Trace Theorems for Sobolev Spaces of Variable Order of Differentiation

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Abstract
We prove a trace theorem and an extension theorem for Sobolev spaces of variable order of differentiation which are defined by pseudodifferential operators.

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Contents

Introduction

Trace and extension theorems for the classical Sobolev spaces have been known for well over 30 years. They go back to Aronszajn, Besov, Slobodeckij, Stein, and many others. Kaljabin obtained corresponding results for the so-called spaces of functions of generalized smoothness.

The theory of pseudodifferential operators led to yet another type of function spaces, where the smoothness varies in space. Although these spaces have been investigated for almost three decades, cf. Unterberger and Bokobza [23], it seems that no general trace or extension theorems have been proven. One reason might be that the standard pseudodifferential methods fail in this situation. The present paper gives a solution for a class of such spaces.

For certain hypoelliptic pseudodifferential symbols a we define the Sobolev space $W_2^{1,a}(\mathbb{R}^n)$ by

$$W_2^{1,a}(\mathbb{R}^n) = \left\{u \in S'(\mathbb{R}^n) : u \in L^2(\mathbb{R}^n) \,\text{and}\, Au \in L^2(\mathbb{R}^n)\right\};$$

here, $A = Op\,a$. We then show that the restrictions of these functions to the hyperplane $\{x_n = 0\}$ belong to $W_2^{1,a'}(\mathbb{R}^{n-1})$, where $a'(x',\xi') = \langle\xi'\rangle^{-1/2} \# a(x',0,\xi',0)$, and that the restriction operator is in fact surjective. Except for the required hypoellipticity, the essential assumption for the trace theorem is a conormal ellipticity condition guaranteeing a minimal growth for the quotient $a(x,\xi)/a(x,\xi',0)$; for details see Theorem 2.1. Conversely, in order to obtain the extension theorem, we have to control the quotient $a(x',0,\xi)/a(x',0,\xi',0)$ from above, cf. Theorem 2.2. Both conditions are quite natural from the point of view of B.-W. Schulze's wedge Sobolev spaces, cf. [21], [20], although, for technical reasons, this relationship is not exploited in the present proof.

An example for a symbol a where all assumptions are fulfilled is $a(x,\xi) = \langle\xi\rangle^{\sigma(x)}(1 + \ln\langle\xi\rangle)^t$ where σ is a smooth function with all derivatives bounded and $\inf\sigma(x) > 1/2$; $t \in \mathbb{R}$.

This includes the classes considered by Unterberger-Bokobza and Beauzamy, where σ was the sum of a constant and a rapidly decreasing function. In addition, symbols of the form $a(x,\xi) = \langle\xi\rangle^{\sigma(x)}$ recently have gained interest as generators of Feller semi-groups and Markov processes [8], [18], [11]. The present results should provide a further step towards the understanding of boundary value problems also for operators with symbols such as these.

We reduce the proof of the trace theorem to a boundedness result for a composition of pseudodifferential operators on standard Sobolev spaces over $\mathbb{R}^n$. Although simple in spirit, this requires a careful analysis since the symbols a and a' live on spaces of different dimension.

Vice versa, the extension operator is constructed in a straightforward way. Again we reduce the problem to showing the boundedness of a composition of operators on Sobolev spaces. This time, however, we have to show continuity from $H^k(\mathbb{R}^{n-1})$ to $H^k(\mathbb{R}^n)$ for large k, and the operators are partly pseudodifferential, partly potential operators in Boutet de Monvel's calculus.

Using different methods, Pesenson [19] has recently proven trace and extension theorems for a certain class of non-isotropic Sobolev spaces on the Heisenberg group, given by vector fields satisfying Hörmander's condition.

The present proof does not carry over to the case of Besov spaces $B^s_{p,q}$ with $p, q \neq 2$. The reason is the well-known limitation of the boundedness results for standard pseudodifferential operators: For $p \neq 2$, the continuity of operators with symbols in $S^0_{\rho,\delta}$ on L^p

Sobolev spaces requires that $\rho = 1$, cf. [6], while here we take advantage of the fact that, for $p = 2$, the much weaker condition $0 \le \delta \le \rho \le 1, \delta < 1$ is sufficient. The basic concept of reducing both the trace and the extension theorem to boundedness results seems to be quite general. We therefore hope that, for a restricted class of symbols, trace and extension theorems can be obtained also in the case $p \neq 2$ using more refined continuity results.

1 Pseudodifferential Operators and Sobolev Spaces of Variable Order of Differentiation

A function $p(x,\xi)$ belongs to the class $S^m_{\rho,\delta}$, $-\infty < m < \infty$, $0 \le \delta$, $\rho \le 1$, $\delta < 1$, provided that for all multi-indices α, β there is a constant $c_{\alpha\beta}$ such that

$$(1) \qquad \left|p^{(\alpha)}_{(\beta)}(x,\xi)\right| \le c_{\alpha\beta} \langle\xi\rangle^{m-\rho|\alpha|+\delta|\beta|}$$

for all $x, \xi \in I\!R^n$. Here $p^{(\alpha)}_{(\beta)} = \partial^\alpha_\xi D^\beta_x p(x,\xi)$ and $\langle\xi\rangle = (1+|\xi|^2)^{1/2}$.
The set of all functions $p(x,\xi)$ such that (1) only holds for $|\alpha| \le I$ and $|\beta| \le J$ will be denoted by $S^m_{\rho,\delta}(I,J)$, $I, J \in I\!N$.
For $p \in S^m_{\rho,\delta}$ or $p \in S^m_{\rho,\delta}(I,J)$ define the semi-norms

$$|p|^{(m)}_{(l,k)} = \max_{\substack{|\alpha|\le l \\ |\beta|\le k}} \sup_{x,\xi} \left\{ \left|p^{(\alpha)}_{(\beta)}(x,\xi)\right| \langle\xi\rangle^{-m+\rho|\alpha|-\delta|\beta|} \right\} .$$

The pseudodifferential operator $p(x,D_x)$ or $Op\,p$ with symbol $p(x,\xi)$ is defined by

$$[p(x,D_x)\,u](x) = \int e^{ix\xi}\, p(x,\xi)\, \hat{u}(\xi)\, đ\xi$$

for u in the Schwartz space $\mathcal{S}(I\!R^n)$; $\hat{u}(\xi) = \int e^{-ix\xi}u(x)\,dx$ is the Fourier transform of u, and $đ\xi = (2\pi)^{-n}d\xi$. For $p \in S^m_{\rho,\delta}$,

$$p(x,D_x) : \mathcal{S}(I\!R^n) \to \mathcal{S}(I\!R^n)$$

is continuous and can be extended to a continuous operator from $\mathcal{S}'(I\!R^n)$ to $\mathcal{S}'(I\!R^n)$. Concerning the composition we will need the following observation.

1.1 Lemma. *Let $p_1 \in S^{m_1}_{\rho_1,\delta_1}$, $p_2 \in S^{m_2}_{\rho_2,\delta_2}$. Then $p_1(x,D_x)p_2(x,D_x) = p(x,D_x)$ has a symbol $p(x,\xi)$ with the following expansion*

$$(2) \qquad p(x,\xi) = \sum_{|\alpha|\le N} \frac{1}{\alpha!} p_1^{(\alpha)}(x,\xi)\, p_{2(\alpha)}(x,\xi) + r_N(x,\xi),$$

where

$$r_N(x,\xi) = (N+1) \sum_{|\gamma|=N+1} \int_0^1 \frac{(1-\theta)^N}{\gamma!} r_{\gamma,\theta}(x,\xi)\,d\theta$$

and

$$(3) \qquad r_{\gamma,\theta}(x,\xi) = (2\pi)^{-n}\mathrm{Os}\!-\!\iint e^{-iy\eta}\, p_1^{(\gamma)}(x,\xi+\theta\eta)\, p_{2(\gamma)}(x+y,\xi)\,dy\,d\eta$$

We shall write $p = p_1 \# p_2$.

As usual, cf. [12, Chapter 2, Theorem 3.1], this follows from Taylor's formula combined with integration by parts for the particular form of $r_{\gamma,\theta}$.
For $\rho_1 = \rho_2 = \rho$, $\delta_1 = \delta_2 = \delta$ and $\delta \le \rho$ we have $p \in S^{m_1+m_2}_{\rho,\delta}$ and $r_N \in S^{m_1+m_2-(N+1)(\rho-\delta)}_{\rho,\delta}$.

1.2 Theorem. *Let $p \in S^m_{\rho,\delta}$, $0 \le \delta \le \rho \le 1$, $\delta < 1$, and $s \in I\!R$. Then there exist integers I, J such that*

$$\|p(x,D_x)|\mathcal{L}(H^s_2(I\!R^n), H^{s-m}_2(I\!R^n))\| \le C|p|^m_{(I,J)} \quad .$$

$H^s_2(I\!R^n)$ is the usual Sobolev space of all distributions u with $(1-\Delta)^{\frac{s}{2}}u \in L_2(I\!R^n)$.

The theorem goes back to Calderón and Vaillancourt, cf. [12, Chapter 7], Theorem 1.6; in fact it is sufficient to ask that $p \in S^m_{\rho,\delta}(I,J)$.

In [14] and [15], Besov and Sobolev spaces of variable order of differentiation were defined.

