Atiyah–Jones conjecture for blown-up surfaces

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Abstract

We show that if the Atiyah–Jones conjecture holds for a surface X, then it also holds for the blow-up of X at a point. Since the conjecture is known to hold for $\mathbb{P}^2$ and for ruled surfaces, it follows that the conjecture is true for all rational surfaces.

Given a 4-manifold X, let $\mathcal{MI}_k(X)$ denote the moduli space of rank 2 instantons on X with charge k and let $\mathcal{C}_k(X)$ denote the space of all connections on X with charge k. In 1978, Atiyah and Jones [AJ] conjectured that the inclusion $\mathcal{MI}_k(X) \to \mathcal{C}_k(X)$ induces an isomorphism in homology and homotopy through a range that grows with k. The original statement of the conjecture was for the case when X is a sphere, but the question readily generalizes for other 4-manifolds.

The stable topology of these moduli spaces was understood in 1984, when Taubes [Ta] constructed instanton patching maps $t_k\colon \mathcal{MI}_k(X) \to \mathcal{MI}_{k+1}(X)$ and showed that the stable limit $\lim\limits_{k\to\infty} \mathcal{MI}_k$ indeed has the homotopy type of $\mathcal{C}_k(X)$. However, understanding the behavior of the maps t_k at finite stages is a finer question. Using Taubes' results, to prove the Atiyah–Jones conjecture it would now suffice to show that the maps t_k induce isomorphism in homology and homotopy through a range.

In 1993, Boyer, Hurtubise, Milgram and Mann [BHMM] proved that the Atiyah–Jones conjecture holds for the sphere S^4 and in 1995, Hurtubise and Mann [HM] proved that the conjecture is true for ruled surfaces.

In this paper we show that if the Atiyah–Jones conjecture holds true for a complex surface X then it also holds for the surface $\widetilde{X}$ obtained by blowing-up X at a point. In particular, it follows that the conjecture holds true for all rational surfaces.

Kobayashi and Hitchin gave a one-to-one correspondence between instantons on a topological bundle E over X and holomorphic structures on E, see [LT]. Using this correspondence we translate the Atiyah–Jones conjecture into the language of holomorphic bundles and compare the topologies of the moduli spaces $\mathfrak{M}_k(X)$ and $\mathfrak{M}_k(\widetilde{X})$ of stable holomorphic bundles on X, resp. $\widetilde{X}$, having $c_1 = 0$ and $c_2 = k$.

The structure of the proof is the following. First we give a concrete description of instantons on $\widetilde{\mathbb{C}^2}$ (section 1) and compute its numerical invariants (section 2). We then study moduli spaces $\mathfrak{M}_k(\widetilde{X})$ of instantons or equivalently stable bundles on $\widetilde{X}$, with respect to a polarization $\widetilde{\mathcal{L}} = N\mathcal{L} - \ell$ where $\mathcal{L}$ is a polarization on X. We show that removing the singular points of this moduli space does not affect our homology calculations and thereafter work only with its smooth points. Moreover, the direct image of a stable bundle might yield an unstable bundle, which does not correspond to an instanton on X. We show that removing such unstable bundles from $\mathfrak{M}_k(X)$ does not affect our homology calculations either (section 3).

We show that any instanton on $\widetilde{X}$ is uniquely determined by holomorphic patching of an instanton on X and an instanton on $\widetilde{\mathbb{C}^2}$ (section 4). We then prove that the moduli space of (framed) instantons on $\widetilde{\mathbb{C}^2}$ with charge i is smooth and of complex dimension not greater than $r_i(2r_i+1)$ where $r_i = \lceil\sqrt{i}\rceil$ (section 5). As a consequence we have a stratifications

$$\mathfrak{M}_k^f(\widetilde{X}) \simeq \bigcup_{i=0}^{k} \mathfrak{M}_{k-i}^f(X) \times \mathcal{N}_i^f$$

and we show that Taubes' map $\mathfrak{M}_k \to \mathfrak{M}_{k+1}$ is homotopically equivalent to a map that preserves the stratifications (section 6). Using the Leray spectral sequences for these stratifications, we then show that Atiyah–Jones conjecture for $\widetilde{X}$ follows from the corresponding statement for X.

1. Moduli of bundles on $\widetilde{\mathbb{C}^2}$ with fixed splitting type

Let $\pi\colon \widetilde{X} \to X$ be the blow-up of a point x on a compact surface X and let ℓ be the exceptional divisor. In this paper all bundles have rank 2 and $c_1 = 0$. Given a bundle E over X we pull it back to $\widetilde{X}$ and then modify it by gluing in new data near the exceptional divisor to construct a bundle $\widetilde{E}$ satisfying $\widetilde{E}|_{\widetilde{X}-\ell} \simeq E|_{X-x}$. The difference of Chern classes $c_2(\widetilde{E}) - c_2(E)$ depends only on the restriction of $\widetilde{E}$ to a small neighborhood $N(\ell)$ of ℓ. To begin with, suppose that $N(\ell)$ is isomorphic to the blow-up of $\mathbb{C}^2$ at the origin, denoted $\widetilde{\mathbb{C}^2}$. Given a rank two bundle V on $\widetilde{\mathbb{C}^2}$ with vanishing first Chern class, there exists an integer $j \geq 0$ determined by the restriction of V to the exceptional divisor, such that $V|_\ell \simeq \mathcal{O}_{\mathbb{P}^1}(j) \oplus \mathcal{O}_{\mathbb{P}^1}(-j)$. This j is called the *splitting type* of the bundle.

THEOREM [G3] *Every holomorphic rank two bundle V over $\widetilde{\mathbb{C}^2}$ with vanishing* *first Chern class and splitting type j is an algebraic extension of the form*

$$0 \to \mathcal{O}_{\widetilde{\mathbb{C}^2}}(-j) \to V \to \mathcal{O}_{\widetilde{\mathbb{C}^2}}(j) \to 0$$

and the extension class can be represented as a polynomial of degree at most $2j-2$.

It follows that the local (= over $N(\ell)$) moduli problem can be studied by considering extensions of line bundles modulo bundle isomorphism even though this is definitely not the case for bundles on the compact surface $\widetilde{X}$. The fact that V is determined by an extension of degree $2j-2$ implies that V is determined over $N(\ell)$ by its restriction to the $2j-2$nd formal neighborhood of the exceptional divisor. Hence, our initial assumption that $N(\ell) \simeq \widetilde{\mathbb{C}^2}$ is not restrictive. Consider the quotient space

$$\mathcal{M}_j \colon = \{\textit{bundles on } \widetilde{\mathbb{C}^2} \textit{ with splitting type } j\}/\sim,$$

where $\sim$ denotes bundle isomorphism. $\mathcal{M}_j$ is topologized as follows: We first choose canonical coordinates for $N(\ell) = U \cup V$, where $U \simeq \mathbb{C}^2$ has coordinates (z,u) and $V \simeq \mathbb{C}^2$ has coordinates (ξ, v) with $(\xi, v) = (z^{-1}, zu)$ in $U \cap V \simeq \mathbb{C} - 0 \times \mathbb{C}$. Then, in these coordinates, we write the extension class as a complex polynomial with $n = 2j(j+1)$ coordinates (see [G3]). In such canonical coordinates, elements of $\mathcal{M}_j$ are represented by transition matrices of the form

$$\begin{pmatrix} z^j & p \\ 0 & z^{-j} \end{pmatrix}. \qquad (*)$$

We set $p \sim p'$ if the corresponding bundles are holomorphically equivalent. This construction identifies $\mathcal{M}_j$ to a quotient of $\mathbb{C}^n$ under a group action and endows $\mathcal{M}_j$ with the structure of a quotient stack. For more on the structure of $\mathcal{M}_j$ see [G3] and [G5]. For a decomposition of $\mathcal{M}_j$ into Hausdorff strata see [BG2]. To study the moduli problem we calculate numerical invariants.

