Constructible functions on 2−dimensional analytic manifolds

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Abstract: We give a characterization of sums of signs of global analytic functions on a real analytic manifold M of dimension two. We use it to study analytically constructible functions. On the real spectrum of the field of meromorphic functions on M, our characterization shows that unbounded orderings are not relevant to recognize signatures of quadratic forms.

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Introduction.

Let M be a paracompact real analytic manifold. A function $\varphi : M \to \mathbb{Z}$ is said to be *semianalytically constructible* if it is constant on each element of a finite partition of M into global semianalytic sets.

The aim of this paper is to give, if $\dim M = 2$, a characterization of the semianalytically constructible functions which are finite sums of signs of global analytic functions. We prove that in dimension two, the obstructions for a semianalytically constructible function φ to be a sum of signs of global analytic functions lie in the boundaries of the $\varphi^{-1}(n)$ for $n \in \mathbb{Z}$. We do it in two steps: first we reduce our problem to the characterization of functions which are generically sums of signs of analytic functions, and then we give such a characterization. We get it as a consequence of a characterization of principal open sets, i.e. sets of the form $\{x \in M \mid f(x) > 0\}$ where f is a global analytic function on M.

These results are similar to the ones for sums of signs of polynomials in [B1], and for sums of signs of Nash functions in [B2], except that here we do not need any hypothesis of compactness. Indeed, φ is a sum of signs of global analytic functions on M if and only if it is true on each compact subset of M. Compare with [BP] for similar results.

Our motivation is to study *analytically constructible functions,* which are defined as linear combinations, with integer coefficients, of Euler characteristic of fibres of proper analytic morphisms. We show that in general, the two classes of analytically constructible and semianalytically constructible functions are not contained in each other, even if M is compact of dimension 2. However, we prove that, if $\dim\ M = 2$, the intersection between these two classes coincides exactly with the class of constructible functions which are sums of signs of global analytic functions on M. Similar results exist in the algebraic case, cf. [CK], [PS], and in the Nash case for a compact set, cf. [B2]. We also prove that the compactness assumption is not necessary in dimension 2 for the Nash case.

Assume that M is connected of dimension 2, and denote by $\mathcal{M}(M)$ the field of meromorphic functions on M. Artin-Lang property gives a correspondence between semianalytically constructible functions on M (considered generically) and constructible functions on the real spectrum of $\mathcal{M}(M)$, i.e. functions $\mathrm{Spec}_{\mathrm{r}}\mathcal{M}(M) \to \mathbb{Z}$ which are continuous with respect to the constructible topology of $\mathrm{Spec}_{\mathrm{r}}\mathcal{M}(M)$ and the discrete topology of $\mathbb{Z}$. Sums of signs of global analytic functions correspond to signatures of quadratic forms over $\mathrm{Spec}_{\mathrm{r}}\mathcal{M}(M)$. This allows us to bound the minimal number of analytic functions we need to write generically a sum of signs of analytic functions. A characterization of signatures of quadratic forms over the real spectrum is given by the Representation Theorem of Becker and Bröcker. We prove that, when working on $\mathcal{M}(M)$, it is not necessary to consider unbounded orderings in this characterization.

The paper is organized as follows. In the first section, we define the functions we work on, and we give some results in dimension one and two. The second section is devoted to a characterization of principal open sets (in any dimension), and a characterization of sums of signs of analytic functions in dimension two. In the third part, we study analytically constructible and Nash constructible functions in dimension two. The fourth part deals with quadratic forms over the real spectrum.

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1 Definitions and first results

1.1 Definitions and notations

Let M be a paracompact real analytic manifold. We denote by $\mathcal{O}(M)$ the ring of global analytic functions on M. If M is connected, we denote by $\mathcal{M}(M)$ the field of meromorphic functions on M, i.e. the quotient field of $\mathcal{O}(M)$.

We will say that a property is true *generically* on M if it is true on M outside a global analytic subset of M of strictly smaller dimension. We will write $=_{gen}$ for an equality holding generically.

A subset $S \subset M$ is called *global semianalytic* if it can be written in the form

$$S = \bigcup_{i=1}^{p} \{x \in M \mid f_i(x) = 0, g_{i,1}(x) > 0, \dots, g_{i,r_i}(x) > 0\}$$

where $f_i, g_{i,j} \in \mathcal{O}(M)$. It is said to be *basic open* if $S = \{x \in M \mid g_1(x) > 0, \dots, g_r(x) > 0\}$ with $g_i \in \mathcal{O}(M)$, and to be *principal open* if $S = \{x \in M \mid g(x) > 0\}$ with $g \in \mathcal{O}(M)$.

The *stability index* of M is the smallest integer s such that every basic open set is an intersection of at most s principal open sets, if such an integer exists. Otherwise, this stability index is ∞.

We define *semianalytically constructible functions* on M as functions $\varphi : M \to \mathbb{Z}$ which admit a presentation as a finite sum

$$\varphi = \sum_i m_i \mathbf{1}_{X_i}, \tag{1.1}$$

where the m_i's are integers, the X_i's are global semianalytic subsets of M and where $\mathbf{1}_{X_i}$ denotes the characteristic function of the subset X_i. In other words, a function $\varphi : M \to \mathbb{Z}$ is semianalytically constructible if it is constant on each element of a finite partition of M into global semianalytic sets. More generally, if $S \subset M$ is a semianalytic subset of M, we say that $\varphi : S \to \mathbb{Z}$ is *semianalytically constructible on* S if φ is the restriction to S of a semianalytically constructible function on M.

Remark 1.1. Our definition of semianalytically constructible functions can be compared to the definition of semialgebraically constructible functions in the algebraic setting. If $M \subset \mathbb{R}^N$ is an affine algebraic (or semialgebraic) set, a function $\varphi : M \to \mathbb{Z}$ is *semialgebraically constructible* if it admits a presentation (1.1) where the X_i's are semialgebraic subsets of M, cf. [MP1].

Note also that in our definition, the adjective "semianalytically" is used to stress the difference with the classical definition of constructible function in the analytic setting, cf. [S], [KS]. There, a function $\varphi : M \to \mathbb{Z}$ on a real analytic manifold M is said to be *constructible* if for each $m \in \mathbb{Z}$ the set $\varphi^{-1}(m)$ is subanalytic and the family $\{\varphi^{-1}(m)\}_{m\in\mathbb{Z}}$ is locally finite. These constructible functions form a wider class than our semianalytically constructible functions.

Let $X \subset \mathbb{R}^k$ be a real analytic set. We follow [MP1] and [S]. If φ is a semianalytically constructible function on X with compact support, there is a presentation (1.1) where the X_i's are global compact semianalytic subsets of X, by [ABR, Th. 7.2]. Then, the *Euler integral* of φ on X is defined by

$$\int_X \varphi = \sum_i m_i \chi(X_i),$$

where χ denotes the Euler characteristic. This number depends only on φ.

If $f : Y \to X$ is a proper analytic map from a real analytic set $Y \subset \mathbb{R}^l$, and φ is semianalytically constructible on Y, the *pushforward* of φ by f is the constructible function $f_*\varphi$ on X defined by

$$\forall x \in X,\ f_*\varphi(x) = \int_Y \varphi \mathbf{1}_{f^{-1}(x)}.$$

The function $f_*\varphi$ does not need to be semianalytically constructible (cf. Examples 3.1 and 3.2).

We define analytically constructible functions following [MP2]:

Definition 1.2. A function $\varphi : X \to \mathbb{Z}$ is *analytically constructible* if it admits a presentation as a finite sum

$$\varphi = \sum m_i f_{i*} \mathbf{1}_{Z_i} \tag{1.2}$$

where the m_i's are integers, the Z_i's are real analytic sets, and each $f_i : Z_i \to X$ is a proper analytic map. Then, for $x \in X$, we have $\varphi(x) = \sum m_i \chi\big(f_i^{-1}(x)\big)$.

Remark 1.3. Assume that X is an affine real algebraic set. A function $\varphi : X \to \mathbb{Z}$ is *algebraically constructible* if it admits a presentation (1.2) where the Z_i's are real algebraic sets, and the f_i's are proper polynomial maps. The function φ is said to be *Nash constructible* if it admits a presentation (1.2) where the Z_i's are connected components of real algebraic sets, and the f_i's are proper polynomial maps.

Algebraically constructible and Nash constructible functions have been introduced by McCrory and Parusiński in [MP1] to study the topology of real algebraic sets and semialgebraic arc-symmetric sets respectively.

Algebraically constructible functions coincide with sums of signs of polynomials, see [CK] or [PS]. Similarly, Nash constructible functions on X are sums of signs of semialgebraic arc-analytic functions if X is compact or $\dim X = 2$, cf. [B2] and Section 3. (A function on an analytic set is *arc-analytic* if it becomes analytic when composed with any analytic arc on the set, cf. [K].)

Note that the algebraic setting and the analytic one are different: the class of algebraically constructible functions is included in the class of the Nash constructible ones, and all these functions are semialgebraically constructible.

