+1}}, \tag{18}$$

$$a_{2m,\varepsilon}^{2h,\delta}(q) = \frac{|\mathrm{O}_{2m}^{\varepsilon}(\mathbb{F}_q)|}{|\mathrm{O}_{2h}^{\delta}(\mathbb{F}_q)||\mathrm{O}_{2m-2h}^{\delta\varepsilon}(\mathbb{F}_q)|} = \frac{p_{2m,\varepsilon}}{p_{2h,\delta}\, p_{2m-2h,\delta\varepsilon}}. \tag{19}$$

Below we will show that, in fact, also in characteristic 2 one obtains the same polynomials $a_{\mathcal{V}}^j(q)$ and $a_{\mathcal{V}}^{j,\delta}(q)$. Thus, by induction, the formulae for $a_{\mathcal{V}}^J(q)$ and $\alpha_{\mathcal{V}}^J(q^{-1})$, which we initially derived only under the extra assumption $\operatorname{char} F \neq 2$, also remain valid in characteristic 2.

So suppose that $\operatorname{char} F = 2$, and let $j \in [n-1]$, $\delta \in \{1,-1\}$. Write $j = 2h+1$, if j is odd, and $j = 2h$, if j is even. The orders of the respective orthogonal groups are now

$$|\mathrm{O}_{2m+1}(\mathbb{F}_q)| = \frac{p_{2m+1}}{2}, \qquad |\mathrm{O}^{\varepsilon}_{2m}(\mathbb{F}_q)| = p_{2m,\varepsilon},$$

and Witt's Extension and Cancellation Theorem still applies to non-defective subspaces; cf. [6, Theorems 3.12 and 14.48] and [2, Theorem 3.15]. Therefore we immediately obtain the counterparts of (18) and (19),

$$a^{2h,\delta}_{2m+1}(q) = \frac{|\mathrm{O}_{2m+1}(\mathbb{F}_q)|}{|\mathrm{O}^{\delta}_{2h}(\mathbb{F}_q)||\mathrm{O}_{2m-2h+1}(\mathbb{F}_q)|} = \frac{p_{2m+1}}{p_{2h,\delta}\, p_{2m-2h+1}},$$
$$a^{2h,\delta}_{2m,\varepsilon}(q) = \frac{|\mathrm{O}^{\varepsilon}_{2m}(\mathbb{F}_q)|}{|\mathrm{O}^{\delta}_{2h}(\mathbb{F}_q)||\mathrm{O}^{\delta\varepsilon}_{2m-2h}(\mathbb{F}_q)|} = \frac{p_{2m,\varepsilon}}{p_{2h,\delta}\, p_{2m-2h,\delta\varepsilon}}.$$

Next we suppose that $n = 2m+1$ is odd and compute $a^{2h+1}_{2m+1}(q)$. If $h = 0$, then we are to count anisotropic lines in $\mathcal{V}$. It is well-known that the polar space associated to $\mathcal{V}$ has $(q^{2m}-1)/(q-1)$ points, each corresponding to an isotropic line; cf. [2, Theorem 3.13]. So we deduce that

$$a^{1}_{2m+1}(q) = \frac{q^{2m+1}-1}{q-1} - \frac{q^{2m}-1}{q-1} = \frac{2\, q^m\, p_{2m+1}}{p_1\, p^{\sharp}_{2m}}.$$

In general, choosing a $(2h+1)$-dimensional non-degenerate subspace U in $\mathcal{V}$ can be split into two parts: first pick a $2h$-dimensional non-degenerate (hence non-defective) subspace U_0 of type 1, then complement your choice by picking an anisotropic line A in $U_0^{\perp}$ to obtain $U = U_0 + A$. Applying Witt's Extension and Cancellation Theorem, we obtain the counterpart of (16),

$$\begin{aligned} a^{2h+1}_{2m+1}(q) &= \frac{a^{2h,1}_{2m+1}(q)\; a^{1}_{2m-2h+1}(q)}{a^{2h,1}_{2h+1}(q)} \\ &= \frac{(p_{2m+1}\cdot 2\, q^{m-h}\, p_{2m-2h+1})/(p_{2h,1}\, p_{2m-2h+1}\cdot p_1\, p^{\sharp}_{2m-2h})}{(p_{2h+1})/(p_{2h,1}\, p_1)} \\ &= \frac{2\, q^{m-h}\, p_{2m+1}}{p_{2h+1}\, p^{\sharp}_{2m-2h}}. \end{aligned}$$

A similar computation yields the counterpart of (17). □

Definition 8 (Refinements of compositions)**.** Let $C_1 = (x_1, \ldots, x_\kappa)$ and $C_2 = (y_1, \ldots, y_\lambda)$ be compositions. A *refinement of a truncation of* C_1 *by* C_2 is a triple $(C_1, C_2, \boldsymbol{\xi})$ such that $\boldsymbol{\xi} = (\xi_1, \ldots, \xi_\kappa) \in [\lambda]_0{}^{\kappa}$ satisfies

$$\xi_1 \le \cdots \le \xi_\kappa = \lambda \quad \text{and} \quad \forall i \in [\kappa] : \; y_{\xi_{i-1}+1} + \cdots + y_{\xi_i} \le x_i,$$

where $\xi_0 := 0$. By slight abuse of terminology, we also call the κ-tuple $\boldsymbol{\xi}$ a refinement of a truncation of C_1 by C_2. For $G, H \subseteq [m]$, the number of refinements of truncations of $C(G)$ by $C(H)$ is denoted by $c_{G,H} := c^{(m)}_{G,H}$.