2. Computation of local numerical invariants

In §1 we saw that an element $V \in \mathcal{M}_j$ is determined by its splitting type j and extension class p. We write $V = V(j,p)$. It is very important to note that the bundle associated to V is trivial on the complement of ℓ.

Lemma [G2] *Every holomorphic bundle on $\widetilde{\mathbb{C}^2}$ with vanishing first Chern class is trivial on $\widetilde{\mathbb{C}^2}$ minus the exceptional divisor.*

It follows that the local bundle $V(j,p)$ can be glued to any bundle E pulled back from X to form a new bundle $\widetilde{E}$ on $\widetilde{X}$. We also note that the topology of $\widetilde{E}$ does not depend on the attaching map. In fact, the attaching is given by a holomorphic map $\psi\colon \mathbb{C}^2 - \{0\} \to Sl(2,\mathbb{C})$. By Hartog's theorem ψ extends to the origin, and therefore has degree zero, thus not contributing to the Chern numbers of $\widetilde{E}$. Hence 2 pairs of data (E,V,p,ψ) and (E,V,p,ψ') determine the same topological bundle, although in general they determine distinct holomorphic bundles.

LEMMA [G4] *Every holomorphic rank two bundle $\widetilde{E}$ on $\widetilde{X}$ with $c_1(\widetilde{E}) = 0$ is topologically determined by a triple (E,j,p) where E is a bundle over X, j is a non-negative integer and p is a polynomial.*

Here $E = \pi_*(\widetilde{E})^{\vee\vee}$ and the Chern class difference $c = c_2(\widetilde{E}) - c_2(E)$ depends only on the local data $V(j,p)$. We call c the *charge* of the bundle near the exceptional divisor. The terminology comes from instanton moduli spaces, where c represents the change in topological charge of the instanton on $\widetilde{X}$ given by the patching of $V(j,p)$. Friedman and Morgan [FM] give the bounds $j \leq c_2(\widetilde{E}) - c_2(E) \leq j^2$.

THEOREM [G1] *The bounds $j \leq c_2(\widetilde{E}) - c_2(E) \leq j^2$ are sharp.*

Using elementary transformations, we obtain that all intermediate values occur. The following result implies non-emptiness of the strata appearing in the stratification on §6.

THEOREM [BG1] *For every integer k satisfying $j \leq k \leq j^2$ there exists a holomorphic bundle on $\widetilde{X}$ with splitting type j and such that $c_2(\widetilde{E}) - c_2(E) = k$.*

For the global moduli problem on $\widetilde{X}$ we need to identify the spaces $\mathcal{N}_i$ of isomorphism classes of bundles on a neighborhood of the exceptional divisor that have fixed charge i. We define

$$\mathcal{N}_i = \{bundles\ on\ \widetilde{\mathbb{C}^2}\ having\ charge\ i\}/\sim .$$

PROPOSITION 2.1 *$\mathcal{N}_i$ has complex dimension $2i-3$.*

Proof. When $j = i$ the generic sets of $\mathcal{M}_j$ and $\mathcal{N}_i$ are the same, and are identified with $\mathbb{P}^{2i-3}$ minus a complex subvariety of codimension at least two

[BG3,Thm. 3.5]. The set of all additional points on $\mathcal{N}_i$ has strictly lower dimension [BG1, Thm. 4.1]. $\square$

REMARK: Although being quite distinct, $\mathcal{M}_j$ and $\mathcal{N}_j$ share the same generic set. On the former j denotes the splitting type, whereas on the latter j denotes the charge addition. If one desires to have a concept of stability for bundles on $N(\ell)$, a good definition would be to call a bundle on $N(\ell)$ *stable*, if its charge equals its splitting type. This certainly gives moduli spaces. The definition proposed earlier in [Kn] turned out to be satisfied by every bundle on $N(\ell)$.

3. Moduli of bundles on $\widetilde{X}$

For the moduli problem on $\widetilde{X}$ we need stability conditions. If $\mathcal{L}$ is an ample divisor on X then, for large N, the divisor $\widetilde{\mathcal{L}} = N\mathcal{L} - \ell$ is ample on $\widetilde{X}$. We fix, once and for all, polarizations $\mathcal{L}$ and $\widetilde{\mathcal{L}}$ on X and $\widetilde{X}$ respectively. For a polarized surface Y, the notation $\mathfrak{M}_k(Y)$ stands for moduli of rank 2 bundles on Y, slope stable with respect to the fixed polarization, with vanishing first Chern class and second Chern class k. If E is $\mathcal{L}$-stable on X, then $\pi^*(E)$ is $\widetilde{\mathcal{L}}$-stable on $\widetilde{X}$ [FM, Thm. 5.5]. Hence, the pull back map induces an inclusion of moduli spaces $\mathfrak{M}_k(X) \hookrightarrow \mathfrak{M}_k(\widetilde{X})$.

Our objective is to show that given a map $\mathfrak{M}_k(X) \to \mathfrak{M}_{k+1}(X)$ inducing a homology equivalence through a range, there is a map $\mathfrak{M}_k(\widetilde{X}) \to \mathfrak{M}_{k+1}(\widetilde{X})$ inducing a homology equivalence through a comparable range, and that both ranges go to ∞ with k. Given that the homology equivalence for the case of $X = \mathbb{P}^2$ holds up to dimension $\lfloor k/2 \rfloor - 1$, we expect to obtain an equivalence range at best equal to this one. From here on, we will feel free to modify $\mathfrak{M}_k(\widetilde{X})$ in any way that does not change topology up to dimension $k/2 = c_2/2$. Firstly, we remove singularities of the moduli spaces; secondly, we remove semistable bundles on $\mathfrak{M}_i(X)$ that are not stable and thirdly we add frames. We then obtain a stratification of the framed moduli space $\mathfrak{M}_k^f(\widetilde{X})$ by smooth submanifolds.

3.1 Removing singularities

We remove the singular points of the moduli space, in order to work only with smooth manifolds. We use the following results of Kirwan and Donaldson.

THEOREM ([Ki], Cor. 6,.4) *Let X be a quasi-projective variety and m a non-negative integer such that every $x_0 \in X$ has a neighborhood in X isomorphic* *to*

$$\{x \in \mathbb{C}^N | f_1(x) = \cdots = f_M(x) = 0\}$$

for some integers N, M, and holomorphic functions f_i depending on x_0 with $M \leq m$. If Y is a closed subvariety of codimension k in X, then for $q < k-m$,

$$H_q(X - Y, \mathbb{Z}) \simeq H_q(X, \mathbb{Z}).$$

Given a complex surface S with polarization H, let $\Sigma_k \subset \mathfrak{M}_k(S)$ denote the algebraic subvariety representing bundles E with $H^2(\mathrm{End}_0 E) \neq 0$. A local model of the moduli space is given by the kernel of the Kuranishi map $Sym^2 H^1(\mathrm{End} E) \to H^2(\mathrm{End}_0 E)$ parametrizing small deformations of E. In case $H^2(\mathrm{End}_0 E) = 0$, small deformations of E are unobstructed and E is a smooth point of the moduli space. Therefore, singular points satisfy $H^2(\mathrm{End}_0 E) \neq 0$ and the singular part of $\mathfrak{M}_k$ is contained in Σ_k.

