Deformation of Resurgent Representations and Exact Asymptotics of Solutions to Differential Equations

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Abstract

A general resurgent functions theory in the framework of the abstract algebras is introduced. On the basis of this notion the problem of deformations of differential equations is investigated. The deformations of the corresponding integral transforms (representations) is used as the main tool of this investigation. The rebuildings of the type of the asymptotic expansions arising in the process of the above mentioned deformations are described.

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Contents

Introduction

The resurgent analysis, that is, the analysis of resurgent functions introduced by Jean Ecalle [1], at present proved to be a powerful tool in the asymptotic theory of both ordinary and partial differential equations (see [2] — [11], and the bibliography therein.)

When one makes the first acquaintaince with the asymptotic aspects of the theory of resurgent functions, it appears to be quite refined and complete. Actually, we found out that there exists an algebra isomorphism between the multiplicative algebra $\mathcal{E}_1$ of functions of exponential growth of degree one at the origin (determined in some sector of the complex plane) and the convolutive algebra $\mathcal{H}$ of hyperfunctions of exponential growth. This isomorphism (given by the Borel-Laplace transform) establishes the one-to-one correspondence between the asymptotic behavior of functions near the origin and the smoothness properties of the corresponding hyperfunctions. Later on, there exists two asymptotic scales in $\mathcal{E}_1$, one generated by the multiplication by exponentials $e^{\omega x}$ with different ωs, and the other — by multiplication by different powers of x. So, the asymptotic expansion of any function of the algebra $\mathcal{E}_1$ must be

$$u(x) = \sum_j e^{\omega_j x} \sum_{k=0}^{\infty} c_{jk} x^k. \tag{0.1}$$

Clearly, since the second scale is more weak than the first one, to interpret expansion (0.1) one have to resummate series in x involved in the latter expansion, and the Borel-Laplace transform provides one with the tool of such a resummation.

The mentioned two asymptotic scales are taken by Borel-Laplace transform into the scales generated by shifts in the p-plane, and by the operator d/dp, respectively. The latter fact leads to the microlocalization of expansion (0.1), since different terms of the outer sum in the outer sum in (0.1) is taken over different points of the p-plane. So, the examination of the behavior of functions $u(x)$ is taken into the local investigation of smoothness properties of the corresponding hyperfunctions[1].

However, it is evident that this clear scheme does not cover all possible applications of the resurgent functions theory. Actually, the exponential growth of order one is not the only type of behavior of solutions to differential equations. For example, they (solutions) can have exponential growth of order different than one.

Clearly, in the resurgent functions theory there exists a generalization of the theory, the so-called *k-Borel-Laplace transform*, which can sufficiently deal with exponential functions of order more than one. And, in spite of the fact that the function algebra under investigation was changed (now it is the algebra $\mathcal{E}_k$ of functions with exponential growth of order $k \neq 1$), *the same picture takes place in the p-plane*. More than that — all the investigation of singularities of the corresponding hyperfunctions makes no difference with the case of functions of exponential growth of order 1.

In fact, the resurgent functions theory can be of use also in the investigation of Fuchs-type equations. In this case we have met again another algebra of functions (the algebra $\mathcal{P}$ of functions of power growth), which needs another transform (Borel-Mellin transform this time) to investigate this algebra. And again we arrive at the same space of hyperfunctions we have met earlier.

Moreover, the similar situation takes place also in the investigation of *deformations* of differential equations. For example, let us see how the exponential asymptotics for differential equation

$$H\left(x, x^{1+\alpha}\frac{d}{dx}\right) u(x) = 0$$

becomes a power asymptotics as $\alpha \to 0$.

[1]We recall that hyperfunctions, that is, the quotient classes of ramifying analytic functions modulo entire functions, naturally arise in the theory of the Borel-Laplace transform (see, e. g. [9]).

So, we have come to the natural problem to work out some *general scheme* including as its specialization all known in present resurgent theories of different function algebras.

This paper is an attempt to construct this general scheme and apply it, in particular, to problem of deformations of differential equations.

The outline of the paper is as follows. We begin with the simplest examples of the Borel-Laplace and the Borel-Mellin transforms in order to understand the framework of the future general theory (Subsection 1.1). Here we determine the objects to appear in this general theory and detemine the degree of generality nesessary for its construction.

Later on, we construct the general theory (Subsections 1.2 and 1.3), and show how specifications of this theory lead to the above two important particular cases — to Borel-Laplace and Borel-Mellin transforms (Subsection 1.4).

In Section 2, we present the method of investigating the asymptotic behavior for solutions of equations in the framework of the above constructed general scheme. Here we show that all the work in constructing asymptotic expansions goes in the convolutive algebra of hyperfunctions in the p-plane, and the results of this investigation are lead to the description of asymptotic behavior of the initial objects.

After this we consider the case when equations in question depend on the additional parameter. Here we investigate the two cases of rebuilding of the asymptotic expansions, and investigate these rebuildings from the general viewpoint (Subsection 3.1).

Finally (Subsection 3.2), we describe the above mentioned rebuilding on the two concrete examples concerning the confluence of the Fuchs-type equations (and, in particular, the confluence of hypergeometric equations, Mathie equations, etc.), as well as the deformation (homotopy) of equations connecting equations with irregular singularities with that with regular (Fuchs-type) singular point. Clearly, the most interesting thing here is the catastrophe occuring when the equation changes its type and the asymptotics changes by jump.

Let us make one more important remark. For simplicity, we consider here the case of scalar differential equations. At the same time, the more general situation occurs to be important in applications. For example, in the theory of equations on manifolds with singularities the equations with operator-valued symbols arise (see, e. g. [12].) The application of the theory developed in the present paper to such a situation allows to obtain new interesting results in the deformation problem in the elliptic theory on manifolds with singularities. Namely, it is well-known that to obtain the asymptotics of solutions for simplest (conical) singularities of an underlying manifold, one can use standard tools such that the Mellin transform and the residue theory. At the same time, in the case of cusp-type singularities, to obtain asymptotic expansions one have to apply the resurgent analysis method (see [13], [14]). Here we show, in particular, how the expansions of one (cusp) type are transformed to the expansions of the another (conical) type.

Acknowledgements. The authors are grateful to Professor Bert-Wolfgang Schulze for fruitful discussions.

1 Resurgent analysis

1.1 Preliminaries

In this section, we present a general sceme of resurgent functions theory allowing one to consider different transforms, and even deformation of such transforms in the framework of the resurgent theory.

Let us briefly illustrate the method of application of the resurgent functions theory in the simplest situations to asymptotic investigation of ordinary differential equations.

Suppose that we need to investigate the asymptotic behavior of solutions to the equation

$$H\left(x^{-1}, -\frac{d}{dx}\right) u\left(x\right) = f\left(x\right) \tag{1.1}$$

as $x \longrightarrow \infty$.

It is well-known that the application of the resurgent analysis is based on the procedure of representation of solutions to differential equation with the help of the Laplace transform

$$u\left(x\right) = \int\limits_{\gamma} e^{-px} U\left(p\right)\, dp. \tag{1.2}$$

Here $U(p)$ is a function in the dual space, and γ is some contour in the complex plane with coordinate p.

Since the operators (d/dp) and p are taken into the operators x and $-d/dx$ under the action of the representation (1.2), equaton (1.1) is transformed into the equation

$$H\left(\left(d/dp\right)^{-1}, p\right) U\left(p\right) = F\left(p\right) \tag{1.3}$$

in the p-space. Later on, transform (1.2) establishes the correspondence between the smoothness of the function $U\left(p\right)$ and the behavior of the solution $u\left(x\right)$ to equation (1.1) at infinity.

So, the reduction of equation (1.1) *to equation* (1.3) *with the help of transform* (1.2) *allows one to obtain the information about the behavior of* $u\left(x\right)$ *at infinity from the smoothness properties of the function* $U\left(p\right)$.

From the other hand, if one investigates the behavior of solutions to differential equation

$$H\left(x, -x^2\frac{d}{dx}\right)u(x) = f(x)$$

near the singular point $x = 0$, he needs another representation of solutions given by the integral

$$u(x) = \int\limits_{\gamma} e^{-p/x} U(p)\, dp$$

(this representation is also called the Laplace transform) In this case, the operators d/dp and p are taken into the operators x and $-x^2 d/dx$, and the singularities of the function $U(p)$ describe the behavior of the initial function $u(x)$ near $x = 0$.

Finally, investigation of equations of the Fuchs type

$$H\left(x, -x\frac{d}{dx}\right)u(x) = f(x)$$

requires the representation given by Borel-Mellin transform

$$u(x) = \int\limits_{\gamma} x^{-p} U(p)\, dp.$$

All the three cases considered have the following mutual features:

- Each transform used for the representation of solutions to the initial equation is determined by some analytic group in p as a kernel:

$$G_x(p_1 + p_2) = G_x(p_1)G_x(p_2) \tag{1.4}$$

So,

– the first transform is determined by the group

$$G(x,p) = e^{-px},$$

– the second — by

$$G(x,p) = e^{-p/x},$$

– the third — by

$$G(x,p) = x^{-p}.$$

- In each case the large parameter of the corresponding asymptotic expansion is equal to

$$G^{-1}(x,p)\frac{dG(x,p)}{dp}$$

(due to the group property, this element does not depend[2] on p). So, for the first case the parameter is x, for the second case it is x^{-1}, and in the third case we have $\ln x$ as a parameter. This correlate with the fact that in the first case one investigates the behavior of solutions at infinity, and in the last two cases — at $x = 0$.

- For each transform one can define the operator $\hat{p}$ corresponding to the operator of multiplication by p in the dual space. These operators are

$$-\frac{d}{dx},\ -x^2\frac{d}{dx},\ \text{and}\ -x\frac{d}{dx}$$

in the three above cases, respectively.

As it can be seen from the examples considered, the key point in construction of resurgent representation is to define the kernel of the corresponding representation. Since we want this representation to be an algebra homomorphism[3], this kernel $G_x(p)$ must be a representation of the a representation of the (additive) group $\mathbf{C}$ in the corresponding algebra $\mathcal{U}$.

We remark also, that both the large parameter and the operator $\hat{p}$ can be used for the determination of the group $G_x(p)$ in question (though in the last case there is some ambiguity in this determination).

So, for consideration different integral representation, or, what is more, some their deformations, some general theory of integral representations of the above described type is nesessary. Such a theory will be constructed in the following subsection.

[2]Actually, differentiating the relation (1.4) and putting $p_2 = 0$ and $p_1 = p$, we obtain the relation

$$G_x^{-1}(p)\frac{dG_x(p)}{dp} = \frac{dG_x}{dp}(0).$$

[3]The fact that the integral representation under construction determines an algebra homomorphism leads to the fact that the corresponding resummation operator (see below) is, in turn, an algebra homomorphism commuting with the differentiation operator. This fact is important for the investigation of asymptotic solutions of differentiial equations

1.2 Analytic groups and integral representations

Let us introduce objects playing essential role in the further constructions.

