Kontsevich integral

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This is a purely expository paper. It grew out of several lectures given by the authors in different places (Pereslavl-Zalessky, Russia; Pisa, Italy; Aizu-Wakamatsu, Japan; Utrecht, Netherlands). Its purpose is to explain the famous contribution of Maxim Kontsevich to Vassiliev knot invariant theory. More specifically, our aim is to present a proof of Kontsevich's fundamental theorem which states that for each weight system there exists a Vassiliev invariant whose symbol is exactly the given weight system, or, in other words, that the only relations in the graded algebra of Vassiliev knot invariants are the 1- and 4-term relations.

Recently, several papers dedicated to Kontsevich integral have appeared: see [AF, BNS, Be, Hu] etc. We hope, however, that our text will be useful because we try to present a self-contained exposition clarifying some vague, but important points left in the shade by the previous authors.

In our approach, we mostly build on D. Bar-Natan's paper [BN1], the only essential difference being that in the proof of the main (Kontsevich's) theorem we follow the original paper [K] using the plain Stokes theorem instead of the flatness of Knizhnik–Zamolodchikov connection.

The paper consists of three sections. In Section 1 we give an introduction to Vassiliev invariant theory following [CD, CDL1]. Section 2 is about Kontsevich integral: it contains Kontsevich's formula with comments, the proof of the invariance properties, the construction of the universal Vassiliev invariant via Kontsevich integral and examples of computation. Finally, in Section 3 we discuss two operations on knots — mutation and change of orientation — in their relation to the Kontsevich integral; as an application, we give a counterexample to our intersection graph conjecture [CDL1] constructed by T. Q. T. Le using mutations.

One of the authors (S. Ch.) would like to extend his gratitude to Dirk Siersma, Michiel Hazewinkel and Ieke Moerdijk, organizers of the Workshop on Discriminant Spaces, Utrecht, August 1996, for the invitation and the proposal to transform his lecture notes into a self-contained text. Many thanks go to the Max-Planck-Institut für Mathematik in Bonn, where a part of this text was written, for a worm hospitality and stimulating atmosphere.

1 Introduction to Vassiliev theory

The ingenious idea of V. Vassiliev [V] was to prolong knot invariants to singular knots according to the rule (*Vassiliev's skein relation*)

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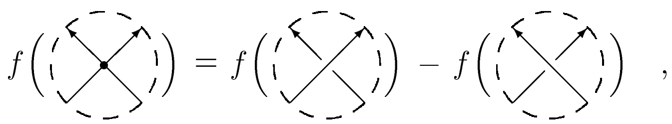
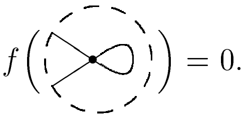


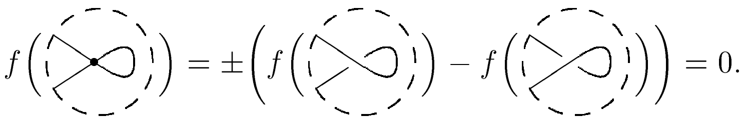
where f is a knot invariant initially defined only on nonsingular knots.

By a singular knot we understand a knot which may have a finite number of simple double points as singularities. For simplicity we consider $\mathbf{C}$-valued knot invariants but almost all results carry over to any field of characteristic zero.

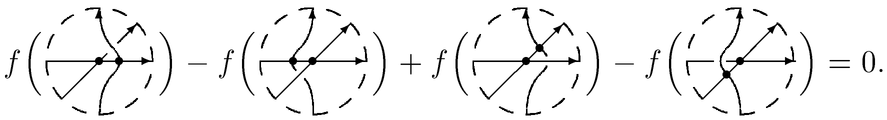
The prolongation of any knot invariant f satisfies two important sets of relations.