Let $I, J \subseteq [n-1]$ such that $I \subseteq J$, and put $G := \varphi(I)$, $H := \varphi(J)$. Clearly, $C(J, n)$ can be regarded as a refinement of $C(I, n)$. Applying the bisecting map, we obtain a refinement of a truncation of $C(G)$ by $C(H)$ as follows.

The sets $[n-1] \setminus I$ and $[n-1] \setminus J$ decompose uniquely into disjoint unions

$$[n-1] \setminus I = \mathcal{I}_{I,1} \,\dot\cup\, \ldots \,\dot\cup\, \mathcal{I}_{I,\|G\|}, \qquad [n-1] \setminus J = \mathcal{I}_{J,1} \,\dot\cup\, \ldots \,\dot\cup\, \mathcal{I}_{J,\|H\|}$$

of intervals $\mathcal{I}_{I,i}$ (respectively $\mathcal{I}_{J,j}$) of natural numbers such that $\max \mathcal{I}_{I,i} < \min \mathcal{I}_{I,i+1}$ (respectively $\max \mathcal{I}_{J,j} < \min \mathcal{I}_{J,j+1}$) for all admissible values of i (respectively j).

The *refinement of a truncation of G by H induced from $I \subseteq J$* is the $\|G\|$-tuple $\boldsymbol{\xi}(I,J) = (\xi_1, \dots, \xi_{\|G\|})$ defined by $\xi_{\|G\|} := \|H\|$ and

$$\forall i \in [\|G\|] : \ \mathcal{I}_{J,\xi_{i-1}+1} \,\dot{\cup}\, \dots \,\dot{\cup}\, \mathcal{I}_{J,\xi_i} \subseteq \mathcal{I}_{I,i},$$

where $\xi_0 := 0$. We remark that, starting from $G, H \subseteq [m]$, every refinement of a truncation of $C(G)$ by $C(H)$ is induced by suitable $I, J \subseteq [n-1]$ with $I \subseteq J$.

We illustrate these notions by an example.

Example. Set $n = 11$ so that $m = 5$. The subsets $G = \{1,3,4\}$, $H = \{2,4,5\} \subseteq [m]$ induce compositions $C(G) = (1,2,1,1)$ of $N(G) = 5$ and $C(H) = (2,1)$ of $N(H) = 3$, respectively. Note that $\|G\| = 4$ and $\|H\| = 2$. Among the seven 'truncations' of $(1,2,1,1)$ to 'pre-compositions' of 3,

$$\begin{array}{lllll} (1,2,0,0), & (1,1,1,0), & (1,1,0,1), & (1,0,1,1), & (0,1,1,1), \\ (0,2,0,1), & (0,2,1,0), & & & \end{array}$$

only the $2 = c_{G,H}$ last ones yield the composition $C(H) = (2,1)$. They are encoded in the tuples $\boldsymbol{\xi} = (0,1,1,2)$ and $\boldsymbol{\xi} = (0,1,2,2)$, respectively.

Define subsets

$$I := \{2,7,9\}, \qquad J_1 := \{1,2,7,8,9\}, \qquad J_2 := \{1,2,3,7,9,10\}$$

of $[n-1] = [10]$. The set I induces the composition $C(I,n) = (2,5,2,2)$ of $N(I,n) = 11$. Thus $\mathrm{cut}(I) = 1+2+1+1 = 5$ and $\varphi(I) = G$. Similarly, $\mathrm{cut}(J_1) = \mathrm{cut}(J_2) = 2+1 = 3$ and $\varphi(J_1) = \varphi(J_2) = H$. We have $\boldsymbol{\xi}(I,J_1) = (0,1,1,2)$ and $\boldsymbol{\xi}(I,J_2) = (0,1,2,2)$.

We are now ready to prove Proposition 2.

Proof of Proposition 2 (i). Let $H \subseteq [m]$. From the definition of $F_{\varphi^{-1}(H)}(\mathbf{X})$ and the fact that $\mathbf{F}$ has the inversion property (IP) we obtain

$$F_{\varphi^{-1}(H)}(\mathbf{X}^{-1}) = \sum_{J \in \varphi^{-1}(H)} (-1)^{|J|} \sum_{I \subseteq J} F_I(\mathbf{X}).$$

Thus it is enough to show that for $I \subseteq [n-1]$ with $\varphi(I) = G$,

$$(-1)^{n-1+\|H\|} \sum_{\substack{J \in \varphi^{-1}(H) \\ I \subseteq J}} (-1)^{|J|} = c_{G,H}.$$

This is certainly the case if $c_{G,H} = 0$, as then the sum on the left hand side is empty. Now suppose that $c_{G,H} \neq 0$ and fix a refinement $\boldsymbol{\xi}$ of a truncation of G by H; put $\xi_0 := 0$. It suffices to show that

$$(-1)^{n-1+\|H\|} \sum_{\substack{J \in \varphi^{-1}(H) \\ I \subseteq J, \ \boldsymbol{\xi}(I,J) = \boldsymbol{\xi}}} (-1)^{|J|} = 1. \tag{20}$$

Decompose $[n-1] \setminus I = \mathcal{I}_{I,1} \,\dot\cup\, \ldots \,\dot\cup\, \mathcal{I}_{I,\|G\|}$ into a disjoint union of intervals $\mathcal{I}_{I,i}$ as in Definition 8, and write $C(H) = (y_1, \ldots, y_{\|H\|})$. We claim that

$$\sum_{\substack{J \in \varphi^{-1}(H) \\ I \subseteq J,\ \boldsymbol{\xi}(I,J) = \boldsymbol{\xi}}} (-1)^{|J|-|I|} = \prod_{i=1}^{\|G\|} \sum_{k=0}^{\xi_i - \xi_{i-1}} \binom{|\mathcal{I}_{I,i}| - \sum_{j=\xi_{i-1}+1}^{\xi_i}(2y_j - 1) - k + 1}{\xi_i - \xi_{i-1}} \cdot \binom{\xi_i - \xi_{i-1}}{k} (-1)^{|\mathcal{I}_{I,i}| - \sum_{j=\xi_{i-1}+1}^{\xi_i}(2y_j-1) - k}. \tag{21}$$