1. Let $\mathcal{U}$ be a commutative topological algebra with unity. Consider the *analitic group* in $\mathcal{U}$, that is, such a mapping

$$G : \mathbf{C} \to \mathcal{U}. \tag{1.5}$$

that

1) the function $G(p)$ is holomorphic in the whole complex plane $\mathbf{C}$;

2) the group property

$$G(p_1 + p_2) = G(p_1)\, G(p_2)$$

takes place.

In essence, mapping (1.5) is a homomorphism of the additive group $\mathbf{C}$ of complex numbers into the multiplicative group of the algebra $\mathcal{U}$.

2. Consider the set $\mathcal{U}^*$ of linear functionals over $\mathcal{U}$ (these functionals are not continuous, and, in particular, they can be defined not on all $\mathcal{U}$.) We shall say that $G^*(p)$ is an *analytic family of functionals* over $\mathcal{U}$ if

$$G^* : \mathbf{C} \to \mathcal{U}$$

is such a mapping that

1) for any element $u \in \mathcal{U}$ the set

$$\Omega_u = \{p \,|\langle G^*(p), u\rangle \text{ is defined}\}$$

is an open set[4] in the plane $\mathbf{C}$;

2) the function

$$U_u(p) = \langle G^*(p), u\rangle \tag{1.6}$$

is holomorphic on Ω_u for any $u \in \mathcal{U}$.

Remark 1 In the sequel, only quotient classes of functions $U_u(p)$ modulo entire functions will be essential for us, not the functions $U_u(p)$ themselves. So, function (1.6) will be thought as a *hyperfunction*.

3. Any analytic family of functionals determines a filtration in the algebra $\mathcal{U}$. Namely, let us denote by $\mathcal{U}_R$ the subset of elements $u \in \mathcal{U}$ such that the set Ω_u contains the complement $\mathbf{C}^R$ in $\mathbf{C}$ of some sector of magnitude less than π with the vertex at point $p = R$ bissected by the direction of the positive real axis (see Figure 1). Clearly,

[4]We do not exclude the case when the set Ω_u is empty for some $u \in \mathcal{U}$.

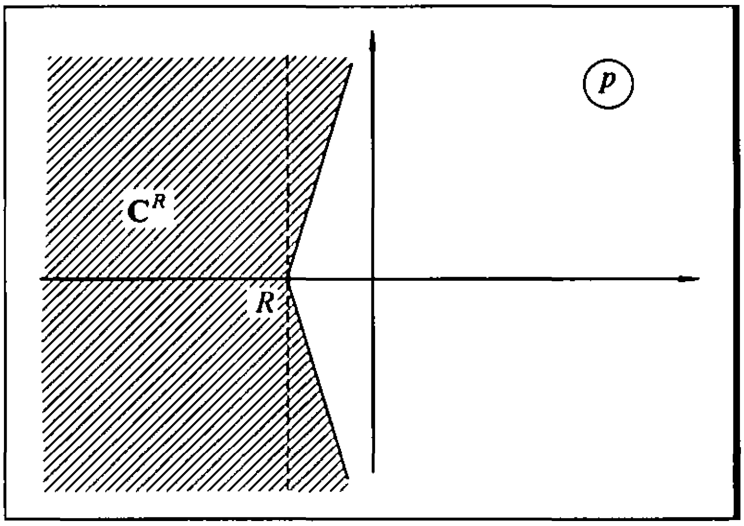

Figure 1. The set $\mathbf{C}_R$.

$$\mathcal{U}_{R_1} \subset \mathcal{U}_{R_2} \text{ if } R_1 < R_2.$$

Besides, it is clear that $\mathcal{U}_R$ is a linear subspace in $\mathcal{U}$ for any $R \in \mathbf{R}$.

Denote also

$$\mathcal{U}_\infty = \bigcap_{R \in \mathbf{R}} \mathcal{U}_R.$$

The set $\mathcal{U}_\infty$ is a set of infinitely smooth elements of the algebra $\mathcal{U}$ with respect to the introduced filtration.

Remark 2 We have supposed that each sector $\mathbf{C}^R$ is bissected by the direction of the positive real axis just to be definite. This direction can be replaced by any other direction without essential changes in the theory.

We suppose that the following condition is fulfilled:

Condition 1 The filtration $\{\mathcal{U}_R,\ R \in \mathbf{R}\}$ is complete. This means that for any family $\{u_R,\ R \in \mathbf{R}\}$ such that $u_R - u_{R'} \in \mathcal{U}_{\min(R,R')}$ there exists a $u \in \mathcal{A}$ such that $u - u_R \in \mathcal{U}_R$ for each $R \in \mathbf{R}$.

4. Let some analytic family of functionals $G^*(p)$ be fixed. Denote by $\mathrm{spec}_G\ (u)$ the minimal set such that the function (1.6) can be analytically continued in the whole plane $\mathbf{C}$ except for $\mathrm{spec}_G\ (u)$ as a ramifying function. The set $\mathrm{spec}_G\ (u)$ will

be called the *G-spectrum*[5] of the element $u \in \mathcal{U}$. As we shall see in the sequel, the set $\operatorname{spec}_G(u)$ determines the asymptotic expansion of the element u with respect to the filtration $\mathcal{U}_R$.

Let us introduce the main definition of this subsection.

Definition 1 The group (1.5) is called *a generating group of integral representation* if the following conditions hold:

- The element $A = dG/dp$ is invertible in the algebra $\mathcal{U}$.
- There exists an analytic family of functionals $G^*(p)$ of the above described type such that
$$\langle G^*(p), G(q)\rangle \equiv \frac{1}{2\pi i\,(p-q)}$$
modulo holomorphic functions.

From the viewpoint of hyperfunctions, the second requirement can be rewritten in the form
$$\langle G^*(p), G(q)\rangle = \delta(p-q),$$
since the analytic function $\frac{1}{2\pi i}(p-q)^{-1}$ determines the delta function in the space of hyperfunctions (see Remark 1 above.)

Let $G(p)$ be a generating group of integral representation. We are interested in the investigation of the integral representations of elements from $\mathcal{U}$ having the form
$$u = \mathcal{R}_G(U) = \int_\gamma G(p)\,U(p)\,dp, \tag{1.7}$$
where $U(p)$ is a function holomorphic in the domain Ω_U such that the intersection of the complement $\mathbf{C} \setminus \Omega_U$ with any right half-plane $\mathbf{C}^R$ is a compact set, and the contour γ is encircling this set, as it is shown on Figure 2.

Consider the question of the convergence of integral (1.7). To do this, we denote by γ_R the intersection of the contour γ with the left half-plane $\operatorname{Re} p < R$ (see Figure 2). The following assertion takes place:

Proposition 1 *The spectrum of the element*
$$u_R = \int_{\gamma_R} G(p)\,U(p)\,dp \tag{1.8}$$
is contained into the curve γ_R.

[5]In the sequel, we shall name G-spectrum simply by "spectrum". One should have in mind that the spectrum of an element $u \in \mathcal{U}$ is, in essence, the set on the Riemannian surface of the function (1.6) rather then in $\mathbf{C}$.

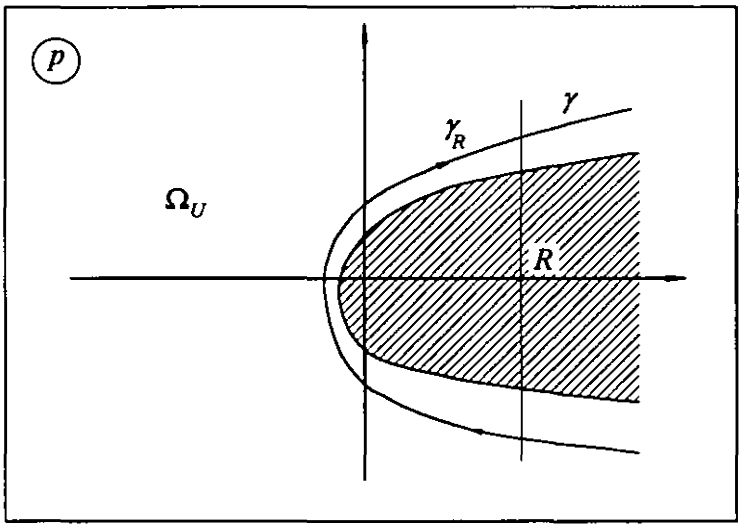

Figure 2. Domain of definition of $U(p)$ and the integration contour.

Proof. Let us apply a functional $G^*(q)$ to the element u_R given by (1.8). We arrive at the relation

$$\langle G^*(q), u_R\rangle = \int_{\gamma_R} \langle G^*(q), G(p)\rangle\, U(p)\, dp.$$

Due to the second condition of Definition 1, the latter relation can be rewritten as

$$\langle G^*(q), u_R\rangle = \frac{1}{2\pi i}\int_{\gamma_R} \frac{U(p)}{q-p}\, dp.$$

The latter integral is an integral of the Cauchy type with the support in γ_R, and, hence, it is holomorphic in the whole plane outside γ_R. This completes the proof of the proposition.

Now, let us consider the family u_R of elements of the algebra $\mathcal{U}$ given by formula (1.8). Proposition 1 shows that

$$u_R - u_{R'} \in \mathcal{U}_{\min(R,R')}.$$

Due to the completeness of the filtration $\mathcal{U}_R$ (see Condition 1 above), there exists a (unique modulo $\mathcal{U}_\infty$) element $u \in \mathcal{U}$ such that

$$u - u_R \in \mathcal{U}_R$$

for any $R \in \mathbf{R}$. By definition, this element is exactly the value of integral (1.7). So, this integral is defined modulo $\mathcal{U}_\infty$.

Remark 3 It is easy to see that if the function $U(p)$ is an entire function, then the corresponding integral vanishes. This observation shows that $U(p)$ must be interpreted not as a function but as a hyperfunction in p.

The following statement shows that the function $U(p)$ is defined by $\mathcal{R}(U)$ in the unique way.

Theorem 1 (on inversion) *If $u = \mathcal{R}_G(U)$, then*

$$U(p) \equiv \langle G^*(p), u \rangle$$

modulo entire functions.

Proof. For Rep sufficiently large in module and negative we have

$$\begin{aligned} \langle G^*(p), u \rangle &= \int_\gamma \langle G^*(p), G(q) \rangle \, U(q) \, dq \\ &= \frac{1}{2\pi i} \int_\gamma \frac{U(q)}{p-q} \, dq. \end{aligned}$$

Deforming the contour γ into the contour γ' drawn on Figure 3 and calculating the residue at the point p, we arrive at the relation

$$\langle G^*(p), u \rangle = U(p) + \frac{1}{2\pi i} \int_{\gamma'} \frac{U(q)}{p-q} \, dq.$$

The integral on the right in the latter formula is, clearly, an entire function in the variable p. This completes the proof of the Theorem.

The following result is a direct consequence of the last theorem.

Theorem 2 *The equality*

$$\operatorname{spec} \mathcal{R}_G(U) = \operatorname{sing} U$$

takes place. Here $\operatorname{sing} U$ *is the set of singularities of the function* $U(p)$.