One-term relation for knots: 

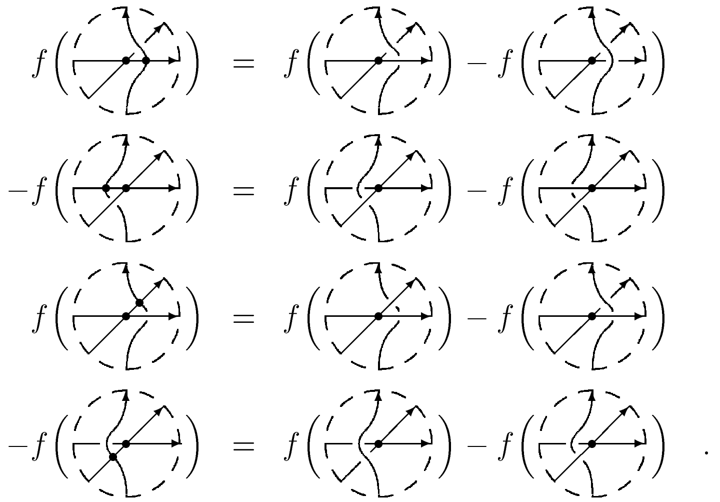
Indeed, two different resolutions of this double point yield one and the same knot:



Four-term relation for knots:



Indeed, resolving one of the two double points, we get:



In the right-hand parts of these equalities, every term appears twice with the opposite signs, so that everything cancels.

1.1 Vassiliev invariants

Definition. A *Vassiliev invariant* of order no greater than n is a knot invariant whose prolongation vanishes on any knot with more than n double points.

The set of all Vassiliev invariants of order no greater than n is a vector space. Denoting this space by $\mathcal{V}_n$, we obtain a filtration

$$\mathcal{V}_0 \subseteq \mathcal{V}_1 \subseteq \mathcal{V}_2 \subseteq \cdots \subseteq \mathcal{V}_n \subseteq \cdots \subset \mathcal{V}$$

in the space $\mathcal{V}$ of Vassiliev invariants of finite order. Using this filtration, we can study a more simple graded space

$$\operatorname{gr}\mathcal{V} = \mathcal{V}_0 \oplus \mathcal{V}_1/\mathcal{V}_0 \oplus \mathcal{V}_2/\mathcal{V}_1 \oplus \ldots \oplus \mathcal{V}_n/\mathcal{V}_{n-1} \oplus \ldots$$

The structure of each component $\mathcal{V}_n/\mathcal{V}_{n-1}$ can be best understood in terms of *chord diagrams*.

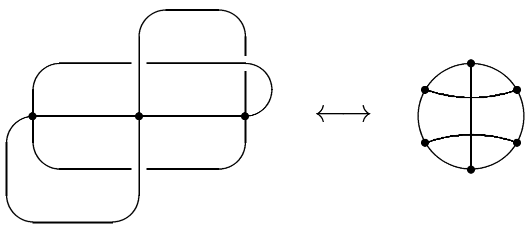
Definition. A *chord diagram* of order n is a circle with a distinguished set of n unordered pairs of points, regarded up to an orientation preserving diffeomorphism of the circle.

A chord diagram may be depicted as a circle with a set of n chords all of whose endpoints are distinct. The chords are drawn as curves whose actual shape is irrelevant; what matters is the way they bind their endpoints into pairs. A chord diagram of order n is thus the same thing as a cyclic word in n letters in which each letter appears twice. The use of diagrams is preferable because of their visual convenience.

Example. The total number of chord diagrams of order 1, 2 and 3 is 1, 2 and 5, respectively. We suggest that the reader who has never done it before should draw all of them now as an exercise, before proceeding with this text.

Definition. The *chord diagram of a singular knot* $S^1 \to \mathbf{R}^3$ is the oriented circle S^1 with the preimages of every double point connected by a chord.

Example. Here is a singular knot and its chord diagram:



Let F_n be the linear space of all $\mathbf{C}$-valued functions on the set of chord diagrams of order n.