Theorem ([Do], Thm. 5.8) *There are constants a, b depending only on S and the ray spanned by H in $H_2(S)$ such that:*

$$\dim_{\mathbb{C}} \Sigma_k \leq a + b\sqrt{k} + 3k.$$

Proposition 3.1 *For large k, removing the singularity set $Sing$ of $\mathfrak{M}_k(S)$ does not change homology in dimension less than k. That is, for $q < k$*

$$H_q(\mathfrak{M}_k(S)) = H_q(\mathfrak{M}_k(S) - Sing).$$

Proof. By Kuranishi theory, points $E \in \mathfrak{M}_k(S)$ satisfying $H^2(\mathrm{End}_0 E) = 0$ are smooth points. Therefore, the singularity set of $\mathfrak{M}_k(S)$ is contained in Σ_k. Moreover, the moduli space is defined on a neighborhood of a singular point by $\dim H^2(\mathrm{End}_0 E)$ equations. In fact, in the proof of his theorem cited above, Donaldson shows that $\dim H^2(\mathrm{End}_0 E) \leq a + b\sqrt{k} + 3k$. Therefore, using Kirwan's result we have that $H_q(\mathfrak{M}_k(S)) = H_q(\mathfrak{M}_k(S) - Sing)$ for $q < \dim \mathfrak{M}_k(S) - 2(a + b\sqrt{k} + 3k) < \dim \mathfrak{M}_k(S) - 7k = 8k - 3(1 - b_1 + b^+) - 7k < k$ for large k. □

We now consider only the smooth part of $\mathfrak{M}_k(\widetilde{X})$, which in what follows stands for just the set of its smooth points. Similarly, when considering $\mathfrak{M}_k(X)$ we will consider only its set of smooth points, keeping in mind that this does not change the homology up to dimension k.

3.2 Removing unstable bundles

Given an $\widetilde{\mathcal{L}}$-stable bundle $\widetilde{E}$, the bundle $E = (\pi_* \widetilde{E})^{\vee\vee}$ is $\mathcal{L}$-semistable [FM, Thm. 5.5]. We want to remove the set of strictly semistable bundles that appear in this process, leaving only stable bundles over X, as these are the ones corresponding to instantons.

PROPOSITION 3.2 *There is a bound, depending only on X, on the dimension of the set of semistable bundles on X that are not stable.*

Proof. Let E be a semistable bundle on X that is not stable. Since we are only considering bundles with $c_1 = 0$ we must have that there is a destabilizing line bundle L with $\deg(L) = 0$. Hence, E is an extension

$$0 \to L \to E \to \mathcal{F} \to 0, \tag{1}$$

where $\mathcal{F}$ is a rank 1 sheaf satisfying $\mathcal{F}^{\vee\vee} = L^{-1}$ and fitting into a short exact sequence

$$0 \to \mathcal{F} \to \mathcal{F}^{\vee\vee} \to Q \to 0,$$

with Q supported only at points. It follows that the dimension of the set $\mathfrak{M}^{su} \colon = \mathfrak{M}^{ss} - \mathfrak{M}^{s}$ of semistable bundles on X that are not stable is given by

$$\dim \mathfrak{M}^{su} = \dim(\mathrm{Ext}^1(\mathcal{F}, L) \times \mathrm{Jac}(X)).$$

It is a basic fact that $\dim \mathrm{Jac}(X) = h^1(X, \mathcal{O}_X)$ is the irregularity of X.
CLAIM: $\dim \mathrm{Ext}^1(\mathcal{F}, L) = h^1(X, \mathcal{O}_X) - h^2(X, \mathcal{O}_X) + \epsilon$, where $\epsilon = 0$ or 1.
In fact, taking $\mathrm{Hom}(., L)$ on the short exact sequence (1) gives

$$0 \to \mathrm{Hom}(\mathcal{F}, L) \to \mathrm{Hom}(\mathcal{F}^{\vee\vee}, L) \to \mathrm{Hom}(Q, L) \to$$

$$\to \mathrm{Ext}^1(\mathcal{F}, L) \to \mathrm{Ext}^1(\mathcal{F}^{\vee\vee}, L) \to \mathrm{Ext}^1(Q, L) \to .$$

But, because Q is supported at points, we have $\mathrm{Hom}(Q, L) = \mathrm{Ext}^1(Q, L) = 0$, and consequently

$$\mathrm{Ext}^1(\mathcal{F}, L) = \mathrm{Ext}^1(\mathcal{F}^{\vee\vee}, L) = H^1(L^2).$$

An application of Hirzebruch–Riemann–Roch gives

$$h^1(L^2) = h^0(L^2) - (c_1^2 + c_2)$$

and since X is a surface $c_1 = -K$, we have

$$h^1(L^2) = h^0(L^2) - (K^2 + c_2) = h^0(L^2) - (1 + p_a),$$

where $p_a = h^2(X, \mathcal{O}_X) - h^1(X, \mathcal{O}_X)$ is the arithmetic genus of X. To calculate $h^0(L^2)$ there are two possibilities, either L^2 has no holomorphic sections, in which case, $h^0(L^2) = 0$; or else, suppose s is a holomorphic section of L^2 and let S its divisor of zeros. Since the divisor $\mathcal{L}$ defining the polarization is ample, it has positive intersection with nonzero divisors. But, $\deg(L) = 0$, hence $S \cdot \mathcal{L} = 0$ forcing $S \equiv 0$. Therefore, L^2 is trivial and in this case $h^0(L^2) = 1$. It follows that

$$\dim \mathfrak{M}^{su} = \dim \mathrm{Ext}^1(\mathcal{F}, L) + \dim \mathrm{Jac}(X) = 2h^1(X, \mathcal{O}_X) - h^2(X, \mathcal{O}_X) + \epsilon$$

where $\epsilon = 0$ or 1. $\square$

Based on the results of this section, we assume in what follows that the moduli spaces $\mathfrak{M}_i$ are smooth and contain only stable points.

4. The gluing construction

We show that a framed bundle on $\widetilde{X}$ is uniquely determined by a pair of framed bundles on X and $\widetilde{\mathbb{C}^2}$. We fix a neighborhood $N(\ell)$ of the exceptional divisor inside of $\widetilde{X}$ and write $\widetilde{X} = (\widetilde{X} - \{\ell\}) \cup N(\ell)$. Put $N_0 \colon = (\widetilde{X} - \{\ell\}) \cap N(\ell)$. The blow-up map gives an isomorphism $i_1 \colon \widetilde{X} - \{\ell\} \to X - \{x\}$. Based on the results of §1, we can assume that there is an isomorphism $i_2 \colon N(\ell) \to \widetilde{\mathbb{C}^2}$. Over the intersection N_0 we have isomorphisms $N(x) - \{x\} \xleftarrow{i_1} N_0 \xrightarrow{i_2} \widetilde{\mathbb{C}^2} - \{0\}$, which we still denote by the same letters i_1 and i_2. Using this isomorphisms we write framed bundles on $\widetilde{X}$ as pairs of framed bundles on X and $\widetilde{\mathbb{C}^2}$.

