THE TWO-PARAMETER SEGAL-BARGMANN TRANSFORM: UNITARITY AND UNIQUENESS

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Abstract. A proof is presented of the fact that the two-parameter Segal-Bargmann transform is a unitary isomorphism. The extent to which $S_{s,t}$ is specified by its fundamental properties is determined exactly.

1. Introduction

In the paper [5], Hall has generalized the classical Segal-Bargmann transform and the Hall transform [4] to a unitary transform $S_{s,t}$, dependent on two positive parameters s and t satisfying $s > t/2$, in the context of Lie groups of compact type. The transform $S_{s,t}$ maps an L^2–space unitarily onto a space of holomorphic square-integrable functions over a complex space. In this article we deal with $S_{s,t}$ in the context of linear spaces, and we

- present a different proof of the fact that $S_{s,t}$ is a unitary isomorphism
- determine the exact extent to which $S_{s,t}$ is specified by its fundamental properties (unitarity, naturality, and a factorization property)

This paper is essentially self-contained.

In the elegant paper [3], Driver and Hall have studied the transform $S_{s,t}$ on the infinite-dimensional Wiener space of paths on a Lie algebra and have deduced from it the corresponding unitary transform on a compact Lie group. The case $s = t$, for linear spaces, corresponds to the classical Segal-Bargmann transform ([7] and [2]). In [5] Hall has shown that there is a special string of transformations which can be used to reduce the two-parameter case for finite-dimensional linear spaces to the classical one-parameter case. The reduction does not work in the infinite-dimensional case, nor for groups. Thus we may justifiably view the transforms $S_{s,t}$ as constituting a larger family of transforms of which the classical case is a special instance. The approach in [3], and

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one of the approaches used in [5], uses Hermite polynomials to prove unitarity of $S_{s,t}$.

In this article we present a direct proof of the fact that $S_{s,t}$ is a unitary isomorphism, without using Hermite polynomials or using a reduction to the $s = t$ case. We also determine the extent of "rigidity" of the unitary transform $S_{s,t}$, i.e. we determine the groups of unitary isomorphisms on the domain and on the range of $S_{s,t}$ which, when composed with $S_{s,t}$, result in transforms which have the same fundamental properties as $S_{s,t}$.

In the last section we recast several of the results in category-theoretic language, which permits clean and efficient formulations of the results.

2. Results

Let s, t be two positive real numbers with $s > t/2$, and let

$$r = 2s - t \text{ and } \rho = \frac{s-t}{s} \tag{2.1}$$

The following relations are useful

$$1 - \rho = \frac{t}{s},\ 1 + \rho = \frac{r}{s},\ \text{and } \sqrt{rt} = s(1-\rho^2)^{1/2} \tag{2.2}$$

Note also that the conditions $s > t/2 > 0$ imply that $-1 < \rho < 1$.

Let V be a real vector space, equipped with an inner-product, of finite (positive) dimension $\dim V$. The inner-product of $x, y \in V$ will be denoted xy (thus x^2 is the inner product of x with itself). Let P_s be the Gaussian probability measure on V specified by

$$dP_s(x) = (2\pi s)^{-\dim V/2} e^{-\frac{x^2}{2s}}\, dx \tag{2.3}$$

where dx denotes the lebesgue measure on V.

Next let V_c be the complexification of V, and let $M_{s,t}$ be the Gaussian measure on V_c given by

(2.4)

$$dM_{s,t}(z) = \frac{1}{(\pi r)^{\dim V/2}(\pi t)^{\dim V/2}} e^{-\frac{x^2}{r}-\frac{y^2}{t}} dz = \frac{1}{[\pi s(1-\rho^2)^{1/2}]^{\dim V}} e^{-\frac{x^2}{r}-\frac{y^2}{t}} dz$$

where $z = x + iy$, with $x, y \in V$, and $dz = dxdy$ is lebesgue measure on V_c relative to the inner-product on $V_c = V^2$, taken for the moment as a real vector space with the inner-product induced from V.

The inner-product on V induces a complex-bilinear map $V_c \times V_c \to \mathbf{C} : (z, w) \mapsto zw$. If $z = x + iy$, with $x, y \in V$, then we have the conjugate

$$\overline{z} = x - iy$$

Let $\mathcal{H}L^2(V_c, M_{s,t})$ be the set of holomorphic functions on V_c which are square-integrable with respect to $M_{s,t}$.

For any $r > 0$ we have the (analytically continued) Gaussian function

$$\rho_r(z) = (2\pi r)^{-\dim V/2} e^{-z^2/(2r)} \text{ for all } z \in V_c \tag{2.5}$$

For f a function on V, and $z \in V_c$ define

(2.6)

$$\boxed{S^V_{s,t} f(z) = \int_V (2\pi t)^{-\dim V/2} e^{-\frac{(z-y)^2}{2t}} f(y)\, dy = \int_V f(y)\rho_t(z-y)\, dy}$$

When the vector space V is clear from the context we will drop it from the notation $S^V_{s,t}$ and simply write $S_{s,t}$.

It can be readily checked by Cauchy-Schwarz that if $f \in L^2(P_s)$ then $S_{s,t}f(z)$ is meaningful for every $z \in V_c$, and then it follows by Morera's theorem that the function $S_{s,t}f$ is holomorphic. Moreover, by Cauchy-Schwarz,

$$|S_{s,t}f(z)| \leq C|f|_{L^2(P_s)} \tag{2.7}$$

Here $C = \left|\rho_t(z-y)/\rho_s(y)\right|_{L^2(dP_s(y))}$ is finite because $s > t/2$.

The inner-product on V induces, naturally, a complex-bilinear form $(z, w) \mapsto zw$ on V_c. Each vector $x \in V$ induces a linear functional

$$\hat{x} : V \to \mathbf{R} : y \mapsto \langle x, y\rangle \tag{2.8}$$

and each $z \in V_c$ induces a complex-linear functional

$$\tilde{z} : V_c \to \mathbf{C} : w \mapsto zw \tag{2.9}$$

Let $U(1) = \{\lambda \in \mathbf{C} : |\lambda| = 1\}$ be the group of unit-modulus complex numbers, forming a group under multiplication.

Let $\mathrm{Aut}(L^2(P_s))_V$ be the set of maps $U : L^2(V, P_s) \to L^2(V, P_s)$ satisfying:

1. Unitarity: U is a unitary isomorphism
2. Functoriality: if n is any positive integer $\leq \dim V$, there is a unitary isomorphism $L^2(\mathbf{R}^n, P_s) \to L^2(\mathbf{R}^n, P_s) : f \mapsto \hat{f}$ such that for any $u_1, ..., u_n$ orthonormal in V, the image of the function $F_f = f(\hat{u}_1, ..., \hat{u}_n)$ under U is the function $F_{\hat{f}} = \hat{f}(\hat{u}_1, ..., \hat{u}_n)$
3. Factorizability: if $u_1, ..., u_{n+m}$ (with n, m being any positive integers) are orthonormal vectors in V and $f \in L^2(\mathbf{R}^n, P_s)$, $g \in L^2(\mathbf{R}^m, P_s)$, and $F_{f,g} \stackrel{\text{def}}{=} f(\hat{u}_1, ..., \hat{u}_n)g(\hat{u}_{n+1}, ..., \hat{u}_{n+m})$ then $UF_{f,g}$ is of the form $F_{\hat{f},\hat{g}}$, where $\hat{f}$ and $\hat{g}$ are as given in Condition 2.

It is readily verified that $\mathrm{Aut}(L^2(P_s))_V$ is a group under composition.

Theorem 2.1. *Suppose V is a finite-dimensional real inner-product space of dimension ≥ 2. Then there is an isomorphism of groups:*

$$U(1) \to Aut(L^2(P_s))_V : a \mapsto \mathcal{U}_a^V \tag{2.10}$$

where $\mathcal{U}_a^V$ is the unique unitary isomorphism $L^2(V, P_s) \to L^2(V, P_s)$ specified by the requirement that its action on the exponential functions $e^{\hat{x}}$ be given as follows:

$$\mathcal{U}_a^V e^{\hat{x}} = e^{a\hat{x} + \frac{1}{2}(1-a^2)sx^2} \tag{2.11}$$

for every $x \in V$.