Indeed, specifying $J \in \varphi^{-1}(H)$ with $I \subseteq J$ and $\boldsymbol{\xi}(I,J) = \boldsymbol{\xi}$ is equivalent to the following task: for each $i \in [\|G\|]$ choose $k_i \in [\xi_i - \xi_{i-1}]_0$ and single out a disjoint union $\mathcal{I}_{J,\xi_{i-1}+1} \,\dot\cup\, \ldots \,\dot\cup\, \mathcal{I}_{J,\xi_i} \subseteq \mathcal{I}_{I,i}$ of intervals $\mathcal{I}_{J,j}$ such that

(a) $\max \mathcal{I}_{J,j} < \min \mathcal{I}_{J,j+1}$ for all admissible values of j,
(b) $|\mathcal{I}_{J,j}| = 2y_j$ for exactly k_i values of j and $|\mathcal{I}_{J,j}| = 2y_j - 1$ for the remaining values of j.

Moreover, the cardinality of the set J corresponding to such a choice of k_i and such a choice of intervals $\mathcal{I}_{J,j} \subseteq \mathcal{I}_{I,i}$ is

$$|I| + \sum_{i=1}^{\|G\|} \left(|\mathcal{I}_{I,i}| - \sum_{j=\xi_{i-1}+1}^{\xi_i} (2y_j - 1) - k_i \right).$$

As $n - 1 + \|H\| = |I| + \sum_{i=1}^{\|G\|} |\mathcal{I}_{I,i}| + \sum_{i=1}^{\|G\|} (\xi_i - \xi_{i-1})$, equation (21) implies that the left hand side of (20) is equal to

$$\prod_{i=1}^{\|G\|} \sum_{k=0}^{\xi_i - \xi_{i-1}} \binom{|\mathcal{I}_{I,i}| - \sum_{j=\xi_{i-1}+1}^{\xi_i}(2y_j - 1) + 1 - k}{\xi_i - \xi_{i-1}} \binom{\xi_i - \xi_{i-1}}{k} (-1)^k.$$

This does indeed equal 1, because for any positive integers $M \leq N$,

$$\sum_{k=0}^{M} \binom{N-k}{M} \binom{M}{k} (-1)^k = 1$$

(cf. [5, p. 169, (5.25)]) and hence each of the $\|G\|$ factors already equals 1. $\square$

Proof of Proposition 2 (ii) *for $n = 2m+1$ odd.* For $G \subseteq [m]$, we are looking to prove

$$\alpha_{2m+1}^{\uparrow G}(q) = (-1)^m q^{m^2+m} \sum_{H \subseteq [m]} (-1)^{\|H\|} c_{G,H}\, \alpha_{2m+1}^{\uparrow H}(q^{-1}). \tag{22}$$

First we deal with the case $N(G) < m$, i.e. $m \in G$. Writing $G' := G \setminus \{m\}$ and $Y := q^{-2}$, we see from Proposition 3 (i) that in this case

$$\alpha_{2m+1}^{\uparrow G}(q^{-1}) = (1 - Y^m)\alpha_{2m-1}^{\uparrow G'}(q^{-1}).$$

If $H \subseteq [m]$ with $c_{G,H} \neq 0$, then $N(H) \leq N(G) < m$, hence $m \in H$, hence we obtain $\|H\| = \|H'\|_{m-1}$ and $c_{G,H} = c_{G',H'}^{(m-1)}$ for $H' := H \setminus \{m\}$. With these observations (22) follows by induction:

$$\begin{aligned}\alpha_{2m+1}^{\uparrow G}(q) &= (1-Y^{-m})(-1)^{m-1}q^{(m-1)^2+(m-1)}\\ &\quad\cdot \sum_{H'\subseteq[m-1]} (-1)^{\|H'\|_{m-1}} c_{G',H'}^{(m-1)}\ \alpha_{2m-1}^{\uparrow H'}(q^{-1})\\ &= (-1)^m q^{m^2+m} \sum_{H'\subseteq[m-1]} (-1)^{\|H'\|_{m-1}} c_{G',H'}^{(m-1)}\ (1-Y^m)\alpha_{2m-1}^{\uparrow H'}(q^{-1})\\ &= (-1)^m q^{m^2+m} \sum_{H\subseteq[m]} (-1)^{\|H\|} c_{G,H}\ \alpha_{2m-1}^{\uparrow H}(q^{-1}).\end{aligned}$$

It remains to consider the case $N(G)=m$, i.e. $G\subseteq[m-1]$. Again set $Y:=q^{-2}$, and write $C(G)=(x_1,\dots,x_{k+1})$. Proposition 3 (i) shows that in this case

$$\alpha_{2m+1}^{\uparrow G}(q^{-1}) = \binom{m}{G}_Y,$$

in particular, as $\deg_Y \binom{m}{G}_Y = \binom{m+1}{2} - \sum_{\kappa\in[k+1]}\binom{x_\kappa+1}{2}$,

$$\alpha_{2m+1}^{\uparrow G}(q) = \alpha_{2m+1}^{\uparrow G}(q^{-1})\ Y^{-\binom{m+1}{2}+\sum_{\kappa\in[k+1]}\binom{x_\kappa+1}{2}}.$$

We shall show below that

$$\alpha_{2m+1}^{\uparrow G}(q^{-1})\ Y^{\sum_{\kappa\in[k+1]}\binom{x_\kappa+1}{2}} = \sum_{H\subseteq[m]} (-1)^{m+\|H\|} c_{G,H}\ \alpha_{2m+1}^{\uparrow H}(q^{-1}). \tag{23}$$

From these equations (22) follows readily.