1.3 Definition. Let $0 \le \delta < 1$ and $0 < m' \le m$. A symbol $a(x,\xi) \in S^m_{1,\delta}$ belongs to the class $S(m,m';\delta)$ if there is a constant $R \ge 0$ with the following properties

(i) for all multi-indices α, β, and all $(x,\xi) \in I\!R^n \times I\!R^n$ with $|\xi| \ge R$

$$\left|a^{(\alpha)}_{(\beta)}(x,\xi)\right| \le c_{\alpha\beta}|a(x,\xi)|\,\langle\xi\rangle^{-|\alpha|+\delta|\beta|}; \tag{4}$$

(ii) there exist constants $c, c' > 0$ such that for all $(x,\xi) \in I\!R^n \times I\!R^n$ with $|\xi| \ge R$

$$c'\,\langle\xi\rangle^{m'} \le |a(x,\xi)| \le c\,\langle\xi\rangle^{m}. \tag{5}$$

The symbols of the class $S(m,m';\delta)$ are a good substitute for the symbol $|\xi|^2$ of the Laplacian which is used in the definition of the usual function spaces. For examples see [14], [15], [12].

1.4 Remark.
(i) If $a \in S(m,m';\delta)$, $b \in S(\tilde m, \tilde m'; \tilde\delta)$, then

$$ab \in S(m+\tilde m, m'+\tilde m'; \max(\delta,\tilde\delta)).$$

(ii) The symbol $c(x',\xi') = a(x',0,\xi',0)$ belongs to $S(m,m',\delta)$ with respect to $I\!R^{n-1}$.
(iii) The elements of $S(m,m',\delta)$ are hypoelliptic. Given $a \in S(m,m',\delta)$ we can construct a parametrix $b(x,D_x)$ to $a(x,D_x)$; i.e. $b(x,D_x)a(x,D_x)-I$ and $a(x,D_x)b(x,D_x)-I$ both have symbols in $S^{-\infty} = \bigcap S^m_{\rho,\delta}$. Moreover, b will satisfy the estimates

$$\left|b^{(\alpha)}_{(\beta)}(x,\xi)\right| \le c_{\alpha\beta}|a(x,\xi)|^{-1}\,\langle\xi\rangle^{-|\alpha|+\delta|\beta|} \tag{6}$$

for all $(x,\xi) \in I\!R^n \times I\!R^n$, $|\xi| \ge R$, cf. [12, Chapter 2, §5].

1.5 Definition. For $j = 1,2,\ldots$, and $a \in S(m,m';\delta)$ let $A = Op\,a$ and

$$W^{j,a}_2(I\!R^n) = \left\{u \in S'(I\!R^n) : \left\|u\,\middle|W^{j,a}_2(I\!R^n)\right\| < \infty\right\},$$

where

$$\left\|u\,\middle|W^{j,a}_2(I\!R^n)\right\| = \left\|A^j u\right| L_2\| + \|u\,|L_2\|.$$

The definition extends to $1 < p < \infty$, cf. [14]. It shows a convenient way to relate function spaces and pseudodifferential operators, cf. [22], [24],[25] [26], [1]. We shall need the following fact.

1.6 Theorem. *$W_2^{j,a}(\mathbb{R}^n)$ is a Hilbert space, and $\mathcal{S}(\mathbb{R}^n)$ is a dense subset. For $p \in S^0_{1,\delta}$ we have $p(x, D_x) \in \mathcal{L}(W_2^{j,a}(\mathbb{R}^n))$. If $\tilde{a}$ is an other symbol and $a - \tilde{a} \in S^{-\infty}$ then $W_2^{j,a}(\mathbb{R}^n) = W_2^{j,\tilde{a}}(\mathbb{R}^n)$.*

For further results see [15].

1.7 Lemma. $W_2^{j,a}(\mathbb{R}^n) = W_2^{1,a^j}(\mathbb{R}^n)$.

P r o o f. By Remark 1.4, $a^j \in S(jm, jm'; \delta)$. Let $B_{(j)} = Op\, b_{(j)}$ be a parametrix to $Op\, a^j$. Then

$$\| A^j u |\, L_2 \| \leq \| A^j B_{(j)} Op\, a^j u |\, L_2 \| + \| A^j R_{(j)} u |\, L_2 \|$$

with $R_{(j)} \in S^{-\infty}$, so that the last term can be estimated by $c\|u|L_2\|$. Applying (6), a careful analysis of $a\#\ldots\#a\#b_{(j)}$ gives $A^j B_{(j)} \in S^0_{1,\delta}$. By Theorem 1.6 we get

$$\| A^j u |\, L_2 \| \leq c \left\| u \left| W_2^{1,a^j} \right. \right\| .$$

Vice versa, let $B = Op\, b$ be a parametrix to $A = Op\, a$. Then B^j is a parametrix to A^j, and

$$\| Op\, a^j u |\, L_2 \| \leq \| Op\, a^j B^j A^j u |\, L^2 \| + \| Op\, a^j R_j u |\, L^2 \|$$

with $R_j \in S^{-\infty}$. Again, the last term can be estimated by $c\,\|u\,|L_2\,\|$, and just like before $Op\, a^j B^j \in S^0_{1,\delta}$. Hence

$$\| Op\, a^j u |\, L_2 \| \leq C\, \| u\, | W_2^{j,a} \| .$$

2 The Main Results

As usual we write the variables in $\mathbb{R}^n$ in the form $x = (x', x_n)$, $\xi = (\xi', \xi_n)$.

2.1 Theorem. *(Trace Theorem)*
Let $a \in \bigcap_{\delta > 0} S(m, m'; \delta)$, $m' > \frac{1}{2}$, satisfy

$$\left| \frac{a(x', x_n, \xi', 0)}{a(x', x_n, \xi', \xi_n)} \right| \leq C \left(\frac{\langle \xi' \rangle}{\langle \xi \rangle} \right)^{\frac{1}{2} + \varepsilon^*} \tag{1}$$

for some $\varepsilon^* > 0$. Then the restriction operator

$$\gamma_0 : W^{1,a}(\mathbb{R}^n) \to W^{1,a'}(\mathbb{R}^{n-1})$$

is continuous. Here, $a'(x', \xi') = \langle \xi' \rangle^{-1/2} \# a(x', 0, \xi', 0)$.

The restriction operator is in fact surjective. This follows from

2.2 Theorem. *(Extension Theorem)*
Let $a \in \bigcap_{\delta>0} S(m, m'; \delta)$, $m' > \frac{1}{2}$, *satisfy*

$$\left| \frac{a(x', 0, \xi', \xi_n)}{a(x', 0, \xi', 0)} \right| \le C \left(\frac{\langle \xi \rangle}{\langle \xi' \rangle} \right)^{\kappa} \tag{2}$$

for some $\kappa > 0$. *Then there is a bounded operator*

$$E : W^{1,a'}(\mathbb{R}^{n-1}) \to W^{1,a}(\mathbb{R}^n)$$

with $\gamma_0 E = id$. *Again,* $a'(x', \xi') = \langle \xi' \rangle^{-1/2} \# a(x', 0, \xi', 0)$.

2.3 Remark. In view of the fact that we may change the symbol a for small $|\xi|$ without changing the space $W_2^{1,a}(\mathbb{R}^n)$, cf. Theorem 1.6, it is sufficient to ask the estimates (1) and (2) hold for $|\xi| \ge R$.
Likewise it is obviously sufficient to ask that estimate (1) holds in a small strip $\{|x_n| < \varepsilon\}$ only.

2.4 Remark. For symbols of the form

$$a(x, \xi) = \langle \xi \rangle^{\sigma(x)} \tag{3}$$

the spaces $W^{1,a}(\mathbb{R}^n)$ were considered by Unterberger and Bokobza [23], Višik and Eskin [24], [25], Beauzamy [2]. Here, $\sigma(x) = s + \psi(x)$ with a constant s, and $\psi \in \mathcal{S}(\mathbb{R}^n)$.
If $t \in \mathbb{R}$, then the spaces $W^{1,a}(\mathbb{R}^n)$ associated with

$$a(x, \xi) = \langle \xi \rangle^{\sigma(x)} (1 + \ln \langle \xi \rangle)^t \tag{4}$$

were treated in Unterberger-Bokobza [23] and Unterberger [22].
It is easy to check that the symbols in (3) and (4) belong to $\bigcap_{\delta>0} S(m, m', \delta)$ and satisfy (1) and (2) provided $\inf \sigma(x) > 1/2$. This is also true for a generalization of (3) used in [11], where $\sigma(x)$ is a function belonging to $B^\infty(\mathbb{R}^n)$ - it is $\sigma(x)$ and all its derivatives are bounded on $\mathbb{R}^n$. For all these examples, trace and extension theorems are so far unknown.

3 Proof of the Trace Theorem

In the following we shall assume that $a \in \bigcap_{\delta>0} S(m, m', \delta)$, $m' > \frac{1}{2}$, satisfies inequality (1) in Theorem 2.1.

3.1 Lemma. *Let* $u \in W_2^{1,a}(\mathbb{R}^n)$. *Then* $\gamma_0 u \in L^2(\mathbb{R}^{n-1})$.

P r o o f. Since A is hypoelliptic, there is a parametrix B with symbol in $S_{1,\delta}^{-m'}(\mathbb{R}^n)$ such that $BA = I + R$ with $R \in S^{-\infty}$. Now $u = BAu - Ru$. By assumption $Au \in L^2(\mathbb{R}^n)$ so that $u \in H_2^{m'}(\mathbb{R}^n)$ by Theorem 1.2. The classical trace theorem now gives the assertion, since $m' > 1/2$.