We have also the analogous result for $\mathcal{H}L^2(V_c, M_{s,t})$. Let $\mathrm{Aut}(\mathcal{H}L^2(M_{s,t}))_V$ be the set of maps $W : \mathcal{H}L^2(V_c, M_{s,t}) \to \mathcal{H}L^2(V_c, M_{s,t})$ satisfying:

1. Unitarity: W is a unitary isomorphism
2. Functoriality: if n is any positive integer $\leq \dim V$, there is a unitary isomorphism $\mathcal{H}L^2(\mathbf{C}^n, M_{s,t}) \to \mathcal{H}L^2(\mathbf{C}^n, M_{s,t}) : f \mapsto \tilde{f}$ such that for any $u_1, ..., u_n$ orthonormal in V, the image of the function $F_f = f(\tilde{u}_1, ..., \tilde{u}_n)$ under W is the function $F_{\tilde{f}} = \tilde{f}(\tilde{u}_1, ..., \tilde{u}_n)$
3. Factorizability: if $u_1, ..., u_{n+m}$ (with n, m being any positive integers) are orthonormal vectors in V and $f \in \mathcal{H}L^2(\mathbf{C}^n, M_{s,t})$, $g \in \mathcal{H}L^2(\mathbf{C}^m, M_{s,t})$, and $F_{f,g} \stackrel{\text{def}}{=} f(\tilde{u}_1, ..., \tilde{u}_n)g(\tilde{u}_{n+1}, ..., \tilde{u}_{n+m})$ then $WF_{f,g}$ is of the form $F_{\tilde{f},\tilde{g}}$, where $\tilde{f}$ and $\tilde{g}$ are as given in Condition 2.

$\mathrm{Aut}(\mathcal{H}L^2(M_{s,t}))_V$ is a group under composition.

Theorem 2.2. *Suppose V is a finite-dimensional real inner-product space of dimension ≥ 2. Then there is an isomorphism of groups:*

$$U(1) \to Aut(\mathcal{H}L^2(V_c, M_{s,t}))_V : a \mapsto \mathcal{W}_a^V \tag{2.12}$$

where $\mathcal{W}_a^V$ is the unique unitary isomorphism $\mathcal{H}L^2(V_c, M_{s,t}) \to \mathcal{H}L^2(V_c, M_{s,t})$ specified by the requirement that its action on the exponential functions $e^{\tilde{w}}$ be given as follows:

$$\mathcal{W}_a^V e^{\tilde{w}} = e^{a\tilde{w} + \frac{1}{2}(1-a^2)(s-t)} \tag{2.13}$$

for every $w \in V$.

Our main result shows that $S_{s,t}$ is the unique, up to an action of $U(1)$, unitary isomorphism $L^2(V, P_s) \to \mathcal{H}L^2(V_c, M_{s,t})$ having certain properties.

Theorem 2.3. *Suppose V is a finite-dimensional real inner-product space of positive dimension.*

The transform $S_{s,t}^V : L^2(V, P_s) \to \mathcal{H}L^2(V_c, M_{s,t})$ has the following properties:

1. Unitarity : *$S^V_{s,t}$ is a unitary isomorphism of $L^2(V, P_s)$ onto $\mathcal{H}L^2(V_c, M_{s,t})$*
2. Naturality : *if n is any positive integer $\leq \dim V$, and $u_1, ..., u_n$ orthonormal in V, the image of the function $F_f = f(\hat{u}_1, ..., \hat{u}_n)$ under $S^V_{s,t}$ is the function $(S^{\mathbf{R}^n} f)(\tilde{u}_1, ..., \tilde{u}_n)$*
3. Factorizability: *for any orthonormal vectors $u_1, ..., u_n, u_{n+1}, ..., u_{n+m} \in V$ and any functions $f \in L^2(\mathbf{R}^n, P_s)$ and $g \in L^2(\mathbf{R}^m, P_s)$, the image of the product function $f(\hat{u}_1, ..., \hat{u}_n) g(\hat{u}_{n+1}, ..., \hat{u}_{n+m})$ under $S^V_{s,t}$ is the product function $(S^{\mathbf{R}^n}_{s,t} f)(\tilde{u}_1, ..., \tilde{u}_n)(S^{\mathbf{R}^n}_{s,t} g)(\tilde{u}_{n+1}, ..., \tilde{u}_{n+m})$.*

Conversely, assume that $\dim V \geq 2$*, and suppose $\tilde{S}_{s,t} : L^2(V, P_s) \to \mathcal{H}L^2(V_c, M_{s,t})$ is a mapping satisfying the following properties:*

1. *$\tilde{S}_{s,t}$ is a unitary isomorphism of $L^2(V, P_s)$ onto $\mathcal{H}L^2(V_c, M_{s,t})$*
2. *for each $f \in L^2(\mathbf{R}, P_s)$ there is a function $\tilde{f}$ on $\mathbf{C}$ such that for any unit vector $u \in V$, $\tilde{S}_{s,t}[f(\hat{u})] = \tilde{f}(\tilde{u})$*
3. *for any orthonormal vectors $u_1, u_2 \in V$ and any functions $f, g \in L^2(\mathbf{R}, P_s)$, the operator $\tilde{S}_{s,t}$ carries the function $f(\hat{u}_1) g(\hat{u}_2)$ to a function of the form $\tilde{f}(\tilde{u}_1)\tilde{g}(\tilde{u}_2)$, where $\tilde{f}$ and $\tilde{g}$ are as in Condition 2.*

Then there is a unique $a \in U(1)$ such that:

$$\tilde{S}_{s,t} = \mathcal{W}^V_a \circ S^V_{s,t} = S^V_{s,t} \circ \mathcal{U}^V_a \tag{2.14}$$

The fact that $S_{s,t}$ is a unitary isomorphism was first proved by Hall in [5].

Perhaps, in the preceding results, the first two hypotheses imply the third.

We shall also prove that the transform $S_{s,t}$ is a meaningful unitary isomorphism even if V is infinite-dimensional.

In the last section we shall give a reformulation of several of our results in a category-theoretic framework.

Two kernel functions R and K, play an important role in proving unitarity of $S_{s,t}$. These functions are defined as follows. For $x \in V$ and $w \in V_c$ we set

(2.15)

$$\boxed{R(x, w) = (1-\rho)^{-\dim V/2} \exp\left[-\tfrac{1}{2t}\{\rho x^2 + w^2 - 2xw\}\right] = \tfrac{\rho_t(x-w)}{\rho_s(x)}}$$

and, for $z, w \in V_c$ we set

(2.16)

$$K(z, w) = (1-\rho^2)^{-\dim V/2} \exp\left[-\frac{\rho}{2(1-\rho^2)s}(z^2 + w^2) + \frac{1}{(1-\rho^2)s} zw\right]$$

which can also be expressed as

$$K(z,w) = \int_V \frac{\rho_t(z-x)\rho_t(w-x)}{\rho_s(x)}\,dx \tag{2.17}$$

3. Proofs I

Let V be a finite-dimensional real vector space of positive dimension, equipped with an inner-product. The inner-product of $v, w \in V$ will be denoted vw. This induces, naturally, a complex bilinear form $(w,z) \mapsto wz$ on the complexification V_c of V. We write a generic element of V_c as $z = x + iy$, where $x, y \in V$; then the conjugate $\bar{z}$ is defined to be $x - iy$.

Let $C(\cdot,\cdot)$ be the holomorphic quadratic polynomial on V_c given by

$$C(z,w) = A(z^2 + w^2) + 2Bzw \tag{3.1}$$

where

$$A = -\frac{\rho}{2(1-\rho^2)s} \text{ and } B = \frac{1}{2(1-\rho^2)s} \tag{3.2}$$

where r, s, t, ρ are as in equations (2.1) and (2.2).

We begin with two observations which may be proven by simple algebra. Recall the functions R and K defined in equations (2.15) and (2.16).

Lemma 3.1. *For every $z, w \in V_c$,*

$$C(z,w) = \frac{1}{4r}(z+w)^2 - \frac{1}{4t}(z-w)^2 \tag{3.3}$$

and

$$C(z,\bar{z}) = \frac{x^2}{r} + \frac{y^2}{t} \tag{3.4}$$

where $z = x + iy$, with $x, y \in V$. Thus

$$K(z,w) = (1-\rho^2)^{-\dim V/2} e^{C(z,w)}, \tag{3.5}$$

(3.6)

$$K(z,\bar{z}) = (1-\rho^2)^{-\dim V/2} \exp\left[\frac{x^2}{r} + \frac{y^2}{t}\right] = (1-\rho^2)^{-\dim V/2} e^{C(z,\bar{z})}$$

and

$$dM_{s,t}(z) = [\pi s(1-\rho^2)^{1/2}]^{-\dim V} e^{-C(z,\bar{z})}\,dz \tag{3.7}$$

The following simple algebraic observation will be useful in proving that the integral operator with kernel $K(z,w)$ yields the "restriction functional" on holomorphic functions.

Lemma 3.2. *For $z, w \in V_c$,*

$$C(z,w) + C(\bar z, \bar w) = C(z,\bar z) + C(w,\bar w) - 2B|w-\bar z|^2 \tag{3.8}$$

and

$$C(z,\bar w + \bar z) - C(w+z,\bar w+\bar z) = C(z,\bar z) - C(\bar z, w+z) - 2B|w|^2 \tag{3.9}$$

Equation (3.9) follows by writing $\bar w + \bar z$ in place of w in (3.8).

Next we check that $K(z,\cdot)$ is square-integrable with respect to the measure $M_{s,t}$.