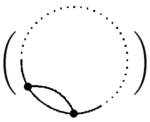
Definition. The *symbol* of a Vassiliev invariant $f \in \mathcal{V}_n$ is the function $[f] \in F_n$ defined by $[f](D) = f(K)$ where K is an arbitrary knot whose chord diagram is D.

The fact $f(K)$ depends only on the chord diagram of K can be checked as follows (see [CDL1] for details). If K_1 and K_2 are two knots with the same chord diagram, then they can be conventionally drawn in the plane in such a way that the pictures coincide everywhere except for a finite number of crossing points in which K_1 may have an undercrossing and K_2

an overcrossing or vice versa. But a flip between an undercrossing and an overcrossing does not change the value of $f(K)$ due to Vassiliev's skein relation.

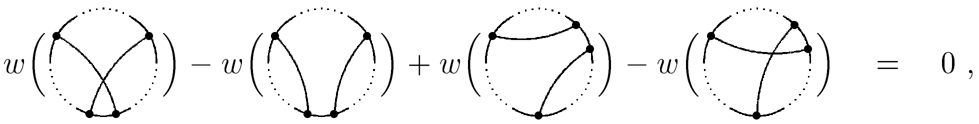
By definition, the kernel of the mapping $\mathcal{V}_n \to F_n : f \mapsto [f]$ coincides with the subspace $V_{n-1} \subset \mathcal{V}_n$. We thus obtain an inclusion $\iota_n : V_n/V_{n-1} \to F_n$. What is the image of ι_n, or, in other words, what are the conditions that distinguish the symbols of Vassiliev invariants among all functions on the set of chord diagrams?

It turns out that any function that belongs to the image of ι_n satisfies the following two sets of relations:

One-term relation: $w\Big($  $\Big) = 0.$

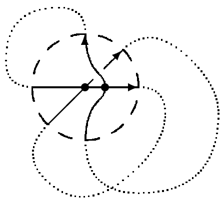
Here and below, dotted arcs suggest that there might be further chords attached to their points, while on the solid portions of the circle all the endpoints are explicitly shown.

Four-term relation:



for an arbitrary fixed position of $(n-2)$ chords (which are not drawn here) and the two additional chords positioned as shown in the picture.

These two facts easily follow from the corresponding relations for knots. Indeed, the chord diagram of the one-term relation can be presented by a singular knot with a small curl like in the one-term relation for knots. (Remind that we have the freedom to choose the knot with the given chord diagram arbitrarily). The first diagram appearing in the four-term relation can be represented by the knot



which has a fragment shown above in the four-term relation for knots. The three remaining knots of that relation will produce the other diagrams in question.

1.2 Kontsevich theorem

This is the main theorem of the article. It says that one- and four-term relations are the only relations for symbols of Vassiliev invariants.

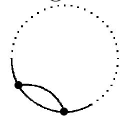
Definition. A *weight system* is a function on chord diagrams satisfying one- and four-term relations.

Let $W_n \subset F_n$ denote the space of all weight systems of order n. In the previous section we have seen that symbols of Vassiliev invariants are weight systems.

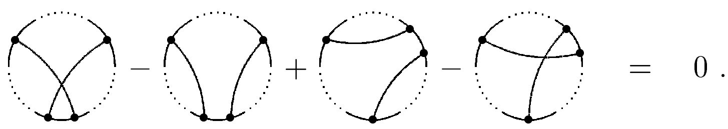
Kontsevich theorem [K]. *Each weight system is the symbol of an appropriate Vassiliev invariant.*

The theorem will be proved in Sec. 2.8. It theorem gives a purely combinatorial description of the graded space $\mathrm{gr}\,\mathcal{V}$ as the space of weight systems.