Lemma 4.1 *There is a one-to-one correspondence between holomorphic vector bundles on $\widetilde{X} - \{\ell\}$ and holomorphic vector bundles on X.*

Proof. It is trivial that there is a one-to-one correspondence between bundles on $\widetilde{X} - \{\ell\}$ and bundles on $X - x$, because these are isomorphic surfaces. If E is a holomorphic bundle on $X - x$ it extends uniquely to a bundle on X by Hartog's because X is 2-dimensional. $\square$

Definitions

- A *framed* bundle $\pi_{\widetilde{E}} \colon \widetilde{E}^f \to \widetilde{X}$ is a bundle $\widetilde{E}$ on $\widetilde{X}$ together with a pair of pointwise linearly independent sections $(f_1, f_2) \colon N_0 \to \pi_{\widetilde{E}}^{-1}(N_0)$.

- A *framed* bundle $\pi_V \colon V^f \to \widetilde{\mathbb{C}^2}$ is a bundle V on $\widetilde{\mathbb{C}^2}$ together with a pair of pointwise linearly independent sections $(t_1, t_2)\colon \widetilde{\mathbb{C}^2} - \{0\} \to \pi_V^{-1}(\widetilde{\mathbb{C}^2}) - \{0\}$.

- A *framed* bundle $\pi_E\colon E^f \to X$ is a bundle E on X together with a pair of pointwise linearly independent sections $(s_1, s_2)\colon N(x) \to \pi_E^{-1}(N(x))$, where $N(x)$ is a small neighborhood of x. We will always consider $N(x) = \pi(N(\ell))$.

Proposition 4.2 *An isomorphism class $[\widetilde{E}]$ of a framed bundle on $\widetilde{X}$ is uniquely determined by a pair of isomorphism classes of framed bundles $[E^f]$ on X and $[V^f]$ on $\widetilde{\mathbb{C}^2}$. We write $\widetilde{E}^f = (E^f, V^f)$.*

Proof. By construction $\widetilde{E} = E \coprod_{(s_1,s_2)=(t_1,t_2)} V$ is made by identifying the bundles as well as the sections over N_0, so that the bundles satisfy $\widetilde{E^f}|_{N_0} = i_1^*(E|_{N(x)}) = i_2^*(V|_{\widetilde{\mathbb{C}^2}-\{0\}})$ and the framing (f_1, f_2) of $\widetilde{E}$ satisfies $(f_1, f_2) = (s_1, s_2) \circ i_1 = (t_1, t_2) \circ i_2$.

Let $\phi\colon E^f \to E'^f$ be an isomorphism such that $\phi \circ (s_1, s_2) = (s_1', s_2')$ and let $\xi\colon V^f \to V'^f$ be an isomorphism such that $\xi \circ (t_1, t_2) = (t_1', t_2')$. We have the following diagrams of bundle maps:

$$\begin{array}{ccccccc}
\widetilde{E^f}|_{\widetilde{X}-\{\ell\}} & \to & E^f & \stackrel{\phi}{\to} & E'^f & \leftarrow & \widetilde{E'^f}|_{\widetilde{X}-\{\ell\}} \\
\downarrow \pi_{\widetilde{E}} & & \downarrow \pi_E & & \downarrow \pi_{E'} & & \downarrow \pi_{\widetilde{E'}} \\
\widetilde{X} - \{\ell\} & \stackrel{i_1}{\to} & X - \{x\} & = & X - \{x\} & \stackrel{i_1}{\leftarrow} & \widetilde{X} - \{\ell\}
\end{array}$$

Hence,

$$\widetilde{E'}^f|_{\widetilde{X}-\{\ell\}} = i_1^*(E'^f|_{X-\{x\}}) = i_1^* \circ \phi(E^f|_{X-\{x\}}) = i_1^* \circ \phi \circ i_{1*}(\widetilde{E}^f|_{\widetilde{X}-\{\ell\}}) \quad (1)$$

showing that $i_1^* \circ \phi \circ i_{1*}$ is an isomorphism of $\widetilde{E}$ and $\widetilde{E'}$ over $\widetilde{X} - \{\ell\}$ such that

$$\phi \circ (f_1, f_2) = \phi \circ (s_1, s_2) \circ i_1 = (s_1', s_2') \circ i_1 = (f_1', f_2'). \quad (2)$$

On the other hand we have the second diagram of bundle maps:

$$\begin{array}{ccccccc}
\widetilde{E^f}|_{N(\ell)} & \to & V^f & \stackrel{\xi}{\to} & V'^f & \leftarrow & \widetilde{E'^f}|_{N(\ell)} \\
\downarrow \pi_{\widetilde{E}} & & \downarrow \pi_V & & \downarrow \pi_{V'} & & \downarrow \pi_{\widetilde{V'}} \\
N(\ell) & \stackrel{i_2}{\to} & \widetilde{\mathbb{C}^2} & = & \widetilde{\mathbb{C}^2} & \stackrel{i_2}{\leftarrow} & N(\ell)
\end{array}.$$

Therefore,

$$\widetilde{E'}^f|_{N(\ell)} = i_2^*(V'^f|_{\widetilde{\mathbb{C}^2}}) = i_2^* \circ \xi(V^f|_{\widetilde{\mathbb{C}^2}}) = i_2^* \circ \xi \circ i_{2*}(\widetilde{E}^f|_{N(\ell)}) \qquad (3),$$

showing that $i_2^* \circ \xi \circ i_{2*}$ is an isomorphism of $\widetilde{E}$ and $\widetilde{E'}$ over $N(\ell)$ such that

$$\xi \circ (f_1, f_2) = \xi \circ (t_1, t_2) \circ i_2 = (t_1', t_2') \circ i_2 = (f_1', f_2'). \qquad (4)$$

These isomorphisms agree over the intersection N_0, in fact, by (1) and (3)

$$i_1^* \circ \phi \circ i_{1*}(\widetilde{E^f}|_{N_0}) = i_1^* \circ \phi(E^f|_{N(x)}) = i_1^*(E'^f|_{N(x)}) = i_2^*(V'^f|_{\widetilde{\mathbb{C}^2}-\{0\}}) =$$

$$= i_2^* \circ \xi(V^f|_{\widetilde{\mathbb{C}^2}-\{0\}}) = i_2^* \circ \xi \circ i_{2*}(\widetilde{E^f}|_{N_0})$$

and moreover they also preserve the framings over the intersection, since over N_0, we have, by (2) and (4)

$$\phi \circ (f_1, f_2) = (f_1', f_2') = \xi \circ (f_1, f_2).$$

By the gluing lemma this gives an isomorphism over the entire $\widetilde{X}$ and we get $\widetilde{E'} \simeq \widetilde{E}$. □

5. Framed moduli spaces

NOTATION: With the added superscript $\mathfrak{M}^f(\widetilde{X})$, $\mathfrak{M}^f(X), \mathcal{M}^f$ and $\mathcal{N}^f$ denote the framed versions of $\mathfrak{M}(\widetilde{X})$, $\mathfrak{M}(X), \mathcal{M}$ and $\mathcal{N}$ respectively, framed according to the definitions of section 4.