Lemma 3.3. *For every $z \in V_c$, the function $K(z,\cdot)$ belongs to $L^2(V_c, M_{s,t})$.*

Proof. From the expressions for $K(z,w)$ and $M_{s,t}(w)$ given in (3.6) and (3.7) we see that the integral $\int_{V_c} |K(z,w)|^2\, dM_{s,t}(w)$ is a finite constant times

$$\int_{V_c} e^{C(z,w)+C(\bar z,\bar w)-C(w,\bar w)}\, dw \tag{3.10}$$

where dw is Lebesgue integration over V_c. By Equation (3.8), the integrand is constant times $e^{-2B|w-\bar z|^2}$, and so, since $B>0$, this integral is convergent. (The exact value of the L^2–norm of $K(z,\cdot)$ can be determined easily this way, but we shall see what its value is in a more elegant way below.) $\square$

Now we can prove the fact that $K(z,w)$ gives the integral kernel for the evaluation map $f \mapsto f(z)$ for holomorphic functions f in $L^2(M_{s,t})$.

Proposition 3.4. *For any $f \in \mathcal{H}L^2(V_c, M_{s,t})$, and any $z \in V_c$,*

$$\int_{V_c} K(z,\bar w) f(w)\, dM_{s,t}(w) = f(z) \tag{3.11}$$

Proof. Since $K(z,\cdot)$ and f are both in $L^2(M_{s,t})$, the function $K(z,\cdot)f(\cdot)$ is $M_{s,t}$–integrable. Then, using the expressions for $K(z,w)$ and $M_{s,t}(w)$ given in (3.6) and (3.7), we have

$$\int_{V_c} K(z,\bar w) f(w)\, dM_{s,t}(w) = \left[(1-\rho^2)^{1/2}\pi s(1-\rho^2)^{1/2}\right]^{-\dim V} \int_{V_c} f(w) e^{C(z,\bar w)-C(w,\bar w)}\, dw \tag{3.12}$$

The constant outside the integral on the right is readily seen to be equal to $\left(\frac{\pi}{2B}\right)^{-\dim V}$. Now we replace w in the integration by $w+z$, and using Equation (3.9), we have

$$\int_{V_c} K(z,\bar w) f(w)\, dM_{s,t}(w) = (2B/\pi)^{\dim V} \int_{V_c} f(w+z) e^{C(z,\bar z)-C(\bar z,w+z)-2B|w|^2}\, dw \tag{3.13}$$

Note that $e^{-2B|\lambda|^2|}\frac{2B}{\pi}d\lambda$ is a Gaussian measure on $\mathbf{C}$ with total mass one, rotationally invariant about the origin. Thus the right side of the above equation is the (finite) mean of a holomorphic function with respect to a probability measure which, in each component (with respect to an orthonormal basis in V), is rotationally symmetric about the origin. Thus the value of the right side of (3.13) equals the value of the integrand at $w = 0$, i.e. it equals $f(z)$. $\square$

The Proposition has the following consequence:

Corollary 3.5. *For any $w_1, w_2 \in V_c$,*

$$\langle K(\cdot, w_1), K(\cdot, w_2)\rangle_{L^2(M_{s,t})} = K(w_1, \bar{w}_2) \tag{3.14}$$

For any $z = x + iy \in V_c$, with $x, y \in V$,

$$\|K(z, \cdot)\|^2_{L^2(M_{s,t})} = K(z, \bar{z}) = (1 - \rho^2)^{-\dim V/2} e^{\frac{x^2}{r} + \frac{y^2}{t}} \tag{3.15}$$

The norm of the functional $L^2(M_{s,t}) \to \mathbf{C} : f \mapsto K_z f$, where

$$K_z f \stackrel{\text{def}}{=} \int_{V_c} K(z, \bar{w}) f(w)\, dM_{s,t}(w),$$

is $K(z, \bar{z})^{1/2}$ and the inequality

$$|K_z f| \le K(z, \bar{z})^{1/2} |f|_{L^2(M_{s,t})} \tag{3.16}$$

is an equality (even without the absolute values on the left) when f is the holomorphic function $K(\cdot, \bar{z})$.

Proof. The first equality follows from the Proposition by taking $f = K(\cdot, w_2)$ and using the symmetry property $K(z, w) = K(w, z)$. The second follows from the first and Equation (3.6). The results concerning K_z now follow directly. $\square$

We then have the result:

Proposition 3.6. *The set $\mathcal{H}L^2(M_{s,t})$ is a closed subspace of $L^2(V_c, M_{s,t})$.*

Proof. Let $f \in \mathcal{H}L^2(M_{s,t})$. Then $K_z f = f(z)$. So, by (3.15) and (3.16), we have

$$|f(z)| \le (1 - \rho^2)^{-\dim V/4} e^{\left(\frac{x^2}{r} + \frac{y^2}{t}\right)/2} |f|_{L^2(M_{s,t})} \tag{3.17}$$

where $z = x + iy$, with x and y in V. This shows that if f_n is Cauchy in $\mathcal{H}L^2(M_{s,t})$ then $f_n(z)$ is uniformly Cauchy for z running over any compact subset of V_c, and so the L^2–limit of f_n is a holomorphic function. $\square$

We also have :

Proposition 3.7. *The functions $K(z, \cdot)$, as z runs over V_c, span a dense subspace of $\mathcal{H}L^2(M_{s,t})$.*

Proof. Any function in $\mathcal{H}L^2(M_{s,t})$ orthogonal to the function $K(z,\cdot)$ must, by Equation (3.11), be zero at $\bar{z}$; hence if $f \in \mathcal{H}L^2(M_{s,t})$ orthogonal to all the functions $K(z,\cdot)$, with z running over V_c, then $f = 0$. $\square$

We turn now to studying the kernel $R(x,w)$.

Lemma 3.8. *For every $w \in V_c$, the function $R(\cdot,w)$ belongs to $L^2(V,P_s)$. If $w_1, w_2, \in V_c$, then*

$$\langle R(\cdot,w_1), R(\cdot,w_2)\rangle_{L^2(P_s)} = K(w_1,\bar{w}_2) \tag{3.18}$$

Proof. Recall that $\rho_r(z) = (2\pi r)^{-\dim V/2} e^{-z^2/(2r)}$, for $z \in V_c$ and $r > 0$. Examining

$$\frac{\rho_t(x-w_1)}{\rho_s(x)} \frac{\rho_t(x-w_2)}{\rho_s(x)} \rho_s(x)$$

we see, by completing the squares, that it is equal to $K(w_1,w_2)$ times the normalized Gaussian factor

$$(2\pi ts/r)^{-\dim V/2} \exp\left[-\frac{r}{2ts}\left\{x - \frac{s}{r}(w_1+w_2)\right\}^2\right]$$

Thus

$$\int_V R(x,w_1)R(x,w_2)\, dP_s(x) = K(w_1,w_2) \tag{3.19}$$

In particular, $R(\cdot,w) \in L^2(P_s)$. $\square$

Next we prove that $S^V_{s,t}$ takes the function $R(\cdot,w)$ to $K(\cdot,w)$:

Proposition 3.9. *For every $w \in V_c$*

$$[S^V_{s,t} R(\cdot,w)](z) = K(z,w) \tag{3.20}$$

Proof. Since $R(x,w) = \rho_t(x-w)/\rho_s(x)$, we have

$$\begin{aligned} \big(S_{s,t}R(\cdot,w)\big)(z) &= \int_V \frac{\rho_t(x-w)}{\rho_s(x)} \rho_t(x-z)\, dx \\ &= \int_V R(x,w)R(x,z)\, dP_s(x) \\ &= K(z,w) \end{aligned}$$

$\square$

Before proving unitarity of $S_{s,t}$ we need one last observation:

Lemma 3.10. *The linear span of the set $\{R(\cdot,w) : w \in V\}$ is dense in $L^2(P_s)$.*

Proof. Consider a function $f \in L^2(P_s)$ which is orthogonal to each function $R(\cdot, w)$, as w runs over $V \subset V_c$. The form of the function $R(\cdot, w)$ then shows that this means, for all $k \in V$,

$$\int e^{kx} e^{-\frac{\rho}{2t}x^2} f(x) e^{-\frac{x^2}{2s}}\, dx = 0$$

It can be verified by Cauchy-Schwarz that the function $f(x)e^{-\frac{x^2\rho}{2t}}$ is in $L^2(V, dP_s(x))$. Thus the function $f(x)e^{-\frac{x^2\rho}{2t}}$ is orthogonal to the subspace spanned by the functions e^{kx}, as k runs over V, in $L^2(dP_s(x))$. The latter subspace is known to be dense in $L^2(V, P_s)$, and so $f(x)e^{-\frac{x^2\rho}{2t}} = 0$, and so $f = 0$. Thus $\{R(\cdot, w) : w \in V\}$ spans a dense linear subspace of $L^2(P_s)$. □

Finally, we can prove that the two-parameter Segal-Bargmann transform is a unitary isomorphism:

Theorem 3.11. *For every $f \in L^2(P_s)$, the function $S_{s,t}f$ is in $\mathcal{H}L^2(V_c, M_{s,t})$, and the operator $S_{s,t}$ is a unitary isomorphism*

$$S_{s,t} : L^2(V, P_s) \to \mathcal{H}L^2(V_c, M_{s,t}) \tag{3.21}$$

Proof. We know that: (i) $S_{s,t}$ carries each $R(\cdot, w)$ to $K(\cdot, w)$, (ii) the $L^2(P_s)$–inner-product of $R(\cdot, w_1)$ and $R(\cdot, w_2)$ is equal to the $L^2(M_{s,t})$–inner-product of $K(\cdot, w_1)$ and $K(\cdot, w_2)$, and (iii) the functions $R(\cdot, w)$ span a dense subspace of $L^2(P_s)$ and the functions $K(\cdot, w)$ span a dense subspace of $\mathcal{H}L^2(M_{s,t})$. Therefore, there is a unique unitary isomorphism $S^1_{s,t} : L^2(V, P_s) \to \mathcal{H}L^2(V_c, M_{s,t})$ which agrees with $S_{s,t}$ on the linear span of the functions $R(\cdot, w)$, as w runs over V. On the other hand, for each fixed $z \in V_c$, the linear functional $A_z : L^2(P_s) \to \mathbf{C} : f \mapsto S_{s,t}f(z)$ is bounded (see Equation (2.7)) and the functional $B_z : L^2(P_s) \to \mathbf{C} : f \mapsto (S^1_{s,t}f)(z)$, being the composite of the unitary map $S^1_{s,t}$ and the bounded evaluation functional $F \mapsto K_zF = F(z)$, is also bounded. Since A_z and B_z agree on the dense subspace spanned by the functions $R(\cdot, w)$, with w running over V, it follows that they agree everywhere. Thus $S_{s,t}$ is, in fact, the same as $S^1_{s,t}$, and so, in particular, is a unitary isomorphism. □

Since we will encounter numerous Gaussian integrals, we record here a simple observation which is very useful in computing such integrals. If X is a real Gaussian random variable and $\lambda \in \mathbf{C}$ then

$$E\left(e^{\lambda X}\right) = e^{\lambda E(X) + \frac{\lambda^2}{2}\mathrm{var}(X)} \tag{3.22}$$

where $E(\cdot)$ is the expectation and $\mathrm{var}(\cdot)$ the variance.

We know that for each $x \in V$ there is $\hat{x} \in L^2(V, P_s)$. So for $z = x+iy$, with $x, y \in V$, we have the function

$$\hat{z} \stackrel{\text{def}}{=} \hat{x} + i\hat{y}$$

which is also an element of $L^2(V, P_s)$.

The following observation will be useful:

Proposition 3.12. *For $x \in V$,*

$$S_{s,t}e^{\hat{x}} = e^{\tilde{x}+\frac{1}{2}tx^2} \tag{3.23}$$

More generally, for $z \in V_c$,

$$S_{s,t}e^{\hat{z}} = e^{\tilde{z}+\frac{1}{2}tz^2} \tag{3.24}$$

The set $\{e^{\hat{x}} : x \in V\}$ spans a dense linear subspace of $L^2(V, P_s)$, and the set $\{e^{\tilde{x}} : x \in V\}$ spans a dense linear subspace of $\mathcal{H}L^2(V_c, M_{s,t})$.

Proof. The formula (3.24) is obtained from the definition of $S_{s,t}e^{\hat{z}}$ using the integration formula (3.22). The fact that $\{e^{\hat{x}} : x \in V\}$ spans a dense linear subspace of $L^2(V, P_s)$ is well known. Applying the unitary isomorphism $S_{s,t}$, which maps *onto* $\mathcal{H}L^2(V_c, M_{s,t})$, and using equation (3.23), we see that $\{e^{\tilde{x}} : x \in V\}$ spans a dense linear subspace of $\mathcal{H}L^2(V_c, M_{s,t})$. □

We list some fundamental properties of $S_{s,t}$ which are verified directly from the definition of $S_{s,t}$, using simply properties of the Gaussian functions.

Proposition 3.13. *Suppose V is a finite-dimensional real inner-product space of positive dimension.*

The mapping $S^V_{s,t} : L^2(V, P_s) \to \mathcal{H}L^2(V_c, M_{s,t})$ has the following properties:

1. <u>Unitarity</u> *: $S^V_{s,t}$ is a unitary isomorphism of $L^2(V, P_s)$ onto $\mathcal{H}L^2(V_c, M_{s,t})$*
2. <u>Naturality</u> *: if n is any positive integer $\leq \dim V$, $f \in L^2(\mathbf{R}^n, P_s)$ and $u_1, ..., u_n$ orthonormal in V, the image of the function $F_f = f(\hat{u}_1, ..., \hat{u}_n)$ under $S^V_{s,t}$ is the function $(S^{\mathbf{R}^n}_{s,t} f)(\tilde{u}_1, ..., \tilde{u}_n)$*
3. <u>Factorizability</u>*: for any orthonormal vectors $u_1, ..., u_n, u_{n+1}, ..., u_{n+m} \in V$ and any functions $f \in L^2(\mathbf{R}^n, P_s)$ and $g \in L^2(\mathbf{R}^m, P_s)$, the image of the product function $f(\hat{u}_1, ..., \hat{u}_n)g(\hat{u}_{n+1}, ..., \hat{u}_{n+m})$ under $S^V_{s,t}$ is the product function $(S^{\mathbf{R}^n}_{s,t} f)(\tilde{u}_1, ..., \tilde{u}_n)(S^{\mathbf{R}^m}_{s,t} g)(\tilde{u}_{n+1}, ..., \tilde{u}_{n+m})$.*

4. Proofs II

We turn now to proving the results which show that $S_{s,t}$ is unique up to a factor depending on a unit modulus complex number. In this section the vector spaces V and W have positive but finite dimension.

Let $\mathrm{Aut}(\mathcal{H}L^2(M_{s,t}))_V$ be the set of maps $W : \mathcal{H}L^2(V_c, M_{s,t}) \to \mathcal{H}L^2(V_c, M_{s,t})$ satisfying:

1. Unitarity: W is a unitary isomorphism
2. Functoriality: if n is any positive integer $\leq \dim V$, there is a unitary isomorphism $\mathcal{H}L^2(\mathbf{C}^n, M_{s,t}) \to \mathcal{H}L^2(\mathbf{C}^n, M_{s,t}) : f \mapsto \tilde{f}$ such that for any $u_1, ..., u_n$ orthonormal in V, the image of the function $F_f = f(\tilde{u}_1, ..., \tilde{u}_n)$ under W is the function $F_{\tilde{f}} = \tilde{f}(\tilde{u}_1, ..., \tilde{u}_n)$
3. Factorizability: if $u_1, ..., u_{n+m}$ (with n, m being any positive integers) are orthonormal vectors in V and $f \in \mathcal{H}L^2(\mathbf{C}^n, M_{s,t})$, $g \in \mathcal{H}L^2(\mathbf{C}^m, M_{s,t})$, and $F_{f,g} \stackrel{\text{def}}{=} f(\tilde{u}_1, ..., \tilde{u}_n)g(\tilde{u}_{n+1}, ..., \tilde{u}_{n+m})$ then $WF_{f,g}$ is of the form $F_{\tilde{f},\tilde{g}} = \tilde{f}(\tilde{u}_1, ..., \tilde{u}_n)\tilde{g}(\tilde{u}_{n+1}, ..., \tilde{u}_{n+m})$, where $\tilde{f}$ and $\tilde{g}$ are as given in Condition 2.

It is clear that $\mathrm{Aut}(\mathcal{H}L^2(M_{s,t}))_V$ is a group under composition.

Let $U(1) = \{\lambda \in \mathbf{C} : |\lambda| = 1\}$ be the group of unit modulus complex numbers.

Proposition 4.1. *For each complex number $a \in U(1)$ there is a unique unitary isomorphism*

$$\mathcal{W}_a^V : \mathcal{H}L^2(V_c, M_{s,t}) \to \mathcal{H}L^2(V_c, M_{s,t}) \tag{4.1}$$

such that for any $z \in V$

$$\mathcal{W}_a^V(e^{\tilde{z}}) = e^{a\tilde{z}+bz^2} \tag{4.2}$$

where

$$b = \frac{1}{2}(1-a^2)(s-t) \tag{4.3}$$

The formula (4.2) is also valid for all $z \in V_c$.

Each $\mathcal{W}_a^V$ is in $Aut(\mathcal{H}L^2(M_{s,t}))_V$ and the map $\mathcal{W}^V : U(1) \to Aut(\mathcal{H}L^2(M_{s,t}))_V : a \mapsto \mathcal{W}_a^V$ is an injective group homomorphism.