1.2 First results in dimension 1 and 2

Let M be a paracompact real analytic manifold. If $f \in \mathcal{O}(M)$, the function sign $f : M \to \mathbb{Z}$ is defined as sign $f = \mathbf{1}_{\{f>0\}} - \mathbf{1}_{\{f<0\}}$, so it is semianalytically constructible. But there are semianalytically constructible functions that can not be written as sums of signs of global analytic functions. For instance, a sum of signs of global analytic functions on M has to be generically constant mod 2 on each connected component of M (since the sign of a global analytic function is so). In dimension one, this condition is sufficient for a semianalytically constructible function to be a sum of signs of global analytic functions. More precisely:

Proposition 1.4. *Let $W \subset M$ be an analytic set of dimension one, and let $\varphi : W \to \mathbb{Z}$ be a semianalytically constructible function. Then φ is a finite sum of signs of global analytic functions on W if and only if φ is generically constant* mod 2 *on each irreducible component of W of dimension one.*

Proof. The "only if" part is clear. For the "if" part, assume first W to be irreducible. Changing φ into $\varphi - 1$ if necessary, we may assume that φ is generically even. Let r be the half of the maximal positive generic value of φ (or 0 if φ is generically non positive), and let $-s$ be the half of the minimal negative generic value of φ (or 0 if φ is generically non negative). Set $X_j = \cup_{n \geq j} \mathrm{Int}_W\big(\varphi^{-1}(2n)\big)$ for

$j = 1, \ldots, r$ and $Y_k = \cup_{n \geq k} \mathrm{Int}_W\big(\varphi^{-1}(-2n)\big)$ for $k = 1, \ldots, s$, where Int_W means the interior taken in W. There exists a discrete set of points $D \subset W$ such that $\varphi = 2\mathbf{1}_{X_1} + \ldots + 2\mathbf{1}_{X_r} - (2\mathbf{1}_{Y_1} + \ldots + 2\mathbf{1}_{Y_s})$ on $W \setminus D$.

By [DC-A, Prop. 4.3], any open semianalytic subset of W is global principal. So, for $j = 1, \ldots, r$ and $k = 1, \ldots, s$, there exist $f_j, f'_k \in \mathcal{O}(W)$ such that $X_j = \{x \in W | f_j(x) > 0\}$ and $Y_k = \{x \in W | f'_k(x) > 0\}$. Then we have

$$\varphi = (1 + \text{sign } f_1) + \ldots + (1 + \text{sign } f_r) - (1 + \text{sign } f'_1) - \ldots - (1 + \text{sign } f'_s),$$

on $W \setminus D'$, where $D' = D \cup \bigcup_i \{f_i = 0\} \cup \bigcup_k \{f'_k = 0\} \subset W$ is a discrete set of points. So φ is generically a sum of signs of global analytic functions on W.

From now on we do not assume that W is irreducible. Let $W = \cup_{i \in I} W_i$ be the decomposition of W into irreducible components. Denote $I' = \{i \in I \mid \dim W_i = 1\}$ and $W' = \cup_{i \in I'} W_i$. By the irreducible case, we know that for each $i \in I'$, there are $f_1^i, \ldots, f_{r_i}^i \in \mathcal{O}(W_i)$ such that $\varphi_{|W_i} =_{gen} \text{sign } f_1^i + \ldots + \text{sign } f_{r_i}^i$ on W_i. Since φ assumes only a finite number of values, we can assume that all r_i's are equal, say $r_i = r$ for any $i \in I'$. For $i \in I'$, let $t_i \in \mathcal{O}(W)$ be a positive equation for $\overline{W \setminus W_i}$. Then $\varphi_{|W_i} =_{gen} \text{sign } (t_i f_1^i) + \ldots + \text{sign } (t_i f_r^i)$ on W_i.

For $j = 1, \ldots, r$, set $g_j = \sum_{i \in I} f_j^i t_i$. These are global analytic functions on W since the sum is locally finite. We have $\varphi =_{gen} \text{sign } g_1 + \ldots + \text{sign } g_r$ on W. Then, we can write $\varphi - \sum_{j=1}^r \text{sign } g_j = k_1 \mathbf{1}_{D_1} + \ldots + k_m \mathbf{1}_{D_m}$, where $k_1, \ldots, k_m$ are integers, and $D_1, \ldots, D_m$ are discrete subsets of W. For $l = 1, \ldots, m$, let $h_l \in \mathcal{O}(W)$ be a positive equation of D_l. Then, it is easy to check that $\varphi = \sum_{j=1}^r \text{sign } g_j + k_1(1 - \text{sign } h_1) + \ldots + k_m(1 - \text{sign } h_m)$. $\square$

Remark 1.5. Since W is coherent, it follows from Cartan's Theorem B that φ is the restriction to W of a sum of signs of global analytic functions on M if and only if it is a sum of signs of global analytic functions on W.

Remark 1.6. If the dimension of M is ≥ 2, there are semianalytically constructible functions which are generically constant mod 2, but which are not generically sums of signs of global analytic functions. For instance, the function $2\mathbf{1}_{\{x>0,y>0\}}$ is not generically a sum of signs of global analytic functions on $\mathbb{R}^2$, since if the sign of a global analytic function changes crossing $\{y = 0, x < 0\}$, it changes also crossing $\{y = 0, x > 0\}$.

Proposition 1.4 solves completely the problem for semianalytically constructible functions on an analytic set of dimension one. We come now to the case of dimension two.

We first prove an analogous in dimension two of the previous proposition. We will use this result in the following section.

Proposition 1.7. *Assume that the dimension of M is two. Let $\varphi : M \to \mathbb{Z}$ be a semianalytically constructible function which is generically constant* mod 4. *Then φ is generically a sum of signs of global analytic functions on M.*

Proof. Let $S \subset M$ be a global semianalytic subset of M, we have to prove that $4\mathbf{1}_S$ is generically a finite sum of signs of global analytic functions on M. By [DC-A], the set S is generically the union of two basic open global semianalytic sets S_1 and S_2. Then $\mathbf{1}_S =_{gen} \mathbf{1}_{S_1} + \mathbf{1}_{S_2} - \mathbf{1}_{S_1 \cap S_2}$. Since a finite intersection of basic open sets is still basic open, we can assume that S is basic open. Again by [DC-A], the stability index of M is two. So there exist $f_1, f_2 \in \mathcal{O}(M)$ such that $S = \{x \in M \mid f_1(x) > 0, f_2(x) > 0\}$. Then we have $4\mathbf{1}_S =_{gen} \text{sign } f_1 + \text{sign } f_2 + \text{sign } (f_1 f_2) + 1$. □

Remark 1.8. The value of the stability index of M is not known if $\dim M > 2$. So we do not know if, in dimension n, a semianalytically constructible function which is generically constant mod 2^n is generically a sum of signs of analytic functions.

Proposition 1.4 shows that, in dimension one, a semianalytically constructible function is a sum of signs of global analytic functions if and only if this is true generically. But in dimension ≥ 2, there are semianalytically constructible functions which are sums of signs of global analytic functions only generically. In Proposition 1.9, we characterize, for a manifold of dimension two, the "smallest" set outside which a semianalytically constructible function that is generically a sum of signs, is a sum of signs.

Let M be of any dimension. If $S \subset M$ is a semianalytic set, we denote by $\partial_Z S$ the Zariski closure of the Euclidean boundary ∂S of S, i.e. the smallest global analytic subset of M containing ∂S. If S is global, then ∂S is contained in the union of the zero sets of the functions appearing in a description of S, so the set $\partial_Z S$ is proper. Note that, if $\dim M > 2$, it is not known if ∂S is still a global semianalytic set.

Proposition 1.9. *Assume that the dimension of M is two. Let $\varphi : M \to \mathbb{Z}$ be a semianalytically constructible function which is generically a sum of signs of global analytic functions on M. Let X be the union of the irreducible components of dimension one of $\bigcup_{n\in\mathbb{Z}} \partial_Z\big(\varphi^{-1}(n)\big)$ on which φ is not generically constant* mod 2. *Then, there exist $f_1, \ldots, f_k \in \mathcal{O}(M)$ such that $\varphi_{|M\setminus X} = (\text{sign } f_1 + \ldots + \text{sign } f_k)_{|M\setminus X}$.*

Proof. By hypothesis we can write $\varphi_{|M\setminus W} = (\text{sign } f_1 + \ldots + \text{sign } f_k)_{|M\setminus W}$ where $f_1, \ldots, f_k \in \mathcal{O}(M)$ and $W \subset M$ is an analytic set of dimension one. We can always suppose $(\text{sign } f_1 + \ldots + \text{sign } f_k)_{|W} = 0$. Write $W = \bigcup_{i\in I} W_i \cup D$, where D is a discrete set, the W_i's are distinct irreducible analytic sets of dimension one, and $\bigcup_i W_i \cap D = \emptyset$.

Set $W' = \cup_{i\in I'} W_i$ where $I' = \{i \in I | \varphi_{|W_i} \text{is generically constant } \bmod 2\}$. Note that for each $i \in I \setminus I'$, the component W_i is contained in $\bigcup_{n\in\mathbb{Z}} \partial_Z\big(\varphi^{-1}(n)\big)$. Indeed, let $i \in I$ such that $W_i \not\subset \bigcup_{n\in\mathbb{Z}} \partial_Z\big(\varphi^{-1}(n)\big)$. Then $W_i \cap \bigcup_{n\in\mathbb{Z}} \partial_Z\big(\varphi^{-1}(n)\big)$ is a discrete subset of W_i, and outside this subset the parity of $\varphi_{|W_i}$ is the generic parity of φ on the connected component of M containing W_i. So i belongs to I', and we have $X = \cup_{i\in I\setminus I'} W_i$.

By Proposition 1.4 and Remark 1.5 there exist $s_1, \ldots, s_r \in \mathcal{O}(M)$ such that $\varphi_{|W'\cup D} = (\text{sign } s_1 + \ldots + \text{sign } s_r)_{|W'\cup D}$. Let $t \in \mathcal{O}(M)$ be a positive equation for

$W' \cup D$. Then $\varphi_{|M\setminus X} = (\text{sign } f_1 + \ldots + \text{sign } f_k - \text{sign } ts_1 - \ldots - \text{sign } ts_r + \text{sign } s_1 + \ldots + \text{sign } s_r)_{|M\setminus X}$. ☐

Corollary 1.10. *Assume* $\dim M = 2$. *Let* $\varphi : M \to \mathbb{Z}$ *be a semianalytically constructible function, which is generically a sum of signs of global analytic functions on* M. *Then,* φ *is a sum of signs of global analytic functions everywhere on* M *if and only if* φ *is generically constant* mod 2 *on each global irreducible component of dimension one of* $\bigcup_{n\in\mathbb{Z}} \partial_Z\big(\varphi^{-1}(n)\big)$.