We denote by $\mathcal{U}^G$ the set of elements of $\mathcal{U}$ representable in the form (1.7). It occurs that the set $\mathcal{U}^G$ is a subalgebra of the algebra $\mathcal{U}$:

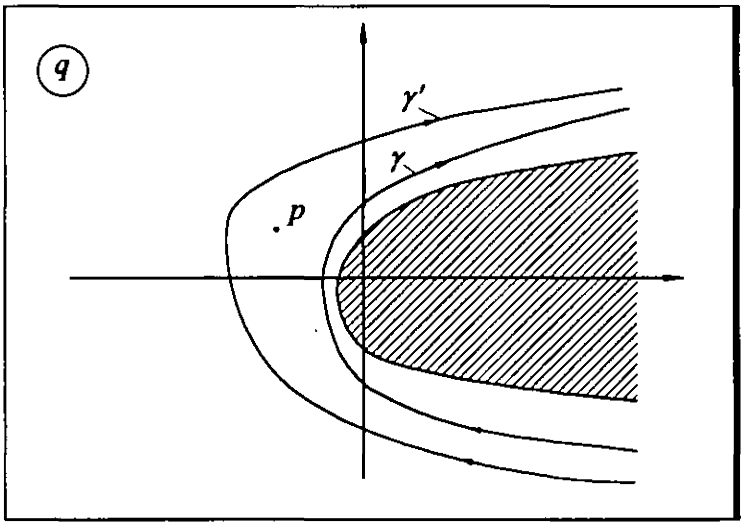

Figure 3. Deformation of the integration contour.

Theorem 3 *If $G(p)$ is a generating group of a integral representation, then the corresponding set $\mathcal{U}^G$ is a subalgebra of the algebra $\mathcal{U}$. The representation $\mathcal{R}_G$ is an algebra homomorphism from the convolutive algebra of hyperfunctions to $\mathcal{U}^G$.*

Proof of this affirmation is based on the fact that

$$\mathcal{R}_G(U_1)\mathcal{R}_G(U_2) = \mathcal{R}_G(U_1 * U_2), \tag{1.9}$$

where $*$ is the convolution of hyperfunctions. The proof of the latter fact uses the group property $G(p_1+p_2) = G(p_1)G(p_2)$ of the kernel and goes quite similar to the proof of the corresponding fact for usual Borel-Laplace transform (see [9]), and we omit it.

Now, let us define the operator $\hat{p}$ on the algebra $\mathcal{U}^G$ by

$$\hat{p}\mathcal{R}_G(U) = \mathcal{R}_G(pU). \tag{1.10}$$

Due to Theorem 1 this operator is uniquely defined. Formula (1.9) together with the fact that the multiplication by p is a differentiation of the convolutive algebra of hyperfunctions shows that the following affirmation is valid

Theorem 4 *The operator $\hat{p}$ is a differentiation of the algebra $\mathcal{U}^G$.*

Moreover, it is easy to see that the multiplication by the "large parameter" A (see Definition 1) corresponds to the operator d/dp from the viewpoint of representation $\mathcal{R}_G$:

$$A\mathcal{R}_G(U) = \mathcal{R}_G\left(\frac{dU}{dp}\right). \tag{1.11}$$

Remark 4 As it was mentioned in the end of Subsection 1.1, the group $G(p)$ can be determined (possibly, non-uniquely) either by the operator $\hat{p}$ or by the element A. In the first case, one uses the relation

$$\hat{p}G(p) = pG(p),$$

(this relation expresses the fact that $G(p)$ is an eigenfunction of the operator $\hat{p}$ with the eigenvalue p In this case $G(p)$ is determined up to a multiplicative constant $C(p)$ subject to the condition $C(p_1 + p_2) = C(p_1)C(p_2)$. In the second case $G(p)$ can be (uniquely) determined as a solution to the problem

$$\begin{cases} \dfrac{dG(p)}{dp} = AG(p), \\ G(0) = 1. \end{cases} \tag{1.12}$$

1.3 Resurgent elements of the algebra

1. We begin with the definition of resurgent elemens of the subalgebra $\mathcal{U}^G$ of the algebra $\mathcal{U}$ with respect to the generating group $G(p)$.

Definition 2 The element $u \in \mathcal{U}^G$ is called to be a *resurgent element* of the algebra $\mathcal{U}^G$, if the corresponding hyperfunction

$$U(p) = \langle G^*(p), u\rangle$$

is endlessly continuable[6] function in the variable p.

Since the set of endlessly continuable (hyper)functions is closed with respect to the convolution (see, e. g. [9]), and taking into account relation (1.9), we arrive to the following statement:

Theorem 5 *The set $\mathcal{U}^{G,r}$ of resurgent elements form a subalgebra of the algebra $\mathcal{U}^G$.*

[6]The exact definition of the notion of endless continuability the reader can find, for instance, in the book [9]. Roughly speaking, the function is endlessly-continuable if this function can be expanded up to a (ramifying) function having a discrete set of singularities on its Riemannian surface.

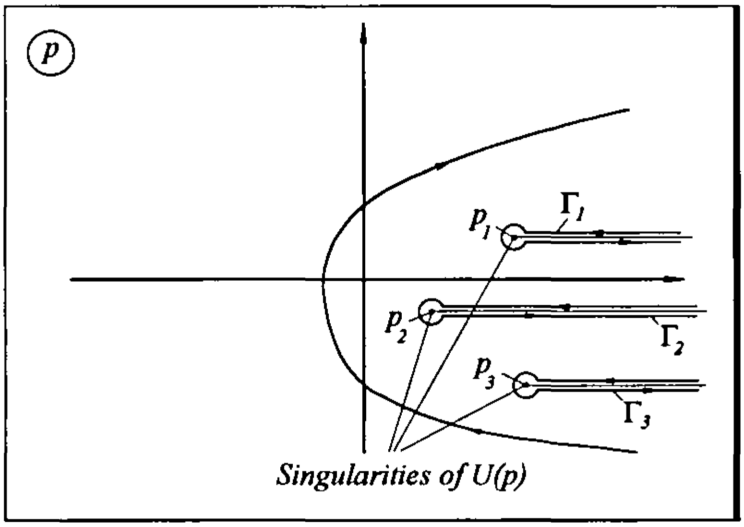

Figure 4. Reduction to the standard contours

If u is some resurgent element of $\mathcal{U}^G$, there exist the *canonical representation* of this element. To obtain this representation, one should decompose the homology class determined by the integration contour γ in the representation (1.7) into the sum of homology classes corresponding to standard contours Γ_j (see Figure 4). Each contour Γ_j surrounds one of the singularities of the function $U(p)$ counterclockwise and comes to infinity (over different sheets of the Riemannian surface of this function) in the direction of the positive real axis.The result of this decomposition is

$$u = \sum_j \int_{\Gamma_j} G(p)\, U(p)\, dp, \tag{1.13}$$

where the summation is fulfilled over all singular poins p_j which can be seen from the original domain of the function $U(p)$ along the direction parallel to the real positive axis. The set of singularities involved into the decomposition (1.13) of the element u is called to be the *support* of this element. As it follows from the definition of the resurgent element, the support of any such element is contained as a whole in some sector with angle less than π bisected by the direction of the real positive axis (with arbitrary origin).

2. Let us stand now on one more point which is of extreme importance in the asymptotic resurgent theory. This point is a notion of the resurgent element with simple singularities.

We recall (see, e. g. [9]), that the endlessly continuable function $U(p)$ is said to have *simple singularities* if this function can be represented in the form

$$U(p) = \frac{c_0}{p - p_0} + \ln(p - p_0) \sum_{k=0}^{\infty} \frac{(p - p_0)^k}{j!} c_{k+1} \tag{1.14}$$

near each point p_0 of its singularity. Here the series on the right in the latter relation is supposed to be convergent in a sufficiently small neighborhood of the point p_0.

Definition 3 The element $u \in \mathcal{U}^{G,r}$ is said to be a *resurgent element with simple singularities* if the corresponding function

$$U(p) = \langle G^*(p), a \rangle$$

has simple singularities in the sense (1.14).

As it is usual in the resurgent function theory, resurgent elements with simple singularities possess some standard asymptotic expansions (which are far away generaliyations of the WKB-expansions in the semi-classical approximation in quantum mechanics).

Let us derive this expansion. First of all, it is evident that we can consider distinctly each term on the right in the decomposition (1.13). Computing the integral

$$u_k = \int_{\Gamma_j} G(p) U(p)\, dp = \int_{\Gamma_j} G(p) \left[\frac{c_0}{p - p_k} + \ln(p - p_k) \sum_{k=0}^{\infty} \frac{(p - p_k)^k}{k!} c_{k+1} \right] dp$$

term-by-term, we obtain:

- For the first term the integral can be computed with the help of the usual residue theorem
$$c_0 \int_{\Gamma_j} G(p) \frac{dp}{p - p_j} = c_0 G(p_j).$$

- Due to relation (1.12), for the second term one has
$$\begin{aligned} c_1 \int_{\Gamma_j} G(p) \ln(p - p_j)\, dp &= c_1 \int_{\Gamma_j} A^{-1} \frac{dG(p)}{dp} \ln(p - p_j)\, dp \\ &= -c_1 \int_{\Gamma_j} A^{-1} G(p) \frac{dp}{p - p_j} dp = -c_1 A^{-1} G(p_j) \end{aligned}$$

  (we have used the integration by parts, and, again, the residue formula.)

- The iteration of the procedure used for the computation of the second term leads us to the following expression for all subsequent terms

$$c_{k+1} \int\limits_{\Gamma_j} G(p) \ln (p - p_j) \frac{(p-p_j)^k}{k!} \, dp = (-1)^{k+1} c_{k+1} A^{-k-1} G(p_j),$$

where $k = 1, 2, \ldots$.

So, for each term of the decomposition (1.13) we obtain the expansion

$$u_j = \int\limits_{\Gamma_j} G(p) \, U(p) \, dp \simeq G(p_j) \sum_{k=0}^{\infty} A^{-k} c_k. \qquad (1.15)$$

3. Let us consider examples of the corresponding asymptotic expansions.

i) First, let us consider the algebra $\mathcal{E}_1$ of functions of exponential growth of order 1 and use the Borel-Laplace representation, that is, the representation with the kernel

$$G_x(p) = e^{-\frac{p}{x}}.$$

Then expansion (1.15) becomes in this case

$$u_j(x) = e^{-\frac{p_j}{x}} \sum_{k=0}^{\infty} c_k x^k,$$

since the values of the group $G_x(p)$ for fixed $p = p_j$ are $e^{-p_j/x}$, and the large parameter of the expansion is x^{-1}.

The same situation takes place for WKB-expansions in quantum mechanics, where we must just replace x by h. The corresponding expansion is

$$u_j(h, x) = e^{-\frac{p_j(x)}{h}} \sum_{k=0}^{\infty} c_k(x) h^k,$$

and x plays a role of parameter (additional variable).

ii) For the k-Borel-Laplace representations one uses the group

$$G_x(p) = e^{-\frac{p}{kx^k}},$$

so that the expansion (1.15) becomes

$$u_j(x) = e^{-\frac{p_j}{kx^k}} \sum_{l=0}^{\infty} c_l x^{kl}$$

(here one can use also the expansion in fractional powers of the "large parameter" $A = x^{-k}$ to obtain the usual Taylor series in the expansion).

iii) Finally, for the Borel-Mellin representation with the kernel

$$G_x(p) = x^p,$$

one arrives to the asymptotic expansions of the form

$$u_j(x) = x^{p_j} \sum_{l=0}^{\infty} c_l \ln^{-l} x$$

since in this case the "large parameter" equals $\ln x$.