We denote by Λ and Λ' the operators with symbols $\langle \xi \rangle$ and $\langle \xi' \rangle$, respectivly. As in Theorem2.1, A' is the operator with the symbol $\langle \xi' \rangle^{-1/2} \# a(x', 0, \xi', 0)$, while A_0 is the operator with the symbol $a(x', x_n, \xi', 0)$.

3.2 Lemma. *Fix $k = \frac{1+\varepsilon^*}{2}$, cf. Theorem 2.1. Without loss of generality we may and will assume that $k < m'$. Let B, R be as in the proof of Lemma 3.1. For the proof of Theorem 2.1 it is sufficient to show that*

$$(1) \qquad \Lambda'^{-k} A_0 B \Lambda^k \in \mathcal{L}(H_2^k(I\!R^n)).$$

P r o o f. We have $\gamma_0 u \in L^2(I\!R^{n-1})$ by Lemma 3.1. It remains to show that $A'\gamma_0 u \in L^2(I\!R^{n-1})$. For $u \in W_2^{1,a}(I\!R^n)$ we have

$$(2) \qquad A'\gamma_0 u = \Lambda'^{k-1/2}\gamma_0 \left(\Lambda'^{-k} A_0 B \Lambda^k\right) \Lambda^{-k} A u - A'\gamma_0 R u,$$

since $\gamma_0 \Lambda'^{-k+1/2} A_0 = \Lambda'^{-k} A' \gamma_0$. The second summand on the right hand side of (2) certainly is a function in $L^2(I\!R^{n-1})$, since $R \in S^{-\infty}$ and $A' \in S(m-1/2, m'-1/2; \delta)$. By assumption, $Au \in L^2(I\!R^n)$, hence $\Lambda^{-k} A u \in H^k(I\!R^n)$. If (1) holds, the trace of $\left(\Lambda'^{-k} A_0 B \Lambda^k\right) \Lambda^{-k} A u$ is in $H^{k-1/2}(I\!R^{n-1})$. So the first summand also belongs to $L^2(I\!R^{n-1})$.

3.3 Lemma. *Let b denote the symbol of B and*

$$(3) \qquad q(x,\xi) = \langle \xi' \rangle^{-k} \# a(x', x_n, \xi', 0) \# b(x,\xi) \# \langle \xi \rangle^k$$

with the fixed k of Lemma 3.2. In order to prove the boundedness in Lemma 3.2, it is sufficient to show that $q = q_1 + q_2$, where q_2 induces an $H_2^k(I\!R^n)$-bounded operator, while for every $\delta > 0$, q_1 satisfies the estimates

$$(4) \qquad \left| q_{1\,(\beta)}^{(\alpha)}(x,\xi) \right| \le C_{\alpha\beta} \langle \xi \rangle^{2[m]\delta - \frac{\varepsilon^*}{2} + \delta|\beta|} \langle \xi' \rangle^{-2[m]\delta + \frac{\varepsilon^*}{2} - |\alpha|}$$

for all α, β with $|\alpha| \le I$, $|\beta| \le J$. Here I and J denote the number of derivatives required in Theorem 1.2 for $s = k$.

P r o o f. The right hand side of (4) equals

$$(5) \qquad C_{\alpha\beta\tau} \langle \xi \rangle^{-\rho|\alpha| + \delta|\beta|} \langle \xi' \rangle^{|\alpha|(\rho - 1)} \left(\frac{\langle \xi' \rangle}{\langle \xi \rangle} \right)^{\frac{\varepsilon^*}{2} - 2[m]\delta - \rho|\alpha|}.$$

By assumption, $a \in S(m, m'; \delta)$ for all $\delta > 0$. Choose $\rho = \min\left(\frac{\varepsilon^*}{4I}, 1\right)$, $\delta < \min\left(\rho, \frac{\varepsilon^*}{8[m]+1}\right)$. Then (5) implies that $q_1 \in S^0_{\rho,\delta}(I, J)$ and Theorem 1.2 shows the boundedness in $H_2^k(I\!R^n)$.

3.4 Lemma. *The symbol q in (3) has a decomposition $q = q_1 + q_2$ as in Lemma 3.3.*

P r o o f. The symbol of $b(x,\xi) \# \langle \xi \rangle^k$ is $b(x,\xi) \langle \xi \rangle^k$. Choose $M = [m]$. The composition formula for pseudodifferential symbols gives

$$(6) \qquad a(x,\xi',0) \# b(x,\xi) \# \langle \xi \rangle^k = \sum_{|\mu| \le M} \frac{1}{\mu!} a^{(\mu)}(x,\xi',0) b_{(\mu)}(x,\xi) \langle \xi \rangle^k + r_M(x,\xi),$$

where the precise form of the remainder term $r_M(x,\xi)$ is given in Lemma 1.1. Let us first look at the terms under the summation. For arbitrary α and β,

$$\left[a^{(\mu)}(x,\xi',0) b_{(\mu)}(x,\xi) \langle \xi \rangle^k \right]^{(\alpha)}_{(\beta)} \le C_{\alpha\beta} \langle \xi' \rangle^{-|\mu| - |\alpha|} \langle \xi \rangle^{|\mu|\delta + |\beta|\delta + k} |a(x,\xi',0)| \, |a(x,\xi)|^{-1}.$$

Here we have used the estimates (4) and (6) of Section 1. Summing for all $|\mu| \le M$, we obtain the estimate

$$(7) \qquad \left[a^{(\mu)}(x,\xi',0)b_{(\mu)}(x,\xi)\langle\xi\rangle^k\right]^{(\alpha)}_{(\beta)}$$

$$\le C\langle\xi'\rangle^{-|\alpha|}\langle\xi\rangle^{|\beta|\delta+k}\langle\xi'\rangle^{-M\delta}\langle\xi\rangle^{M\delta}|a(x,\xi',0)|\,|a(x,\xi)|^{-1}\max_{|\mu|\le M}\left(\langle\xi'\rangle^{M\delta-|\mu|}\langle\xi\rangle^{-M\delta+|\mu|\delta}\right)$$

where the maximum is less than or equal 1. In order to control the remainder term we consider

$$r_{\mu,\theta}(x,\xi) = \mathrm{Os}-\iint e^{-iy\eta}a^{(\mu)}(x,\xi'+\eta\theta)b_{(\mu)}(x+y,\xi)\langle\xi\rangle^k\,dyd\eta$$

for $|\mu| = M+1$, and we identify ξ' and $(\xi',0)$.
The estimate is quite technical. It will therefore be given in Lemma 3.5, below. We will show that, for all multi-indices α, β, with $|\alpha| \le I$ and $|\beta| \le J$, with the numbers I and J of Lemma 3.3, there exists an $L \in \mathbb{N}$, independent of δ, such that

$$(8) \qquad \left|r^{(\alpha)}_{\mu,\theta(\beta)}(x,\xi)\right| \le C\langle\xi\rangle^{k-m'+|\mu|\delta+2L\delta+|\beta|\delta}\langle\xi'\rangle^{-|\alpha|}.$$

Since δ can be chosen arbitrarily small, we may assume that

$$0 \le ([m]+1+2L+J)\delta < m'-k.$$

Then (8) implies that $r_M \in S^0_{0,0}(I,J)$, so that $Op\, r_M \in \mathcal{L}(H^k_2(\mathbb{R}^n))$ by Theorem 1.2.
By $c_M(x,\xi)$ denote the sum on the right hand side of (6). Consider

$$\langle\xi'\rangle^{-k}\#a(x,\xi',0)\#b(x,\xi)\#\langle\xi\rangle^k = \langle\xi'\rangle^{-k}\#(c_M(x,\xi)+r_M(x,\xi)).$$

Clearly, $\langle\xi'\rangle^{-k}\#r_M(x,\xi)$ induces a bounded operator on $H^k_2(\mathbb{R}^n)$. Using the composition formula for $N=[m]$

$$(9) \qquad \langle\xi'\rangle^{-k}\#c_M(x,\xi) = \sum_{|\nu|\le N}\frac{1}{\nu!}\left(\langle\xi'\rangle^{-k}\right)^{(\nu)}c_{M(\nu)}(x,\xi)+\tilde r_N(x,\xi).$$

Proceeding as before and using (7) and (1)

$$\sum_{|\nu|\le N}[\left(\langle\xi'\rangle^{-k}\right)^{(\nu)}c_{M(\nu)}(x,\xi)]^{(\alpha)}_{(\beta)}$$

$$\le \quad c_{\alpha\beta}\langle\xi'\rangle^{-k-|\alpha|}\langle\xi\rangle^{|\beta|\delta+k}\langle\xi'\rangle^{-(M+N)\delta}\langle\xi\rangle^{(M+N)\delta}|a(x,\xi',0)|\,|a(x,\xi)|^{-1}$$

$$\le \quad c'_{\alpha\beta}\langle\xi'\rangle^{-k-|\alpha|+1/2+\varepsilon^*-(M+N)\delta}\langle\xi\rangle^{k+|\beta|\delta+(M+N)\delta-1/2-\varepsilon^*}.$$

Since $k \le \frac{1+\varepsilon^*}{2}$, $M = N = [m]$, the summation on the right hand side of (9) satisfies precisely the estimates required for the function q_1 in (4). We may therefore set $q_2(x,\xi) = \langle\xi'\rangle^{-k}\#r_M(x,\xi)+\tilde r_N(x,\xi)$, and we only have to show that $\tilde r_N(x,\xi)$ induces a bounded operator on $H^k_2(\mathbb{R}^n)$. Like before, we introduce $\tilde r_{\gamma,\theta}$ and show in Lemma 3.5, below, that for all α, β with $|\alpha| \le I$, $|\beta| \le J$, there exists an $\tilde L$ independent of δ such that

$$(10) \qquad \left|\tilde r^{(\alpha)}_{\gamma,\theta(\beta)}(x,\xi)\right| \le C\langle\xi\rangle^{k-m'-|\mu|\delta+M\delta+2\tilde L\delta+|\beta|\delta}\langle\xi'\rangle^{-|\alpha|}.$$

We may decrease δ again and achieve that the first exponent in (10) is negative for all $|\beta| \le J$. This will imply that $Op\,\tilde r_N \in \mathcal{L}(H^k_2(\mathbb{R}^n))$ and complete the proof of the lemma.