It remains to prove (23). For this we need the following formulae.

(i) For all $i\in\mathbb{N}_0$: $Y^{\binom{i+1}{2}} = \sum_{j=0}^{i}\binom{i}{[i-j,i-1]}_Y (Y-1)^j Y^{\binom{i-j}{2}}$.

(ii) For all $i\in\mathbb{N}_0$: $Y^{\binom{i+1}{2}} = \sum_{I\subseteq[i]}\binom{i+1}{I}_Y(-1)^{i-|I|}$.

Part (i) is easily proved inductively (see the end of this proof), part (ii) is a well-known fact about Gaussian polynomials. With the formulae (i), (ii) at our disposal, the left hand side of (23) can be written as

(24)

$$\begin{aligned}&\alpha_{2m+1}^{\uparrow G}(q^{-1})\ Y^{\sum_{\kappa\in[k+1]}\binom{x_\kappa+1}{2}}\\ &\quad= \binom{m}{G}_Y \prod_{\kappa\in[k+1]} Y^{\binom{x_\kappa+1}{2}}\\ &\quad= \binom{m}{G}_Y \prod_{\kappa\in[k+1]} \left(\sum_{j=0}^{x_\kappa}\binom{x_\kappa}{[x_\kappa-j,x_\kappa-1]}_Y (Y-1)^j\, Y^{\binom{x_\kappa-j}{2}}\right)\\ &\quad= \binom{m}{G}_Y \prod_{\kappa\in[k+1]} \left(\sum_{j=0}^{x_\kappa}\binom{x_\kappa}{[x_\kappa-j,x_\kappa-1]}_Y (1-Y)^j\right.\\ &\qquad\qquad\left.\cdot \sum_{K\subseteq[x_\kappa-j-1]}\binom{x_\kappa-j}{K}_Y(-1)^{x_\kappa+|K|+\delta(\text{`}j\neq x_\kappa\text{'})}\right),\end{aligned}$$

where the Kronecker-delta $\delta(\text{`}j \neq x_\kappa\text{'}) \in \{1, 0\}$ reflects whether or not the inequality $j \neq x_\kappa$ holds. On the other hand, setting

$$\Xi := \bigcup_{H \subseteq [m]} \{(H, \boldsymbol{\xi}) \mid \boldsymbol{\xi} \text{ a refinement of a truncation of } G \text{ by } H\},$$

the right hand side of (23) can be written as

$$\sum_{H \subseteq [m]} (-1)^{m+\|H\|} c_{G,H}\ \alpha_{2m+1}^{\uparrow H}(q^{-1}) = \sum_{(H,\boldsymbol{\xi}) \in \Xi} (-1)^{m+\|H\|} \alpha_{2m+1}^{\uparrow H}(q^{-1}). \tag{25}$$

Now we explain why the last sum is indeed equal to the right hand side of (24). Choosing an element $(H, \boldsymbol{\xi}) \in \Xi$, so that $\boldsymbol{\xi} = (\xi_1, \ldots, \xi_{k+1})$ is a refinement of a truncation of $C(G) = (x_1, \ldots, x_{k+1})$ by $C(H) = (y_1, \ldots, y_\lambda)$, is the same as fixing for each $\kappa \in [k+1]$ a truncation length $j_\kappa \in [x_\kappa]_0$ and a subset $K_\kappa \subseteq [x_\kappa - j_\kappa - 1]$, corresponding to a composition $(y_{\xi_{\kappa-1}+1}, \ldots, y_{\xi_\kappa})$ of $x_\kappa - j_\kappa$. Moreover, the summands attached to the data $(H, \boldsymbol{\xi})$ in (25) and $(j_\kappa, K_\kappa)_{\kappa \in [k+1]}$ in (24) respectively agree:

$$(-1)^{m+\|H\|} = (-1)^{\sum_{\kappa \in [k+1]} x_\kappa + \sum_{\kappa \in [k+1]} (|K_\kappa| + \delta(\text{`}j \neq x_\kappa\text{'}))}$$

and

$$\begin{aligned}
\alpha_{2m+1}^{\uparrow H}(q^{-1}) &= \frac{(1-Y^m)(1-Y^{m-1})\cdots(1-Y)}{\prod_{\iota \in [\lambda]} (1-Y^{y_\iota})(1-Y^{y_\iota - 1})\cdots(1-Y)} \\
&= \frac{(1-Y^m)(1-Y^{m-1})\cdots(1-Y)}{\prod_{\kappa \in [k+1]} \prod_{\iota=\xi_{\kappa-1}+1}^{\xi_\kappa} (1-Y^{y_\iota})(1-Y^{y_\iota - 1})\cdots(1-Y)} \\
&= \frac{(1-Y^m)(1-Y^{m-1})\cdots(1-Y)}{\prod_{\kappa \in [k+1]} (1-Y^{x_\kappa - j_\kappa})(1-Y^{x_\kappa - j_\kappa - 1})\cdots(1-Y)} \prod_{\kappa \in [k+1]} \binom{x_\kappa - j_\kappa}{K_\kappa}_Y \\
&= \frac{(1-Y^m)(1-Y^{m-1})\cdots(1-Y)}{\prod_{\kappa \in [k+1]} (1-Y^{x_\kappa})(1-Y^{x_\kappa - 1})\cdots(1-Y)} \\
&\quad \cdot \prod_{\kappa \in [k+1]} \binom{x_\kappa}{[x_{\kappa - j_\kappa}, x_\kappa - 1]}_Y (1-Y)^{j_\kappa} \binom{x_\kappa - j_\kappa}{K_\kappa}_Y \\
&= \binom{m}{G}_Y \prod_{\kappa \in [k+1]} \binom{x_\kappa}{[x_{\kappa - j_\kappa}, x_\kappa - 1]}_Y (1-Y)^{j_\kappa} \binom{x_\kappa - j_\kappa}{K_\kappa}_Y.
\end{aligned}$$