2 The characterizations

2.1 A criterion for principality

Let M be a paracompact real analytic manifold of dimension n, and let $S \subset M$ be a global semianalytic subset. Denote $S^* = \text{Int}\big(\text{Adh}(S)\big)$. This is a semianalytic set, although its global character is known only in dimension one and two by [C]. Note that $\partial S^* \subset \partial S$, so $\partial_Z S^*$ has codimension one in M. We define a *wall of S* as an irreducible component of $\partial_Z S^*$ of codimension one in M. The notion of wall is used in [ABV], [1] and [BP] to study basicness and separation of semialgebraic and global semianalytic sets.

Definition 2.1. Let W be a wall of S. We say that W is *even* if $\dim(W \cap S^*) = n-1$ or if $\dim\big(W \cap (M\setminus S)^*\big) = n-1$, and we say that W is *odd* if $\dim\big(W \cap \text{Adh}(S^*) \cap \text{Adh}((M\setminus S)^*)\big) = n-1$, cf. [1], [AB]. (The dimension of a semianalytic set T is the maximum integer d such that T contains regular points of dimension d.)

Proposition 2.2. *Let $S \subset M$ be a global semianalytic set. Then $S \setminus \partial_Z S$ is principal open if and only if no wall of S is simultaneously odd and even.*

Proof. Assume $S \setminus \partial_Z S$ is principal open, i.e. there exists $f \in \mathcal{O}(M)$ such that $S \setminus \partial_Z S = \{f > 0\}$. Let W be wall of S. Denote by $t \in \mathcal{O}(M)$ an uniformizer of $\mathcal{O}(M)_{\mathcal{I}(W)}$, which is a discrete valuation ring of rank one by [BP, Prop.1.5]. Then $f = t^k g$ for some integer k and $g \in \mathcal{O}(M)$ not vanishing identically on W. The number k is even (respectively odd) if W is so, and W cannot be simultaneously odd and even.

Since $S\setminus\partial_Z S$ is principal open if and only if it is principal open in each component of M, it enough to prove the converse implication if M is connected. Assume that no wall is simultaneously odd and even. Write $\partial_Z S = Y \cup Z$ where Y is the union of the odd walls, and Z is a global analytic set. If $f \in \mathcal{O}(M)$ is such that $S\setminus\partial_Z S = \{f > 0\}$, then f has to vanish on Y and also on the components of Z which are adherent to $S \setminus \partial_Z S$. We will construct an f such that its zero set coincides exactly with $\partial_Z S$.

Let $p \in \mathcal{O}(M)$ be a positive equation for Z. Denote by $\mathcal{I}$ the ideal sheaf induced by $\mathcal{I}(Y)$. Note that it is enough to prove that $\mathcal{I}$ is principal. Indeed if $f \in \mathcal{O}(M)$ generates $\mathcal{I}(Y)$, then, by construction we get $S \setminus \partial_Z S = \{pf > 0\}$ or $S \setminus \partial_Z S = \{pf < 0\}$.

By [BP, Lemma 1.2], the ideal sheaf $\mathcal{I}$ is locally principal, so for each $x \in M$ we can choose a generator h_x of $\mathcal{I}_x$. Fix $x \in M$, we consider the analytic components of the analytic set germ induced by $\partial_Z S$ at x. The definition of odd and even walls still makes sense, and by the hypothesis no wall of S_x can be simultaneously odd and even. Note that the components of codimension one of Y_x are odd. So the analytic function germ $h_x p_x$ is such that $(S \setminus \partial_Z S)_x = \{h_x p_x > 0\}$ or $(S \setminus \partial_Z S)_x = \{h_x p_x < 0\}$. Changing h_x into $-h_x$ if necessary, we can assume that $(S \setminus \partial_Z S)_x = \{h_x p_x > 0\}$. So we obtain local generators $\{h_i\}$ for $\mathcal{I}$ on a locally finite open covering $\{U_i\}$ of M such that $h_{i_{|U_i \cap U_j}}$ and $h_{j_{|U_i \cap U_j}}$ have the same sign, i.e the cocycle $\{h_i/h_j\}$ is zero in $H^1(M, \mathbb{Z}_2)$. Now we get the thesis by general theory of sheaves.

The exponential map and the associated usual exact sequence

$$0 \to \mathcal{O}_M \to \mathcal{O}_M^* \to \mathcal{O}_M^*/\mathcal{O}_M^+ = \mathbb{Z}_2 \to 0$$

induce an isomorphism between $H^1(M, \mathbb{Z}_2)$ and $H^1(M, \mathcal{O}_M^*)$, so the cocycle $\{h_i/h_j\}$ is zero also in $H^1(M, \mathcal{O}_M^*)$.

Then, using the exact sequence

$$0 \to \mathcal{O}_M^* \to \mathcal{M}_M^* \to \mathcal{M}_M^*/\mathcal{O}_M^* \to 0,$$

we see that the global section $x \mapsto h_{i,x}$ of the sheaf of the divisors lies in the image of the map $\Gamma(M, \mathcal{M}_M^*) \to \Gamma(M, \mathcal{M}_M^*/\mathcal{O}_M^*)$. This means that there exists a non zero global meromorphic function f on M such that $f_x\mathcal{O}^* = h_{i,x}\mathcal{O}^*$. In particular, f lies in $\mathcal{O}(M)$ since f_x lies in $\mathcal{O}_{M,x}$ for each $x \in M$, and f generates $\mathcal{I}(Y)$. The proof is done. $\square$

Remark 2.3. Proposition 2.2 shows in particular that S is generically principal if and only if $S \setminus \partial_Z S$ is principal open. It shows also that S is generically principal if and only if for each compact set $K \subset M$, the set $S \cap K$ is generically principal.

2.2 A characterization of sums of signs in dimension 2

Let M be a paracompact real analytic manifold of dimension 2, and let $\varphi : M \to \mathbb{Z}$ be a semianalytically constructible function. We want to know if φ is a finite sum of signs of global analytic functions. By Corollary 1.10 we can reduce ourselves to study a generic problem. A *wall of the function* φ is a wall of a $\varphi^{-1}(n)$ for $n \in \mathbb{Z}$.

Let $W \subset M$ be a wall of φ, and denote $X_n = \varphi^{-1}(n)$ for $n \in \mathbb{Z}$. Only a finite number of X_n are non empty. Then, we can define the function $\partial_W \varphi : W \to \mathbb{Z}[1/2]$ in this way:

$$\partial_W \varphi = \sum_{n \in \mathbb{Z}} \frac{n}{2} \mathbf{1}_{\partial(X_n^*) \cap W} + n \mathbf{1}_{X_n^* \cap W}.$$

The function $\partial_W \varphi$ is constant on each element of a finite semianalytic partition of W, but a priori it is not integer-valued. If φ is generically constant mod 2 on each connected component of M, then $\partial_W \varphi$ is generically semianalytically constructible.

Let x be a point in $W \cap \operatorname{Reg}\left(\bigcup_n \partial_Z X_n\right)$. Then φ is constant on each of the two connected components of $(M \setminus W)_x$, and $\partial_W \varphi(x)$ coincides with the media of the values of φ on these two components.

In the algebraic or Nash context, the functions $\partial_W \varphi$ have been used to study algebraically constructible and Nash constructible functions defined in Remark 1.3 (cf. [B1], [B2]).

Remark 2.4. If M is embedded in some $\mathbb{R}^k$, and φ is a semianalytically constructible function on M, the *link* of φ is the constructible function $\Lambda\varphi$ on M defined by

$$\forall x \in M,\ (\Lambda\varphi)(x) = \int_M \varphi \mathbf{1}_{S(x,\varepsilon)},$$

where $\varepsilon > 0$ is small enough, and $S(x,\varepsilon)$ denotes the sphere with center x and radius ε in M, cf. [MP1]. This value depends only on φ and x.

It can be checked that we have $\partial_W \varphi =_{gen} (\varphi - \frac{1}{2}\Lambda\varphi)_{|W}$.

Theorem 2.5. *Let M be a paracompact real analytic manifold of dimension two. A semianalytically constructible function $\varphi : M \to \mathbb{Z}$ is generically a sum of signs of global analytic functions if and only if, for each wall W of φ, the function $\partial_W \varphi : W \to \mathbb{Z}$ is generically integer-valued and generically constant* mod 2.