For every n the space of weight systems W_n can be represented as the dual of the linear space $\mathcal{A}_n$ generated by chord diagrams of order n considered modulo

one-term relations:  $= \quad 0$ and

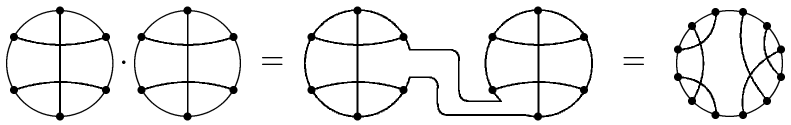
four-term relations:



The direct sum

$$\mathcal{A} = \mathcal{A}_0 \oplus \mathcal{A}_1 \oplus \mathcal{A}_2 \oplus \dots$$

is in fact a graded algebra. The multiplication defined by the connected sum of chord diagrams:



is well-defined thanks to four-term relations. In fact $\mathcal{A}$ is even a Hopf algebra but we will not need the comultiplication here.

The proof of Kontsevich's theorem is based on the construction of the so called universal Vassiliev invariant. It is a knot invariant $I = \{I_n\}$ with values in the graded completion $\overline{\mathcal{A}}$ of the algebra $\mathcal{A}$ of chord diagrams with the following property: *for any weight system $w \in W_n$ the value of w on $I_n(K)$ is a Vassiliev invariant of K whose symbol is exactly w.* (By the *graded completion* we understand an extension of the graded algebra which, apart from the proper elements of the algebra, also contains series with an infinite number of nonzero terms.)

The main tool to construct the universal Vassiliev invariant is the famous Kontsevich integral described in the next section.

2 Kontsevich integral

Represent the three-space $\mathbf{R}^3$ as a direct product of a complex line $\mathbf{C}$ with coordinate z and a real line $\mathbf{R}$ with coordinate t. Consider a knot K embedded in $\mathbf{R}^3 = \mathbf{C}_z \times \mathbf{R}_t$ in such a way that the coordinate t is a Morse function on K (all critical points are nondegenerate and all critical levels are different).

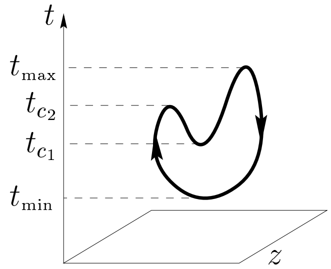
Definition. The *Kontsevich integral* of K is the following element of the graded completion $\overline{\mathcal{A}}$ of the algebra of chord diagrams:

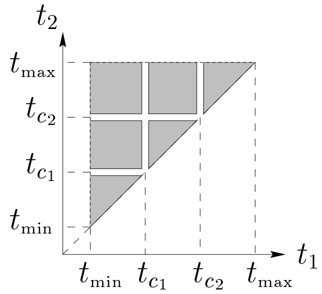
$$Z(K) = \sum_{m=0}^{\infty} \frac{1}{(2\pi i)^m} \int\limits_{\substack{t_{\min}<t_1<\ldots<t_m<t_{\max} \\ t_j \text{ are noncritical}}} \sum_{P=\{(z_j,z_j')\}} (-1)^{\downarrow} D_P \bigwedge_{j=1}^{m} \frac{dz_j - dz_j'}{z_j - z_j'} .$$

2.1 Who is who in Kontsevich integral

Real numbers $t_{\min}$ and $t_{\max}$ are the minimum and the maximum of the function t on K.

The integration domain is the m-dimensional simplex $t_{\min} < t_1 < \ldots < t_m < t_{\max}$ divided by the critical values into a certain number of *connected components*. For example, for the following embedding of the unknot and $m = 2$ the corresponding integration domain has six connected components and looks like