4. Clearly, the series on the right in the latter relation are not, as a rule, convergent in $\mathcal{U}$, and the expression

$$u_j = \int_{\Gamma_j} G(p)\, U(p)\, dp$$

supply us with the resummation procedure for series (1.15). However, one should define, in what sence the series on the right in (1.15) is asymptotic. In other words, we must present in the explicit way the asymptotic scales used for exact asymptotic expansions.

As one can see from the above considerations, there are the *two* asymptotic scales present.

The first one was introduced above and is determined by the filtration

$$\mathcal{U}_R^{G,r} = \mathcal{U}_R \cap \mathcal{U}^{G,r}.$$

This asymptotic scale is generated by multiplication by elements of the form $G(p)$ with different p since such elements correspond to *shifts* by p in the p-plane.

The other scale is determined on the set of resurgent elements having the support in a single point p_j. Clearly, it is sufficient to describe this scale for $p_j = 0$, which is a good choice since the set ${}^0\mathcal{U}^{G,r}$ of elements from $\mathcal{U}^{G,r}$ with supports at $p = 0$ form a subalgebra in $\mathcal{U}^{G,r}$. For such elements we use the filtration

$$\mathcal{F}_\alpha = \left\{ U(p) \middle|\, |U(p)| \le C_\varepsilon\, |p|^{\alpha-\varepsilon} \text{ for each } \varepsilon > 0 \right\}, \ \alpha \in \mathbf{R}$$

in the space of endlessly continuable functions with singularity at $p = 0$ (cf. [15]). Denoting this filtration by $\mathcal{U}\mathcal{F}_\alpha \subset {}^0\mathcal{U}^{G,r}$, one can see that it is generated by the operator A, since this operator corresponds to the differentiation operator in the space of microfunctions supported at $p = 0$ (see (1.11)). For example, in the series

$$\sum_{l=0}^{\infty} A^{-l} c_l$$

the main term (corresponding to $l = 0$) belongs to the space $\mathcal{UF}_{-1}$, since the unit element of the algebra $\mathcal{U}$ corresponds to the function

$$U(p) = c_0 [2\pi i p]^{-1},$$

the first term belongs to $\mathcal{AF}_0$: it corresponds to

$$U(p) = c_1 (2\pi i)^{-1} \ln p,$$

etc.

1.4 Examples

In this subsection, we shall show how the classical Borel-Laplace and Borel-Mellin transforms are embedded into the above described general scheme. To do this, we consider the algebra $\mathcal{U}$ of functions holomorphic in the sector

$$S(\varepsilon, R) = \{x \in \mathbf{C}_x | -\varepsilon < \arg x < \varepsilon,\ 0 < |x| < R\}$$

in the complex plane $\mathbf{C}$ with the topology of uniform convergence on compact subsets (we denote by x the coordinate in this plane).

1. Borel-Laplace transform. As it was already mentioned in Subsection 1.1, the corresponding group can be defined with the help of the operator

$$\hat{p} = x^2 \frac{d}{dx}. \tag{1.16}$$

Namely, the function $G(p, x)$ is defined for any fixed p as an eigenfunction of the operator (1.16) corresponding to the eigenvalue p. In other words, $G(p, x)$ satisfies the following equation:

$$x^2 \frac{d}{dx} G(p, x) = pG(p, x).$$

This equation can be easily solved and the general solution is given by

$$G(p, x) = C(p) e^{-\frac{p}{x}},$$

where $C(p)$ is an arbitrary constant (in x) subject to the condition

$$C(p_1) C(p_2) = C(p_1 + p_2).$$

So, as we have already remarked, operator (1.16) determines *not a unique group*, but the *family of such groups* parametrized by all complex-valued representations $C(p)$ of the additive group $\mathbf{C}$.

If we put $C(p) = 1$, we obtain the corresponding integral representation in the form[7]

$$\mathcal{R}_G(U) = \int\limits_{\gamma} e^{-\frac{p}{x}} U(p)\, dp,$$

which coincides with the standard definition of the complex Laplace transform. Here γ is a contour drawn on Figure 2. From the theory of the Borel-Laplace transform, it follows that the corresponding subalgebra $\mathcal{U}^G$ is simply an algebra $\mathcal{E}_1(S(\varepsilon, R))$ of functions of exponential growth with degree 1 defined in the sector $S(\varepsilon, R)$:

$$\mathcal{U}^G = \mathcal{E}_1(S(\varepsilon, R)) = \left\{ f(x) \in \mathcal{U} \mid |f(x)| \le C e^{a|x|} \text{ with some } C > 0 \text{ and } a \right\},$$

and the corresponding analytic family of functionals is given by the relation

$$\langle G^*(p), f(x) \rangle = \frac{1}{2\pi i} \int\limits_{\gamma_A} e^{\frac{p}{x}} f(x)\, dx, \tag{1.17}$$

where the contour γ_A is drawn on Figure 5 (it is easy to see that changing the initial point A of this contour leads to appearence of an additive entire function, and, hence, the right-hand part of formula (1.17) is well-defined as a hyperfunction, as required). Relation (1.17) is exactly the definition of the Borel transform.

To conclude this example, we remark that the large parameter corresponding to the group

$$G(p, x) = e^{-\frac{p}{x}}$$

is simply $A = x^{-1}$.

Clearly, all k-Borel transforms for any k can be described in the same way. This will be one of the topics of our consideration below.

2. Borel-Mellin transform. This transform can be determined by the operator

$$\hat{p} = x \frac{d}{dx}.$$

The equation for the corresponding group $G(p, x)$ is

$$x \frac{d}{dx} G(p, x) = p G(p, x),$$

and the expression for $G(p, x)$ can be obtained by resolving this equation:

$$G(p, x) = C(p)\, x^p,$$

[7] As we shall see below, the choice $C(p) \neq 1$ is not meaningless!

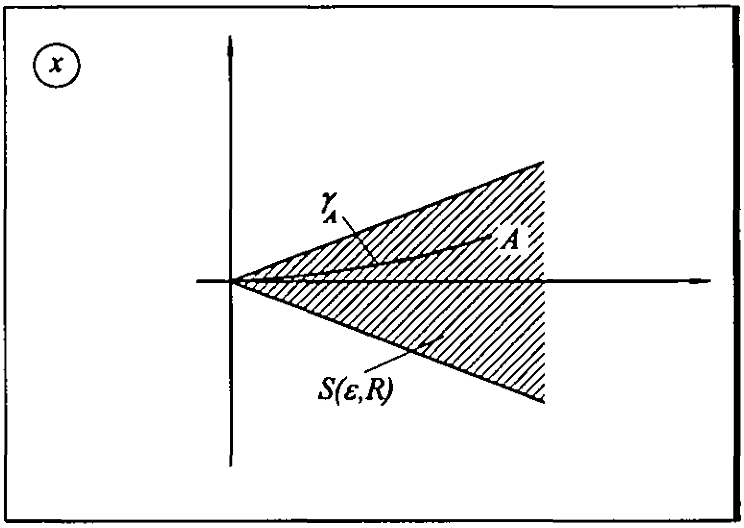

Figure 5. The form of the contour γ_A.

where $C(p)$ is as above. Again, putting $C(p) = 1$, one obtains the corresponding integral representation

$$\mathcal{R}_G(U) = \int_\gamma x^p U(p)\, dp,$$

which coincides with the inverse Borel-Mellin transform. In this case it is easy to compute that the corresponding subalgebra $\mathcal{U}^G$ is a subalgebra of functions of power growth, and the corresponding large parameter is $A = \ln x$.

Leaving the refinement of the other details to the reader, we consider here only one additional question. The matter is that the standard Fuchsian operator

$$H\left(x, x\frac{d}{dx}\right) \tag{1.18}$$

is transformed into the operator

$$H(T_1, p), \tag{1.19}$$

where T_1 is the shift by 1 in the p-plane:

$$(T_1U)(p) = U(p-1).$$

Clearly, operator (1.19) is not an operator of the form

$$H\left(\frac{d}{dp}, p\right)$$

(the latter corresponds to the operator of the form

$$H\left(\ln^{-1} x, x\frac{d}{dx}\right)$$

in the dual space.) *So, the models for Fuchsian operators in the dual space are not differential, but difference operators.*

The last consideration shows that in investigating differential equations one should start with operators of more general form than that involved into formula (1.3). Namely, one should consider operators

$$H(\hat{a}, p),$$

where $\hat{a}$ is an operator corresponding to an element a of the corresponding algebra $\mathcal{U}^G$. Such operators will be considered in the next section.

1.5 The parametric case and the Stokes phenomenon

Here we shall consider the case when the hyperfunctions in the dual space (and, hence, the corresponding elements of the algebra $\mathcal{U}^{G,r}$) depend on an additional parameter $x \in \mathbf{C}^n$. This means that we consider here the representations of the form[8]

$$u(x) = R_G[U(x,p)] = \sum_j \int_{\Gamma_j} G(p)\, U(x,p)\, dp. \tag{1.20}$$

Suppose that the hyperfunction $U(x,p)$ is endlessly continuable with respect to the variable p for any given value of x. Then the singulatiy set of the function $U(x,p)$ can be described by the equation

$$p = S(x),$$

where the function $S(x)$ is, as a rule, ramifying function of the variable x. The points of ramification of this function will be called focal points of the element (1.20). The set of focal points of the element $u(x)$ will be denoted by $\mathcal{F}_u$, or simply by $\mathcal{F}$ if this does not lead to misunderstanding.

The interest to the parametric case can be explained by considerations of exact asymptotics of solutions to the so-called A-differential equations [16], [17]

$$H\left(x, A^{-1}\frac{\partial}{\partial x}\right) u(x) = 0. \tag{1.21}$$

[8] It is possible to consider the case when the variable x is changed in some open set in the space $\mathbf{C}^n$; for simplicity we shall not consider this case here, the more that all needed changes for consideration of this more general case are quite obvious.

For example, exact semi-classical approximations for quantum mechanics equations leads to equation (1.21) with i/h as the operator A.

The consideration of representation of the type (1.20) leads to the so-called Stokes phenomenon, which can be described and investigated in the framework of the above constructed general scheme quite similar to the investigaion of this phenomenon in the classical theory of resurgent functions. To do this, we suppose that the element (1.20) has simple singularities. This means that the function $U(x,p)$ has the asymptotic expansion (in smoothness) of the type

$$U(x,p)=\frac{c_0(x)}{p-S(x)}+\ln(p-S(x))\sum_{j=0}^{\infty}\frac{(p-S(x))^j}{j!}c_j(x)$$

near each point of its singularity in p for each $x\notin\mathcal{F}$.