This section contains a series of lemmas, which prove the following result.

THEOREM 5.1 *$\mathcal{N}_i^f$ is a smooth complex manifold of dimension not greater than $r_i(2r_i+1)$.*

We identify the space $\mathcal{N}_i^f$ with the moduli space $M_i(\Sigma)$ of bundles on the first Hirzebruch surface $\Sigma = \mathbb{P}(\mathcal{O}(1) \oplus \mathcal{O}(-1))$ having second Chern class i, and that are trivial (and framed) over the line at infinity ℓ_∞. We then show that $M_i(\Sigma)$ is smooth. Recall N_0 is the deleted neighborhood of the exceptional divisor, that is, $N_0 = N(\ell) - \ell$.

LEMMA 5.2 *For all E, E' in $\mathcal{N}_i^f$, the map*

$$H^0(\mathrm{Hom}(E, E')) \to H^0(\mathrm{Hom}(E|_{N_0}, E'|_{N_0})$$

is injective.

Proof. N_0 is open in $\widetilde{\mathbb{C}^2}$, hence two holomorphic functions that coincide in N_0 and are globally defined must be equal. $\square$

LEMMA 5.3 *In order to study endomorphisms of a bundle $V \in \mathcal{N}_i$ it suffices to chose a fixed transition matrix T for V and then to consider endomorphisms fixing T.*

Proof. In fact, suppose there is an automorphism ϕ of V taking T to $\bar{T}$. Then, $\phi = (X, Y)$ is given by a pair of transition matrices $X \in \Gamma(U)$ and $Y \in \Gamma(V)$ such that $YTX = \bar{T}$. But, since T and $\bar{T}$ represent the same bundle V, there are change of coordinates $A \in \Gamma(U)$ and $B \in \Gamma(V)$ such that $T = B\bar{T}A$. Now define a new endomorphism of V by $\bar{\phi} = (XA, BY)$, then $\bar{\phi}(T) = BYTXA = B\bar{T}A = T$. $\square$

LEMMA 5.4 *An automorphism of a framed bundle $(V, f) \in \mathcal{N}_i^f$ over N_0 can be written in the form $T \mapsto X^{-1}TX$ with $X \in \Gamma(U_0) \cap \Gamma(V_0)$ and $X = \left(\begin{smallmatrix} a & b \\ 0 & d \end{smallmatrix}\right)$.*

Proof. Let $\phi(V) = (X, Y)$ given by $YTX = T$ be an automorphism of V so that $\phi(V, f) = (V, \bar{f})$ where f and $\bar{f}$ are equivalent framings of V. Let $f = (A, B)$ and $\bar{f} = (\bar{A}, \bar{B})$ be the expressions for these framings in coordinate charts, with $A, \bar{A} \in \Gamma(U_0)$ and $B, \bar{B} \in \Gamma(V_0)$, where $U_0 = U \cap N_0$ and $V_0 = V \cap N_0$. Hence $\phi(A) = XA = \bar{A}$ and $\phi(B) = YB = \bar{B}$.

On the other hand, since f and $\bar{f}$ are trivializations of the trivial bundle on N_0, there exists $G \in \Gamma(sl(2, \mathbb{C}), N_0)$ such that $\bar{A} = GA$ and $\bar{B} = G^{-1}B$. Therefore on U_0 we have $\bar{A} = GA = XA$ and on V_0 we have $\bar{B} = G^{-1}B = YB$. Now A and B are invertible matrices, hence $X^{-1} = G^{-1} = Y$ on $U_0 \cap V_0$.

Suppose $X = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$. From the equality $XT = TX$ we get $-z^{-j}c = z^j c$, which immediately implies that $c = 0$. $\square$

LEMMA 5.5*An automorphism of a framed bundle $(V, f) \in \mathcal{N}_i^f$ over $N(\ell) \simeq \widetilde{\mathbb{C}^2}$ can be written in the form $T \mapsto X^{-1}TX$ with $X = \left(\begin{smallmatrix} a_0 + a & b \\ 0 & d_0 + d \end{smallmatrix}\right)$ where $a, b, d \in \Gamma(U_0) \cap \Gamma(V_0)$ and a, b and d vanish over the exceptional divisor and a_0, d_0 are constants..*

Proof. By lemma 0.3 $X = \left(\begin{smallmatrix} a & b \\ 0 & d \end{smallmatrix}\right)$ over N_0 and we must check what are the possible extensions of X to the full coordinate charts U and V. Now consider any possible holomorphic extension $\left(\begin{smallmatrix} \bar{a} & \bar{b} \\ \bar{c} & \bar{d} \end{smallmatrix}\right)$ to the entire $U-$ chart. We claim the extension is also of the form $X = \left(\begin{smallmatrix} \bar{a} & \bar{b} \\ 0 & \bar{d} \end{smallmatrix}\right)$. In fact, $U \simeq \mathbb{C}^2$ and $\bar{c} = c = 0$

on U_0 which is an open subset of U hence $\bar{c} = 0$ everywhere on U. Similarly on V we have the form $X = \left(\begin{smallmatrix} \bar{\alpha} & \bar{\beta} \\ 0 & \bar{\delta} \end{smallmatrix}\right)$. Write $\bar{x} = x_o(z) + x(z,u)$ for $x \in \{a,b,d\}$ and write $\bar{y} = \bar{y}_0(z^{-1}) + y(z^{-1}, zu)$ for $y = \{\alpha, \beta, \delta\}$. Over the exceptional divisor one has $p = 0 = u$ and the equality $(*)$ becomes

$$\begin{pmatrix} z^j a_0 & z^j b_0 \\ 0 & z^{-j} d_0 \end{pmatrix} = \begin{pmatrix} z^j \alpha_0 & z^{-j} \beta_0 \\ 0 & z^{-j} \delta_0 \end{pmatrix}$$

where a_0, b_0, d_0 are holomorphic in z whereas $\alpha_0, \beta_0, \delta_0$ are holomorphic in z^{-1}. It immediately follows that $b_0 = \beta_0 = 0$ and that $a_0 = \alpha_0$ and $d_0 = \delta_0$ are constants. □

Lemma 5.6 *A framed bundle $(V, f) \in \mathcal{N}_i^f$ has no traceless automorphisms, unless V splits.*

Proof. The equality $trace(\lambda X) = \lambda^2 trace(X)$ holds for any $X \in GL(2, \mathbb{C})$ and constant λ. Therefore, the bundle V has traceless $GL(2, \mathbb{C})$ automorphisms if and only if it has traceless $SL(2, \mathbb{C})$ automorphims. So, we assume $det(X) = 1$. Over the exceptional divisor $X|_\ell = \left(\begin{smallmatrix} a_0 & 0 \\ 0 & d_0 \end{smallmatrix}\right)$. Hence $a_o d_0 = 1$. If $trace(X)$ is zero, then we also have $a_0 + d_0 = 0$. It follows that $a_0 = \pm i$. Suppose $a_0 = i$, the other case is analogous. Since X is traceless, it then follows that $X = \left(\begin{smallmatrix} i+a & b \\ 0 & -i-a \end{smallmatrix}\right)$. On the other hand $det(X) = -(i+a)^2 = 1$ implies that either $a = 0$ or $a = -2i$. Therefore $X = \pm \left(\begin{smallmatrix} i & b \\ 0 & -i \end{smallmatrix}\right)$. Now the equality $XT = TX$ gives

$$\begin{pmatrix} iz^j & ip + z^{-j} b \\ 0 & -iz^{-j} \end{pmatrix} = \begin{pmatrix} iz^j & -ip + z^j b \\ 0 & -iz^{-j} \end{pmatrix}$$

implying

$$2ip = (z^j - z^{-j})b.$$

But b has only positive powers of z, hence the r.h.s. contains powers of z greater of equal to j whereas the l.h.s. only has powers of z strictly smaller than j. Hence $b = 0 = p$. We conclude that (V, f) has no traceless automorphisms unless V splits, in which case the automorphisms are multiples of $\left(\begin{smallmatrix} i & 0 \\ 0 & -i. \end{smallmatrix}\right)$. □