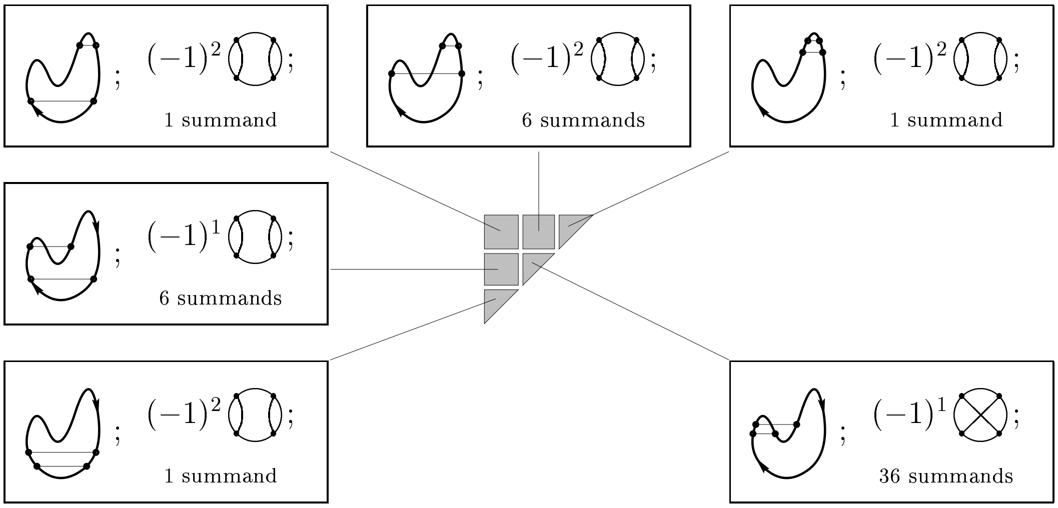




The number of summands in the integrand is constant in each connected component of the integration domain, but can be different for different components. In each plane $\{t = t_j\} \subset \mathbf{R}^3$ choose an unordered pair of distinct points (z_j, t_j) and (z_j', t_j) on K. We denote by $P = \{(z_j, z_j')\}$ a set of such pairs for $i = 1, \ldots, m$. The integrand is the sum over all choices P. In the example above for the component $\{t_{\min} < t_1 < t_{c_1}, t_{c_2} < t_2 < t_{\max}\}$ we have only one possible pair of points on the levels $\{t = t_1\}$ and $\{t = t_2\}$. Therefore, the sum over P for this component consists of only one summand. Unlike this, in the component $\{t_{\min} < t_1 < t_{c_1}, t_{c_1} < t_2 < t_{c_2}\}$ we still have only one possibility for the level $\{t = t_1\}$, but the plane $\{t = t_2\}$ intersects our knot K in four points. So we have $\binom{4}{2} = 6$ possible pairs (z_2, z_2') and the total number of summands is six.

For a pairing P the symbol '$\downarrow$' denotes the number of points (z_j, t_j) or (z_j', t_j) in P where the coordinate t decreases along the orientation of K.

Fix a pairing P. Consider our knot K as an oriented circle and connect the points (z_j, t_j) and (z_j', t_j) by a chord. We obtain a chord diagram with m chords. The corresponding element of the algebra $\mathcal{A}$ is denoted by D_P. In the picture below, for each connected component in our example, we show one of the possible pairings, the corresponding chord diagram with the sign $(-1)^{\downarrow}$ and the number of summands of the integrand (some of which are equal to zero in $\mathcal{A}$ due to one-term relation; see the discussion in Sec. 2.6).



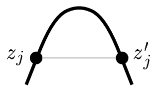
Over each connected component z_j and z'_j are continuous functions in t_j. By $\bigwedge_{j=1}^{m} \frac{dz_j - dz'_j}{z_j - z'_j}$ we mean the pullback of this form to the integration domain of variables $t_1, \ldots, t_m$.

By convention, the term in the Kontsevich integral corresponding to $m = 0$ is the (only) chord diagram of order 0 with coefficient one. It represents the unit of the algebra $\mathcal{A}$.