Proof. Unless there is possibility for confusion, we will simply write $\mathcal{W}_a$ to mean $\mathcal{W}_a^V$. We know that the unitary transformation $S_{s,t}^V$ carries the function $e^{\hat{x}} \in L^2(V, P_s)$ to $e^{\tilde{x}+\frac{1}{2}tx^2}$, for every $x \in V$, and, further, that the linear span of $\{e^{\tilde{x}} : x \in V\}$ spans a dense subspace of $\mathcal{H}L^2(V_c, M_{s,t})$. So, to prove that $\mathcal{W}_a^V$ is a unitary isomorphism, it will suffice to prove that $\langle \mathcal{W}_a(e^{\tilde{w}_1}), \mathcal{W}_a(e^{\tilde{w}_2})\rangle_{\mathcal{H}L^2(M_{s,t})}$ equals $\langle e^{\tilde{w}_1}, e^{\tilde{w}_2}\rangle_{\mathcal{H}L^2(M_{s,t})}$, for all $w_1, w_2 \in V_c$. Now, using the Gaussian integration formula (3.22), we have

$$\begin{aligned}\langle e^{\tilde{w}_1}, e^{\tilde{w}_2}\rangle_{\mathcal{H}L^2(M_{s,t})} &= \int_{V^2} e^{(w_1+\overline{w}_2)x+i(w_1-\overline{w}_2)y-\frac{x^2}{r}-\frac{y^2}{t}} \frac{dxdy}{(\sqrt{\pi r}\sqrt{\pi t})^{\dim V}} \\ &\stackrel{(3.22)}{=} e^{\frac{1}{2}(w_1+\overline{w}_2)^2\frac{r}{2}-\frac{1}{2}(w_1-\overline{w}_2)^2\frac{t}{2}} \end{aligned} \tag{4.4}$$

On the other hand,

$$\begin{aligned} \langle \mathcal{W}_a(e^{\tilde{w}_1}), \mathcal{W}_a(e^{\tilde{w}_2}) \rangle &= \langle e^{a\tilde{w}_1+bw_1^2}, e^{a\tilde{w}_2+bw_2^2} \rangle_{\mathcal{H}L^2(M_{s,t})} \\ &= e^{w_1^2 b+\overline{w}_2^2\overline{b}+\frac{1}{2}(aw_1+\overline{aw}_2)^2\frac{r}{2}-\frac{1}{2}(aw_1-\overline{aw}_2)^2\frac{t}{2}} \end{aligned} \tag{4.5}$$

To see that the exponents in (4.4) and (4.5) are equal, observe first that the difference $(w_1 \pm \overline{w}_2)^2 - (aw_1 \pm \overline{aw}_2)^2$ equals $(1-a^2)w_1^2 + (1-\overline{a}^2)\overline{w}_2^2$, because $|a| = 1$. So the difference in the exponents is:

$$(1-a^2)\frac{(s-t)}{2}w_1^2 + (1-\overline{a}^2)\frac{(s-t)}{2}\overline{w}_2^2 - bw_1^2 - \overline{b}\overline{w}_2^2$$

and this is, by definition of b, equal to 0 [of course, this is why b has been defined by (4.3)].

Thus, by the usual density argument, $\mathcal{W}_a$ extends uniquely to a unitary isomorphism of $\mathcal{H}L^2(V_c, M_{s,t})$ onto itself, and is uniquely specified by (4.2) for $z \in V$.

Let $u_1, ..., u_n$ be orthonormal in V. Then the relation (4.2) shows that

$$\mathcal{W}_a^V(f(\tilde{u}_1, ..., \tilde{u}_n)) = (\mathcal{W}_a^{\mathbf{R}^n} f)(\tilde{u}_1, ..., \tilde{u}_n)$$

holds if f is any holomorphic exponential function (i.e. a function on $\mathbf{C}^n$ of the form $e^{\tilde{\lambda}}$ for any $\lambda \in \mathbf{C}^n$) and so, by the density argument, for all $f \in L^2(\mathbf{R}^n, P_s)$.

If w_1, w_2 are orthogonal in V then it is clear that

$$\mathcal{W}_a^V(e^{\tilde{w}_1}e^{\tilde{w}_2}) = \mathcal{W}_a^V(e^{\tilde{w}_1})\mathcal{W}_a^V(e^{\tilde{w}_2}) \tag{4.6}$$

Let $u_1, ..., u_{n+m}$ be orthonormal in V, $f \in \mathcal{H}L^2(\mathbf{C}^n, M_{s,t})$ and $g \in \mathcal{H}L^2(\mathbf{C}^m, M_{s,t})$. By the density of holomorphic exponentials (Proposition 3.12), f is the $L^2(\mathbf{C}^n, M_{s,t})$–limit $\lim_{k\to\infty} f_k$, where each f_k is a linear combination of functions of the form $e^{\tilde{\lambda}}$, with $\lambda \in \mathbf{R}^n$. So $f(\tilde{u}_1, ..., \tilde{u}_n)$ is the $\mathcal{H}L^2(V_c, M_{s,t})$–limit $\lim_{k\to\infty} f_k(\tilde{u}_1, ..., \tilde{u}_n)$, where each $f_k(\tilde{u}_1, ..., \tilde{u}_n)$ is a linear combination of functions of the form $e^{\tilde{w}_1}$, with $w_1 \in V$ a linear combination of the vectors $\tilde{u}_1, ..., \tilde{u}_n$. Similarly, $g(\tilde{u}_{n+1}, ..., \tilde{u}_{n+m})$ is the $\mathcal{H}L^2(V_c, M_{s,t})$–limit of of functions of the form $e^{\tilde{w}_2}$, with $w_2 \in V$ a linear combination of the vectors $\tilde{u}_{n+1}, ..., \tilde{u}_{n+m}$. Moreover, orthogonality implies that $\{\tilde{u}_1, ..., \tilde{u}_n\}$ and $\{\tilde{u}_{n+1}, ..., \tilde{u}_{n+m}\}$ are mutually independent under $M_{s,t}$. Using this, it follows that the product $f(\tilde{u}_1, ..., \tilde{u}_n)g(\tilde{u}_{n+1}, ..., \tilde{u}_{n+m})$ is the $\mathcal{H}L^2(V_c, M_{s,t})$–limit of the corresponding linear combination of products $e^{\tilde{w}_1}e^{\tilde{w}_2}$. Then, by the property (4.6), and continuity of $\mathcal{W}_a$, we conclude that $\mathcal{W}_a^V$ has the Factorizability property. Thus:

$$\mathcal{W}_a^V \in \mathrm{Aut}(\mathcal{H}L^2(M_{s,t}))_V$$

Next, for any $w \in V$,

$$\mathcal{W}_{a_1}(\mathcal{W}_{a_2}e^{\tilde{w}}) = \mathcal{W}_{a_1}(e^{a_2\tilde{w}+b_2w^2}) = e^{b_2w^2}\mathcal{W}_{a_1}(e^{a_2\tilde{w}}) = e^{a_1a_2\tilde{w}+(b_1a_2^2+b_2)w^2}$$

and so, since $b_1a_2^2+b_2$ is readily verified to be equal to $\frac{1}{2}(1-a_1^2a_2^2)(s-t)$, which is the b corresponding to a_1a_2, $\mathcal{W}_{a_1}(\mathcal{W}_{a_2}e^{\tilde{w}})$ equals $\mathcal{W}_{a_1a_2}(e^{\tilde{w}})$. So, by the usual density arguments, $\mathcal{W}_{a_1a_2} = \mathcal{W}_{a_1}\mathcal{W}_{a_2}$. □

Using the isomorphism $S_{s,t} : L^2(V, P_s) \to \mathcal{H}L^2(V_c, M_{s,t})$, which satisfies unitarity, naturality, and factorizability, we obtain the following result:

Proposition 4.2. *There is an injective homomorphism of groups*

$$\mathcal{U}^V : U(1) \to Aut(L^2(P_s))_V : a \mapsto \mathcal{U}_a^V$$

such that for each $a \in U(1)$ and $x \in V$:

$$\mathcal{U}_a^V(e^{\hat{x}}) = e^{a\hat{x}+\frac{1}{2}(1-a^2)sx^2} \tag{4.7}$$

The map $\mathcal{U}_a^V : L^2(V, P_s) \to L^2(V, P_s)$ is uniquely specified by the requirement that it is a bounded linear map and that it satisfies equation (4.7) for every $x \in V$.

Proof. A straightforward calculation verifies that $S_{s,t}^{-1}\mathcal{W}_a^V S_{s,t}(e^{\hat{x}})$ is given by (4.7), and so we get the result by defining $\mathcal{U}_a^V$ to be $S_{s,t}^{-1}\mathcal{W}_a^V S_{s,t}$. Since the exponential functions $e^{\hat{x}}$ span a dense subspace of $L^2(V, P_s)$, equation (4.7) and the requirement that $\mathcal{U}_a^V$ is a bounded linear map specify $\mathcal{U}_a^V$ uniquely. □

The relationship between $\mathcal{W}$, $\mathcal{U}$, and $S_{s,t}$ is worth recording separately:

Proposition 4.3. *For every $a \in U(1)$,*

$$S_{s,t}^{-1}\mathcal{W}_a S_{s,t} = \mathcal{U}_a \tag{4.8}$$

Finally we turn to our main result.