This finishes the proof of (23). For later use we record

$$\alpha_{2m+1}^{\uparrow G}(q^{-1})\ Y^{\sum_{\kappa \in [k+1]} \binom{x_\kappa}{2}} = \sum_{\substack{H \subseteq [m] \\ N(H) = m}} (-1)^{m+\|H\|} c_{G,H}\ \alpha_{2m+1}^{\uparrow H}(q^{-1}). \tag{26}$$

Indeed, summing only over those $H \subseteq [m]$ such that $N(H) = m$ is achieved by setting persistently $j = j_\kappa = 0$ in the above formulae. Clearly, under the restriction $j = 0$ the term in the third line of (24) reduces to the left hand side of (26).

Finally, we supply the proof of the formulae (i) above. We argue by induction on $i \in \mathbb{N}_0$. For $i = 0$ we have

$$Y^{\binom{1}{2}} = 1 = \binom{0}{\varnothing}_Y (Y-1)^0 Y^{\binom{0}{2}},$$

and for $i>0$ we obtain, by induction,

$$\begin{aligned}
&\sum_{j=0}^{i}\binom{i}{[i-j,i-1]}_Y (Y-1)^j Y^{\binom{i-j}{2}}\\
&= Y^{\binom{i}{2}} + \sum_{j\in[i]}\binom{i}{[i-j,i-1]}_Y (Y-1)^j Y^{\binom{i-j}{2}}\\
&= Y^{\binom{i}{2}} + \sum_{j\in[i]}\binom{i-1}{[i-j,i-2]}_Y \binom{i}{i-1}_Y (Y-1)^j Y^{\binom{i-j}{2}}\\
&= Y^{\binom{i}{2}} + (Y-1)\binom{i}{i-1}_Y \sum_{j\in[i]}\binom{i-1}{[i-j,i-2]}_Y (Y-1)^{j-1} Y^{\binom{i-j}{2}}\\
&= Y^{\binom{i}{2}} + (Y-1)\frac{(Y^i-1)}{(Y-1)} Y^{\binom{i}{2}} = Y^{i+\binom{i}{2}} = Y^{\binom{i+1}{2}}.
\end{aligned}$$

□

Proof of Proposition 2 (ii) *for $n=2m$ even.* For $G\subseteq[m]$ we are looking to prove

$$\alpha^{\uparrow G}_{2m,\varepsilon}(q) = \varepsilon(-1)^m q^{m^2}\sum_{H\subseteq[m]}(-1)^{\|H\|}c_{G,H}\,\alpha^{\uparrow H}_{2m,\varepsilon}(q^{-1}). \tag{27}$$

Again by an inductive argument, analogous to the case $n=2m+1$, we may assume that in fact $N(G)=m$, i.e. $G\subsetneq[m-1]$. Write $Y:=q^{-2}$ and $C(G)=(x_1,\dots,x_{k+1})$. Proposition 3 (ii) shows that in this case

$$\alpha^{\uparrow G}_{2m,\varepsilon}(q^{-1}) = \alpha^{\uparrow G}_{2m+1}(q^{-1}) = \binom{m}{G}_Y,$$

in particular, as $\deg_Y\binom{m}{G}_Y = \binom{m}{2} - \sum_{\kappa\in[k+1]}\binom{x_\kappa}{2}$,

$$\alpha^{\uparrow G}_{2m,\varepsilon}(q) = \alpha^{\uparrow G}_{2m,\varepsilon}(q^{-1})\,Y^{-\binom{m}{2}+\sum_{\kappa\in[k+1]}\binom{x_\kappa}{2}}.$$

We shall show below that

$$\alpha^{\uparrow G}_{2m,\varepsilon}(q^{-1})\,Y^{\sum_{\kappa\in[k+1]}\binom{x_\kappa}{2}}q^{-m} = \varepsilon\sum_{H\subseteq[m]}(-1)^{m+\|H\|}c_{G,H}\,\alpha^{\uparrow H}_{2m,\varepsilon}(q^{-1}) \tag{28}$$

From these equations (27) follows readily.

It remains to prove (28). An easy computation gives

$$\begin{aligned}
Y^{\sum_{\kappa\in[k+1]}\binom{x_\kappa}{2}}q^{-m} &= \varepsilon\frac{q^{-2m}+\varepsilon q^{-m}}{1+\varepsilon q^{-m}}Y^{\sum_{\kappa\in[k+1]}\binom{x_\kappa}{2}}\\
&= \varepsilon\frac{1}{1+\varepsilon q^{-m}}\left(Y^{\sum_{\kappa\in[k+1]}\binom{x_\kappa+1}{2}} + \varepsilon q^{-m}Y^{\sum_{\kappa\in[k+1]}\binom{x_\kappa}{2}}\right)\\
&= \varepsilon\left(\frac{Y^{\sum_{\kappa\in[k+1]}\binom{x_\kappa+1}{2}} - Y^{\sum_{\kappa\in[k+1]}\binom{x_\kappa}{2}}}{1+\varepsilon q^{-m}} + Y^{\sum_{\kappa\in[k+1]}\binom{x_\kappa}{2}}\right).
\end{aligned}$$