Proposition 5.7$M_i(\Sigma)$ *is smooth.*

Proof. By [L] Theorem 1.1, given the conditions of lemma 0.1 above, $M_i(\Sigma)$ is smooth at E provided that $H^2(sl(E) \otimes \mathcal{O}_\Sigma(-\ell_\infty)) = 0$, where $sl(E)$ is the bundle of traceless endomorphisms of E. But lemma 5.6 implies that there are no traceless endomorphisms of E, unless E splits, in which case $sl(E)$ is trivial. The same conclusions then holds also for bundles on $M(\Sigma)$. Therefore $H^2(sl(E) \otimes \mathcal{O}_\Sigma(-\ell_\infty)) = 0$. □

NOTATION:

- $S_{i,j}$ denotes the space of framed bundles having charge i and splitting type j. Hence $S_{i,j} = \mathcal{N}_i^f \cap \mathcal{M}_j^f$.
- For any integer i, let r_i^2 be the smallest square not lower than i. Hence $r_i = \lceil \sqrt{i} \rceil$.

PROPOSITION 5.8 *$dim\mathcal{N}_i^f$ is not greater than $r_i(2r_i + 1)$.*

Proof. The dimension of $\mathcal{N}_i^f$ is calculated in the following sequence of 3 lemmas, showing that

$$\dim \mathcal{N}_i^f \leq \dim \mathcal{M}_{r_i}^f = \dim S_{r_i^2, r_i} = r_i(2r_i + 1).$$

□

LEMMA 5.9 $\dim \mathcal{N}_i^f \leq \dim \mathcal{M}_{r_i}^f$. *In particular,* $\dim S_{i,j} \leq \dim S_{i,r_i}$ *for all* $j \geq r_i$.

Proof. By [G1] the inequality $j \leq i \leq j^2$ between the charge i and the splitting type j is sharp. Therefore, bundles on $\mathcal{N}_i^f$ have splitting type j varying as $r_i \leq j \leq i$.

Let R be a bundle on $\mathcal{N}_i^f$ with splitting type r_i. The dimension of $\mathcal{N}_i^f$ at R is given by $\dim Def_{\mathcal{N}i}(R) + \dim Fram(R)$, where $Def_{\mathcal{N}_i}(R)$ is the space of local deformations of R inside $\mathcal{N}_i$ and $Fram(R)$ is the space of inequivalent reframings of R.

We claim that $Def_{\mathcal{N}i}(R) \subset Def_{\mathcal{M}_{r_i}}(R)$. In fact, by [G3] R can be written as an extension of $\mathcal{O}(r_i)$ by $\mathcal{O}(-r_i)$, and it follows that any small deformation of R has splitting type smaller or equal to r_i. But deformations with splitting type lower than r_i fall outside of $\mathcal{N}_i$. Hence $Def_{\mathcal{N}i}(R) \subset Def_{\mathcal{M}_{r_i}}(R)$ and consequently $\dim \mathcal{N}_i^f \leq \dim \mathcal{M}_{r_i}^f$. □

LEMMA 5.10 $\dim \mathcal{M}_j^f = \dim S_{j^2,j}$. *In particular,* $\dim S_{k,j} \leq \dim S_{j^2,j}$, *for all* $k \leq j^2$. □

Proof. By]G3] elements of $\mathcal{M}_j$ are determined by pairs (j, p) where p has the form $p = \sum_{i=1}^{2j-2} \sum_{l=i-j+1}^{j-1} p_{il} z^l u^i$. Consequently, $\mathcal{M}_j$ is parametrized by the N

coefficients of the polynomial and is identified with a quotient $\mathbb{C}^N/\sim$ where $\sim$ is the equivalence determined by bundle isomorphism. The relation $p \sim \lambda p$ holds for all $\lambda \neq 0$ and for all p and this implies that any neighborhood of $0 \in \mathbb{C}^N/\sim$ contains the entire space. By [BG2, thm. 4.1] the space of framed isomorphism classes of bundles having underlying data $(j, 0)$ equals $S_{j^2,j.}$ It follows that $\dim S_{k,j} \leq \dim S_{j^2,j} = \dim \mathcal{M}_j$. □

LEMMA 5.11 $\dim S_{j^2,j} = j(2j+1)$.

Proof. $S_{j^2,j}$ is the space of inequivalent framings of the split bundle $\mathcal{O}(j) \oplus \mathcal{O}(-j)$. In canonical coordinates as in $(*)$ the split bundle is represented by the transition matrix $T = \left(\begin{smallmatrix} z^j & 0 \\ 0 & z^{-j} \end{smallmatrix}\right)$ and has trivializing section on N_0 given by $s_1 = (u^j, 0)$ and $s_2 = (0, u^{-j})$. To calculate the dimension of the space of inequivalent framings, we use the method of [Hu, p. 116]. The space of all possible framings is obtained from the canonical one by the action of the group $Sl(2, \mathbb{C})_{N_0}$ of holomorphic maps $N_0 \to Sl(2, \mathbb{C})$. The space of inequivalent framings will be the quotient of $Sl(2, \mathbb{C})_{N_0}$ by the subgroup H_{N_0} of maps that extend to an isomorphism also over the exceptional divisor. We set $A = \left(\begin{smallmatrix} u^j & 0 \\ 0 & u^{-j} \end{smallmatrix}\right)$ and $B = \left(\begin{smallmatrix} zu^j & 0 \\ 0 & zu^{-j} \end{smallmatrix}\right)$. We have the following diagram of transition matrices

$$\begin{array}{ccc} U \times \mathbb{C} & \xrightarrow{T} & V \times \mathbb{C} \\ A \downarrow & & \downarrow B \\ U \times \mathbb{C} & \xrightarrow{I} & V \times \mathbb{C} \\ G \downarrow & & \downarrow G \\ U \times \mathbb{C} & \xrightarrow{I} & V \times \mathbb{C} \\ A^{-1} \downarrow & & \downarrow B^{-1} \\ U \times \mathbb{C} & \xrightarrow{T} & V \times \mathbb{C} \end{array}$$

The condition that $G \in HN_0$ is then that $A^{-1}GA$ be holomorphic nonsingular in $U \cap N_0$ and that $B^{-1}GB$ be holomorphic nonsingular in $V \cap N_0$.

$$A^{-1}GA = \begin{pmatrix} G_{11} & u^2 G_{12} \\ u^{-2j} G_{21} & G_{22} \end{pmatrix}.$$

The condition $u^{-2j}G_{21} \in \Gamma(U)$ implies that the part of G_{21} that does not extend to an isomorphism over the divisor has the form $\sum_{i=0}^{2j-1} \sum_{l \leq i} g_{il} z^l u^i$. There are $j(2j+1)$ such coefficients. It follows that $\dim S_{j^2,j} = j(2j+1)$. □

6. Topology of $\mathfrak{M}_k(\widetilde{X})$

We assume that the Atiyah–Jones conjecture holds true for X, that is, we assume that there exist maps $r_k\colon \mathfrak{M}^f_k(X) \to \mathfrak{M}^f_{k+1}(X)$ inducing isomorphisms in homology $r_{k*}\colon H_q(\mathfrak{M}^f_k(X)) \to H_q(\mathfrak{M}^f_{k+1}(X))$ for $q \leq \lfloor k/2 \rfloor - c$ where c is a constant.