Proof. Assume there exist $f_1, \dots, f_r \in \mathcal{O}(M)$ such that $\varphi =_{gen} \operatorname{sign} f_1 + \ldots + \operatorname{sign} f_r$ on M. Let $t \in \mathcal{O}(M)$ be an uniformizer of $\mathcal{O}(M)_{\mathcal{J}(W)}$, which is a discrete valuation ring of rank one by [BP]. For $i = 1, \dots, r$, we can write $f_i = t^{k_i} g_i$ where k_i is an integer and $g_i \in \mathcal{O}(M)$ is not identically zero on W. Then on W we have

$$\partial_W \varphi =_{gen} \sum_{i \,|\, k_i \text{ even}} \operatorname{sign} g_i.$$

We now prove the converse implication. Assume that on each wall W of φ, the function $\partial_W \varphi : W \to \mathbb{Z}$ is generically integer-valued and generically constant mod 2. We first prove that φ is generically constant mod 2 on each connected component of M. Let $\bar{\varphi} : M \to \{0,1\}$ be a semianalytically constructible function such that $\varphi - \bar{\varphi}$ is generically even. If $\bar{\varphi}$ had a wall W, then $\partial_W \bar{\varphi}$ would be generically $1/2$ on a non empty open subset of W. As $\partial_W(\varphi - \bar{\varphi})$ is generically integer-valued and as we have $\partial_W \varphi =_{gen} \partial_W(\varphi - \bar{\varphi}) + \partial_W \bar{\varphi}$, it would mean that $\partial_W \varphi$ is not generically integer-valued on W. So $\bar{\varphi}$ has no wall, it is generically constant on each connected component of M, and φ is generically constant mod 2 on each connected component. We can write $\varphi = \varphi_1 + \varphi_2 + \mathbf{1}_C$ where $\varphi_1, \varphi_2 : M \to \mathbb{Z}$ are semianalytically constructible functions such that φ_1 is generically constant mod 4, φ_2 takes only the two values 1 and -1 and C is a union of connected components of M.

By Proposition 1.7, we only have to prove that the function φ_2 is generically a sum of signs of analytic functions. It is enough to show that $S = \varphi_2^{-1}(1)$ is generically principal. Indeed, if $S =_{gen} \{f > 0\}$, then $\varphi_2 =_{gen} \operatorname{sign} f$. By Proposition 2.2 we know that S is generically principal if and only if no wall of S is simultaneously odd and even. Let W be a wall of S. It is also a wall of φ. Remark that $\partial_W\varphi_2 =_{gen} 0$ on $W\cap$ $\mathrm{Adh}(S^*)\cap$ $\mathrm{Adh}\big((M\setminus S)^*\big)$, and $\partial_W\varphi_2 =_{gen} 1$ (respectively -1) on $W\cap S^*$ (respectively on $W\cap(M\setminus S)^*$). As we have $\partial_W\varphi_2 =_{gen} \partial_W\varphi - \partial_W\varphi_1$ if $W \not\subset C$ and $\partial_W\varphi_2 =_{gen} \partial_W\varphi - \partial_W\varphi_1 + 1$ if $W\subset C$, the function $\partial_W\varphi_2$ is generically constant mod 2 on W, so the wall W of S cannot be simultaneously odd and even. The proof is done. □

Corollary 2.6. *Let M be a paracompact real analytic manifold of dimension two. A semianalytically constructible function $\varphi : M\to\mathbb{Z}$ is a sum of signs of global analytic functions if and only if, for each compact set $K\subset M$, the function $\varphi_{|K}$ is the restriction to K of a finite sum of signs of global analytic functions on M.*

3 Analytically constructible functions in dimension two

3.1 The analytic case

Let X be a real analytic subset of $\mathbb{R}^k$, and let φ be an analytically constructible function on X (cf. Definition 1.2). As remarked before, φ is not necessarily semianalytically constructible, even in dimension 1.

Example 3.1. Set $X=\bigcup_{n\geq 0}\big\{(x,y)\in\mathbb{R}^2 | x-n=\frac{y^2}{2^n}\big\}$. The set X is a locally finite union of global analytic subsets of $\mathbb{R}^2$, so it is global analytic. Let $\pi : X\to\mathbb{R}$ be the projection on the first component: $\pi(x,y)=x$. Then, $\pi_*\mathbf{1}_X:\mathbb{R}\to\mathbb{Z}$ is not semianalytically constructible since it is not finitely valued. Indeed, for each $n\in\mathbb{N}$, we have $\pi_*\mathbf{1}_X(x)=2n$ for any $x\in]n-1,n[$.

Note that if $X\subset\mathbb{R}^k$ is a compact analytic set, a constructible function on X has to be finitely valued, i.e. it has to be constant on a finite partition of X into subanalytic sets. If $\dim X=1$, any subanalytic set is semianalytic and any semianalytic set is global. So, any constructible function on a compact analytic curve is semianalytically constructible.

But if $\dim X>1$, the compactness assumption is not sufficient: even if X is compact and smooth, there are analytically constructible functions on X which are not semianalytically constructible. The reason of this fact is the presence of compact semianalytic subsets of X which are not global.

Example 3.2. Set $Z=\{(x,y)\in\mathbb{R}^2|(x-\frac{3}{2})^2+y^2=\frac{1}{4}\}$ and let $f:Z\to\mathbb{R}^2$ be defined by $f(x,y)=(x,\sin\frac{1}{x})$. This is an analytic morphism since the circle Z does not intersect the y-axis and it is proper since Z is compact. The values of the function $\varphi=f_*\mathbf{1}_Z:\mathbb{R}^2\to\mathbb{Z}$ are:

$$\varphi(x,y)=\begin{cases} 1 & \text{if } (x,y)=(1,\sin 1) \text{ or } (x,y)=(2,\sin\frac{1}{2}),\\ 2 & \text{if } 1<x<2 \text{ and } y=\sin\frac{1}{x},\\ 0 & \text{else.}\end{cases}$$

The function φ is finitely valued, but it is not semianalytically constructible, not even generically. Indeed, the set $\varphi^{-1}(2)$ is not global. We have $\varphi^{-1}(2) = \{(x,y) \in \mathbb{R}^2 | 1 < x < 2, y = \sin\frac{1}{x}\}$, and any global analytic function on $\mathbb{R}^2$ vanishing on $\varphi^{-1}(2)$ has to be identically zero. So the Zariski closure of $\varphi^{-1}(2)$ coincides with the whole $\mathbb{R}^2$.

Let $g : \mathbb{R}^2 \to S^2$ be the one-point compactification, then $(g \circ f)_* \mathbf{1}_Z : S^2 \to \mathbb{Z}$ is an example of analytically constructible function on a two-dimensional compact analytic manifold which is not semianalytically constructible.

Vice-versa, it is easy to find examples of semianalytically constructible functions which are not analytically constructible. For instance, analytically constructible functions must verify some parity conditions, see Lemma 3.3. We prove that, on a two-dimensional analytic manifold, a semianalytically constructible function is analytically constructible if and only if it is a sum of signs of global analytic functions (and equivalently an analytically constructible function is semianalytically constructible if and only if it is such a sum of signs).

Lemma 3.3. *Let X be a real analytic subset of $\mathbb{R}^k$, and let φ be an analytically constructible function on X.*

1. *If $Y \subset \mathbb{R}^l$ is a real analytic set, and $f : Y \to X$ is an analytic map, then the function $\varphi \circ f$ is analytically constructible on Y.*
2. *If X is an irreducible curve (maybe singular), then φ is generically constant* mod 2 *on X.*
3. *The function $\frac{1}{2}\Lambda\varphi$ is analytically constructible on X.*
4. *If f is an analytic function on X, then* sign f *is analytically constructible on X.*

Proof. 1. This point can be proved as for algebraically constructible and Nash constructible functions in [MP1] and [B2].

2. There exists $\pi : X' \to X$ a locally finite sequence of blowings-up of points such that X' is a non singular connected curve. The function $\varphi \circ \pi$ is analytically constructible on X' by point *1.*, therefore by [MP1, Prop.5.1] it is generically constant mod 2 on X', and so is φ on X. (Here generically means outside a discrete set of points.)

3. We can copy the proof of Theorem 2.5 in [MP1]. The main element of this proof is the desingularization of analytic sets, which is possible in our context using global blowings-up by [H].

4. Denote $Y = \{(x,t) \in X \times \mathbb{R} \mid f(x) = t^2\} \subset \mathbb{R}^k \times \mathbb{R}$, and let $\pi : Y \to X$ be the projection on the first component. Then we have $\text{sign} f = \pi_* \mathbf{1}_Y - \mathbf{1}_X$, and sign f is analytically constructible. $\square$

Proposition 3.4. *Let $X \subset \mathbb{R}^k$ be an analytic curve, not necessarily smooth. Let φ be a semianalytically constructible function on X, i.e. a finitely valued constructible* *function on X. The function φ is analytically constructible if and only if it is a sum of signs of global analytic functions on X.*

Proof. By Proposition 1.4, the function φ is a sum of signs of global analytic functions on X if and only if it is generically constant mod 2 on each irreducible component of X of dimension 1. The same condition is necessary (and then, sufficient) for φ to be analytically constructible. $\square$

Theorem 3.5. *Let M be a paracompact real analytic manifold of dimension two. A semianalytically constructible function $\varphi : M \to \mathbb{Z}$ is analytically constructible if and only if it is a sum of signs of global analytic functions on M.*

Proof. The "if" part is given by point *4.* in Lemma 3.3. For the "only if" part, assume that φ is analytically constructible. We use Theorem 2.5 and Corollary 1.10 to prove that φ is a sum of signs. Let W be a wall of φ. As analytically constructible functions are preserved by the operator $\frac{1}{2}\Lambda$, the function $\partial_W \varphi$ is generically analytically constructible by Remark 2.4. In particular, $\partial_W \varphi$ is generically integer-valued and generically constant mod 2 on W. By Theorem 2.5, the function φ is generically a sum of signs of global analytic functions on M.

Let W' be an irreducible component of dimension one of $\bigcup_{n\in\mathbb{Z}} \partial_Z\left(\varphi^{-1}(n)\right)$. The function $\varphi_{|W'}$ is analytically constructible on W', so it is generically constant mod 2 on W'. Using Corollary 1.10, we conclude that φ is a sum of signs of global analytic functions on M. $\square$

Remark 3.6. The result of Theorem 3.5 is no more true if $\dim M \geq 3$, or if we work on a singular analytic set, cf. the examples in Remark 3.5 in [B2].