From Proposition 3 we see that for $H\subseteq[m]$,

$$\alpha^{\uparrow H}_{2m,\varepsilon}(q^{-1}) = \begin{cases}\alpha^{\uparrow H}_{2m+1}(q^{-1}) & \text{if } N(H)=m,\\ \alpha^{\uparrow H}_{2m+1}(q^{-1})/(1+\varepsilon q^{-m}) & \text{otherwise.}\end{cases}$$

In view of (23) and (26), we thus obtain

$$
\begin{aligned}
&\alpha^{\uparrow G}_{2m,\varepsilon}(q^{-1})Y^{\sum_{\kappa\in[k+1]}\binom{x_\kappa}{2}}q^{-m}\\
&\quad= \varepsilon\alpha^{\uparrow G}_{2m+1}(q^{-1})\left(\frac{Y^{\sum_{\kappa\in[k+1]}\binom{x_\kappa+1}{2}}-Y^{\sum_{\kappa\in[k+1]}\binom{x_\kappa}{2}}}{1+\varepsilon q^{-m}}+Y^{\sum_{\kappa\in[k+1]}\binom{x_\kappa}{2}}\right)\\
&\quad= \varepsilon\sum_{H\subseteq[m],N(H)\neq m}(-1)^{m+\|H\|}c_{G,H}\;\alpha^{\uparrow H}_{2m+1}(q^{-1})/(1+\varepsilon q^{-m})\\
&\qquad+\;\varepsilon\sum_{H\subseteq[m],N(H)=m}(-1)^{m+\|H\|}c_{G,H}\;\alpha^{\uparrow H}_{2m+1}(q^{-1})\\
&\quad= \varepsilon\sum_{H\subseteq[m]}(-1)^{m+\|H\|}c_{G,H}\;\alpha^{\uparrow H}_{2m,\varepsilon}(q^{-1}).
\end{aligned}
$$

This proves (28). □

5. A conjecture for the orthogonal case

In this section we discuss in more detail Conjecture C. As in Section 4, let $\mathcal{V}=(V,B,f)$ be an n-dimensional, non-degenerate quadratic space over the finite field $F=\mathbb{F}_q$. Our aim is to give, for $J\subseteq[n-1]$, an expression for the polynomial $\alpha^J_{\mathcal{V}}(q^{-1})$ in terms of parabolic length functions on the Coxeter group W of type A_{n-1}. If Conjecture C holds, the orthogonal case of Theorem A follows directly from Theorem 1.

Fix the Coxeter system (W,S) where $W=\mathcal{S}_n$ and $S=\{s_1,\dots,s_{n-1}\}$ denotes the standard set of Coxeter generators $s_i=(i\;\;i+1)$, $i\in[n-1]$. A crucial role is played by the following statistic on W.

Definition 9 (Length L)**.** Recalling the notation from Section 2, for $w\in W$ set

$$L(w):=\mathbf{b}\cdot\mathbf{l}_{\mathrm{R}}(w),\quad\text{where }\mathbf{b}=(b_I)_{I\subseteq S}=\left((-1)^{|I|}2^{|S|-|I|-1}\right)_{I\subseteq S}. \tag{29}$$

It is well-known that the ordinary Coxeter length of a permutation $w\in W$ is equal to the number of inversion pairs associated to w, i.e. $l(w)=|\mathcal{I}(w)|$ where

$$\mathcal{I}(w):=\{(i,j)\mid 1\leq i<j\leq n, i^w>j^w\}.$$

The parabolic length function L also has a simple interpretation in terms of inversion pairs.

Lemma 4. *For each $w\in W$,*

$$L(w)=|\{(i,j)\in\mathcal{I}(w)\mid i\not\equiv j \mod 2\}|.$$

Proof. Let $w\in W$ and note that for any $I\subseteq[n-1]$,

$$l^I_{\mathrm{R}}(w)=|\{(i,j)\in\mathcal{I}(w)\mid[i,j-1]\not\subseteq I\}|.$$

From this we derive

$$\begin{aligned}
L(w) &= \frac{1}{2}\sum_{I\subseteq[n-1]}(-1)^{|I|}2^{|S|-|I|}l_{\mathrm{R}}^{I}(w)\\
&= \frac{1}{2}\sum_{(i,j)\in\mathcal{I}(w)}\sum_{I\subseteq[n-1]}(-1)^{|I|}2^{|S|-|I|}\,\delta(\text{‘}[i,j-1]\not\subseteq I\text{’})\\
&= \frac{1}{2}\sum_{(i,j)\in\mathcal{I}(w)}\left(\sum_{I\subseteq[n-1]}(-1)^{|I|}2^{|S|-|I|}-\sum_{[i,j-1]\subseteq I}(-1)^{|I|}2^{|S|-|I|}\right)\\
&= \frac{1}{2}\sum_{(i,j)\in\mathcal{I}(w)}\left((2-1)^{|S|}-(-1)^{j-i}(2-1)^{|S|-(j-i)}\right)\\
&= \frac{1}{2}\sum_{(i,j)\in\mathcal{I}(w)}\left(1-(-1)^{j-i}\right),
\end{aligned}$$

where the Kronecker-delta $\delta(\text{‘}[i,j-1]\not\subseteq I\text{’})\in\{1,0\}$ reflects whether or not the inclusion $[i,j-1]\not\subseteq I$ holds. □

Definition 10 (Chessboard elements)**.** We say that $w\in W$ is a *chessboard element* if $i+i^w\equiv j+j^w$ modulo 2 for all $i,j\in[n]$. Clearly, the set $\mathcal{C}_n$ of chessboard elements forms a subgroup of W. Note that $\mathcal{C}_n$ contains a subgroup $\mathcal{C}_{n,0}$ consisting of elements w such that $i\equiv i^w$ modulo 2 for all $i\in[n]$. If $n=2m+1$ is odd, we have $\mathcal{C}_n=\mathcal{C}_{n,0}\cong\mathcal{S}_{m+1}\times\mathcal{S}_m$. If $n=2m$ is even, we have $\mathcal{C}_n=\langle w_0\rangle\ltimes\mathcal{C}_{n,0}$, where w_0 denotes the longest element of W, and $\mathcal{C}_{n,0}\cong\mathcal{S}_m\times\mathcal{S}_m$.