Theorem 4.4. *Let V be a real inner-product space of finite dimension $\dim V \geq 2$. Suppose that $\tilde{S}_{s,t} : L^2(P_s) \to \mathcal{H}L^2(M_{s,t})$ satisfies the following conditions:*

1. *$\tilde{S}_{s,t}$ is a unitary isomorphism of $L^2(V, P_s)$ onto $\mathcal{H}L^2(V_c, M_{s,t})$*
2. *for each $f \in L^2(\mathbf{R}, P_s)$ there is a function $\tilde{f}$ on $\mathbf{C}$ such that for any unit vector $u \in V$, $\tilde{S}_{s,t}f(\hat{u}) = \tilde{f}(\tilde{u})$*
3. *for any orthonormal vectors $u_1, u_2 \in V$ and any functions $f, g \in L^2(\mathbf{R}, P_s)$, the operator $\tilde{S}_{s,t}$ carries the function $f(\hat{u}_1)g(\hat{u}_2)$ to the function $\tilde{f}(\tilde{u}_1)\tilde{g}(\tilde{u}_2)$, where $\tilde{f}$ and $\tilde{g}$ are as in Condition 2.*

Then there is a unique $a \in U(1)$ such that:

$$\tilde{S}_{s,t} = \mathcal{W}_a \circ S_{s,t} = S_{s,t} \circ \mathcal{U}_a \tag{4.9}$$

Proof. The function $\tilde{f}$ provided by Condition 2 is holomorphic and the mapping $f \mapsto \tilde{f}$ a linear unitary map $L^2(\mathbf{R}, P_s) \to \mathcal{H}L^2(\mathbf{C}, M_{s,t})$.

For $k > 0$, let E_k be the function in $L^2(\mathbf{R}, P_s)$ given by $E_k(x) = e^{kx}$, and let $A_k = \tilde{E}_k$ be the function on $\mathbf{C}$ provided by Condition 2. Then

$$A_k(zu) = \big(\tilde{S}_{s,t} e^{k\hat{u}}\big)(z)$$

for every $z \in V_c$ and every unit vector $u \in V$.

Since $k \mapsto e^{k\hat{u}} \in L^2(V, P_s)$ is continuous and $\tilde{S}_{s,t}$ unitary, and since the evaluation map $\mathcal{H}L^2(V_c, M_{s,t}) \to \mathbf{C} : f \mapsto f(z)$ is continuous, it follows that $\big(\tilde{S}_{s,t} e^{k\hat{u}}\big)(z)$ is continuous in k, for each fixed z. Moreover, $\big(\tilde{S}_{s,t} e^{k\hat{u}}\big)(z)$ is holomorphic in z for each fixed k. Thus $A_k(\lambda)$ is continuous in k (for fixed λ) and holomorphic in $\lambda \in \mathbf{C}$ for fixed k.

Let $u_1, u_2 \in V$ be orthogonal unit vectors and k_1, k_2 positive real numbers. Then

$$e^{k_1 \hat{u}_1} e^{k_2 \hat{u}_2} = e^{\sqrt{k_1^2+k_2^2}\left\langle \cdot, \frac{k_1 u_1 + k_2 u_2}{\sqrt{k_1^2+k_2^2}} \right\rangle} \tag{4.10}$$

Applying $\tilde{S}_{s,t}$ to (4.10), we have for every $z \in V_c$:

$$A_{k_1}(zu_1) A_{k_2}(zu_2) = A_{\sqrt{k_1^2+k_2^2}}\left(z \cdot \left(\frac{k_1 u_1 + k_2 u_2}{\sqrt{k_1^2+k_2^2}} \right) \right) \tag{4.11}$$

Now consider any $\lambda_1, \lambda_2 \in \mathbf{C}$. Let

$$z = \lambda_1 u_1 + \lambda_2 u_2$$

Then (4.11) gives:

$$A_{k_1}(\lambda_1) A_{k_2}(\lambda_2) = A_{\sqrt{k_1^2+k_2^2}}\left(\frac{k_1}{\sqrt{k_1^2+k_2^2}} \lambda_1 + \frac{k_2}{\sqrt{k_1^2+k_2^2}} \lambda_2 \right) \tag{4.12}$$

for all $\lambda \in \mathbf{C}$ and all $k > 0$.

It is possible the show that the solution of the functional equation (4.12) is

$$A_k(\lambda) = e^{ka\lambda + k^2 c} \tag{4.13}$$

where a and c are complex constants. As already noted, $f \mapsto \tilde{f}$ is unitary. So, since $A_k = \tilde{E}_k$:

$$\begin{aligned}
\langle E_{k_1}, E_{k_2}\rangle_{L^2(\mathbf{R},P_s)} &= \int_{\mathbf{C}} A_{k_1}(\lambda)\overline{A_{k_2}(\lambda)}\, dM_{s,t}(\lambda) \\
&= \int_{\mathbf{R}^2} e^{(k_1a+k_2\overline{a})x+i(k_1a-k_2\overline{a})y-\frac{x^2}{r}-\frac{y^2}{t}+k_1^2c+k_2^2\overline{c}}\,\frac{dxdy}{\sqrt{\pi r}\sqrt{\pi t}} \\
&\overset{(3.22)}{=} e^{\frac{1}{2}(k_1a+k_2\overline{a})^2\frac{r}{2}-\frac{1}{2}(k_1a-k_2\overline{a})^2\frac{t}{2}+k_1^2c+k_2^2\overline{c}}
\end{aligned}$$

Now

$$\langle E_{k_1}, E_{k_2}\rangle_{L^2(\mathbf{R},P_s)} = \int_{\mathbf{R}} e^{k_1x+k_2x}\, dP_s(x) = e^{(k_1+k_2)^2\frac{s}{2}} \tag{4.14}$$

So we must have:

$$\frac{(k_1+k_2)^2s}{2} = k_1^2\big[\frac{a^2}{4}(r-t)+c\big] + k_2^2\big[\frac{\overline{a}^2}{4}(r-t)+\overline{c}\big] + \frac{k_1k_2}{2}(r+t)|a|^2$$

Since this must hold for all $k_1, k_2 > 0$, it follows that

$$\frac{s}{2} = \frac{a^2}{4}(r-t)+c = \frac{a^2}{2}(s-t)+c$$

and

$$s = \frac{1}{2}(r+t)|a|^2 = s|a|^2$$

So $|a| = 1$, and

$$c = \frac{1}{2}\big[s - a^2(s-t)\big]$$

So

$$A_k(\lambda) = e^{ka\lambda+k^2c}$$

for all $k > 0$ and $\lambda \in \mathbf{C}$, where $a, c \in \mathbf{C}$ are as above. For any non-zero vector $w \in V$, and $z \in V_c$,

$$\begin{aligned}
(\tilde{S}_{s,t}e^{\hat{w}})(z) &= (\tilde{S}_{s,t}E_{|w|})(z\cdot\frac{w}{|w|}) \\
&= e^{|w|az\cdot\frac{w}{|w|}+|w|^2c} \\
&= e^{az\cdot w+w^2c} \\
&= e^{a\tilde{w}(z)+cw^2} \\
&= \big(\mathcal{W}_a\big(S^V_{s,t}e^{\hat{w}}\big)\big)(z) \qquad (4.15)
\end{aligned}$$

because

$$
\begin{aligned}
\mathcal{W}_a\left(S^V_{s,t}e^{\hat{w}}\right) &= \mathcal{W}_a\left(e^{\tilde{w}+\frac{1}{2}w^2t}\right)\\
&= e^{\frac{1}{2}w^2t+a\tilde{w}+\frac{1}{2}(1-a^2)(s-t)w^2}\\
&= e^{a\tilde{w}+w^2[\frac{1}{2}t+\frac{1}{2}(1-a^2)(s-t)]}\\
&= e^{a\tilde{w}+\frac{w^2}{2}[s-a^2(s-t)]}\\
&= e^{a\tilde{w}+w^2c} \qquad (4.16)
\end{aligned}
$$

Now $w \mapsto e^{\hat{w}}$ and $w \mapsto e^{\tilde{w}}$ are continuous, and so the equality $\mathcal{W}_a S^V_{s,t}e^{\hat{w}} = \tilde{S}_{s,t}e^{\tilde{w}}$ holds also for $w = 0$. Since the linear span of $\{e^{\hat{w}} : w \in V\}$ is dense in $L^2(V, P_s)$, it follows that $\tilde{S}^V_{s,t} = \mathcal{W}_a \circ S^V_{s,t}$.

$\square$

We can now sharpen the descriptions obtained for $\mathrm{Aut}(L^2(P_s))_V$ and $\mathrm{Aut}(\mathcal{H}L^2(M_{s,t}))_V$.