3.2 The Nash case

In this section M is an affine Nash manifold, i.e. a semialgebraic analytic submanifold of some $\mathbb{R}^k$. We denote by $\mathcal{N}(M)$ the ring of Nash functions on M, i.e. analytic and semialgebraic functions on M. Let $S \subset M$ be a semialgebraic set. The Nash-walls of S are defined in [B2] as the Nash-irreducible components of codimension one of the Nash-closure of the Euclidean boundary of S^*. By [CS], a global analytic set that is Nash, and Nash-irreducible, is also irreducible as a global analytic set. So the Nash-walls of S coincide with the (analytic) walls of S we use in this paper.

Theorem 3.7. *Let $M \subset \mathbb{R}^k$ be a Nash manifold and let $S \subset M$ be a semialgebraic set such that no wall of S is simultaneously odd and even. Then $S \setminus \partial_Z S$ is Nash principal open, i.e. $S \setminus \partial_Z S = \{f > 0\}$ with $f \in \mathcal{N}(M)$. In particular, a semialgebraic set is Nash principal open if and only if it is principal open as a global semianalytic set.*

Proof. By Proposition 2.2, we know that there exists $f \in \mathcal{O}(M)$ such that $S \setminus \partial_Z S = \{f > 0\}$. Write $\partial_Z S = Y \cup Z$ where Y is the union of the odd components of $\partial_Z S$ of codimension one in M and Z is a Nash set. By the proof of Proposition 2.2, we can assume that $f = hp$ where $p \in \mathcal{N}(M)$ is a positive equation for Z and $h \in \mathcal{O}(M)$ is a generator for the principal ideal $\mathcal{I}(Y)\mathcal{O}(M)$. Being Y a Nash set, we can find $g \in \mathcal{N}(M)$ a positive equation of Y. Then we have $g \in \mathcal{I}(Y)\mathcal{O}(M)$, and in particular we can write $g = hq$ with $q \in \mathcal{O}(M)$. By the factorization conjecture proved in [CS], we can find $\phi_1, \phi_2 \in \mathcal{O}(M)$ strictly positive analytic functions, and ψ_1, ψ_2 Nash functions on M, such that $\phi_1\phi_2 = 1$, and such that $h = \psi_1\phi_1$ and $q = \psi_2\phi_2$. The function $\psi_1 p$ is Nash on M and sign $\psi_1 =$ sign h on M. Then, $S \setminus \partial_Z S = \{\psi_1 p > 0\}$ is Nash principal. □

We can use this result to study Nash constructible functions. A semialgebraic set $S \subset M$ is *arc-symmetric* if for each analytic arc $\gamma :]-1,1[\to M$ such that $\gamma(]-1,0[) \subset S$, we have $\gamma(]-1,1[) \subset S$ (cf. [K]).

Corollary 3.8. *Let $\varphi : M \to \mathbb{Z}$ be a generically Nash constructible function which assumes only the values 1 and -1. Suppose that the walls of φ are irreducible as semialgebraic arc-symmetric sets. Then, there is $f \in \mathcal{N}(M)$ such that φ is generically* sign f.

Proof. We want to prove that $\varphi^{-1}(1)$ is generically Nash principal. By Theorem 3.7, it is enough to show that no wall of $\varphi^{-1}(1)$ is simultaneously odd and even. Assume that a wall W of $\varphi^{-1}(1)$ is simultaneously odd and even. This would mean that $\partial_W \varphi$ is not generically constant mod 2 on the irreducible arc-symmetric semialgebraic set W. So $\partial_W \varphi$ would not be generically Nash constructible on W by [B2, Prop.1.5]. This contradicts the generic Nash constructibility of φ, since Nash constructible functions are preserved by the operator $\frac{1}{2}\Lambda$. □

Theorem 3.9. *Let M be an affine Nash manifold of dimension two. A function $\varphi : M \to \mathbb{Z}$ is Nash constructible if and only if it is a sum of signs of Nash functions.*

Proof. We can copy the proofs of Theorem 2.5 and Theorem 3.5, since the stability index of $\mathcal{N}(M)$ is two, and since an irreducible analytic curve is irreducible as an arc-symmetric set (cf. [K]). □

Again, this result is false in general for a Nash manifold of dimension ≥ 3 or a singular set, see the examples of [B2, Rk.3.5].

Repeating the proof of Theorem 3.6 in [B2], we get:

Corollary 3.10. *Let $X \subset \mathbb{R}^k$ be a semialgebraic arc-symmetric set of dimension two. A function $\varphi : X \to \mathbb{Z}$ is Nash constructible if and only if φ is a sum of signs of semialgebraic arc-analytic functions on X.*

4 Constructible functions and quadratic forms

4.1 Recalls about real spectrum and Artin-Lang property

Let M be a paracompact real connected analytic manifold. The real spectrum $\mathrm{Spec}_{\mathrm{r}} K$ of a field K is the set of the orderings of the field K. In this section we want to translate the criterion of Theorem 2.5 in terms of the behaviour of fans in the real spectrum of $\mathcal{M}(M)$.

On the real spectrum, the characterization of signatures of quadratic forms and many others problems, as for instance basicness and separability for constructible sets, can be solved in terms of finite spaces of orderings. In the real spectrum of the field of rational functions on an irreducible real algebraic set, the relevant information lies in some particular finite spaces of orderings, that are attached to discrete valuation rings, and therefore behave specially well with respect to geometry (see for instance [1]). Thanks to Artin-Lang property that acts as a translator between constructible sets and semialgebraic sets, we can read the statements proved in the real spectrum as geometric results.

In the global analytic case, in general we cannot use the same technique. For instance, Artin-Lang property is known only for $\mathrm{Spec}_\mathrm{r}\mathcal{M}(M)$ if $\dim M \leq 2$, and it is known that this property does not hold for $\mathcal{O}(M)$ even in dimension one. Another problem is the lackness of a natural compactification (this is one among the reasons why many problems are still open also in the Nash setting).

In this section we assume that the dimension of M is two, so Artin-Lang property holds for $\mathrm{Spec}_\mathrm{r}\mathcal{M}(M)$ by [C].

Let K be a field. A *constructible subset* of $\mathrm{Spec}_\mathrm{r}K$ is a set of the form $\cup_{i=1}^{p}\{\sigma \in \mathrm{Spec}_\mathrm{r}K \mid \sigma(g_{i,1}) > 0, \ldots, \sigma(g_{i,j_i}) > 0\}$ with $g_{i,j} \in K$. The constructible subsets of $\mathrm{Spec}_\mathrm{r}K$ form a basis for the *constructible topology* of $\mathrm{Spec}_\mathrm{r}K$.

A *constructible function* on the real spectrum of K is defined as a function $\mathrm{Spec}_\mathrm{r}K \to \mathbb{Z}$, continuous with respect to the constructible topology of $\mathrm{Spec}_\mathrm{r}K$ and the discrete topology of $\mathbb{Z}$. Note that such a function can assume only a finite number of values, since $\mathrm{Spec}_\mathrm{r}K$ is compact with respect to the constructible topology.

Assume $K = \mathcal{M}(M)$. The so called *tilde map* associates to a global semianalytic set $S \subset M$ the constructible set $\widetilde{S} \subset \mathrm{Spec}_\mathrm{r}\mathcal{M}(M)$ with the same description. Artin-Lang property means that this map is well defined, and that if S is a global semianalytic set such that $\widetilde{S}$ is non empty, then S is non empty.

A semianalytically constructible function $\varphi = \sum_i n_i \mathbf{1}_{S_i}$ on M induces a constructible function $\widetilde{\varphi} = \sum_i n_i \mathbf{1}_{\widetilde{S}_i}$ on $\mathrm{Spec}_\mathrm{r}\mathcal{M}(M)$. If we change the presentation of φ, by Artin-Lang property we get the same function $\widetilde{\varphi}$ on $\mathrm{Spec}_\mathrm{r}\mathcal{M}(M)$. So the function $\widetilde{\varphi}$ is well-defined.

Two semianalytically constructible functions induce the same function on the real spectrum of $\mathcal{M}(M)$ if and only if they are generically equal. We get a "generic" correspondence between semianalytically constructible functions on M and constructible functions on the real spectrum $\mathrm{Spec}_\mathrm{r}\mathcal{M}(M)$. Artin-Lang property implies directly that a semianalytically constructible function $\varphi : M \to \mathbb{Z}$ is generically the sum of the signs of the global analytic functions $f_1, \ldots, f_r$ on M if and only if the associated function $\widetilde{\varphi}$ on $\mathrm{Spec}_\mathrm{r}\mathcal{M}(M)$ is the signature of the quadratic form $\langle f_1, \ldots, f_r \rangle$ over $\mathrm{Spec}_\mathrm{r}\mathcal{M}(M)$.

4.2 A bound of the number of functions

Let M be a paracompact real connected analytic manifold of dimension 2, and let $\varphi : M \to \mathbb{Z}$ be a semianalytically constructible function. Assume that φ is a sum of signs of global analytic functions. We are interested in the minimal number $n(\varphi)$ of functions necessary to write φ generically, that is

$$n(\varphi) = \min\big\{r \in \mathbb{N} \mid \exists f_1, \dots, f_r \in \mathcal{O}(M), \quad \varphi =_{gen} \text{sign } f_1 + \ldots + \text{sign } f_r\big\}.$$

Denote by $m(\varphi)$ the maximal generic value of $\mid \varphi \mid$. Clearly we have $n(\varphi) \geq m(\varphi)$.