We write $\sigma: W\to\{1,-1\}$, $w\mapsto(-1)^{l(w)}$ for the *sign character*, and $\tau:\mathcal{C}_n\to\{1,-1\}$ for the linear character with $\ker(\tau)=\mathcal{C}_{n,0}$. Recall from the introduction that, in the even-dimensional case, we attach a sign $\varepsilon\in\{1,-1\}$ to $\mathcal{V}$. Observing that τ is trivial for n odd, we define

$$\chi_\varepsilon:\mathcal{C}_n\to\{1,-1\},\chi_\varepsilon(w):=\begin{cases}\sigma(w) & \text{if } n \text{ is odd, or if } n \text{ is even and } \varepsilon=1,\\ \sigma(w)\tau(w) & \text{if } n \text{ is odd, or if } n \text{ is even and } \varepsilon=-1.\end{cases}$$

Conjecture C. *For each $J\subseteq[n-1]$,*

$$\alpha_{\mathcal{V}}^{J}(q^{-1})=\alpha_{n,\varepsilon}^{J}(q^{-1})=\sum_{\substack{w\in\mathcal{C}_n\\ D_{\mathrm{L}}(w)\subseteq J}}\chi_\varepsilon(w)q^{-L(w)}. \tag{30}$$

Note that, if Conjecture C holds, the orthogonal case of Theorem A follows from Theorem 1, equation (5), with $W'=\mathcal{C}_n$, $\mathbf{b}$ as defined in (29) and $\chi=\chi_\varepsilon$. Conjecture C has been confirmed for $|J|\le 1$ and verified for $n\le 13$.

Example. For $n=3$, we have

$J\subseteq[2]$	$\varnothing$	$\{1\}$	$\{2\}$	$\{1,2\}$
$a_3^J(q)$	1	q^2	q^2	q^3-q
$\alpha_3^J(q^{-1})$	1	1	1	$1-q^{-2}$

For $w\in W=\langle s_1,s_2\rangle=\mathcal{S}_3$, the statistic $L(w)=2l(w)-l_{\mathrm{R}}^{\{1\}}(w)-l_{\mathrm{R}}^{\{2\}}(w)$, the character $\chi_\varepsilon(w)=\sigma(w)=(-1)^{l(w)}$ and the left descent set $D_{\mathrm{L}}(w)$ take the values

w	id	s_1	s_2	s_1s_2	s_2s_1	$s_1s_2s_1$
$L(w)$	0	1	1	1	1	2
$\chi_\varepsilon(w)$	1	-1	-1	1	1	-1
$D_{\mathrm{L}}(w)$	$\varnothing$	$\{1\}$	$\{2\}$	$\{1\}$	$\{2\}$	$\{1,2\}$

If n is odd or if n is even and $\varepsilon = 1$, the character χ_ε naturally extends to the sign character on the whole group W. Interestingly, in this case also a modified version of equation (30) seems to hold, where χ_ε is replaced by σ and one sums over *all* elements $w \in W$. In fact, we originally introduced chessboard elements in an attempt to control cancellation in this larger sum. Evidently, the contributions of any two elements $w_1, w_2 \in W$ with $w_1^{-1}w_2 \in S$ and $L(w_1) = L(w_2)$ cancel each other. Therefore we were led to sum over the set

$$\mathcal{M} := \{w \in W \mid \forall s \in S : D_{\mathrm{L}}(w) \neq D_{\mathrm{L}}(ws) \text{ or } L(w) \neq L(ws)\}.$$

The set $\mathcal{M}$ is easily seen to be closed under right-multiplication by the longest element w_0 and might indeed coincide with $\mathcal{C}_n$. Aided by computer evidence, we distilled Conjecture C out of this circle of ideas.

Appendix A. Explicit examples

Here we collect a few examples of Igusa-type functions for the family

$$\mathbf{F} = (F_J(\mathbf{X}))_{J\subseteq[n-1]} = \left(\prod_{i\in J} \frac{X_i}{1-X_i}\right)_{J\subseteq[n-1]}.$$

A.1. **A flag of alternating bilinear forms.** Let V be a vector space of dimension $n = 6$, equipped with a flag of alternating bilinear forms $\boldsymbol{B}$ of type $I = \{4\}$. This kind of example was considered immediately after Definition 4. We have

$J \subseteq [4]$	$\varnothing$	$\{2\}$	$\{4\}$	$\{2,4\}$
$a^J_{6,\{4\}}(q)$	1	$q^8+q^4+q^2$	q^8+q^6+1	$q^2(q^{12}-1)/(q^2-1)$
$\alpha^J_{6,\{4\}}(q^{-1})$	1	$1+q^{-4}+q^{-6}$	$1+q^{-2}+q^{-8}$	$(1-q^{-12})/(1-q^{-2})$