6.1 Statement of the conjecture

For a 4-manifold X, let $\mathcal{MI}_k(X)$ denote the moduli space of rank 2 instantons on E with charge k and let $\mathcal{B}^\epsilon_k(X)$ denote the space of connections on X whose anti-self-dual part of the curvature has norm less than ϵ. Given a point $x_o \in X$, Taubes [Ta] constructed a map

$$t^{x_0}_k\colon \mathcal{MI}_k(X) \to \mathcal{B}^\epsilon_{k+1}(X)$$

by patching a small instanton on a neighborhood of x_0. The map $t^{x_0}_k$ takes an instanton on X with charge k to a connection on X with charge $k+1$ whose anti-self-dual part of the L^2 norm of its curvature is less than ϵ. Taubes also constructed a strong deformation retract

$$\tau_{k+1}\colon \mathcal{B}^\epsilon_{k+1}(X) \to \mathcal{MI}_{k+1}(X).$$

Based on Taubes' results, the Atiyah–Jones conjecture in homology is equivalent to the statement that the maps $i_k = \tau_{k+1} \circ t^{x_0}_k$ induce isomorphisms in homology

$$i_{k*}\colon H_q(\mathcal{MI}_k(X)) \to H_q(\mathcal{MI}_{k+1}(X))$$

through a range $q(k)$ increasing with k.

For a Kähler surface X, Kobayashi and Hitchin gave a one-to-one correspondence between instantons on a topological bundle E on X and holomorphic structures on E, see [LT]. The correspondence is made by taking the holomorphic part of an anti-self-dual connection. Let

$$\mathcal{K}_X\colon \mathcal{MI}_k(X) \to \mathfrak{M}_k(X)$$

denote the Kobayashi–Hitchin map on X given by $\mathcal{K}(\nabla = \partial + \overline{\partial}) = \overline{\partial}$, and let $H_X\colon = D_X^{-1}$ be the inverse map. For the noncompact surface $X = \widetilde{\mathbb{C}^2}$ this correspondence takes an instanton to a holomorphic bundle with an added trivialization at infinity [Ki]. In terms of instantons proposition 4.2 can be restated as:

PROPOSITION 6.1 *Every instanton on $\widetilde{X}$ is obtained by holomorphic patching and instanton on X to an instanton on $\widetilde{\mathbb{C}^2}$.*

6.2 Stratifications

DEFINITION A smooth manifold M is L-stratified if there is a decomposition of M into disjoint submanifolds $M(K)$ such that
(1) The index set $\mathcal{K} = \{K\}$ is finite with a given fixed well ordering $\leq$.
(2) If K_0 is the smallest element in $(\mathcal{K}, \leq)$, then $M(K_0)$ is an open-dense subset of M.
(3) For all $K \in \mathcal{K}$ the union of the submanifolds of the same or smaller order

$$Z(K) = \cup_{K' \leq K} M(K')$$

is an open-dense submanifold of M.
(4) For all $K \in \mathcal{K}$ the normal bundle, $\nu(K)$, of $M(K)$ in M is orientable.

PROPOSITION 6.1 *There is an L-stratification of the moduli space of framed bundles on $\widetilde{X}$ as*

$$\mathfrak{M}_k^f(\widetilde{X}) \simeq \bigcup_{i=0}^{k} \mathfrak{M}_{k-i}^f(X) \times \mathcal{N}_i^f. \qquad (*)$$

Proof. The existence of the decomposition follows directly from the definition of the $\mathcal{N}_i$. The first stratum $K_0 = \mathfrak{M}_k^f(X) \times \mathcal{N}_0^f$ equals the set of pull-back bundles and is dense in $\mathfrak{M}_k^f(\widetilde{X})$. To show this gives an L-stratification we need to prove that the normal bundle of each stratum $K_i = \mathfrak{M}_{k-i}^f(X) \times \mathcal{N}_i^f$ is orientable. Using the results from sections 3.1 and 3.2, we may assume that $\mathfrak{M}_k^f(\widetilde{X})$ is smooth and that $\mathfrak{M}_i^f(X)$ is smooth and contains only stable bundles. By theorem 5.1 $\mathcal{N}_i^f$ is a complex manifold. It then follows that each stratum K_i is a complex submanifold of $\mathfrak{M}_k^f(\widetilde{X})$ and therefore has orientable normal bundle. □

6.3 Maps between spectral sequences

Whereas it is not strictly true that Taubes map on $\widetilde{X}$ preserves the stratification $(*)$, we show that the map on $\mathfrak{M}_k(\widetilde{X})$ induced by Taubes map is homotopy equivalent to a map that preserves the stratification. Recall that we are considering the stratifications;

$$\mathfrak{M}_k^f(\widetilde{X}) = \bigoplus K_i, \text{ and } \mathfrak{M}_{k+1}^f(\widetilde{X}) = \bigoplus K_i'$$

where $K_i = \mathfrak{M}_{k-i}^f(X) \times \mathcal{N}_i^f$ and $K_i' = \mathfrak{M}_{k-i+1}^f(X) \times \mathcal{N}_i^f$.

Let $d\colon \mathfrak{M}_k^f(\widetilde{X}) \to \mathfrak{M}_k^f(\widetilde{X})$ be the map that re-writes a bundle into its two components of the decomposition, $d(\widetilde{E}) = (E, V)$ as in proposition 4.2. We choose a point x_0 in X that is far from the point x we blew-up, in the sense that $x_0 \notin \pi_*(N(\ell))$ and we set $\widetilde{x_0} = \pi^*(x_0)$. Let

$$t_k^{x_o} \colon \mathcal{MI}_k(X) \to \mathcal{B}^\epsilon_{k+1}(X) \text{ and } t_k^{\widetilde{x_o}} \colon \mathcal{MI}_k(\widetilde{X}) \to \mathcal{B}^\epsilon_{k+1}(\widetilde{X})$$

denote Taubes patching on a small neighborhood of x_o in X and of $\widetilde{x}_0$ in $\widetilde{X}$ respectively, taking an instanton of charge k to a connection of charge $k+1$ whose anti-self-dual part of the L^2 norm of its curvature is less than ϵ. Taubes also constructed strong deformation retracts

$$\tau_{k+1} \colon \mathcal{B}^\epsilon_{k+1} \to \mathcal{MI}_{k+1}.$$

The idea now is that choosing x to be far from the exceptional divisor, the the patching remains far from the exceptional divisor. From the point of view of vector bundles, this implies that we can choose the map so that it does not alter the bundle near the divisor. We now make this statement precise.