3.5 Lemma. *Let $0 \le \delta < 1$ and let p, q be two symbols for which the following estimates hold*

$$\left|p^{(\alpha)}_{(\beta)}(x,\xi)\right| \le c^p_{\alpha\beta} \langle\xi\rangle^{s+|\beta|\delta} \langle\xi'\rangle^{s'-|\alpha|} \tag{11}$$

$$\left|q^{(\alpha)}_{(\beta)}(x,\xi)\right| \le c^q_{\alpha\beta} \langle\xi\rangle^{t+|\beta|\delta} \langle\xi'\rangle^{t'-|\alpha|} \tag{12}$$

For $|\theta| \le 1$ and a multi-index γ define

$$r_{\gamma,\theta}(x,\xi) = \text{Os-}\iint e^{-iy\eta} p^{(\gamma)}(x,\xi'+\theta\eta) q_{(\gamma)}(x+y,\xi) dy đ\eta. \tag{13}$$

Here we write ξ' instead of $(\xi',0)$. Moreover, let I, $J \in \mathbb{N}$ be given. Then there exists an L, depending on I, J, s, s', t' and n, but independent of δ and θ, such that

$$\left|r^{(\alpha)}_{\gamma,\theta\,(\beta)}(x,\xi)\right| \le c^r_{\alpha\beta\gamma} \langle\xi\rangle^{t+|\gamma|\delta+2L\delta+|\beta|\delta} \langle\xi'\rangle^{-|\alpha|} \tag{14}$$

for all $|\gamma| \ge s+s'+t'$ and all α, β with $|\alpha| \le I$, $|\beta| \le J$.

P r o o f. The proof employs a technique of Kumano-go, [12, Chapter 2, Lemma 2.4]. Due to the more subtle estimates (11) and (12), however, a more careful analysis is required. We first note that

$$r^{(\alpha)}_{\gamma,\theta(\beta)}(x,\xi) = \text{Os-}\iint e^{-iy\eta} \sum_{\substack{\alpha_1+\alpha_2=\alpha \\ \beta_1+\beta_2=\beta}} c_{\alpha_1\alpha_2\beta_1\beta_2}\, p^{(\alpha_1+\gamma)}_{(\beta_1)}(x,\xi'+\theta\eta) q^{(\alpha_2)}_{(\gamma+\beta_2)}(x+y,\xi) dy đ\eta.$$

It is sufficient to consider the terms under the summation separately.
Step 1. Fix γ, α_1, α_2, β_1, β_2 and let

$$I_\theta(x,\xi) = \text{Os-}\iint e^{-iy\eta} p^{(\alpha_1+\gamma)}_{(\beta_1)}(x,\xi'+\theta\eta) q^{(\alpha_2)}_{(\gamma+\beta_2)}(x+y,\xi) dy đ\eta.$$

Now let $l_0 = \left[\frac{n}{2}\right]+1$. The fact that

$$e^{-iy\eta} = \left(1+\langle\xi'\rangle^{2\delta}|y|^2\right)^{l_0} \left(1+\langle\xi'\rangle^{2\delta}(-\Delta_\eta)\right)^{l_0} e^{-iy\eta}$$

together with integration by parts gives

$$I_\theta(x,\xi) = \text{Os-}\iint e^{-iy\eta} h_\theta(x,\xi,y,\eta) dy đ\eta$$

where

$$\begin{aligned} & h_\theta(x,\xi,y,\eta) \\ (15)\quad = {} & \left(1+\langle\xi'\rangle^{2\delta}|y|^2\right)^{-l_0} \left(1+\langle\xi'\rangle^{2\delta}(-\Delta_\eta)\right)^{l_0} \left[p^{(\gamma+\alpha_1)}_{\beta_1}(x,\xi'+\theta\eta) q^{(\alpha_2)}_{(\gamma+\beta_2)}(x+y,\xi)\right]. \end{aligned}$$

Since $2l_0 > n$, integration over y is well-defined.
Let

$$\Omega_1 = \left\{\eta : |\eta| \le \langle\xi'\rangle^\delta/2\right\},\ \Omega_2 = \left\{\eta : \langle\xi'\rangle^\delta/2 < |\eta| < \langle\xi'\rangle/2\right\},\ \Omega_3 = \{\eta : |\eta| \ge \langle\xi'\rangle/2\}.$$

On $\Omega_1 \cup \Omega_2$

$$\frac{\langle\xi'\rangle}{2} \le \langle\xi' + \theta\eta\rangle \le \frac{3}{2}\langle\xi'\rangle \tag{16}$$

and

$$\frac{\langle\xi'\rangle}{2} \le \langle\xi' + \theta\eta'\rangle \le \frac{3}{2}\langle\xi'\rangle \tag{17}$$

We will now analyze the integrals

$$I_{\theta,j} = \mathrm{Os} - \int_{\Omega_j}\int_{R^n} e^{-iy\eta} h_\theta(x,\xi,y,\eta)dy\,đ\eta, \qquad j = 1,2,3.$$

Step 2. On $\Omega_1 \cup \Omega_2$, (11), (12), (16), and (17) imply that

$$\begin{aligned}
&|h_\theta(x,\xi,y,\eta)| \\
&\le \left(1 + \langle\xi'\rangle^{2\delta}|y|^2\right)^{-l_0}\left[\sum_{j=0}^{l_0}\binom{l_0}{j}\langle\xi'\rangle^{2\delta j}\theta^{2j}c^p_{\gamma+\alpha_1,\beta_1}\langle\xi'+\theta\eta\rangle^{s+|\beta_1|\delta}\langle\xi'+\theta\eta'\rangle^{s'-|\alpha_1+\gamma|-2j}\right] \\
&\qquad c^q_{\alpha_2,\gamma_2+\beta_2}\langle\xi\rangle^{t+|\gamma+\beta_2|\delta}\langle\xi'\rangle^{t'-|\alpha_2|} \\
&\le \quad c^{pq1}_{\alpha\beta\gamma}\left(1 + \langle\xi'\rangle^{2\delta}|y|^2\right)^{-l_0}\langle\xi\rangle^{t+|\gamma|\delta+|\beta|\delta}\langle\xi'\rangle^{s+s'+t'-|\gamma|-|\alpha|}.
\end{aligned}\tag{18}$$

Step 3. Since $l_0 > \frac{n}{2}$,

$$\int_{\Omega_1}\int\left(1 + \langle\xi'\rangle^{2\delta}|y|^2\right)^{-l_0}dy\,đ\eta = \langle\xi'\rangle^{-\delta n}\int_{\Omega_1}\int\langle w\rangle^{-2l_0}dw\,đ\eta = c_{l_0 n}$$

independent of δ and ξ', cf. [12, Chapter 2, (2.31)].
Therefore

$$|I_{\theta,1}(x,\xi)| \le c\langle\xi\rangle^{t+|\gamma|\delta+|\beta|\delta}\langle\xi'\rangle^{s+s'+t'-|\gamma|-|\alpha|}.$$

Since we had assumed that $|\gamma| \ge s + s' + t'$, the first term satisfies the desired estimate.
Step 4. In order to estimate $I_{\theta,2}$, we first note that

$$\left|\partial^\mu_y(1 + \langle\xi'\rangle^{2\delta}|y|^2)^{-l_0}\right| \le c_{\mu l_0}\langle\xi'\rangle^{\delta|\mu|}(1 + \langle\xi'\rangle^{2\delta}|y|^2)^{-l_0}. \tag{19}$$

Together with the estimate in (18) this gives

$$|(-\Delta_y)^{l_0}h_\theta(x,\xi,y,\eta)| \le c_{l_0}(1 + \langle\xi'\rangle^{2\delta}|y|^2)^{-l_0}\; c^{pq2}_{\alpha\beta\gamma}\langle\xi\rangle^{t+|\gamma|\delta+|\beta|\delta+2l_0\delta}\langle\xi'\rangle^{s+s'+t'-|\gamma|-|\alpha|}.$$

Integration by parts yields

$$\begin{aligned}
|I_{\theta,2}(x,\xi)| &= \left|\int_{\Omega_2}|\eta|^{-2l_0}\int e^{-iy\eta}(-\Delta_y)^{l_0}h_\theta(x,\xi,y,\eta)dy\,đ\eta\right| \\
&\le c^{pq2}_{\alpha\beta\gamma}c_{l_0 n}\langle\xi\rangle^{t+|\gamma|\delta+|\beta|\delta+2l_0\delta}\langle\xi'\rangle^{s+s'+t'-|\gamma|-|\alpha|-2l_0\delta}
\end{aligned}\tag{20}$$