Thus, in view of Definition 5,

$$\mathrm{Ig}_{(V,\boldsymbol{B})}(q^{-1},\mathbf{X}) = \frac{1+(q^{-4}+q^{-6})X_1+(q^{-2}+q^{-8})X_2+q^{-10}X_1X_2}{(1-X_1)(1-X_2)}.$$

Note that, in accordance with Theorem B,

$$\mathrm{Ig}_{(V,\boldsymbol{B})}(q, X_1^{-1}, X_2^{-1}) = q^{10}\,\mathrm{Ig}_{(V,\boldsymbol{B})}(q^{-1}, X_2, X_1).$$

A.2. **Quadratic spaces.**

A.2.1. *Odd dimension.* Let V be a vector space of odd dimension $n = 3$, equipped with a non-degenerate quadratic form f. We have

$J \subseteq [2]$	$\varnothing$	$\{1\}, \{2\}$	$\{1, 2\}$
$a_3^J(q)$	1	q^2	$q^3 - q$
$\alpha_3^J(q^{-1})$	1	1	$1 - q^{-2}$

In view of Definition 6,

$$\mathrm{Ig}_3(q^{-1}, \mathbf{X}) = \frac{1 - q^{-2}X_1X_2}{(1 - X_1)(1 - X_2)}$$

and this satisfies the functional equation predicted by Theorem A.

A.2.2. *Even dimension.* Let V be a vector space of dimension $n = 4$, equipped with a non-degenerate quadratic form f. We have

$J \subseteq [3]$	$\varnothing$	$\{1\}, \{3\}$	$\{2\}$	$\{1, 2\}, \{2, 3\}, \{1, 3\}$	$\{1, 2, 3\}$
$a_{4,1}^J(q)$	1	$q^3 - q$	$q^4 + q^2$	$q^5 - q^3$	$q^2(q^2 - 1)^2$
$\alpha_{4,1}^J(q^{-1})$	1	$1 - q^{-2}$	$1 + q^{-2}$	$1 - q^{-2}$	$(1 - q^{-2})^2$
$a_{4,-1}^J(q)$	1	$q^3 + q$	$q^4 + q^2$	$q^5 + q^3$	$q^2(q^4 - 1)$
$\alpha_{4,-1}^J(q^{-1})$	1	$1 + q^{-2}$	$1 + q^{-2}$	$1 + q^{-2}$	$1 - q^{-4}$

Note that, for both values of ε, the number of *distinct* polynomials among the $\alpha_{4,\varepsilon}^J(q^{-1})$ is comparatively small. This illustrates that the map $J \mapsto \alpha_{n,\varepsilon}^J(q^{-1})$ factors over the bisecting map φ; cf. Proposition 3. One readily computes

$$\mathrm{Ig}_{4,1}(q^{-1}, \mathbf{X}) = \frac{1 + q^{-2}\left(-X_1X_2 + X_1X_3 - X_2X_3 - X_1 + X_2 - X_3\right) + q^{-4}X_1X_2X_3}{(1 - X_1)(1 - X_2)(1 - X_3)},$$

$$\mathrm{Ig}_{4,-1}(q^{-1}, \mathbf{X}) = \frac{1 + q^{-2}\left(-X_1X_2 - X_1X_3 - X_2X_3 + X_1 + X_2 + X_3\right) - q^{-4}X_1X_2X_3}{(1 - X_1)(1 - X_2)(1 - X_3)}.$$

These Igusa-type functions satisfy the functional equations predicted by Theorem A.

A.3. **A symmetric bilinear space in characteristic 2.** Let $V = \langle e_1, e_2, e_3, e_4 \rangle$ be a 4-dimensional vector space over a finite field $F = \mathbb{F}_q$ with $\mathrm{char}\, F = 2$. Let B denote the non-degenerate symmetric bilinear form on V such that $B(e_i, e_j) = \delta_{ij}$, and define $f : V \to F$ by $f(x) := B(x, x)$. We consider flags of non-degenerate symmetric bilinear subspaces in the non-degenerate symmetric bilinear space $\mathcal{V} := (V, B, f)$. For $J \subseteq \{1, 2, 3\}$ let $a_{\mathcal{V}}^J(q)$ denote the number of non-degenerate flags of type J in $\mathcal{V}$, and define the normalised polynomials $\alpha_{\mathcal{V}}^J(q^{-1})$ and the 'Igusa-type function' $\mathrm{Ig}_{\mathcal{V}}(q^{-1}, \mathbf{X})$ accordingly. Then we have

$J \subseteq [3]$	$\varnothing$	$\{1\}, \{3\}$	$\{2\}$	$\{1, 2\}, \{2, 3\}, \{1, 3\}$	$\{1, 2, 3\}$
$a_{\mathcal{V}}^J(q)$	1	q^3	$q^4 + q^2$	q^5	$q^4(q^2 - 1)$
$\alpha_{\mathcal{V}}^J(q^{-1})$	1	1	$1 + q^{-2}$	1	$1 - q^{-2}$

The associated 'Igusa-type function'

$$\mathrm{Ig}_{\mathcal{V}}(q^{-1}, \mathbf{X}) = \frac{1 + q^{-2} X_2 (1 - X_1 - X_3)}{(1 - X_1)(1 - X_2)(1 - X_3)}$$

does not satisfy a functional equation; this illustrates our remarks at the end of the Introduction.

Acknowledgements. The second author acknowledges support by the Deutsche Forschungsgemeinschaft in form of a Forschungsstipendium and the Max-Planck-Gesellschaft in form of a Fortbildungsstipendium. He gratefully acknowledges the hospitality of the Heinrich-Heine-Universität in Düsseldorf and the Max-Planck-Institut für Mathematik in Bonn during the writing of this paper.

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