If W is an irreducible analytic curve in M, and $\psi : W \to \mathbb{Z}$ is a sum of signs of analytic functions on W, we can define $n(\psi)$ and $m(\psi)$ in the same way as before and we have $n(\psi) = m(\psi)$. This follows from the proof of Proposition 1.4.

The correspondence between sums of signs of global analytic functions and signatures of quadratic forms allows us to prove the following:

Proposition 4.1. *Let M be a paracompact real analytic manifold of dimension 2, and let $\varphi : M \to \mathbb{Z}$ be a sum of signs of global analytic functions on M. Let $k \in \mathbb{N}$, $\delta \in \mathbb{Z}$ be such that the set of the generic values of φ is contained in the interval $[\delta - k, \delta + k]$. Then we have*

$$n(\varphi) \leq \begin{cases} 2k + |\delta| & \text{if } k \text{ is even,} \\ 2k - 1 + |\delta| & \text{if } k \text{ is odd.} \end{cases}$$

Proof. We can assume that M is connected. By Proposition 3 in [B1], there is a quadratic form over $\mathrm{Spec}_r \mathcal{M}(M)$ representing $\widetilde{\varphi}$ and of dimension at most $2^{s-1}k + |\delta|$ if k is even, $2^{s-1}(k-1) + 1 + |\delta|$ if k is odd, where s is the stability index of $\mathcal{M}(M)$. By [DC-A], we know that $s = 2$. $\square$

Example 4.2. If $\varphi : M \to [-1, 1]$ is generically a sum of signs of analytic functions, then it is generically the sign of an analytic function.

Remark 4.3. The bound of Proposition 4.1 is sharp. Indeed, assume k is even and $\delta \geq 0$, and consider $\varphi : \mathbb{R}^2 \to \mathbb{Z}$ defined by $\varphi(x, y) = \delta - k$ if $x < 0$ and $y < 0$, and $\varphi(x, y) = \delta + k$ else. Then φ is a sum of signs of $2k+ \mid \delta \mid$ analytic functions on $\mathbb{R}^2$ and not less, by an argument of [B1, §5.2]. Similarly, if k is odd and $\delta \geq 0$, the function $\varphi : \mathbb{R}^2 \to \mathbb{Z}$ defined by $\varphi(x, y) = \delta + k$ if $y > 0$, $\varphi(x, y) = \delta + k - 2$ if $x > 0$ and $y < 0$, and $\varphi(x, y) = \delta - k$ else, is the sum of the signs of $2k - 1 + |\delta|$ analytic functions, and not less. The proof is similar if $\delta < 0$.

Let W be a wall of φ, and let $t \in \mathcal{O}(M)$ be an uniformizer of $\mathcal{O}(M)_{\mathcal{I}(W)}$. Then we define $\partial_W^t \varphi = \partial_W\big((\text{sign } t)\varphi\big)$. If $t' \in \mathcal{O}(M)$ is another uniformizer of $\mathcal{O}(M)_{\mathcal{I}(W)}$, we have $\partial_W^{t'} \varphi = \big(\text{sign } (tt')\big)\partial_W^t \varphi$, and $n(\partial_W^t \varphi) = n(\partial_W^{t'} \varphi)$. So we will use $\partial_W^t \varphi$ without making precise the chosen uniformizer t. Write $\varphi = \sum_{i=1}^{n(\varphi)} \text{sign } f_i$. For $i = 1, \dots, n(\varphi)$, write $f_i = t^{k_i} g_i$ with $k_i \in \mathbb{N}$ and $g_i \in \mathcal{O}(M) \setminus \mathcal{I}(W)$. Then we have $\partial_W \varphi = \sum_{i | k_i\, even} \text{sign } g_i$ and $\partial_W^t \varphi = \sum_{i | k_i\, odd} \text{sign } g_i$. So we get $n(\varphi) \geq n(\partial_W \varphi) + n(\partial_W^t \varphi)$.

If $\delta = 0$ and $k = 2h$ is even, we have a characterization of the functions which reach the bound. Note that, for such a function φ, neither $\varphi^{-1}(2h)$ nor $\varphi^{-1}(-2h)$ are generically empty. Indeed, if one of these two sets is generically empty, then the values of φ are generically in the interval $[\delta - (2h-1), \delta + (2h-1)]$ with $\delta = 1$ or -1, and Proposition 4.1 gives that $n(\varphi) \leq 2(2h-1) - 1 + \mid \delta \mid = 4h - 2$.

We say that two global semianalytic subsets A and B of M are *generically separable* if there exists a non zero analytic function $f \in \mathcal{O}(M)$ such that $f_{|_A} \geq 0$ and $f_{|_B} \leq 0$.

Proposition 4.4. *Let $\varphi : M \to [-2h, 2h]$ be a sum of signs of global analytic functions, with $h \in \mathbb{N}$. Then we have $n(\varphi) = 4h$ if and only if the two sets $\varphi^{-1}(2h)$ and $\varphi^{-1}(-2h)$ are not generically separable.*

To prove Proposition 4.4 we need the following theorem about separation of global semianalytic sets.

Theorem 4.5. *Let M be a 2-dimensional analytic manifold and let $A, B \subset M$ be global semianalytic subsets. Let $\pi : M' \to M$ be a locally finite sequence of blowings-up of points such that the walls of $\pi^{-1}(A)$ and $\pi^{-1}(B)$ are non singular and with normal crossing intersections. Then, $\pi^{-1}(A)$ and $\pi^{-1}(B)$ are generically separable in M' if and only if A and B are generically separable in M.*

The proof of the theorem will follow from this lemma.

Lemma 4.6. *Let M be a paracompact connected real analytic manifold of dimension two and let $\pi : M' \to M$ be a locally finite sequence of blowings-up of points.*

1. *If X' is an analytic curve in M' not contained in the exceptional divisor, then $\pi(X')$ is an analytic curve in M.*

2. *If Y is an analytic irreducible curve in M, then its strict transform Y' is still irreducible in M'.*

Proof. 1. Since the dimension of M is two, $\pi(X')$ is a semianalytic subset of dimension one by [L]. Then, it is an analytic curve if and only if it is arc-symmetric. Let $\gamma : (-\varepsilon, \varepsilon) \to M$ be an analytic arc such that $\gamma(0, \varepsilon) \subset \pi(X')$, we want to prove that $\gamma(-\varepsilon, \varepsilon)$ is contained in $\pi(X')$. We can assume, up to take a smaller ε, that $\gamma(0, \varepsilon) \subset M \backslash D$, where D is a discrete set of points such that $\pi : M' \backslash \pi^{-1}(D) \to M \backslash D$ is an isomorphism. Let $\gamma' : (-\varepsilon, \varepsilon) \to M'$ be the analytic arc such that $\pi \circ \gamma' = \gamma$. Since $\gamma(0, \varepsilon)$ is contained in $\pi(X') \setminus D$, we have $\gamma'(0, \varepsilon) \subset X'$. As X' is a curve, this implies that $\gamma'(-\varepsilon, \varepsilon) \subset X'$.

2. Assume $Y' = Y'_1 \cup Y'_2$, where both Y'_1 and Y'_2 are analytic curves, we have to prove that $Y' = Y'_1$ or $Y' = Y'_2$. Since Y' does not contain the exceptional divisor, neither Y'_1 nor Y'_2 contains it. So, by point *1.*, the sets $\pi(Y'_1)$ and $\pi(Y'_2)$ are analytic curves such that $Y = \pi(Y'_1) \cup \pi(Y'_2)$. This implies $\pi(Y'_1) = Y$ or $\pi(Y'_2) = Y$. Since π is an isomorphism $M' \setminus \pi^{-1}(D) \to M \setminus D$, we get the thesis. □

Corollary 4.7. *Let π be as in Lemma 4.6. If S' is a global semianalytic subset of M', then $\pi(S')$ is a global analytic subset of M.*

Proof. Since $\pi(S')$ is semianalytic ([L]), by [CA, Lemma 3.1] and [CA, Prop. 3.3], it is enough to prove that the exists a proper analytic set in M containing $\partial\pi(S')$. We have $\partial\pi(S') \subset \pi(\partial S') \subset \pi(\partial_Z S')$, where $\pi(\partial_Z S')$ is a proper analytic set by Lemma 4.6. $\square$

Proof of Theorem 4.5 . If A and B are generically separated by $f \in \mathcal{O}(M)$ then, $f \circ \pi \in \mathcal{O}(M')$ generically separates $\pi^{-1}(A)$ and $\pi^{-1}(B)$.

Let us prove the converse implication. Let $f \in \mathcal{O}(M')$ generically separating $\pi^{-1}(A)$ and $\pi^{-1}(B)$. Denote $S' = \{f > 0\}$. By Corollary 4.7, the set $\pi(S')$ is a global semianalytic subset of M. We prove that it is generically principal open, i.e. there exists $g \in \mathcal{O}(M)$ such that $\pi(S') =_{gen} \{g > 0\}$. Then, by construction, g generically separates A from B.