Here we have used the inequality

$$\int_{\Omega_2}|\eta|^{-2l_0}\int(1 + \langle\xi'\rangle^{2\delta}|y|^2)^{-l_0}dy\,đ\eta \le c_{l_0 n}\langle\xi'\rangle^{-2l_0\delta}.$$

Note that (20) again has the required from.
Step 5. On Ω_3, $\langle\xi' + \theta\eta'\rangle \leq \langle\xi' + \theta\eta\rangle \leq 3|\eta|$. Hence (11), (12), and (19) imply the preliminary estimate

$$|(-\Delta_y)^L\ h_\theta\ (x,\xi,y,\eta)|$$
$$\leq c_{Ll_0}(1 + \langle\xi'\rangle^{2\delta}|y|^2)^{-l_0}\, c^{pq3}_{\alpha\beta\gamma}|\eta|^{s_+ + s'_+ + t'_+ + |\beta|\delta + 2l_0\delta + |\alpha|}\,\langle\xi\rangle^{t+|\gamma|\delta+2L\delta+|\beta|\delta}\,\langle\xi'\rangle^{-|\alpha|}. \tag{21}$$

Here $s_+ = \max\{s, 0\}$, analogously we define s'_+, t'_+; so far $L \in \mathbb{N}$ is arbitrary.
Given I and J, we now choose L so that

$$I + J + 2l_0 + s_+ + s'_+ + t'_+ \leq 2L - n - 1$$

This is independent of γ, δ, and θ. It allows us to estimate the exponent of $|\eta|$ in (21) by $2L - n - 1$. Integrating by parts

$$\begin{aligned} |I_{\theta,3}(x,\xi)| &\leq \left|\int_{\Omega_3} |\eta|^{-2L}\int e^{-iy\eta}(-\Delta_y)^L\, h_\theta(x,\xi,y,\eta)dy\,d\!\!\bar{}\eta\right| \\ &\leq c^{pq4}_{\alpha\beta\gamma L}\int_{\Omega_3}|\eta|^{-n-1}d\!\!\bar{}\eta\int(1+\langle\xi'\rangle^{2\delta}|y|^2)^{-l_0}dy\ \langle\xi\rangle^{t+|\gamma|\delta+2L\delta+|\beta|\delta}\langle\xi'\rangle^{-|\alpha|}. \end{aligned}$$

The integration over y and η gives a finite value, independent of ξ'; so we obtain precisely the estimate required for (14). This concludes the proof.

3.6 Remark. Looking more closely at the estimates, it is easy to check that we need derivatives of p up to order $|\alpha| + |\gamma| + n + 1$ in ξ, and order $|\beta|$ in x. For q we need the derivatives with respect to ξ up to order $|\alpha|$, and with respect to x up to order $|\beta|+|\gamma|+2L$.

4 Proof of the Extension Theorem

The idea in this section is the following. The hypoellipticity of the symbol a gives an embedding of $W_2^{1,a'}(\mathbb{R}^{n-1})$ in $H_2^{m'-1/2}(\mathbb{R}^{n-1})$. On this space, however, there exist many continuous extension operators to $H_2^{m'}(\mathbb{R}^n)$. We choose a particularly simple one. Since $m' > 1/2$, the restriction operator γ_0 is well-defined on $H_2^{m'}(\mathbb{R}^n)$. Restricting the extension operator to $W_2^{1,a'}(\mathbb{R}^{n-1})$, we immediately get a right inverse to γ_0. The difficult task in this section is to show that the extension operator in fact maps $W_2^{1,a'}(\mathbb{R}^{n-1})$ to $W_2^{1,a}(\mathbb{R}^n)$. This again requires a careful analysis. We proceed in a series of lemmata. As before A, A', Λ, and Λ' are the operators with symbols $a(x,\xi)$, $a'(x',\xi') = \langle\xi'\rangle^{-1/2}\#a(x',0,\xi',0)$, $\langle\xi\rangle$ and $\langle\xi'\rangle$, respectively. $B' = Op\,b'$ is a parametrix to A' over $\mathbb{R}^{n-1}$.

4.1 Lemma. *For a fixed $\phi \in C_0^\infty(\mathbb{R})$ with $\phi \equiv 1$ near 0 define*

$$E:\ \mathcal{S}(\mathbb{R}^{n-1}) \to \mathcal{S}(\mathbb{R}^n)$$

by

$$(E\,u)(x', x_n) = \mathcal{F}^{-1}_{\xi'\to x'}\ \phi(\langle\xi'\rangle\, x_n)\,\mathcal{F}_{x'\to\xi'}\,u.$$

Then E extends to a continuous operator

$$W_2^{1,a'}(\mathbb{R}^{n-1}) \to H_2^{m'}(\mathbb{R}^n);$$

moreover, $\gamma_0\, E = id$ on $W_2^{1,a'}(\mathbb{R}^{n-1})$.

P r o o f. The operator E is suggested in [12, Chapter 6, Lemma 2.3]. The hypoellipticity of a implies that $a' \in S(m-1/2, m'-1/2, \delta)$. If $u \in W_2^{1,a'}(\mathbb{R}^{n-1})$, then $A'u \in L^2$, hence $u \in H_2^{m'-1/2}(\mathbb{R}^{n-1})$. It is well-known that $E: H_2^{m'-1/2}(\mathbb{R}^{n-1}) \to H_2^{m'}(\mathbb{R}^n)$ is bounded and that $\gamma_0 E = id$ on $H_2^{m'-1/2}(\mathbb{R}^{n-1})$, a forteriori on $W_2^{1,a'}(\mathbb{R}^{n-1})$.

4.2 Lemma. *Fix some $k > \kappa$. In order to prove Theorem 2.2 it is sufficient to show that*

$$(1) \qquad \Lambda^{-k} A\, E\, B'\, \Lambda'^k \in \mathcal{L}(H_2^k(\mathbb{R}^{n-1}), H_2^k(\mathbb{R}^n)).$$

P r o o f. After Lemma 4.1 it remains to show that for $u \in W_2^{1,a'}(\mathbb{R}^{n-1})$, we have $A\,E\,u \in L^2(\mathbb{R}^n)$; notice that, trivially, $E\,u \in L^2(\mathbb{R}^n)$. In view of the fact that $W_2^{1,a}(\mathbb{R}^n) \hookrightarrow H_2^{m'}(\mathbb{R}^n)$, the continuity of E then is immediate from the closed graph theorem.
On the other hand, $A\,E\,u \in L^2(\mathbb{R}^n)$ iff $\Lambda^{-k} A\,E\,u \in H_2^k(\mathbb{R}^n)$. Now

$$\Lambda^{-k} A\, E u = \Lambda^{-k} A\, E\, B'\, A' u + \Lambda^{-k} A\, E\, R' u\,,$$

where $R' = I - B'\,A' \in S^{-\infty}$. Therefore $\Lambda^{-k} A\,E\,R'u \in H_2^k(\mathbb{R}^n)$; it is in fact much better. Noting that $A'u \in L^2(\mathbb{R}^{n-1})$ and $\Lambda^{-k} A\,E\,B'\,A' = \Lambda^{-k} A\,E\,B'\,\Lambda'^k\,\Lambda'^{-k} A'$ we obtain the assertion.

In the following we will consider the products $\Lambda^{-k} A$ and $E\,B'\,\Lambda'^k$ separately. We will decompose each of them in a finite number of terms which are "easy" to handle and remainders with sufficiently good mapping properties.
For the analysis of $E\,B'\,\Lambda'^k$ we rely on the theory of potential operators in Boutet de Monvel's calculus.

4.3 Theorem. *Let $l = l(x', \xi', x_n)$ be a C^∞ function on $\mathbb{R}^{n-1} \times \mathbb{R}^{n-1} \times \mathbb{R}$, $\mu \in \mathbb{R}$, and suppose that for all $r, r' \in \mathbb{N}_0$, $\alpha, \beta \in \mathbb{N}_0^{n-1}$*

$$(2) \qquad \|x_n^r\, D_{x_n}^{r'}\, D_{\xi'}^\alpha\, D_{x'}^\beta\, l(x', \xi', x_n) \,|\, L^2(\mathbb{R}_{x_n})\| \le C_{\alpha\beta r r'}\, \langle \xi' \rangle^{\mu - |\alpha| - r + r'}\,.$$

Then the operator

$$Op_K l: \; H_2^s(\mathbb{R}^{n-1}) \to H_2^{s-\mu}(\mathbb{R}^n)$$

is bounded for every $s \in \mathbb{R}$. Here the potential operator $Op_K l$ associated with l is defined by

$$(Op_K l) f(x', x_n) = (2\pi)^{\frac{n-1}{2}} \int e^{ix'\xi'} l(x', \xi', x_n)\, \hat{f}(\xi') d\xi'.$$

For every fixed s we only need a finite number of α, β, r, r' in (2) in order to estimate $\|Op_K l \,|\mathcal{L}(H_2^s(\mathbb{R}^{n-1}),\, H_2^{s-\mu}(\mathbb{R}^n))\|$.

P r o o f. This is immediate e.g. from Theorem 2.5.1 and (2.3.26) in [7].