Lemma 6.2 *The maps* $\mathbf{t}_k \colon \mathfrak{M}^f_k(\widetilde{X}) \to \mathfrak{M}^f_{k+1}(\widetilde{X})$ *and* $\mathbf{r}_k \colon \mathfrak{M}^f_k(\widetilde{X}) \to \mathfrak{M}^f_{k+1}(\widetilde{X})$ *given by*

$$\mathbf{t}_k \colon = \mathcal{K}_{\widetilde{X}} \circ \tau_{k+1} \circ t_k^{\widetilde{x}_0} \circ \mathcal{H}_{\widetilde{X}}$$

and

$$\mathbf{r}_k \colon = d^{-1} \circ \mathcal{K}_{\widetilde{X}} \circ \tau_{k+1} \circ (t^k_{x_0}, 1) \circ (\mathcal{H}_X, \mathcal{H}_{\widetilde{\mathbb{C}^2}}) \circ d$$

are homotopically equivalent.

Proof. The following diagram summarizes the situation

$$\begin{tikzcd}
\mathfrak{M}^f_k(\widetilde{X}) \arrow[r, "\mathcal{H}_{\widetilde{X}}"] \arrow[d, "d"] & \mathcal{MI}_k(\widetilde{X}) \arrow[r, "t_k^{\widetilde{x}_0}"] \arrow[d, equal] & \mathcal{B}^\epsilon_{k+1}(\widetilde{X}) \arrow[r, "\tau_{k+1}"] \arrow[d, equal] & \mathcal{MI}_{k+1}(\widetilde{X}) \arrow[r, "\mathcal{K}_{\widetilde{X}}"] \arrow[d, equal] & \mathfrak{M}^f_{k+1}(\widetilde{X}) \\
\mathfrak{M}^f_k(\widetilde{X}) \arrow[r, "(\mathcal{H}_X, \mathcal{H}_{\widetilde{\mathbb{C}}^2})"] & \mathcal{MI}_k(\widetilde{X}) \arrow[r, "(t^k_{x_0}, 1)"] & \mathcal{B}^\epsilon_{k+1}(\widetilde{X}) \arrow[r, "\tau_{k+1}"] & \mathcal{MI}_{k+1}(\widetilde{X}) \arrow[r, "\mathcal{K}_{\widetilde{X}}"] & \mathfrak{M}^f_{k+1}(\widetilde{X}) \arrow[u, "d^{-1}"]
\end{tikzcd}.$$

We need only to show that $t_k^{\widetilde{x}_0} \sim (t^k_{x_0}, 1)$. But the space of connections if an affine space, hence the linear homotopy $st_k^{\widetilde{x}_0} + (1-s)(t^k_{x_0}, 1)$ will do (divide by the norm and multiply by $\epsilon/2$ here if necessary). □

Proposition 6.3 $\mathbf{t}_k \colon \mathfrak{M}^f_k(\widetilde{X}) \to \mathfrak{M}^f_{k+1}(\widetilde{X})$ *is homotopy equivalent to a map that preserves the filtrations.*

Proof. This is just a restatement of lemma 6.1. □

6.4 Proof of AJ for blow-ups

LEMMA 6.4 *The real codimension of the stratum K_i is at least $12i - 6(r_i + 1)$.*

Proof. $K_i = \mathfrak{M}^f_{k-i}(X) \times \mathcal{N}^f_i$. The complex codimension of K_i is

$$\mathrm{cod}K_i = \dim\mathfrak{M}^f_k(\widetilde{X}) - (\dim\mathfrak{M}^f_{k-i}(X) + \dim\mathcal{N}^f_i) =$$

$$= 8k - 3(1 - b_1 + b^+) - (8(k-i) - 3(1 - b_1 + b^+) + \dim\mathcal{N}^f_i).$$

According to theorem 5.1, the complex codimension of $\mathcal{N}^f_i = r_i(2r_i + 1)$. Hence, $\mathrm{cod}K_i = 8i - r_i(2r_i + 1) \geq 6i - 3(r_i + 1)$. □

Let $\mathfrak{S}_k$ denote the Leray spectral sequence associated to the filtration $\{K_i\}$ of $\mathfrak{M}^f_k(\widetilde{X})$ and let $\mathfrak{S}'_{k+1}$ denote the Leray spectral sequence associated to the filtration $\{K'_i\}$ of $\mathfrak{M}^f_{k+1}(\widetilde{X})$.

THEOREM 6.5 *The maps $\mathbf{t}_k \colon \mathfrak{M}^f_k(\widetilde{X}) \to \mathfrak{M}^f_{k+1}(\widetilde{X})$ induce an isomorphisms in homology $H_q(\mathfrak{M}^f_k(\widetilde{X})) \to H_q(\mathfrak{M}^f_{k+1}(\widetilde{X}))$ for $q \leq \lfloor k/2 \rfloor - c$.*

Proof. By proposition 6.3 we may assume that $\mathbf{t}_k \colon \mathfrak{M}^f_k(\widetilde{X}) \to \mathfrak{M}^f_{k+1}(\widetilde{X})$ respects the filtrations and therefore induces a map of spectral sequences $\mathbf{t}_k \colon \mathfrak{S}_k \to \mathfrak{S}'_{k+1}$. The E^1 term of $\mathfrak{S}_k$ is $\bigoplus_{K_i} H_*(T\nu(K_i))$ and the E^1 term of $\mathfrak{S}'_{k+1}$ is $\bigoplus_{K'_i} H_*(T\nu(K'_i))$. By hypothesis Atiyah–Jones holds for X, hence we are assuming that for each k and for $q \leq \lfloor k/2 \rfloor - c$, $H_q(\mathfrak{M}^f_k(X)) = H_q(\mathfrak{M}^f_{k+1}(X))$. Consequently

$$H_q(K_i) = H_q(\mathfrak{M}^f_{k-i}(X) \times \mathcal{N}^f_i) = H_q(\mathfrak{M}^f_{k-i+1}(X) \times \mathcal{N}^f_i) = H_q(K'_i)$$

for $q \leq \lfloor (k-i)/2 \rfloor - c$ (where we might need to take the largest of the c's if they vary with i). The Thom isomorphism gives $H_*(T\nu(K_i)) = H_*(\Sigma^{\tau(K_i)} K_i)$, where $\tau(K_i)$ is the real codimension of K_i and similarly for K'_i. By lemma 6.4 the real codimension of K_i at least $12i - 6(r_i + 1)$. Therefore

$$H_q(T\nu(K_i)) = H_q(T\nu(K'_i)) \text{ for } q \leq \lfloor (k-i)/2 \rfloor - c + 12i - 6(r_i + 1).$$

Since $\lfloor (k-i)/2 \rfloor - c + 12i - 6(r_i + 1) \geq \lfloor k/2 \rfloor - c$, we conclude that $\mathbf{t}_k \colon \mathfrak{S}_k \to \mathfrak{S}'_{k+1}$ induces an isomorphism of E^1 terms for all $E^1_{r,s}$ with $r + s \leq \lfloor k/2 \rfloor - c$. □

COROLLARY 6.4 *The Atiyah–Jones conjecture is true for rational surfaces.*

Proof. Every rational surface is obtained by blowing up points on $\mathbb{P}^2$ or on a rational ruled surface, and on these cases the conjecture holds true by [BHMM] and [HM]. $\square$

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