By Proposition 2.2, it is enough to prove that no global irreducible component of dimension one of $\partial_Z\pi(S')$ is simultaneously odd and even. Assume that there exists such a component, say W. Then, by Lemma 4.6, its strict transform is an irreducible component of codimension one of $\partial_Z S'$, and it is simultaneously odd and even with respect to S'; this contradicts the principality of S'. The proof is done. $\square$

Proof of Proposition 4.4. Assume that $\varphi^{-1}(2h)$ and $\varphi^{-1}(-2h)$ are not generically separable, and that the walls of φ are non singular with normal crossing intersections. Then, by [BP, Th. 4.2], there is a wall W of φ which is simultaneously odd and even for $\varphi^{-1}(2h)$ and $\varphi^{-1}(-2h)$, i.e. $\dim\big(W \cap \overline{\varphi^{-1}(2h)^*} \cap \overline{\varphi^{-1}(-2h)^*}\big) = 1$ and $\dim\big(W \cap \big(\varphi^{-1}(2h)^* \cup \varphi^{-1}(-2h)^*\big)\big) = 1$. On the odd part of W, the generic value of $\partial_W\varphi$ is 0, and the generic values of $\partial_W^t\varphi$ are $2h$ and $-2h$. On the even part of W, the generic values of $\partial_W\varphi$ are $2h$ and $-2h$, and the generic value of $\partial_W^t\varphi$ is 0. So we have $n(\partial_W\varphi) = n(\partial_W^t\varphi) = 2h$, and $n(\varphi) \geq 4h$. As by Proposition 4.1 we have $n(\varphi) \leq 4h$, we conclude that $n(\varphi) = 4h$.

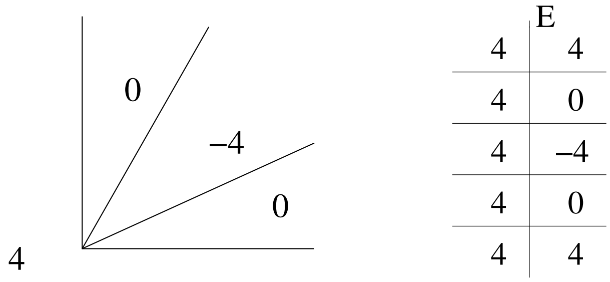
We assume now that $\varphi^{-1}(2h)$ and $\varphi^{-1}(-2h)$ are not generically separable, but now we do not make any hypothesis about the walls. There exists $\pi : M' \to M$ a locally finite sequence of blowings-up of points such that the walls of the function $\varphi \circ \pi$ are non singular with normal crossing intersections. Theorem 4.5 assures us that $(\varphi\circ\pi)^{-1}(2h)$ and $(\varphi\circ\pi)^{-1}(-2h)$ are not generically separable. So we have $n(\varphi\circ\pi) = 4h$. Note that $n(\varphi) \geq n(\varphi\circ\pi)$. Indeed, if we can write $\varphi = \sum_{i=1}^r \text{sign}\ f_i$ with $f_i \in \mathcal{O}(M)$, then we have $\varphi\circ\pi = \sum_{i=1}^r \text{sign}\ (f_i\circ\pi)$ with $f_i \circ \pi \in \mathcal{O}(M')$. So we get $n(\varphi) \geq 4h$, and by Proposition 4.1 we have $n(\varphi) = 4h$.

Conversely, assume that $\varphi^{-1}(2h)$ and $\varphi^{-1}(-2h)$ are generically separable. Let $f \in \mathcal{O}(M)$ be a non zero function such that $f_{|_{\varphi^{-1}(2h)}} \geq 0$ and $f_{|_{\varphi^{-1}(-2h)}} \leq 0$. We consider the function $\psi = \varphi - \text{sign}\ f : M \to [-2h+1, 2h-1]$. By Proposition 4.1, we have $n(\varphi) \leq n(\psi) + 1 \leq 4h - 2$. $\square$

Corollary 4.8. *Let $\varphi : M \to [-2h, 2h]$ be a sum of signs of global analytic functions, with non singular walls with normal crossing intersections. Then we have $n(\varphi) = 4h$ if and only if there is a wall W of φ such that $m(\partial_W\varphi) + m(\partial_W^t\varphi) = 4h$.*

Proof. This is immediate by the proof of Proposition 4.4. ☐

Remark 4.9. If we do not assume that the walls have normal crossing intersections, then the result of Corollary 4.8 is not true. For instance, consider the function on $M = \mathbb{R}^2$ defined on the left-hand side of the following figure:



Since it is constant mod 4, it is a sum of signs of analytic functions on $\mathbb{R}^2$. If $\pi : M' \to \mathbb{R}^2$ is the blowing-up of the point of intersection of the walls, and if E is the exceptional divisor of π, then it can be checked that $m\big(\partial_E(\varphi \circ \pi)\big) = m\big(\partial_E^t(\varphi \circ \pi)\big) = 4$ (see the right-hand side of the figure). So we have $n(\varphi \circ \pi) = 8$, and $n(\varphi) \geq n(\varphi \circ \pi)$ (cf. proof of Proposition 4.4) so $n(\varphi) = 8$. But for each wall W of φ, we have $m(\partial_W \varphi) + m(\partial_W^t \varphi) = 6$.

If $\delta = 0$ and k is odd (with the notations of Proposition 4.1), the situation is quite different. We have the following partial result.

Proposition 4.10. *Let $\varphi : M \to [-2h+1, 2h-1]$ be a sum of signs of global analytic functions, with $h \in \mathbb{N}$. If we have $n(\varphi) = 4h-3$, then the sets $\varphi^{-1}(2h-1) \cup \varphi^{-1}(2h-3)$ and $\varphi^{-1}(-2h+1) \cup \varphi^{-1}(-2h+3)$ are not generically separable.*

Proof. Let $\bar{\varphi} : M \to \{-1, 0, 1\}$ be the function defined by

$$\bar{\varphi}(x) = \begin{cases} \;\;1 & \text{if } \varphi(x) \equiv 2h-1 \mod 4, \\ -1 & \text{if } \varphi(x) \equiv -2h+1 \mod 4 \\ \;\;0 & \text{else} \end{cases}$$

Then $\varphi - \bar{\varphi}$ is generically constant mod 4, so $\bar{\varphi}$ is a sum of signs of analytic functions, and by Example 4.2 we have $n(\bar{\varphi}) = 1$. In order to have $n(\varphi) = 4h-3$, we need that $n(\varphi - \bar{\varphi}) = 4h-4$. As the values of $\varphi - \bar{\varphi}$ are in $[-2h+2, 2h-2]$, by Proposition 4.4 we have $n(\varphi - \bar{\varphi}) = 4h-4$ if and only if $(\varphi - \bar{\varphi})^{-1}(2h-2)$ and $(\varphi - \bar{\varphi})^{-1}(-2h+2)$ are not generically separable. As $(\varphi - \bar{\varphi})^{-1}(2h-2) = \varphi^{-1}(2h-1) \cup \varphi^{-1}(2h-3)$ and $(\varphi - \bar{\varphi})^{-1}(-2h+2) = \varphi^{-1}(-2h+1) \cup \varphi^{-1}(-2h+3)$, the proof is done. ☐

Remark 4.11. The non separability of $\varphi^{-1}(2h-1) \cup \varphi^{-1}(2h-3)$ and $\varphi^{-1}(-2h+1) \cup \varphi^{-1}(-2h+3)$ does not guarantee that $n(\varphi) = 4h-3$. Consider $\varphi : \mathbb{R}^2 \to \mathbb{Z}$ defined by $\varphi = h - 1 + (h-1)\text{sign } y - (h-1)\text{sign } (xy) + (h-2)\text{sign } x$. Then $n(\varphi) = 4h-5$.

2h−1	2h−3
−2h+3	2h−3

Note also that we can have $n(\varphi) = 4h - 3$ even if $\varphi^{-1}(2h-1) = \emptyset$ or $\varphi^{-1}(-2h+1) = \emptyset$, as in the following example. For each wall of φ, we have $m(\partial_W \varphi) + m(\partial_W^t \varphi) = 4h - 3$.

2h−1	2h−1
−2h+3	2h−1

4.3 Fans and obstructions

We recall the definition of a fan:

Definition 4.12. A (non empty) finite subset F of the real spectrum of a field K is a *fan* if for any triple $\sigma_1, \sigma_2, \sigma_3 \in F$, the product $\sigma_1\sigma_2\sigma_3$ is an ordering and belongs to F. Here the product of orderings means the product of signs for these orderings.

If the stability index of K is $k \in \mathbb{N}$, then the cardinality of a fan of K is 2^r with $r \leq k$ ([ABR]). If we work with $K = \mathcal{M}(M)$, then we only have fans with one, two or four elements, since the stability index of $\mathcal{M}(M)$ is two (cf. [DC-A]).

The Representation Theorem of Becker and Bröcker ([BB]) gives a characterization of signatures of quadratic forms in terms of fans:

Theorem 4.13. *A constructible function $\widetilde{\varphi}$ on the real spectrum of a field K is the signature of a quadratic form if and only if for each finite fan F of K, we have $\sum_{\sigma \in F} \widetilde{\varphi}(\sigma) \equiv 0 \mod |F|$.*

If $K = \mathcal{M}(M)$, it is enough to consider 2-element and 4-element fans when we use the Representation Theorem.

Given an irreducible curve $W \subset M$, denote by $\mathcal{M}(W) = \text{q.f.}\big(\mathcal{O}(M)/\mathcal{I}(W)\big)$ the field of meromorphic functions on W. Since Artin-Lang property holds for $\mathcal{M}(W)$ by [DC-A], there is a well-defined tilde map associating to a global semianalytic subset of W the constructible subset of $\operatorname{Spec_r}\mathcal{M}(W)$ with the same description. An ordering $\tau \in \operatorname{Spec_r}\mathcal{M}(W)$ is a *specialization* of $\alpha \in \operatorname{Spec_r}\mathcal{M}(M)$ if any $f \in \mathcal{O}(M)$ such that $f_{|W}$ is strictly positive for τ, is strictly positive for α. We write $\alpha \to \tau$.

Similarly we write $\alpha \to x$ for $x \in M$ if for any $f \in \mathcal{O}(M)$ such that $f(x) > 0$, the function f is positive for α. We define in the same way $\tau \to x$ for $x \in W$.