4.4 Lemma. *Let $\psi \in S(\mathbb{R})$ and $l(x', \xi', x_n) = \psi(\langle \xi' \rangle\, x_n)$. Then l satisfies the estimates in (2) for $\mu = -1/2$.*

P r o o f. This is an easy calculation, noting that $\|\psi(\langle \xi' \rangle\, x_n) \,|\, L^2(\mathbb{R}_{x_n})\| \le C\, \langle \xi' \rangle^{-1/2}$.

4.5 Lemma. *Let $\psi \in \mathcal{S}(I\!R)$, $s \in S^{\mu}_{1,\delta}(I\!R^{n-1})$. Then*

$$Op_K\psi(\langle\xi'\rangle x_n)\circ Op\, s = \sum_{|\gamma|\le N} Op_K l_\gamma + (N+1)\sum_{|\gamma|=N+1} Op_K \int_0^1 \frac{(1-\theta)^N}{\gamma!} r_{\gamma,\theta}(x',\xi',x_n)d\theta$$

with

$$l_\gamma(x',\xi',x_n) = D^\gamma_{\xi'}\,\psi(\langle\xi'\rangle x_n)\partial^\gamma_{x'}\, s(x',\xi')/\gamma!$$

and

$$r_{\gamma,\theta}(\,x',\xi',x_n) = \text{Os-}\iint e^{-iy'\eta'} D^\gamma_{\xi'}\psi(\langle\xi'+\theta\eta'\rangle\, x_n)\partial^\gamma_{x'} s(x'+y',\xi')dy'đ\eta'.$$

Here, $đ\eta = (2\pi)^{1-n}d\eta$.

P r o o f. Writing out the integrals, $Op_K\ \psi(\langle\xi'\rangle x_n)\circ Op\, s = Op_K\, l$ with

$$l(x',\xi',x_n) = \text{Os-}\iint e^{-iy'\eta'}\ \psi(\langle\xi'+\eta'\rangle\, x_n)\ s(x'+y',\xi')\ dy'đ\eta'.$$

Taylor expansion with respect to η' and integration by parts then gives the desired formulas.

It is our next goal to estimate the symbols of the operators in Lemma 4.5. As a preparation we prove the following technical result.

4.6 Lemma. *Let $\chi \in \mathcal{S}(I\!R)$, $p = p(\xi') \in S^{\nu}_{1,0}(I\!R^{n-1})$, $q \in S^{\nu'}_{1,\delta}(I\!R^{n-1})$, $0 \le \theta \le 1$, and*

$$k_\theta(x',\xi',x_n) = \text{Os-}\iint e^{-iy'\eta'}\chi(\langle\xi'+\theta\eta'\rangle\, x_n)p(\xi'+\theta\eta')q(x'+y',\xi')dy'đ\eta'\ .$$

Then

$$\|k_\theta(x',\xi',x_n)|L^2(I\!R_{x_n})\| \le C\,\langle\xi'\rangle^{\nu+\nu'-1/2}$$

with a constant independent of θ.

P r o o f. We again use the technique of the proof of Lemma 3.5; however, now the domain of the integration is $I\!R^{n-1}$ and we need $L^2(I\!R_{x_n})$–estimates.
Fixing $l_0 = [\frac{n-1}{2}]+1$ we have

$$k_\theta(x',\xi',x_n) = \text{Os-}\iint e^{-iy'\eta'} h_\theta(x',\xi',y',\eta',x_n)dy'đ\eta'$$

with

$$h_\theta(x',\xi',y',\eta',x_n) = \left(1+\langle\xi'\rangle^{2\delta}|y'|^2\right)^{-l_0}\left(1+\langle\xi'\rangle^{2\delta}(-\Delta_{\eta'})\right)^{l_0}\left[\chi(\langle\xi'+\theta\eta'\rangle\, x_n)p(\xi'+\theta\eta')q(x'+y',\xi')\right].$$

Let

$$\Omega_1 = \left\{\eta' : |\eta'| \le \langle\xi'\rangle^\delta/2\right\},\ \Omega_2 = \left\{\eta' : \langle\xi'\rangle^\delta/2 < |\eta'| < \langle\xi'\rangle/2\right\},\ \Omega_3 = \{\eta' : |\eta'| \ge \langle\xi'\rangle/2\}\ ,$$

and consider the three integrals similarly as before. The identity

$$D_{\eta_j}\chi(\langle\xi'+\theta\eta'\rangle\, x_n) = \theta\frac{(\xi'+\theta\eta')_j}{\langle\xi'+\theta\eta'\rangle}\langle\xi'+\theta\eta'\rangle^{-1}(tD_t\chi)(\langle\xi'+\theta\eta'\rangle\, x_n)\ ,$$

then leads to the desired estimate, very much like before.

4.7 Corollary. We use the notation of Lemma 4.5. Then

$$\|D^\alpha_{\xi'} D^\beta_{x'} x_n^r D^{r'}_{x_n} r_{\gamma,\theta}(x',\xi',x_n)|L^2(I\!R_{x_n})\| \le C\,\langle\xi'\rangle^{\mu-(N+1)(1-\delta)-|\alpha|+|\beta|\delta-r+r'} \ .$$

The constant is independent of θ.

P r o o f. We have

$$D^\alpha_{\xi'} D^\beta_{x'} x_n^r D^{r'}_{x_n} r_{\gamma,\theta}(x',\xi',x_n) = \sum_{\alpha_1+\alpha_2=\alpha} c_{\alpha_1\alpha_2}\,\mathrm{Os-}\!\!\iint e^{-iy'\eta'} x_n^r D^{r'}_{x_n} D^{\alpha_1+\gamma}_{\xi'}\{\psi(\langle\xi'+\theta\eta'\rangle\,x_n)\} D^{\alpha_2}_{\xi'} D^{\beta+\gamma}_{x'} s(x'+y',\xi')dy'đ\eta' \ .$$

We now have the following identities, $j=1,\dots,n-1$:

$$\begin{aligned} x_n^r D^{r'}_{x_n}\{\psi(\langle\xi'+\theta\eta'\rangle\,x_n)\} &= \langle\xi'+\theta\eta'\rangle^{r'-r} (t^r D_t^{r'}\psi)(\langle\xi'+\theta\eta'\rangle\,x_n)\ , \\ D_{\xi_j}\psi(\langle\xi'+\theta\eta'\rangle\,x_n) &= \frac{(\xi'+\theta\eta')_j}{\langle\xi'+\theta\eta'\rangle}\langle\xi'+\theta\eta'\rangle^{-1}(tD_t\psi)(\langle\xi'+\theta\eta'\rangle\,x_n)\ . \end{aligned} \tag{3}$$

The notation $(t^r D_t^{r'}\psi)(\langle\xi'+\theta\eta'\rangle\,x_n)$ indicates that we evaluate the function $t\mapsto t^r D_t^{r'}\psi(t)$ at $t=\langle\xi'+\theta\eta'\rangle\,x_n$. Therefore $x_n^r D^{r'}_{x_n} D^{\alpha_1+\gamma}_{\xi'}\{\psi(\langle\xi'+\theta\eta'\rangle\,x_n)\}$ is a linear combination of terms of the form $p(\xi'+\theta\eta')\chi(\langle\xi'+\theta\eta'\rangle\,x_n)$ with $p\in S_{1,0}^{-|\alpha_1|-|\gamma|-r+r'}(I\!R^{n-1})$ and $\chi\in\mathcal{S}(I\!R)$. Note that both p and χ depend on the choice of r and r'.
Now the assertion follows from Lemma 4.6.

4.8 Lemma. *The operator $EB'\Lambda'^k$ can be written in the form*

$$EB'\Lambda'^k = \sum_{|\gamma|\le N}\frac{1}{\gamma!}Op_K D^\gamma_{\xi'}\phi(\langle\xi'\rangle\,x_n)\partial^\gamma_{x'}b'(x',\xi')\,\langle\xi'\rangle^k + R_N\ , \tag{4}$$

where $R_N : H_2^k(I\!R^{n-1}) \longrightarrow H_2^m(I\!R^n)$ is bounded, provided N is large enough. Here ϕ is the fixed function of Lemma 4.1

P r o o f. First apply Lemma 4.5 in order to obtain the representation (4). Lemma 4.5 also gives the precise form of the remainder. Corollary 4.7 in connection with Theorem 4.3 implies the boundedness of R_N provided N is large.

4.9 Lemma. *We have the following estimates:*

$$|D^\alpha_{\xi'} D^\beta_{x'} b'(x',\xi')| \le C\,\langle\xi'\rangle^{1/2-|\alpha|+\delta|\beta|}\,|a(x',0,\xi',0)|^{-1}$$

for large $|\xi'|$.

P r o o f. The symbol b' is a parametrix to $a'(x',\xi')=\langle\xi'\rangle^{-1/2}\#a(x',0,\xi',0)\in S\,(m-1/2, m'-1/2,\delta)$. By Remark 1.4(iii) we get the desired estimate.

Now we consider the composition of $\Lambda^{-k}A$ with $EB'\Lambda'^k$. We start with a decomposition of the symbol of $\Lambda^{-k}A$.