If the resurgent element $u(x)$ has simple singularities, then, as this was shown in Subsection 1.3, for each fixed value of x there exists an asymptotic expansion of this element having the form

$$u(x)=\sum_j G(S_j(x))\sum_{k=0}^{\infty}A^{-k}c_{kj}(x), \tag{1.22}$$

where the outer sum is taken over some subset of the set of singular points of the function $U(x,p)$, $S_j(x)$ are different branches of the function $S(x)$, and each inner sum is understood as the result of its resummation

$$\sum_{k=0}^{\infty}A^{-k}c_{kj}(x)=\int_{\Gamma_j}G(p)\,U(x,p)\,dp. \tag{1.23}$$

Here Γ_j is a standard contour encircling the singularity point $p=S_j(x)$ of the integrand. As it is well-known in the classical resurgent functions theory, asymptotic expansion (1.22) can be changed by jump if the parameter x intersects the so-called Stokes lines (this is exactly the *Stokes phenomenon* which is well-known in the classical resurgent function theory; see [9] for details.)

2 Asymptotics of solutions

2.1 Description of the class of equations

Let us fix some generating group $G(p)$ of integral representation in the algebra $\mathcal{U}$. Consider the equation[9]

$$H\left(\overset{2}{a},\overset{1}{\hat{p}}\right)u=0, \tag{2.1}$$

where $H(x,p)$ is a polynomial in p of order m with holomorphic coefficients, and a is some element of the algebra $\mathcal{U}^G$. We suppose that the operator of multiplication by a is an operator of negative order in the double filtration defined by the asymptotic scales

$$\left\{\mathcal{U}_R^{G,r},\ R\in\mathbf{R}\right\} \text{ and } \left\{\mathcal{U}\mathcal{F}_\alpha,\ \alpha\in\mathbf{R}\right\}.$$

Since the first scale is stronger than the second, the last requirement on the operator a means that

- either this operator has negative order with respect to the asymptotic scale $\left\{\mathcal{U}_R^{G,r},\ R\in\mathbf{R}\right\}$,
- or this operator has zero order with respect to $\left\{\mathcal{U}_R^{G,r},\ R\in\mathbf{R}\right\}$ and has the negative order with respect to $\left\{\mathcal{U}\mathcal{F}_\alpha,\ \alpha\in\mathbf{R}\right\}$.

Let us interpret these two requirements in terms of the operator $\hat{a}$ in the dual space.

In the first case the fact that the operator a has negative order in the asymptotic scale $\left\{\mathcal{U}_R^{G,r},\ R\in\mathbf{R}\right\}$ means that the corresponding operator $\hat{a}$ shifts supports of functions $U(p)$ to the right in the p-plane to some positive value (which is interpreted as the negative order of this operator). For example, this operator can be the shift in the p-plane by a complex number p_0 with positive real part. This operator can be represented as the convolution with the function

$$a(p)=\langle G^*(p),a\rangle,$$

namely

$$\hat{a}U(p)=a(p)*U(p),$$

[9] Below we use Feunmann indices over the operators defiing the order of their action, see [16], [18].

and the support of this function must be contained as a whole in the half-plane $\{\mathrm{Re}p > 0\}$ (as well as in some sector with angle less than π bisected with the direction of the positive real axis; see the remark after the definition of a resurgent element from $\mathcal{U}$).

In the second case, we again represent the operator $\hat{a}$ as the convolution with the corresponding function $a(p)$. Since this operator has zero order with respect to the scale $\left\{A_R^{G,r},\ R \in \mathbf{R}\right\}$, the support of the element a is contained as a whole in the half-plane $\{\mathrm{Re}p > 0\}$, and it intersects the line $\{\mathrm{Re}p > 0\}$. We shall restrict ourselves with the most simple but the most interesting case when the support of a consists of a single point $p = 0$. In this case the function $a(p)$ has the only singularity point $p = 0$, and, since $\hat{a}$ is an operator of negative order with respect to the scale $\{AF_\alpha,\ \alpha \in \mathbf{R}\}$, the singularity at this point is weak, that is

$$|a(p)| \leq C\,|p|^\beta$$

with some $C > 0$ and $\beta > -1$.

The process of solving equation (2.1) in these two cases is quite different, and we consider the constructing of an asymptotic expansion in each of these two cases distinctly.

2.2 Asymptotic expansion (first case)

Since the element a involved into equation (2.1) is in some sense small (it has the negative order), it is natural to expand this equation in powers of this element:

$$H(a,\hat{p})\,u = [H_0(\hat{p}) + aH_1(a,\hat{p})]\,u = 0.$$

Due to (1.10), passing to the dual equation with the help of the representation

$$u = \mathcal{R}_G\left[U(p)\right],$$

one arrives at the following equation for the function $U(p)$:

$$\left[H_0(p) + \overset{3}{\hat{a}}\ H_1\left(\overset{2}{\hat{a}},\overset{1}{p}\right)\right] U(p) = 0.$$

Let us search for the solution of the latter equation with the help of the successive approximation method. Neglecting the term $\hat{a}H_1(\hat{a},p)\,U(p)$, we determine the zeroth iteration $U_0(p)$ as a solution to the equation

$$H_0(p)\,U_0(p) = 0. \tag{2.2}$$

Suppose, for simplicity, that all the roots of the polynomial $H_0(p)$ are simple. Then equation (2.2) has exactly m independent solutions of the form

$$U^{(j)}(p) = \frac{1}{2\pi i\,(p - p_j)},\ j = 1, \ldots m,$$

where p_j are different roots of $H_0(p)$ (we recall that all equations in the dual space are to be solved in the spaces of hyperfunctions). Let us fix one of these solutions (say, $U^{(1)}(p)$). Now, searching for the next iteration of the form (we omit the superscript (1))

$$U_1(p) = U_0(p) + V_1(p),$$

we obtain the equation for $V_1(p)$ in the form

$$H_0(p)\,V_1(p) + \overset{3}{\hat{a}}\ H_1\left(\overset{2}{\hat{a}}, \overset{1}{p}\right) U_0(p) + \overset{3}{\hat{a}}\ H_1\left(\overset{2}{\hat{a}}, \overset{1}{p}\right) V_1(p) = 0.$$

Again neglecting the term $\hat{a}H_1(\hat{a}, p)\,V_1(p)$, we arrive at the expression for $V_1(p)$:

$$V_1(p) = -\left[H_0(p)\right]^{-1} \overset{3}{\hat{a}}\ H_1\left(\overset{2}{\hat{a}}, \overset{1}{p}\right) U_0(p).$$

Note, that the support of the function $V_1(p)$ is shifted to the right in the p-plane by $\delta > 0$, where $-\delta$ is an order of the operator $\hat{a}$ in the asymptotic scale $\left\{A_R^{G,r},\ R \in \mathbf{R}\right\}$.

Continuing this process, we shall construct a solution to equation (2.1) in the form

$$U(p) = U_0(p) + V_1(p) + V_2(p) + \ldots. \tag{2.3}$$

Taking into account that

- the supports of only finite number of terms of series (2.3) have a nonempty intersection with any left half-plane $\{\mathrm{Re}\,p < R\}$, and
- the filtration $\left\{A_R^{G,r},\ R \in \mathbf{R}\right\}$ is complete,

one obtains that series (2.3) converge in the space of resurgent elements thus determining a solution to equation (2.1).

So, we have arrived at the following statement:

Theorem 6 *If the operator $\hat{a}$ has a negative order with respect to the asymptotic scale $\left\{A_R^{G,r},\ R \in \mathbf{R}\right\}$, there exist a full system of resurgent solutions to equation (2.1). Each solution of this system is determined by some root p_j of the polynomial $H_0(p)$, and the support of this solution consists of the point p_j itself together with shifts of this point by any p from the support of the element a. The set of singularities of the function $U(p)$ is the union of the set $\{p_1, \ldots, p_m\}$ and all above described shifts of this set.*

2.3 Asymptotic expansion (second case)

The case when the operator a has zero order with respect to the asymptotic scale $\left\{A_R^{G,r},\ R \in \mathbf{R}\right\}$ is a little bit more difficult[10]. However, the initial steps in the constructing an asymptotic solution go quite similar to the above considered case. Namely, we expand the equation in powers of the operator a and transform the obtained equation into the equation for the function

$$U(p) = \langle G^*(p), u\rangle$$

with the help of the representation $\mathcal{R}_G$. Similar to the above case, we get

$$\left[H_0(p) + \overset{2}{\hat{a}}\, H_1\left(\overset{2}{\hat{a}}, \overset{1}{p}\right)\right] U(p) = 0. \tag{2.4}$$

Using again the successive approximation method, we obtain a formal solutions to this equation in the form of the series

$$U(p) = U_0(p) + \sum_{k=0}^{\infty} V_k(p), \tag{2.5}$$

where $U_0(p)$ is one of the functions

$$U_0(p) = U_0^{(j)}(p) = \frac{1}{2\pi i\,(p - p_j)},\ j = 1, \ldots m,$$

and the functions $V_k(p)$ are solutions of the following recurrent system

$$\begin{aligned} H_0(p)\, V_1(p) &= -\overset{2}{\hat{a}}\, H_1\left(\overset{2}{\hat{a}}, \overset{1}{p}\right) U_0(p), \\ H_0(p)\, V_k(p) &= -\overset{2}{\hat{a}}\, H_1\left(\overset{2}{\hat{a}}, \overset{1}{p}\right) V_{k-1}(p),\ k = 2, 3, \ldots. \end{aligned}$$

The difference between the above considered case and this one is that all the functions $V_k(p)$ are supported at one and the same point $p = p_j$, and, hence, we must prove the convergence of series (2.5) in the space of endlessly continuable functions.

Such an affirmation can be proved, if one takes into account the fact that the operator $\hat{a}$ can be represented as the convolution

$$\hat{a}U(p) = a(p) * U(p)$$

[10] We recall that we consider the case when the element a involved into equation (2.1) is supposed to have the support consisting of the single point $p = 0$.

with the function $a(p) = \langle G^{*}(p), a\rangle$ which has a weak singularity at the origin (and the origin is the only singular point of this function). Due to this fact, equation (2.4) is an equation of the Volterra type, and this allows one to obtain the needed estimates (we omit here the details).

So, we arrive at the following statement:

Theorem 7 *Under the above formulated conditions, for any p_j, $j = 1, \dots, m$ there exists a resurgent solution to equation* (2.1) *with support at $p = p_j$. The singularity set of the corresponding function $U(p)$ is $\{p_1, \dots, p_m\}$.*

2.4 A-differential equations

Here we consider the so-called A-differential equations having the form

$$H\left(x, A^{-1}\frac{\partial}{\partial x}\right) u(x) = 0, \tag{2.6}$$

where $x \in \mathbf{C}^n$ is an n-dimensional complex variable, $u(x)$ is a function in x with values in the algebra $\mathcal{U}^{G.r}$. As it was mentioned before (see Subsection 1.5), the model for such equations are equations of quantum mechanics, for which the operator A is simply the multiplication by h^{-1}, where h is a Planck constant. The same method can be applied to the investigation of the Mathieu equation (see [19], [20])

$$\frac{d^2y}{dz^2} + (\lambda - q\cos 2z)\, y = 0$$

and others.