Proof of Theorem 2.1 and Theorem 2.2 . We have the real inner-product space V with $2 \leq \dim V < \infty$. Let $U \in \mathrm{Aut}(L^2(P_s))_V$. Consider the map $\tilde{S}_{s,t} \stackrel{\text{def}}{=} S^V_{s,t} \circ U : L^2(V, P_s) \to \mathcal{H}L^2(V_c, M_{s,t})$. Then $\tilde{S}_{s,t}$ satisfies the three conditions of Theorem 4.4. Therefore there is a $a \in U(1)$ such that $\tilde{S}_{s,t} = S^V_{s,t} \circ \mathcal{U}^V_a$ and so $U = \mathcal{U}^V_a$. Thus the injective group homomorphism $U(1) \to \mathrm{Aut}\big(L^2(P_s)\big)_V : a \mapsto \mathcal{U}^V_a$ is also surjective. A similar argument works for $\mathrm{Aut}(\mathcal{H}L^2(M_{s,t}))_V$. $\square$

5. The Infinite Dimensional Case

In this section V will be a real vector space of positive but possibly infinite dimensions, equipped with an inner-product $V^2 \to \mathbf{R} : (v, w) \mapsto vw$ (it is customary to also assume that the induced topology is separable). If X is a real vector space, of positive dimension, with an inner-product $g(\cdot,\cdot)$ then there is a probability space $(X', \mathcal{F}, P)$ and a linear map $X \to L^2(X', P) : x \mapsto \hat{x}$ such that :(i) each $\hat{x}$ is a Gaussian random variable of mean 0 and variance $g(x, x)$, and (ii) the P–almost-everywhere-defined random variables $\hat{x}$, as x runs over X, generate the completed σ–algebra $\mathcal{F}$. For example, if B is an orthonormal basis of the completion of X then one can take X' to be the product $\mathbf{R}^B$, with each copy of $\mathbf{R}$ equipped with standard Gaussian measure, and define $\hat{x}$ by requiring (linearity, continuity, and) that $\hat{b} : X' \to \mathbf{R}$ is the projection on the b–th coordinate, for each $b \in B$. Applying this procedure to the inner-product on V given by $g(v, w) = svw$, we obtain a probability measure P_s on a space V', and a linear map $V \to L^2(V', P_s) : x \mapsto \hat{x}$ such that the random variables $\hat{x}$, with x running over V, generate the completed σ–algebra on which P_s is defined, and

$$\int_{V'} e^{\lambda\hat{x}}\, dP_s = e^{s\lambda^2 x^2/2} \tag{5.1}$$

for every $\lambda \in \mathbf{C}$. Using this mechanism on $V_c = V^2$, viewed as a real inner-product space, we also obtain a probability space $(V_c', M_{s,t})$ and a linear map $V_c' \to L^2(V_c', M_{s,t}) : z \mapsto \check{z}$ such that the random variables $\check{z}$ generate the completed σ–algebra on which the measure $M_{s,t}$ is defined and

$$\int_{V_c'} e^{\lambda_1\check{x}+\lambda_2\check{(iy)}}\, dM_{s,t} = e^{\frac{\lambda_1^2 r}{4}+\frac{\lambda_2^2 t}{4}} \tag{5.2}$$

for every $\lambda_1, \lambda_2 \in \mathbf{C}$, and $x, y \in V$.

If $z = x + iy \in V_c$, where $x, y \in V$, then we have the complex Gaussian random variable

$$\tilde{z} = \check{x} + i\check{y} \tag{5.3}$$

and the map $V_c \to L^2(V_c', M_{s,t}) : z \mapsto \tilde{z}$ is complex-linear.

A cylinder function on V' is a function which can be expressed in the form $f(\hat{x}_1, ..., \hat{x}_n)$ for some $x_1, ..., x_n \in V$ and some function f on $\mathbf{R}^n$. A cylinder function on V_c' is holomorphic if it can be expressed in the form $f(\tilde{z}_1, ..., \tilde{z}_n)$, for some $z_1, ..., z_n \in V_c$ and holomorphic function f on $\mathbf{C}^n$. We denote by $\mathcal{H}L^2(M_{s,t})$ the closure in $L^2(M_{s,t})$ of the set of functions in $L^2(M_{s,t})$ which are holomorphic cylinder functions.

We have then the standard exponentials which form dense subspaces in $L^2(V', P_s)$ and $\mathcal{H}L^2(V_c', M_{s,t})$:

Lemma 5.1. *The set $\{e^{\hat{x}} : x \in V\}$ spans a dense linear subspace of $L^2(V', P_s)$. The set $\{e^{\tilde{x}} : x \in V\}$ spans a dense linear subspace of $\mathcal{H}L^2(V_c', M_{s,t})$.*

Proof. We know, from Proposition 3.12, that any $f \in L^2(\mathbf{R}^n, P_s)$ is the L^2–limit $\lim_{k\to\infty} f_k$, where each f_k is a linear combination of functions of the form $e^{\hat{\lambda}}$ (with λ running over $\mathbf{R}^n$). Let $u_1, ..., u_n$ be unit vectors in V. Then $f(\hat{u}_1, ..., \hat{u}_n)$ is $\lim_{k\to\infty} f_k(\hat{u}_1, ..., \hat{u}_n)$ in $L^2(V', P_s)$, and each $f_k(\hat{u}_1, ..., \hat{u}_n)$ is a linear combination of fuctions of the form $e^{\hat{v}}$ for $v \in V$. Thus $\{e^{\hat{x}} : x \in V\}$ spans a dense linear subspace of $L^2(V', P_s)$. The argument for $\mathcal{H}L^2(V_c', M_{s,t})$ is analogous. $\square$

Theorem 5.2. *There is a unique unitary isomorphism*

$$S_{s,t} : L^2(V', P_s) \to \mathcal{H}L^2(V_c', M_{s,t}) \tag{5.4}$$

such that for any unit vectors $u_1, ..., u_n \in V$ *and any function* $f \in L^2(\mathbf{R}^n, P_s)$,

$$S_{s,t}[f(\hat{u}_1, ..., \hat{u}_n)] = \big(S^{\mathbf{R}^n}_{s,t} f\big)(\tilde{u}_1, ..., \tilde{u}_n) \tag{5.5}$$

Proof. For $x, y \in V$, we can check that

$$\langle e^{\hat{x}}, e^{\hat{y}}\rangle_{L^2(P_s)} = \langle e^{\tilde{x}+\frac{t}{2}x^2}, e^{\tilde{y}+\frac{t}{2}y^2}\rangle_{\mathcal{H}L^2(M_{s,t})} \tag{5.6}$$

Thus, by the preceding lemma, there is a unique unitary isomorphism $S_{s,t} : L^2(V', P_s) \to \mathcal{H}L^2(V'_c, M_{s,t})$ taking $e^{\hat{x}}$ to $e^{\tilde{x}+\frac{t}{2}x^2}$, for each $x \in V$. This definition of $S_{s,t}$ insures that equation (5.5) holds whenever f is an exponential function on $\mathbf{R}^n$, i.e. a function of the form $e^{\hat{v}}$ for some $v \in \mathbf{R}^n$. Approximating the general function in $L^2(\mathbf{R}^n, P_s)$ by linear combinations of exponentials, and using the observation that $L^2(\mathbf{R}^n, P_s) \to L^2(V', P_s) : f \mapsto f(\hat{u}_1, ..., \hat{u}_n)$ is a linear isometry, we obtain equation (5.5). $\square$

The unitary isomorphisms $\mathcal{U}^V_a$ and $\mathcal{W}^V_a$, defined earlier in the finite-dimensional context, are meaningful as defined even when V is infinite dimensional. It may be of some interest to understand these isomorphisms in a different way. We focus on $\mathcal{U}^V_a$, with $a \in U(1)$.

Equip $\mathcal{T}(V_c)$, the tensor algebra over V_c, with the inner-product specified on n-tensors by

$$\langle u_1 \otimes \cdots \otimes u_n, v_1 \otimes \cdots \otimes v_n\rangle = s^n n! \langle u_1, v_1\rangle \cdots \langle u_n, v_n\rangle$$