4.10 Lemma. *Let $L, M \in I\!N$. Then we can write*

$$\langle\xi\rangle^{-k} \# a(x,\xi) = \sum_{j=0}^{M} x_n^j a_j(x',0,\xi) + x_n^{M+1} a_{M+1}(x,\xi) + r_L(x,\xi),$$

where, for all multi-indices α, β, we have the estimates

$$\begin{array}{llll}
(5) & |D_\xi^\alpha D_x^\beta a_j(x',0,\xi)| \le C|a(x',0,\xi)| \langle\xi\rangle^{-k+j\delta-|\alpha|+|\beta|\delta}, & j=0,\dots,M, & |\xi| \text{ large},\\
(6) & |D_\xi^\alpha D_x^\beta a_j(x',0,\xi)| \le C\langle\xi\rangle^{m-k+j\delta-|\alpha|+|\beta|\delta}, & j=0,\dots,M, & \\
(7) & |D_\xi^\alpha D_x^\beta a_{M+1}(x,\xi)| \le C\langle\xi\rangle^{m-k+(M+1)\delta-|\alpha|+|\beta|\delta}, & & \\
(8) & |D_\xi^\alpha D_x^\beta r_L(x,\xi)| \le C\langle\xi\rangle^{m-k-L(1-\delta)-|\alpha|+|\beta|\delta}. & &
\end{array}$$

P r o o f. We use the asymptotic expansion formula for symbols

$$\langle\xi\rangle^{-k} \# a(x,\xi) = \sum_{|\nu|\le L} \frac{1}{\nu!} (\langle\xi\rangle^{-k})^{(\nu)} a_{(\nu)}(x,\xi) + r_L(x,\xi).$$

In view of the fact that $a \in S(m, m', \delta)$ we have

$$\left|[(\langle\xi\rangle^{-k})^{(\nu)} a_{(\nu)}(x,\xi)]_{(\beta)}^{(\alpha)}\right| \le C\langle\xi\rangle^{-k-|\nu|(1-\delta)-|\alpha|+|\beta|\delta} |a(x,\xi)|, \tag{9}$$

and (8) is the immediate standard estimate for the remainder. Now we use Taylor's formula

$$f(x_n) = \sum_{j=0}^{M} \frac{x_n^j}{j!} f^{(j)}(0) + \frac{x_n^{M+1}}{M!} \int_0^1 (1-\theta)^M f^{(M+1)}(\theta x_n) d\theta$$

for the function $\sum_{|\nu|\le L} \frac{1}{\nu!} (\langle\xi\rangle^{-k})^{(\nu)} a_{(\nu)}(x,\xi)$. We obtain (5) from (9). Estimates (6) and (7) also follow from (9), using that $|a(x,\xi)| \le C\langle\xi\rangle^m$.

4.11 Remark. In order to show the boundedness of $\Lambda^{-k} AEB'\Lambda'^k : H_2^k(I\!R^{n-1}) \longrightarrow H^k(I\!R^n)$ we split the operator up into four parts. In the notation of Lemma 4.8 and Lemma 4.10

$$\begin{aligned}
\Lambda^{-k} AEB'\Lambda'^k = & \sum_{j=0}^{M} \sum_{|\gamma|\le N} Op(x_n^j a_j(x',0,\xi)) Op_K(D_{\xi'}^\gamma \phi(\langle\xi'\rangle x_n) \partial_{x'}^\gamma b'(x',\xi') \langle\xi'\rangle^k)/\gamma! \\
& + \sum_{|\gamma|\le N} Op(x_n^{M+1} a_{M+1}(x,\xi)) Op_K(D_{\xi'}^\gamma \phi(\langle\xi'\rangle x_n) \partial_{x'}^\gamma b'(x',\xi') \langle\xi'\rangle^k)/\gamma! \\
& + \sum_{|\gamma|\le N} Op\, r_L Op_K(D_{\xi'}^\gamma \phi(\langle\xi'\rangle x_n) \partial_{x'}^\gamma b'(x',\xi') \langle\xi'\rangle^k)/\gamma! \\
& + \Lambda^{-k} AR_N\,.
\end{aligned} \tag{10}$$

Obviously the last summand does not require any attention, since $R_N : H_2^k(I\!R^{n-1}) \longrightarrow H_2^m(I\!R^n)$ is bounded.
In order to analyze the others we make the following observations

(i) $Op(x_n^j c(x,\xi)) = \sum_{l=0}^{j} \binom{j}{l} Op((-D_{\xi_n})^l c(x,\xi)) Op(x_n^{j-l})$ for an arbitrary symbol c.

(ii) $D^{\gamma}_{\xi'}\phi(\langle\xi'\rangle x_n)$ is a linear combination of terms of the form $p(\xi')\psi(\langle\xi'\rangle x_n)$ where $p \in S^{-|\gamma|}_{1,0}(\mathbb{R}^{n-1})$ and $\psi \in \mathcal{S}(\mathbb{R})$. Noting that

$$Op(x_n)Op_K(\psi(\langle\xi'\rangle x_n)) = Op_K(\langle\xi'\rangle^{-1}(t\psi)(\langle\xi'\rangle x_n))$$

(for the notation cf. the remark after (3)), the composition

$$Op\,(x_n^{j-l})Op_K(D^{\gamma}_{\xi'}\phi(\langle\xi'\rangle x_n)\partial^{\gamma}_{x'}b'(x',\xi')\langle\xi'\rangle^k)$$

is a linear combination of terms of the form

$$Op_K(p(\xi')\psi(\langle\xi'\rangle x_n)\partial^{\gamma}_{x'}b'(x',\xi')\langle\xi'\rangle^k) \tag{11}$$

with $p \in S^{-|\gamma|-j+l}_{1,0}(\mathbb{R}^{n-1})$ and $\psi \in \mathcal{S}(\mathbb{R})$.

(iii) In view of (ii), 4.9, and Theorem 4.3 we will have

$$Op_K(D^{\gamma}_{\xi'}\phi(\langle\xi'\rangle x_n)\partial^{\gamma}_{x'}b'(x',\xi')\langle\xi'\rangle^k) : H^k_2(\mathbb{R}^{n-1}) \to H^{m'}_2(\mathbb{R}^n)$$

bounded. Moreover $Op\,r_L : H^{m'}_2(\mathbb{R}^n) \to H^k(\mathbb{R}^n)$ will be bounded for large L. Hence the third term in (10) induces a bounded operator from $H^k(\mathbb{R}^{n-1})$ to $H^k(\mathbb{R}^n)$ and this is sufficient for Theorem 2.2 according to Lemma 4.2.

We now treat the second summand in (10).

4.12 Lemma. *For every* γ

$$Op(x_n^{M+1}a_{M+1}(x,\xi))Op_K(D^{\gamma}_{\xi'}\phi(\langle\xi'\rangle x_n)\partial^{\gamma}_{x'}b'(x',\xi')\langle\xi'\rangle^k) : H^k_2(\mathbb{R}^{n-1}) \longrightarrow H^k_2(\mathbb{R}^n)$$

is bounded provided M is sufficiently large.

P r o o f. The considerations in Remark 4.11 show that we have to estimate the norm of an expression of the form

$$Op\;D^l_{\xi_n}a_{M+1}(x,\xi)\,Op_K(p(\xi')\psi(\langle\xi'\rangle x_n)\partial^{\gamma}_{x'}b'(x',\xi')\langle\xi'\rangle^k)$$

where $l = 0,\dots,M+1$, $p \in S^{-M-1-|\gamma|+l}_{1,0}(\mathbb{R}^{n-1})$, and $\psi \in \mathcal{S}(\mathbb{R})$. We now use the estimates of Lemma 4.4 and Lemma 4.9 together with the facts that $|a(x',0,\xi',0)|^{-1} \le C\langle\xi'\rangle^{-m'}$. We get

$$\|D^{\alpha}_{\xi'}D^{\beta}_{x'}x_n^rD^{r'}_{x_n}\{p(\xi')\psi(\langle\xi'\rangle x_n)\partial^{\gamma}_{x'}b'(x',\xi')\langle\xi'\rangle^k\}|L^2(\mathbb{R}_{x_n})\| \le c\langle\xi'\rangle^{-|\alpha|-r+r'-M-1-|\gamma|+l+|\gamma|\delta+|\beta|\delta+k-m'}.$$

Using Theorem 4.3 this will induce a bounded operator from $H^k_2(\mathbb{R}^{n-1})$ to $H^{k+M-l-k+m'}_2(\mathbb{R}^n) = H^{M-l+m'}_2(\mathbb{R}^n)$, noting that δ can be taken arbitrarily small, so that for the maximum $|\beta|$ required for the boundedness estimate we still have $|\beta|\delta < 1/2$.
The operator $Op\,D^l_{\xi_n}a_{M+1}$ maps $H^{M-l+m'}_2(\mathbb{R}^n)$ to $H^{k+M(1-\delta)-\delta-m+m'}_2(\mathbb{R}^n)$ which is a subspace of $H^k_2(\mathbb{R}^n)$, provided M is large.

Now for the terms in the first summand in (10). We start with a simple observation.

4.13 Lemma. *Let* $q \in \bigcap_{\delta>0} S^{\mu}_{1,\delta}(\mathbb{R}^{n-1})$, $\psi \in \mathcal{S}(\mathbb{R})$. *Then*

$$Op_K(\psi(\langle\xi'\rangle\, x_n)q(x',\xi')) = Op\, q(x',\xi')\, Op_K(\psi(\langle\xi'\rangle\, x_n)) \; .$$

Notice that the mapping $(x', x_n, \xi', \xi_n) \mapsto q(x', \xi')$ *defines a symbol in* $S^{\mu_+ + \varepsilon}_{0,0}(\mathbb{R}^n)$ *for every* $\varepsilon > 0$, *so that the operator* $Op\, q$ *is well-defined on each* $H^s_2(\mathbb{R}^n)$.