An ordering $\sigma \in \mathrm{Spec_r}\mathcal{M}(M)$ is called *unbounded* if it does not specialize to any point in M, see for instance [CA], [DC-A]. Here we prove a slight improvement of the Representation Theorem for the special case $K = \mathcal{M}(M)$, namely, that unbounded orderings have no role. This result, with similar ones found in [BP], shows that the unbounded orderings, because of which Artin-Lang property fails in $\mathrm{Spec_r}\mathcal{O}(M)$, are not relevant as far as it concerns many geometric problems.

Denote by $\Sigma_W[\mathbb{Z}_2]$ the set of the orderings in $\mathrm{Spec}_r\mathcal{M}(M)$ specializing to some ordering in $\mathrm{Spec}_r\mathcal{M}(W)$ (see [1]). We recall a well known way to build fans in $\Sigma_W[\mathbb{Z}_2]$. Let τ_1, τ_2 be distinct orderings of $\mathcal{M}(W)$. The ring $\mathcal{O}(M)_{\mathcal{J}(W)}$ is a discrete valuation ring of rank 1, and we can consider $t \in \mathcal{O}(M)_{\mathcal{J}(W)}$ an uniformizer of this ring. Then, each $f \in \mathcal{O}(M)_{\mathcal{J}(W)}$ can be written as $f = t^k u$ where k is the valuation of f and u is a unit in $\mathcal{O}(M)_{\mathcal{J}(W)}$. Denote by $\overline{u}$ the class of u in $\mathcal{M}(W)$. We set:

$$\begin{array}{ll} \alpha_1(f) = \tau_1(\overline{u}) & \alpha_3(f) = (-1)^k \tau_1(\overline{u}) \\ \alpha_2(f) = \tau_2(\overline{u}) & \alpha_4(f) = (-1)^k \tau_2(\overline{u}) \end{array}$$

This defines four orderings $\alpha_1, \alpha_2, \alpha_3, \alpha_4$ of $\mathcal{M}(M)$, and these orderings form a 4-element fan contained in $\Sigma_W[\mathbb{Z}_2]$.

We come back to our signature problem. Let $\varphi : M \to \mathbb{Z}$ be a semianalytically constructible function.

Lemma 4.14. *Assume that there exists a wall W of φ such that $\partial_W\varphi$ is not generically constant* mod 2. *Then there exists a four-element fan $F = \{\alpha_1, \alpha_2, \alpha_3, \alpha_4\} \subset \Sigma_W[\mathbb{Z}_2]$, made of bounded orderings, such that $\sum_i \widetilde{\varphi}(\alpha_i) \not\equiv 0 \mod 4$.*

Proof. There exist Ω_1 and $\Omega_2 \subset W \cap \mathrm{Reg}\left(\bigcup_n \partial_Z(\varphi^{-1}(n))\right)$ non empty open semianalytic sets such that $(\partial_W\varphi)_{|\Omega_1}$ is even and $(\partial_W\varphi)_{|\Omega_2}$ is odd. Consider two points, $x_1 \in \Omega_1$ and $x_2 \in \Omega_2$. Let τ_1 and τ_2 be orderings in $\mathrm{Spec_r}\mathcal{M}(W)$ such that $\tau_1 \to x_1$ and $\tau_2 \to x_2$ in $\mathrm{Spec}_r\mathcal{O}(W)$. Then we have $\tau_1 \in \widetilde{\Omega_1}$ and $\tau_2 \in \widetilde{\Omega_2}$. Let F be the fan defined by τ_1 and τ_2 as above. Then $\alpha_1, \alpha_3 \to x_1$ and $\alpha_2, \alpha_4 \to x_2$, so the α_i's are bounded. Moreover,

$$\widetilde{\partial_W\varphi}(\tau_1) + \widetilde{\partial_W\varphi}(\tau_2) = \frac{\widetilde{\varphi}(\alpha_1) + \widetilde{\varphi}(\alpha_3)}{2} + \frac{\widetilde{\varphi}(\alpha_2) + \widetilde{\varphi}(\alpha_4)}{2}$$

is odd, and this implies that $\sum_i \widetilde{\varphi}(\alpha_i) \not\equiv 0 \mod 4$. □

Similarly we have:

Lemma 4.15. *Assume that there exists a wall W of φ such that $\partial_W\varphi$ is not generically integer-valued. Then there exists a two-element fan $F = \{\alpha_1, \alpha_2\} \subset \Sigma_W[\mathbb{Z}_2]$, made of bounded orderings, such that $\sum_i \widetilde{\varphi}(\alpha_i) \not\equiv 0 \mod 2$.*

Using Theorem 2.5 and these two lemmas, we conclude:

Theorem 4.16. *Let $\varphi : M \to \mathbb{Z}$ be a semianalytically constructible function. It is generically a sum of signs of global analytic functions on M if and only if for each fan $F \subset \mathrm{Spec_r}\mathcal{M}(M)$ made only of bounded orderings, we have $\sum_{\sigma\in F}\widetilde{\varphi}(\sigma) \equiv 0 \mod |F|$.*

This is a generic result. To state a complete result we consider, if $W \subset M$ is an irreducible curve, the function $\widetilde{\varphi_{|W}} : \mathrm{Spec_r}\mathcal{M}(W) \to \mathbb{Z}$ induced by $\varphi_{|W}$. The function $\widetilde{\varphi_{|W}}$ is defined as $\widetilde{\varphi}$ above, using that Artin-Lang property holds for $\mathcal{M}(W)$.

Proposition 4.17. *Let $\varphi : M \to \mathbb{Z}$ be a semianalytically constructible function. It is a sum of signs of global analytic functions on M if and only if for each fan $F \subset \mathrm{Spec_r}\mathcal{M}(M)$ made only of bounded orderings, $\sum_{\sigma\in F}\widetilde{\varphi}(\sigma) \equiv 0 \mod |F|$, and for each irreducible curve W of M contained in $\bigcup_{n\in\mathbb{Z}}\partial\big(\varphi^{-1}(n)\big)$, for each fan $F \subset \mathrm{Spec_r}\mathcal{M}(W)$ made only of bounded orderings, $\sum_{\sigma\in F}\widetilde{\varphi_{|W}}(\sigma) \equiv 0 \mod |F|$.*

This is a consequence of Proposition 1.9.

References

[1] Acquistapace, F., Andradas, C. and Broglia, F.: Separation of semialgebraic sets, *J. Amer. Math. Soc.* **12**(3) (1999) 703-728

[AB] Acquistapace, F., Broglia, F. and Fortuna, E.: A separation theorem in dimension 3, *Nagoya Math. Journal* **143** (1996) 171-193

[ABV] Acquistapace, F., Broglia, F. and Vélez, P.: Basicness of Semialgebraic Sets, *Geom. Dedicata* **78** (1999) 229-240

[ABR] Andradas, C., Bröcker, L. and Ruiz, J.M.: *Constructible Sets in Real Geometry*, Ergebnisse der Math. 3.Folge Vol.33 Springer (1996)

[BB] Becker, E. and Bröcker, L.: On the description of the reduced Witt ring, *J. Alg.* **52** (1978) 328-346

[B1] Bonnard, I.: Un critère pour reconnaître les fonctions algébriquement constructibles *J. reine angew. Math.*, **526** (2000) 61-88

[B2] Bonnard, I.: Nash constructible functions, *Prépublication de l'Université d'Angers* **128** (2001)

[BP] Broglia, F. and Pieroni, F.: Separation of global semianalytic subsets of two dimensional manifolds, *Submitted*

[C] Castilla, A.: Artin-Lang property for analytic manifolds of dimension two, Math. Z. **217**(1994) 5-14

[CA] Castilla, A. and Andradas, C.: Connected components of global semianalytic subsets of 2-dimensional analytic maniflods, *J. Reine Angew. Math.* **475**(1996) 137-148

[CK] Coste, M. and Kurdyka, K.: Le discriminant d'un morphisme de variétés algébriques réelles, *Topology* **37** (1998) 393-400

[CS] Coste, M. and Shiota, M.: Nash functions on noncompact Nash manifolds, *Ann. Scient. Ecole Norm. Sup. (4)* **33** (2000) 139-149

[DC-A] Díaz-Cano, A. and Andradas, C.: Complexity of global semianalytic sets in a real analytic manifold of dimension 2, *J. Reine Angew. Math.* **534** (2001) 195-208

[H] Hironaka, H.: Resolution of singularities of an algebraic variety over a field of characteristic zero I, II, *Ann. of Math.* **79** (1964) 109-326

[KS] Kashiwara, M. and Schapira, P.: *Sheaves on manifolds*, Grundlehren der Mathematischen Wissenschaften, 292 Springer (1994)

[K] Kurdyka, K.: Ensembles semi-algébriques symétriques par arcs, *Math. Ann.* **282** (1988) 445-462

[L] Łojasiewicz, S.: Sur la géométrie semi- et sous-analytique, *Ann. Inst. Fourier* **43** (1993) 1575-1595

[MP1] McCrory, C. and Parusiński, A.: Algebraically constructible functions, *Ann. Scient. Ecole Norm. Sup. (4)* **30** (1997) 527-552

[MP2] McCrory, C. and Parusiński, A.: Algebraically Constructible Functions: Real Algebra and Topology, available on http://www.maths.univ-rennes1.fr/~raag01/Surveys.html

[PS] Parusiński, A. and Szafraniec, Z.: On the Euler characteristic of fibres of real polynomials maps, *Singularities Symposium-Lojasiewicz 70, Banach Center Publications* **44** (1998) 175-182

[S] Schapira, P.: Operations on constructible functions, *J. Pure Appl. Algebra* **72** (1991) 83-93