To apply the general resugrent scheme to this equation, one should determine the corresponding representation from the given "large parameter" A. As it was described above (see Subsection 1.2), the kernel $G(p)$ of the representation $\mathcal{R}_G$ is connected with the large parameter by the relations

$$\frac{dG(p)}{dp} = AG(p), \; G(0) = 1.$$

Now, passing to the resurgent representation,

$$u(x) = \mathcal{R}_G\left[U(x,p)\right]$$

one obtains the following equation for the (hyper)function $U(x,p)$:

$$H\left(x, \left(\frac{\partial}{\partial p}\right)^{-1}\frac{\partial}{\partial x}\right) U(x,p) = 0. \tag{2.7}$$

We remark that the latter equation is *quantized with respect to smoothness.* To investigate the asymptotic solutions to this equations in smoothness, one can use the theory of differential equations on complex-analytic manifolds (see [15]). For one-dimensional equations (that is, for $x \in \mathbf{C}^1$), the existence of an endlessly-continuable solution to equation (2.7) was proved in the book [9] by the authors.

Clearly, in the process of investigation of asymptotic behavior of solutions to equation (2.6) one should investigate the Stokes phenomenon. This can be done quite similar to the investigation fo this phenomenon in the framework of the classical resurgent functions theory, and we shall not present here these considerations (see, e. g. [2], [9]).

3 Deformations of integral transforms and equations

3.1 General theory

In this section, we shall investigate equations of the above described type in the case when these equations depend on some additional parameter. The main question of the theory will be the changes of type of asymptotic expansions taking place at some values of the parameter.

Let us describe the situation in detail. Consider a family of generating groups of integral representations depending on some complex parameter $\alpha \in \mathbf{C}^k$. We suppose that the corresponding function

$$G(p, \alpha), \tag{3.1}$$

as well as the corresponding functionals $G^*(p,\alpha)$ depend on the parameter α analytically in some set Ω in the space $\mathbf{C}^k$. Then all the objects constructed in Section 1 such as the operator $\hat{p}$, "large parameter" A, the subalgebra $\mathcal{U}^{G,r}$, and the above introduced filtrations will depend on the parameter α, in turn. This dependence will be denoted by the subscript α for the corresponding objects. So, $\hat{p}_\alpha$ will be the notation for the operator $\hat{p}$ defined by the group (3.1), $\mathcal{U}^{G,r}_\alpha$ will denote the corresponding subalgebra of resurgent elements, and so on.

Consider the equation

$$H(a_\alpha, \hat{p}_\alpha)\, u = 0, \tag{3.2}$$

where a_α is an element from $\mathcal{U}^{G,r}_\alpha$ such that the multiplication by a_α is the operator of negative order in the double asymptotic scale determined by the group $G(p,\alpha)$. We suppose that the dependence of the element a_α on α is analytic in Ω. Under this condition, the asymptotic solution to equation (3.2) can be obtained literally in the way described in Section 2, and this solution has the form

$$U(p,\alpha) = U_0(p,\alpha) + \sum_{k=0}^{\infty} V_k(p,\alpha), \tag{3.3}$$

and the terms of this series are graduated with respect to the double asymptotic scale determined by the group $G(p,\alpha)$. However, one must have in mind that

- first, this double scale depends on the parameter α and
- second, the operator $\hat{a} = \hat{a}_\alpha$ also depends on this parameter.

So, when the parameter α changes its value, it is possible that expansion (3.3) changes its type by jump (we mean the two types of asymptotic expansion described in Section 2). In this process there are two possible cases. To describe these cases, let us follow the evolution of asymptotic expansion (3.3) along some curve l in the domain Ω where the parameter α have its values.

Case 1. On the initial stage of the deformation (that is, on some initial part of the curve l; see Figure 6) the asymptotic expansion has the first type (that is, the operator $\hat{a}_\alpha$ has a negative order with respect to the filtration $\left\{\mathcal{U}^{G,r}_{R,\alpha},\ R \in \mathbf{R}\right\}$), and at some point α_0 on the curve this expansion becomes an expansion of the second type (this means that the operator $\hat{a}_{\alpha_0}$ has zero order in the filtration $\left\{\mathcal{U}^{G,r}_{R,\alpha_0},\ R \in \mathbf{R}\right\}$ but negative order with respect to the filtration $\{\mathcal{UF}_\alpha,\ \alpha \in \mathbf{R}\}$).

Case 2. The deformation begins with values of α such that the asymptotic expansion has the second type, and for $\alpha = \alpha_0$ this expansion becomes that of the first type.

Let us consider these two cases in more detail.

We recall that the support of the resurgent solution is deternimed by the roots of the polynomial $H_0(p) = H(0,p)$ and by the support of the function

$$a(p,\alpha) = \langle G^*(p,\alpha), a_\alpha \rangle . \tag{3.4}$$

The set of roots of $H_0(p)$ does not depend on α and, hence, it does not affect the type of the asymptotic expansion. On the opposite, the support of function (3.4) changes with the change of α and, hence, is crucial for the determination of the type of the asymptotic expansion in question.

In the first case, the support of the function $a(p,\alpha)$ at the beginning of the deformation process is contained as a whole in the open half-plane $\{\mathrm{Re} p > 0\}$. In the process of deformation, some points of this support approach the line $\{\mathrm{Re} p = 0\}$,

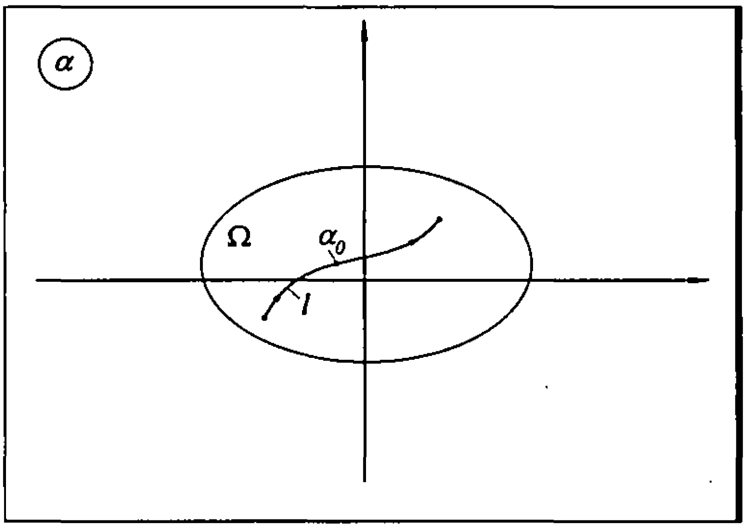

Figure 6. Deformation path.

and, finally, come to this line at $\alpha = \alpha_0$. The examples below show that even infinite number of points of support can come to one and the same point of the line $\{\mathrm{Re}p = 0\}$. So, the expansion initially graded with the help of scale $\left\{\mathcal{U}^{G,r}_{R,\alpha},\ R \in \mathbf{R}\right\}$ can become to be graded with the help of the more weak scale $\{\mathcal{UF}_\alpha,\ \alpha \in \mathbf{R}\}$.

In the second case, at the initial stage of the deformation process the support of the function $a\,(p,\alpha)$ has points on the line $\{\mathrm{Re}p = 0\}$ (this support can even consist of a single point $p = 0$ on this line). Then, at $\alpha = \alpha_0$, the support of $a\,(p,\alpha)$ looses points on the imaginary axis and becomes a set laying as a whole in the half-plane $\{\mathrm{Re}p > 0\}$. Hence, the set of terms of the asymptotic expansion being initially graded with the help of more weak scale $\{\mathcal{UF}_\alpha,\ \alpha \in \mathbf{R}\}$ becomes to be graded with the scale $\left\{\mathcal{U}^{G,r}_{R,\alpha},\ R \in \mathbf{R}\right\}$. We shall see this phenomenon on one of the examples below.

So, the asymptotic expansion can change its type for values of the parameter α *such that the support of the function* $a\,(p,\alpha)$ *comes to the line* $\{\mathrm{Re}p = 0\}$ *or leaves this line.*

3.2 Examples

1. Multiple Mellin transforms (see [21]).

Let us consider the equation

$$H\left(x, x(x+\alpha)\frac{d}{dx}\right)u(x)=0,$$

where $H(x,p)$ is a polynomial in p with coefficients holomorphic near the origin in the complex x-plane. This is an equation of the Fuchsian type with the two singular points $x=0$ and $x=-\alpha$. Our aim is to investigate the uniform asymptotic expansion of solutions to this equation in a neighborhood of the origin in α-plane.

Using the above described scheme of investigation, one should first to construct an integral representation defined by the operator

$$\hat{p}=x(x+\alpha)\frac{d}{dx}.$$

The corresponding group is determined by the requirement that $G(p,x,\alpha)$ is an eigenfunction for the operator $\hat{p}$:

$$x(x+\alpha)\frac{d}{dx}G(p,x,\alpha)=pG(p,x,\alpha).$$

The direct computation gives

$$G(p,x,\alpha)=C(p,\alpha)\left(\frac{x}{x+\alpha}\right)^{p/\alpha}.$$

The constant $C(p,\alpha)$ has to be chosen in such a way that the function $G(p,x,\alpha)$ is regular at the point $\alpha=0$. Since we have

$$\lim_{\alpha\to 0}\left(\frac{x}{x+\alpha}\right)^{p/\alpha}=e^{-p/x},$$

it is sufficient to put $C(p,\alpha)=1$. So, we obtain the representation of the form

$$\mathcal{R}_{G,\alpha}(U)=\int_{\Gamma}\left(\frac{x}{x+\alpha}\right)^{p/\alpha}U(p)\,dp. \tag{3.5}$$

This representation can be named *double Mellin representation* since its kernel has two singular points $x=0$ and $x=-\alpha$, and both these points are of the Mellin (Fuchs) type. Later on, as $\alpha\to 0$ this transform takes the form

$$\lim_{\alpha\to 0}\mathcal{R}_{G,\alpha}(U)=\mathcal{R}_{G,0}(U)=\int_{\Gamma}e^{-p/x}U(p)\,dp$$

of the classical (complex) Laplace representation.