4.14 Lemma. *We use the notation of Lemma 4.10 and Lemma 4.11.*

$$Op(x_n^j a_j(x',0,\xi))Op_K(D^{\gamma}_{\xi'}\phi(\langle\xi'\rangle\, x_n)\partial^{\gamma}_{x'}b'(x',\xi')\,\langle\xi'\rangle^k) : H^k(\mathbb{R}^{n-1}) \longrightarrow H^k(\mathbb{R}^n)$$

is bounded for $0 \le j \le M$ *and all multi-indices* $\gamma, |\gamma| \le N$. *Without loss of generality we assume that* M *and* N *are fixed so large that the boundedness results of Lemma 4.8 and Lemma 4.12 are valid.*

P r o o f. As pointed out in Remark 4.11 it is sufficient to consider

$$Op(D^l_{\xi_n} a_j(x',0,\xi))\, Op_K(p(\xi')\psi(\langle\xi'\rangle\, x_n)\partial^{\gamma}_{x'}b'(x',\xi')\,\langle\xi'\rangle^k)$$

for $l = 0,\dots,j$, $p \in S^{-|\gamma|-j+l}_{1,0}(\mathbb{R}^{n-1})$, $\psi \in \mathcal{S}(\mathbb{R})$. Using the boundedness of

$$Op_K(\langle\xi'\rangle^{1/2}\psi(\langle\xi'\rangle\, x_n)) : H^k(\mathbb{R}^{n-1}) \longrightarrow H^k(\mathbb{R}^n)$$

and Lemma 4.13, all we have to show is the boundedness of

$$Op(D^l_{\xi_n} a_j(x',0,\xi))\, Op(p(\xi')\partial^{\gamma}_{x'}b'(x',\xi')\,\langle\xi'\rangle^{k-1/2}) : H^k(\mathbb{R}^n) \longrightarrow H^k(\mathbb{R}^n) \; .$$

For fixed j, γ and l we denote the first symbol by $q_1(x',\xi)$ and the second by $q_2(x',\xi')$. By Lemma 1.1 we get

$$(q_1 \# q_2)(x',\xi) = \sum_{|\tilde\alpha|\le\tilde N} \frac{1}{\tilde\alpha!} q_1^{(\tilde\alpha)}(x',\xi)q_{2(\tilde\alpha)}(x',\xi') + r_{\tilde N}(x',\xi) \tag{12}$$

where $r_{\tilde N}(x',\xi)$ is given in Lemma 1.1 and depends, of course, on j, γ and l.
Now we show that each term of the sum of the right hand side of (12) defines a bounded operator in $H^k(\mathbb{R}^n)$. The boundedness of the operators defined by the remainder terms in (12) will be shown in Lemma 4.16, below. Consider an arbitrary term under the summation in (12). By Lemma 4.10 we have

$$\begin{aligned}
&|D^{\alpha}_{\xi} D^{\beta}_{x}(\partial^{\tilde\alpha}_{\xi} D^l_{\xi_n} a_j(x',0,\xi)p(\xi')\,\langle\xi'\rangle^{k-1/2}\, D^{\tilde\alpha}_{x}\partial^{\gamma}_{x'}b'(x',\xi'))| \\
\le\;& c|a(x',0,\xi)|\,\langle\xi\rangle^{-k+j\delta-l-|\tilde\alpha|+|\beta|\delta}\,|a(x',0,\xi',0)|^{-1}\,\langle\xi'\rangle^{1/2-|\gamma|-j+l+(k-1/2)+|\tilde\alpha|\delta+|\gamma|\delta-|\alpha|} \\
\le\;& c\left|\frac{a(x',0,\xi',\xi_n)}{a(x',0,\xi',0)}\right| \left(\frac{\langle\xi\rangle}{\langle\xi'\rangle}\right)^{-k+\rho|\alpha|-l+j\delta} \langle\xi\rangle^{-\rho|\alpha|+\delta|\beta|}\,\langle\xi'\rangle^{-|\gamma|(1-\delta)-|\tilde\alpha|(1-\delta)-j(1-\delta)-|\alpha|(1-\rho)} \\
\le\;& c\,\langle\xi\rangle^{-\rho|\alpha|+\delta|\beta|}\left(\frac{\langle\xi\rangle}{\langle\xi'\rangle}\right)^{\kappa-k+\rho|\alpha|+j\delta-l},
\end{aligned}$$

because of the assumption of Theorem 2.2.

Let I and J again denote the numbers of derivatives required in Theorem 1.2 for $s = k$. Then we have for $|\alpha| \le I, |\beta| \le J$

$$
\begin{aligned}
|D_\xi^\alpha D_x^\beta(\partial_\xi^{\tilde\alpha} q_1 D_x^{\tilde\alpha} q_2)| &\le c\,\langle\xi\rangle^{-\rho|\alpha|+\delta|\beta|}\left(\frac{\langle\xi\rangle}{\langle\xi'\rangle}\right)^{\kappa-k+\rho I+\delta M-l} \\
&\le c\,\langle\xi\rangle^{-\rho|\alpha|+\delta|\beta|}
\end{aligned}
$$

provided that we choose $\rho = \min\left(\frac{\kappa-k}{I+M}, 1\right)$ and $\delta < \rho$. This is possible, since we fixed $k > \kappa$ in Lemma 4.2, and since $a(x,\xi)$ belongs to $S(m,m';\delta)$ for all $\delta > 0$ by assumption. Then the last inequality implies that $\partial_\xi^{\tilde\alpha} q_1 D_x^{\tilde\alpha} q_2 \in S^0_{\rho,\delta}(I,J)$ and by Theorem 1.2 we get the boundedness in $H^k(I\!R^n)$ of all summands of the sum in the right hand side of (12). The proof is complete except for the estimates of the remainder terms which will be given in the next two lemmas.

To estimate the terms $r_{\tilde N}(x,\xi)$ we need yet another modification of Lemma 2.4 in [12, Chapter 2].

4.15 Lemma. *Let $0 \le \delta < \frac{1}{2}$ and let q_1, q_2 be two symbols for which the following estimates hold*

$$
\begin{aligned}
|q_{1(\beta)}^{(\alpha)}(x,\xi)| &\le c^1_{\alpha\beta}\,\langle\xi\rangle^{s-|\alpha|+|\beta|\delta} \\
|q_{2(\beta)}^{(\alpha)}(x,\xi)| &\le c^2_{\alpha\beta}\,\langle\xi\rangle^{|\beta|\delta}\,\langle\xi'\rangle^{t'-|\alpha|}\ .
\end{aligned}
$$

For $|\theta| \le 1$ then define

$$
r_{\gamma,\theta}(x,\xi) = \mathrm{Os}\!-\!\iint e^{-iy\eta} q_1^{(\gamma)}(x, \xi+\theta\eta) q_{2(\gamma)}(x+y,\xi')\,dy\,đ\eta\ ;
$$

where $\xi' = (\xi_1, \dots, \xi_{n-1}, 0)$. Moreover, let $I, J \in I\!N$ be given. Then

$$
|q_{\gamma,\theta(\beta)}^{(\alpha)}(x,\xi)| \le c^r_{\alpha\beta\gamma}\,\langle\xi\rangle^{s_+ + t'_+ - |\gamma|(1-\delta)+|\beta|\delta}\,\langle\xi'\rangle^{-|\alpha|}
$$

for all γ with $|\gamma| \ge 2(s_+ + t'_+)$ and all α, β with $|\alpha| \le I$, $|\beta| \le J$. As before we use the notation $\sigma_+ = \max\{\sigma, 0\}$.

We omit the complete proof. It is very similar to that of Lemma 3.5; in fact it even is slightly easier. The inequality $|\gamma| \ge 2(s_+ + t'_+)$ is an analog of the condition $|\gamma| \ge s + s' + t'$ in 3.5; here we want $(1-\delta)|\gamma|$ sufficiently large and estimate it by $|\gamma|/2$.

4.16 Lemma. *We use the notation of Lemma 4.14.*
For $\tilde N$ sufficiently large the remainder terms $r_{\tilde N}(x',\xi)$ in (12) define bounded operators in $H^k(I\!R^n)$. Specifically, it is sufficent if $\tilde N \ge M + I + 2k$.

P r o o f. ¿From Lemma 4.10 and Lemma 4.14 we know that the symbols q_1 and q_2 introduced before fulfill the estimates in Lemma 4.15 with $s = m - k + j\delta - l$ and $t' = -|\gamma| - j + l - m' + |\gamma|\delta + k$. Hence it is sufficient to have $\tilde N \ge 2(s_+ + t'_+)$. By Lemma 4.15 we then have

$$
|r_{\tilde N(\beta)}^{(\alpha)}(x,\xi)| \le c^r_{\alpha\beta}\,\langle\xi\rangle^{(m-k)_+ + M\delta + (l-j)_+ + k - \tilde N(1-\delta) + |\beta|\delta}\,\langle\xi'\rangle^{-|\alpha|}.
$$

In view of the fact that $l \le j$, $0 < \delta \le 1/2$, $k > m$ and $|\beta| \le I$ we only need to ask that $\tilde N \ge M + I + 2k$ in order to make sure that $r_{\tilde N}(x,\xi) \in S^0_{0,0}(I,J)$. By Theorem 1.2 this will guarantee the boundedness of $Op\, r_{\tilde N}$ in $H^k_2(I\!R^n)$.

This completes the proof of Lemma 4.14, hence that of Theorem 2.2.

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