(The inner-product $\langle \cdot, \cdot\rangle$ appearing on the right is the Hermitian inner-product on V_c induced by the inner-product on V.) On 0–tensors, i.e. on $\mathbf{C}$, the inner-product is the usual Hermitian inner-product on $\mathbf{C} : (\lambda, \mu) \mapsto \lambda\overline{\mu}$. Let $\mathcal{S}(V_c)_s$ be the Hilbert space obtained by completing the subspace of symmetric tensors in $\mathcal{T}(V_c)$. This is the symmetric Fock space over V (for the inner-product scaled by the parameter s). For each $z \in V_c$, we have the exponential $\mathrm{Exp}z \stackrel{\text{def}}{=} \sum_{n=0}^{\infty} \frac{1}{n!} z^{\otimes n}$, the series being clearly convergent in $\mathcal{S}(V_c)_s$. It is a fact that $\{\mathrm{Exp}(x) : x \in V\}$ span a dense subspace of $\mathcal{S}(V_c)_s$. It is also readily checked that for any $z, w \in V_c$, the inner-product $\langle e^{\hat{z}-\frac{s}{2}z^2}, e^{\hat{w}-\frac{s}{2}w^2}\rangle_{L^2(V',P_s)}$ and $\langle \mathrm{Exp}(z), \mathrm{Exp}(w)\rangle$ are equal (to $e^{sz\overline{w}}$). So, by the usual density argument, there is a unique unitary isomorphism (the Itô iterated multiple integral isomorphism)

$$\mathcal{I}_s : \mathcal{S}(V_c)_s \to L^2(V', P_s) \tag{5.7}$$

satisfying

$$\mathcal{I}_s\big(\mathrm{Exp}(x)\big) = e^{\hat{x}-\frac{s}{2}x^2} \tag{5.8}$$

for every $x \in V$, the relation being also valid if x is replaced by any $z \in V_c$. The definition, via (4.7), of $\mathcal{U}_a$ implies that it takes $e^{\hat{x}-\frac{s}{2}x^2}$ to $e^{\widehat{ax}-\frac{s}{2}(ax)^2}$. Thus $\mathcal{I}_s^{-1}\mathcal{U}_a\mathcal{I}_s$ is simply the unique unitary isomorphism $\mathcal{S}(V_c)_s \to \mathcal{S}(V_c)_s$ induced by the unitary map $V_c \to V_c : z \mapsto az$.

6. Category theoretic formulation

Some of the results of the preceding two sections can be recast nicely in a category-theoretic framework. We will not write out the proofs of these category-theoretic formulations separately since they are reformulations of results we have already proven.

Let **Euc** be the category whose objects are real inner-product spaces and whose arrows are linear isometries, let **Hilb** be the category whose objects are complex Hilbert spaces and whose arrows are again linear isometries. If V is an object of **Euc** we can associate to it the Hilbert space $L^2(V', P_s)$. If $j : V \to W$ is a morphism in **Euc** then there is a naturally induced isometry $L^2(j, P_s) : L^2(V', P_s) \to L^2(W', P_s)$: it is the unique linear isometry $L^2(V', P_s) \to L^2(W', P_s)$ which takes $e^{\hat{x}}$ to $e^{\widehat{j(x)}}$, for each $x \in W$. It is also readily verified that if $u_1, ..., u_n$ are orthonormal in W and $f \in L^2(\mathbf{R}^n, P_s)$ then $L^2(j, P_s)$ takes the function $f(\hat{u}_1, ..., \hat{u}_n) \in L^2(W', P_s)$ to the function $f(\widehat{j(u_1)}, ..., \widehat{j(u_n)}) \in L^2(V', P_s)$. If $V = \{0\}$, we take $L^2(V, P_s)$ to be the complex line $\mathbf{C}$. Then $L^2(P_s)$, consisting of the assignments $V \mapsto L^2(V', P_s)$ and $j \mapsto L^2(j, P_s)$ is a functor

$$L^2(P_s) : \mathbf{Euc} \to \mathbf{Hilb} \tag{6.1}$$

Analogously, we also have the functor

$$\mathcal{H}L^2(M_{s,t}) : \mathbf{Euc} \to \mathbf{Hilb} \tag{6.2}$$

which takes each object V in **Euc** to the Hilbert space $\mathcal{H}L^2(V_c', M_{s,t})$.

The categories **Euc** and **Hilb** are *monoidal* in that they have a product with certain simple properties: in **Euc** it is the direct sum, and in **Hilb** it is the tensor product. The functors $L^2(P_s)$ and $\mathcal{H}L^2(M_{s,t})$ are *monoidal* functors in the following sense. For $L^2(P_s)$, being a monodial functor means that there is a natural isomorphism θ between the functors $\mathbf{Euc}^2 \to \mathbf{Hilb} : (V, W) \mapsto L^2(V', P_s) \otimes L^2(W', P_s)$ and $\mathbf{Euc}^2 \to \mathbf{Hilb} : (V, W) \mapsto L^2((V \oplus W)', P_s)$. Thus, for each $V, W \in \mathbf{Euc}$, there is a unitary isomorphism

$$\theta_{V,W} : L^2((V \oplus W)', P_s) \to L^2(V', P_s) \otimes L^2(W', P_s)$$

The analogous statement holds for $\mathcal{H}L^2(M_{s,t})$, and, for simplicity, we use the same notation $\theta_{V,W}$ for the corresponding isomorphism.

Now to each object V of **Euc** we have the unitary isomorphism $S^V_{s,t} : L^2(V', P_s) \to \mathcal{H}L^2(V'_c, M_{s,t})$. In this language, we can state the basic properties of $S_{s,t}$ as follows:

Proposition 6.1. *The assignment $V \mapsto S^V_{s,t}$ is a natural isomorphism between the functors $L^2(P_s)$ and $\mathcal{H}L^2(M_{s,t})$. Moreover, this natural isomorphism preserves the monoidal character of the functors $L^2(P_s)$ and $\mathcal{H}L^2(M_{s,t})$:*

$$S^{V\oplus W}_{s,t} = \theta^{-1}_{V,W}(S^V_{s,t} \otimes S^W_{s,t})\theta_{V,W} \tag{6.3}$$

*for every $V, W \in$ **Euc**.*

Some terminology: Let A and B be monoidal categories [for the present purposes, we may think of a monoidal category simply as any one of the two categories **Euc** and **Hilb**] and denote the monoid operation in A by $\oplus$ and the one in B by $\otimes$. Consider a functor $F : A \to B$ between monoidal categories . Then we have two functors $F \circ \oplus : A^2 \to B : (V, W) \mapsto F(V \oplus W)$ and $\otimes \circ F^2 : (V, W) \mapsto F(V) \otimes F(W)$. The functor F is *monoidal* if there is a natural transformation $\theta^F : F \circ \oplus \to \otimes \circ F^2$. Note that this means, in particular, that for every objects V and W in A there is an isomorphism $\theta^F_{VW} : F(V \oplus W) \to F(V) \otimes F(W)$. If $F, G : A \to B$ are two monoidal functors and $\tau : F \to G$ is a natural transformation, we say that τ is *monoidal* if

$$\tau_{V\oplus W} = \left(\theta^G_{VW}\right)^{-1}(\tau_V \otimes \tau_W)\theta^F_{VW} \tag{6.4}$$

Let $Mon(F, G)$ denote the set of all monoidal natural transformations from F to G.

If $F : A \to B$ is a monoidal functor between monoidal categories, we denote by $\mathrm{Aut}(F)$ the group of monoidal natural isomorphisms $F \to F$. Thus $\mathrm{Aut}(F)$ is the group of (monoidal) symmetries of the functor F, and its size describes how "rigid" F is.

It is clear that $\mathrm{Aut}(F)$ acts on $Mon(F, G)$ be pre-composition and $\mathrm{Aut}(G)$ acts on $Mon(F, G)$ by post-composition.

Let $U(1)$ be the group of unit-modulus complex numbers.

Theorem 6.2. *There is a unique isomorphism of groups*

$$U(1) \to Aut(L^2(P_s)) : a \mapsto \mathcal{U}_a$$

such that the unitary isomorphism $\mathcal{U}^{\mathbf{R}}_a : L^2(\mathbf{R}, P_s) \to L^2(\mathbf{R}, P_s)$ carries the function e^{kx} (here k is any real constant and $x \in \mathbf{R}$ the variable) into the function $e^{akx+\frac{1}{2}(1-a^2)sk^2}$, for all $k \in \mathbf{R}$.

There is a unique isomorphsim of groups

$$U(1) \to Aut(\mathcal{H}L^2(M_{s,t})) : a \mapsto \mathcal{W}_a$$

such that the unitary isomorphism $\mathcal{W}_a^{\mathbf{R}} : \mathcal{H}L^2(\mathbf{C}, M_{s,t}) \to \mathcal{H}L^2(\mathbf{C}, M_{s,t})$ *carries the function* e^{kz} *(here* k *is constant and* $z \in \mathbf{C}$ *the variable) into the function* $e^{akz+\frac{1}{2}(1-a^2)(s-t)k^2}$, *for each* $k \in \mathbf{R}$.

The next result determines the "rigidity" of $S_{s,t}$ and shows that, modulo a $U(1)$–factor, $S_{s,t}$ is the unique natural isomorphism $L^2(P_s) \to \mathcal{H}L^2(M_{s,t})$.

Theorem 6.3. *For each* $\tilde{S}_{s,t} \in Mon\big(L^2(P_s), \mathcal{H}L^2(M_{s,t})\big)$ *there is a unique* $a \in U(1)$ *such that*

$$\tilde{S}_{s,t} = \mathcal{W}_a \circ S_{s,t} = S_{s,t} \circ \mathcal{U}_a \tag{6.5}$$

In particular, $Mon\big(L^2(P_s), \mathcal{H}L^2(M_{s,t})\big)/Aut(L^2(P_s))$ *has exactly one element, the (class of the) two-parameter Segal-Bargmann transform.*

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