To determine a point α_0 at which the type of asymptotic behavior of solution is changed, one have to compute the operator $\hat{x}_\alpha$ corresponding to multiplication by x in the x-plane. This can be done with the following procedure. Multiplying the equality (3.5) by $x/(x+\alpha)$ we obtain

$$\begin{aligned}\left(\frac{x}{x+\alpha}\right)\mathcal{R}_{G,\alpha}(U) &= \int_\Gamma \left(\frac{x}{x+\alpha}\right)^{(p+\alpha)/\alpha} U(p)\,dp \\ &= \int_\Gamma \left(\frac{x}{x+\alpha}\right)^{p/\alpha} U(p-\alpha)\,dp = \mathcal{R}_{G,\alpha}(T_\alpha U),\end{aligned}$$

where T_α is a shift operator in the p-plane:

$$(T_\alpha U)(p) = U(p-\alpha).$$

Hence, we have

$$\left(\frac{x}{x+\alpha}\right)^\wedge = \frac{\hat{x}_\alpha}{\hat{x}_\alpha+\alpha} = T_\alpha.$$

Resolving this equation with respect to the operator $\hat{x}_\alpha$, one arrives at the relation

$$\hat{x}_\alpha = \frac{\alpha T_\alpha}{1-T_\alpha}.$$

The operator $(1-T_\alpha)^{-1}$ involved into the latter relation requires interpretation.It can be formally rewritten in the two different ways:

$$(1-T_\alpha)^{-1} = \sum_{j=0}^{\infty} T_{j\alpha}, \tag{3.6}$$

or

$$(1-T_\alpha)^{-1} = -T_{-\alpha}(1-T_{-\alpha})^{-1} = -\sum_{j=1}^{\infty} T_{-j\alpha}. \tag{3.7}$$

To interpret the operator $(1-T_\alpha)^{-1}$ for different values of x and α, we rewrite the expression of the kernel $G(p,x,\alpha)$ of representation (3.5) as

$$G(p,x,\alpha) = \exp\left[\frac{p}{\alpha}\ln\frac{x}{x+\alpha}\right].$$

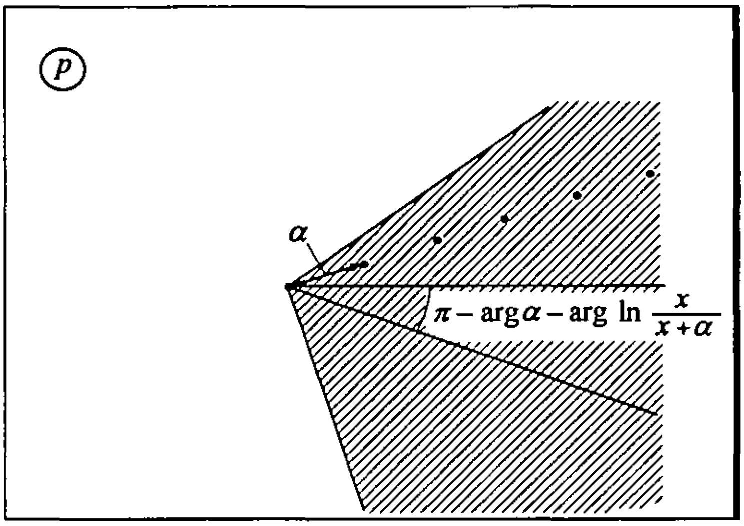

Figure 7. Decomposition of $(1-T_\alpha)^{-1}$ (first case).

Then it is evident that the direction of the steepest descent contour Γ for representation (3.5) is given by the formula

$$\arg\theta = \pi + \arg\alpha - \arg\ln\frac{x}{x+\alpha}. \tag{3.8}$$

So, if the value of α lies in some sector of angle less than π bissected by the direction (3.8) with its vertex in the origin (see Remark 2 above), then the operator T_α is an operator of negative order with respect to the asymptotic scale $\left\{\mathcal{U}^{G,r}_{R,\alpha},\ R\in\mathbf{R}\right\}$, and the decomposition (3.6) is correct. In the opposite case one should use the decomposition (3.7). The geometry of both these cases is drawn on Figures 7 and 8. To be definite, we consider the first possibility.

Then the operator $\hat{x}_\alpha$ is given by

$$\hat{x}_\alpha = \frac{\alpha T_\alpha}{1-T_\alpha} = \alpha T_\alpha\sum_{j=0}^{\infty}T_{j\alpha} = \alpha\sum_{j=1}^{\infty}T_{j\alpha},$$

and the corresponding hyperfunction is

$$X(p,\alpha) = \alpha\sum_{j=1}^{\infty}\frac{1}{2\pi i\,(p-j\alpha)}.$$

One can easily see that the set of singularities of the latter function (or, what is the same in the considered case, the support of the corresponding resurgent element)

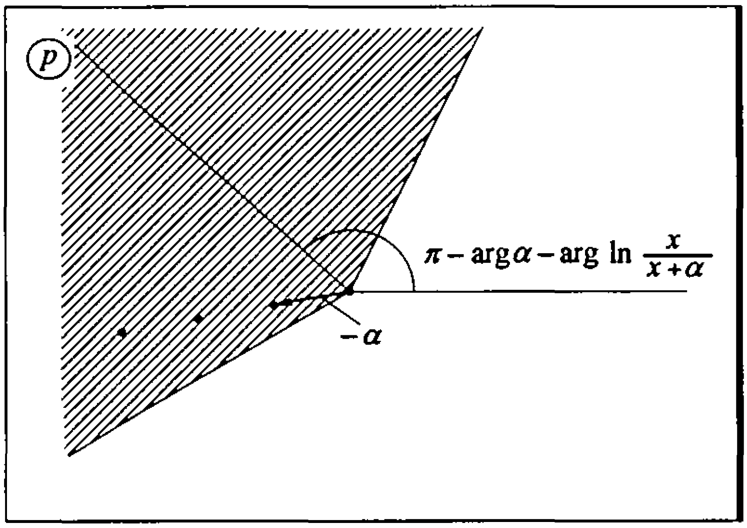

Figure 8. Decomposition of $(1 - T_\alpha)^{-1}$ (second case).

forms a lattice with step α originated from the point α in the p-plane. As α tends to zero, all this lattice shrinks to one and the same point, namely to the origin in the p-plane. The limit of the corresponding operator is

$$\lim_{\alpha\to 0} \hat{x}_\alpha = \lim_{\alpha\to 0} T_\alpha \Delta_\alpha^{-1} = \left(\frac{d}{dp}\right)^{-1},$$

where

$$\Delta_\alpha = \alpha^{-1}(1 - T_\alpha)$$

is the difference derivative with step α. So, the corresponding limit hyperfunction is

$$X(p, 0) = \frac{1}{2\pi i} \ln p,$$

and this function has the only singularity point at $p = 0$.

So, we see that for this example the first case of changing the asymptotic expansion is realized.

One can also consider the triple Mellin transform which arises in the investigation of the equation

$$H\left(x, x(x+\alpha)(x+\beta)\frac{d}{dx}\right)u = 0.$$

We shall not consider this case in detail, but shall discuss briefly the properties of the corresponding representation. This representation has the form

$$\mathcal{R}_{\alpha,\beta}(U) = \int_\Gamma \left[\frac{x^{\frac{1}{\alpha\beta}}}{(x+\alpha)^{\frac{1}{\alpha(\beta-\alpha)}}(x+\beta)^{\frac{1}{\beta(\alpha-\beta)}}}\right]^p U(p)\,dp.$$

This representation can be called *triple Mellin representation* corresponding to the points $x = 0$, $x = -\alpha$, and $x = -\beta$.

It can be shown that:

i) There exists a double limit $\lim\limits_{\alpha,\beta\to 0} \mathcal{R}_{\alpha,\beta}(U)$. This limit coincides with the 2-Laplace transform

$$\mathcal{R}_{0,0}(U) = \int_\Gamma e^{\frac{p}{2x^2}} U(p)\,dp.$$

ii) There exist limits

$$\lim_{\alpha\to 0}\mathcal{R}_{\alpha,\beta}(U) = \mathcal{R}_{0,\beta}(U) = \int_\Gamma e^{\frac{p}{x\beta}}\left(\frac{x}{x+\beta}\right)^{\frac{p}{\beta^2}} U(p)\,dp, \tag{3.9}$$

$$\lim_{\beta\to 0}\mathcal{R}_{\alpha,\beta}(U) = \mathcal{R}_{\alpha,0}(U) = \int_\Gamma e^{\frac{p}{x\alpha}}\left(\frac{x}{x+\alpha}\right)^{\frac{p}{\alpha^2}} U(p)\,dp,$$

and

$$\lim_{\alpha\to \beta}\mathcal{R}_{\alpha,\beta}(U) = \mathcal{R}_{\beta,\beta}(U) = \int_\Gamma e^{\frac{p}{\beta(x+\beta)}}\left(\frac{x}{x+\beta}\right)^{-\frac{p}{\beta^2}} U(p)\,dp,$$

These representations can be considered as the mixed Laplace-Mellin representations. For instance, representation (3.9) is of Laplace type at the point $x = 0$, and of Mellin type at $x = -\beta$.

So, in this case the parameter space can be split into five stratum. First is the origin where the representation has the 2-Laplace type, three others are planes $\alpha = 0$, $\beta = 0$, and $\alpha = \beta$ with origin deleted (here the representation has the mixed Laplace-Mellin type), and the rest is all the space $\mathbf{C}^2$ with all these planes deleted

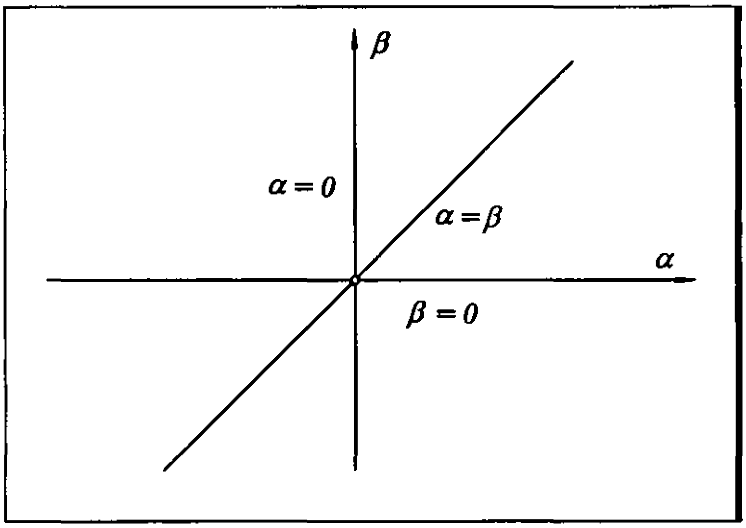

Figure 9. Strata in the parameter space.

(here the representation is of a triple Mellin type). Schematically all these strata are drawn on Figure 9.

Let us present some concrete examples. The above described scheme can be applied to the investigation of the confluent hypergeometric equations (see [22], [23], , [25]), and the Lame equation (see [20]). Let us consider first the case of the hypergeometric function.

It is well-known that the confluent hypergeometric equation appears from the Hauss hypergeometric equation

$$z\left(1-\frac{z}{b}\right)\frac{d^2y}{dz^2}+\left[\gamma-(\alpha+\beta+1)\frac{z}{b}\right]\frac{dy}{dz}-\alpha\frac{\beta}{b}y=0 \tag{3.10}$$

for $b=\beta\to\infty$. We remark that equation (3.10) is an equation of the Fuchsian type with three regular singular points $z=0$, $z=b$, and $z=\infty$. During the mentioned limit the two points $z=b$ and $z=\infty$ coincide with each other forming an irregular singular point at infinity.

To investigate this limit, one can apply the method developed in this paper. We transform equation (3.10) to the form convenent for investigation by the variable change $z=x^{-1}$ with $b=\beta$, thus obtaining the equation

$$\left\{\left[x\left(x-\frac{1}{b}\right)\frac{d}{dx}\right]^2-\left(\gamma x-\frac{\alpha+b}{b}\right)\left[x\left(x-\frac{1}{b}\right)\frac{d}{dx}\right]-\alpha\left(x-\frac{1}{b}\right)\right\}y=0.$$

The latter equation have the form

$$H\left(x, x(x+\beta)\frac{d}{dx}\right)y = 0$$

with $\beta = -b^{-1}$, and, hence, all the examination above can be applied to this equation. The form of the limit equation (confluent hypergeometric equation or the Kummer equation) is

$$\left[\left(x^2\frac{d}{dx}\right)^2 - (\gamma x + 1)\left(x^2\frac{d}{dx}\right) - \alpha x\right]y = 0.$$

We mention that the confluence phenomenon for hypergeometric functions was investigated with the help of different technique by J. Martinet and J.-P. Ramis [26], and H. Kimura and K. Takano [27] (see also the bibliography therein).

Let us examine now the Lamé equation. This equation has the form

$$\begin{aligned}&\frac{d^2\Lambda}{d\lambda^2} + \frac{1}{2}\left[\frac{1}{a^2+\lambda} + \frac{1}{b^2+\lambda} + \frac{1}{c^2+\lambda}\right]\frac{d\Lambda}{d\lambda}\\ &-\frac{n(n+1)\lambda + C}{4(a^2+\lambda)(b^2+\lambda)(c^2+\lambda)}\Lambda = 0,\end{aligned}$$

where a, b, c, n, and C are parameters. The Lamé equation has four regular singular points, namely $\lambda = -a^2$, $\lambda = -b^2$, $\lambda = -c^2$, and $\lambda = \infty$. The behavior of the solution to this equation during the confluence between these four points also can be investigated by our methods. To show this, we shall rewrite the equation in the form

$$\begin{aligned}&\left\{\left[\left(a^2+\lambda\right)\left(b^2+\lambda\right)\left(c^2+\lambda\right)\frac{d}{d\lambda}\right]^2 - \frac{1}{2}\left[\left(b^2+\lambda\right)\left(c^2+\lambda\right) + \left(a^2+\lambda\right)\left(c^2+\lambda\right)\right.\right.\\ &\left.+\left(a^2+\lambda\right)\left(b^2+\lambda\right)\right]\left[\left(a^2+\lambda\right)\left(b^2+\lambda\right)\left(c^2+\lambda\right)\frac{d}{d\lambda}\right] - \frac{1}{4}\left(n(n+1)\lambda + C\right)\\ &\left.\times\left(a^2+\lambda\right)\left(b^2+\lambda\right)\left(c^2+\lambda\right)\right\}\Lambda = 0,\end{aligned}$$

which, in essence, coincide with the above considered equation

$$H\left(x, x(x+\alpha)(x+\beta)\frac{d}{dx}\right)y = 0$$

after the shift by a^2 with $\alpha = b^2 - a^2$ and $\beta = c^2 - a^2$. The variable change $\lambda = x^{-1}$ allows to include into consideration the fourth singular point $\lambda = \infty$.

2. The family of Borel-Laplace transforms.

Here we investigate the equation

$$H\left(x, x^{1+\alpha}\frac{d}{dx}\right)u = 0 \tag{3.11}$$

for $\alpha \in \mathbf{C}$, $\mathrm{Re}\alpha \geq 0$. The most interesting is the investigation of the behavior of the solutions as $\alpha \to 0$, so we shall consider αs sufficiently small in module.

The interest for the investigation of the limit $\alpha \to 0$ is due to the fact that solutions to equation (3.11) has quite different structure in cases $\alpha \neq 0$ and $\alpha = 0$. In the first case each solution (from the full system of resurgent solutions) is a function with support at a single point p_j, where $p_1, \ldots, p_m$ are roots of the polynomial $H(0,p)$. The asymptotics of these solutions have the form

$$u_j \simeq e^{-\frac{p_j}{\alpha x^\alpha}} \sum_{k=0}^{\infty} c_k x^{\gamma_k}, \tag{3.12}$$

where γ_k is some increasing sequence of reals such that $\gamma_k \to +\infty$ as $k \to \infty$. The series on the right in the latter expansion are, as a rule, divergent, and one must use the resurgent analysis for resummating these series.

On the opposite, in the second case (that is, for $\alpha = 0$), equation (3.11) is an equation of Fuchs type, and the solution to this equation has the well-known conormal form

$$u_j \simeq \sum_{k=0}^{\infty} x^{p_{kj}} \sum_{l=0}^{m_{kj}-1} c_{jkl} \ln^l x, \tag{3.13}$$

where

$$\{p_{kj} = p_j + k\} \tag{3.14}$$

are lattices originated from points p_j with step 1, and m_{kj} are corresponding multiplicities. As it can shown, this functions are resurgent with respect to the Borel-Mellin representation; the supports of these functions are given by (3.14), and there are finite number of terms in the expansions corresponding to each point of support, so that the resummation procedure is not needed for the interpretation of this expansion.

One can show, that the question how expansion (3.12) transforms into expansion (3.13) can be solved with the above introduced technique.

Let us compute the corresponding family of groups. Due to the above theory, the corresponding function $G(p, x, \alpha)$ must be a solution to the equation

$$x^{1+\alpha}\frac{d}{dx}G(p, x, \alpha) = pG(p, x, \alpha).$$

The general solution to this equation is

$$G(p,x,\alpha) = C(p,\alpha)\, e^{-\frac{p}{\alpha x^{\alpha}}},$$

where $C(p,\alpha)$ is an arbitrary constant. Since we are interested in investigatopn of the limit $\alpha \to 0$, we must choose this constant in such a way that the function $G(p,x,\alpha)$ is regular at $\alpha = 0$. In contrast to the previous example, we cannot use here $C(p,\alpha) = 1$ since the function $\exp[-p/(\alpha x^{\alpha})]$ has an essential singularity at the point $\alpha = 0$. It is easy to see that to obtain a function $G(p,x,\alpha)$ regular at $\alpha = 0$ it is sufficient to put

$$C(p,\alpha) = e^{\frac{p}{\alpha}}.$$

So, finally we have

$$G(p,x,\alpha) = e^{-p\left(\frac{1}{\alpha x^{\alpha}} - \frac{1}{\alpha}\right)},$$

and the corresponding representation has the form

$$\mathcal{R}_{G,\alpha}(U) = \int_{\Gamma} e^{-p\left(\frac{1}{\alpha x^{\alpha}} - \frac{1}{\alpha}\right)} U(p)\, dp.$$

The limit of this representation as $\alpha \to 0$ is

$$\lim_{\alpha \to 0} \mathcal{R}_{G,\alpha}(U) = \mathcal{R}_{G,0}(U) = \int_{\Gamma} x^{p} U(p)\, dp,$$

which, naturally, coincides with the Borel-Mellin representation. So, we have constructed a deformation of the representations connecting Borel-Laplace representations of different orders α with the Borel-Mellin representation.

Now, one has to investigate the corresponding operator $\hat{x}_{\alpha}$. Denote by $\mathcal{L}_{\alpha}$ the classical Laplace representation of order α:

$$\mathcal{L}_{\alpha}(U) = \int_{\Gamma} e^{-\frac{p}{\alpha x^{\alpha}}} U(p)\, dp.$$

Clearly, we have

$$\mathcal{R}_{G,\alpha}(U) = \mathcal{L}_{\alpha}\left(e^{\frac{p}{\alpha}} U\right).$$

Now, taking into account the relation

$$x\mathcal{L}_{\alpha}(U) = \mathcal{L}_{\alpha}\left(\alpha^{-\frac{1}{\alpha}} \left(\frac{d}{dp}\right)^{-\frac{1}{\alpha}} U\right),$$

one has

$$
\begin{aligned}
x\mathcal{R}_{G,\alpha}(U) &= x\mathcal{L}_\alpha\left(e^{\frac{p}{\alpha}}U\right) = \mathcal{L}_\alpha\left(\alpha^{-\frac{1}{\alpha}}\left(\frac{d}{dp}\right)^{-\frac{1}{\alpha}} e^{\frac{p}{\alpha}}U\right) \\
&= \mathcal{L}_\alpha\left(e^{\frac{p}{\alpha}}\left[e^{-\frac{p}{\alpha}}\alpha^{-\frac{1}{\alpha}}\left(\frac{d}{dp}\right)^{-\frac{1}{\alpha}} e^{\frac{p}{\alpha}}\right]U\right) = \mathcal{R}_{G,\alpha}(\hat{x}_\alpha U),
\end{aligned}
$$

where the operator $\hat{x}_\alpha$ is given by

$$
\hat{x}_\alpha = e^{-\frac{p}{\alpha}}\alpha^{-\frac{1}{\alpha}}\left(\frac{d}{dp}\right)^{-\frac{1}{\alpha}} e^{\frac{p}{\alpha}}. \tag{3.15}
$$

The latter operator can be rewritten as the convolution with some function $X(p,\alpha)$. To do this, we use the relation

$$
\left(\frac{d}{dp}\right)^{-\beta} U(p) = U(p) * \frac{p^{\beta-1}}{\Gamma(\beta)}
$$

valid for $\beta > 0$ (here $\Gamma(\beta)$ is the Gamma function). Using this relation, operator (3.15) can be rewritten in the form

$$
\hat{x}_\alpha U(p) = U(p) * X(p,\alpha),
$$

where

$$
X(p,\alpha) = \frac{\alpha^{-\frac{1}{\alpha}}}{\Gamma\left(\frac{1}{\alpha}\right)} p^{\frac{1}{\alpha}-1} e^{-\frac{p}{\alpha}}.
$$

One can see from the latter relation that the less in module the number α is, the more negative order with respect to the asymptotic scale $\{\mathcal{AF}_\alpha,\ \alpha \in \mathbf{R}\}$ the operator $\hat{x}_\alpha$ has. From the other hand, this operator has zero order with respect to the scale $\left\{\mathcal{U}^{G,r}_{R,\alpha},\ R \in \mathbf{R}\right\}$ for any α with $\mathrm{Re}\alpha > 0$.

The following affirmation is almost evident.

Lemma 1 *The following relation takes place*

$$
\lim_{\alpha \to 0} \hat{x}_\alpha = T_1,
$$

where T_1 is the shift by 1 in the plane $\mathbf{C}$:

$$
T_1 U(p) = U(p-1).
$$

Proof. Since

$$\lim_{\alpha\to 0} \mathcal{R}_{G,\alpha}\left(\hat{x}_\alpha U\left(p\right)\right) = \lim_{\alpha\to 0} x\mathcal{R}_{G,\alpha}\left(U\left(p\right)\right) = x\mathcal{R}_{G,0}\left(U\left(p\right)\right) = \mathcal{R}_{G,0}\left(T_1 U\left(p\right)\right),$$

the required statement follows from the invertibility of the representation $\mathcal{R}_{G,\alpha}$ up to $\alpha = 0$.

Now one can see that the terms of asymptotic expansion (3.3) being concentrated at one and the same point of support p_j of the resurgent solution for $\mathrm{Re}\alpha > 0$, "jump" from this point to points $p_j + k$, $k \in \mathbf{Z}_+$ when α becomes to be equal to zero. This means that we have here the second case of behavior of the asymptotic expansion described above.

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