2.2 Convergence of the integral

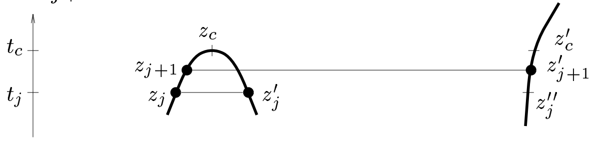
The integrand of Kontsevich integral may have singularities near the boundaries of the connected components. This happens only in the following two cases.

1) An isolated chord (z_j, z'_j) tends to zero:



In this case the corresponding chord diagram D_P is equal to zero in $\mathcal{A}$ by one-term relation.

2) A chord (z_j, z'_j) tends to zero near a critical point but is separated from that point by another, 'long' chord (z_{j+1}, z'_{j+1}):



In this case the integral for $(j+1)$-th chord can be estimated as follows:

$$\left| \int_{t_j}^{t_c} \frac{dz_{j+1} - dz'_{j+1}}{z_{j+1} - z'_{j+1}} \right| \le C \left| \int_{t_j}^{t_c} d(z_{j+1} - z'_{j+1}) \right| = C \left| (z_c - z_j) - (z'_c - z''_j) \right| \le C' |z_j - z'_j|$$

for some positive constants C and C'. So the integral for $(j+1)$-th chord is as small as $z_j - z'_j$ and this smallness compensates the smallness of the denominator corresponding to j-th chord.

This implies the convergence of the Kontsevich integral.

2.3 Invariance of the integral

Theorem. *Kontsevich integral is invariant under the deformations in the class of knots with nondegenerate critical points and the same number of critical points.*

A deformation is a smooth family of diffeomorphisms of $\mathbf{R}^3$ which starts at identity.

Any deformation that keeps a given Morse knot within the class of knots with nondegenerate critical points and preserves the number of critical points can be decomposed into *horizontal deformations* and *movements of critical points.*

A horizontal deformation is a deformation of $\mathbf{R}^3$ which preserves all horizontal planes $\{t = \text{const}\}$ and leaves the critical levels $\{t = t_c\}$ fixed pointwise (in particular, critical points do not move). The invariance under horizontal deformations is the most substantial point of the theory. This class of deformations is natural because such a deformation preserves the connected components of the integration domain. It turns out that the integral over each component is invariant under horizontal deformations. We show this in the next subsection.

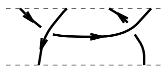
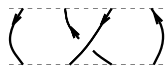
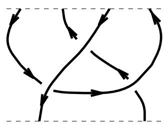
A movement of the critical point C is a deformation which is identical everywhere outside a small neighbourhood of C. In subsection 2.5 we consider invariance under the movements of critical points. As there is no sense to speak about the invariance of the integral over a particular component of the integration domain (the components are not in general preserved), in this case we only claim the invariance of the whole integral over the entire integration domain.

Unfortunately, Kontsevich integral is not invariant under deformations that change the number of critical points. However, there is a formula that shows how the integral changes under such deformations, and this will lead to the construction of the universal Vassiliev invariant in subsection 2.7.

2.4 Invariance under horizontal deformations

To prove the horizontal invariance of the Kontsevich integral, we will use tangles and their properties.

Horizontal planes cut the knot into tangles. A *tangle* is a one-dimensional compact submanifold of the slice $[a, b] \times \mathbf{C} \subset \mathbf{R}^3$ whose boundary points belong to the boundary of the slice, considered up to an isotopy of the slice, identical on the boundary. We also assume that in the points where the tangle meets the boundary planes (the *top* and the *bottom*), it is not tangent to these planes. In the case when the top of one tangle T_1 coincides with the bottom of another tangle T_2, one can define the product $T_1 \cdot T_2$ by putting T_2 on top of T_1:

$T_1 =$  ; $T_2 =$  ; $T_1 \cdot T_2 =$  .

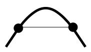
We will often enclose the tangle in a box by adding vertical sides or even in a ball depicted by a dashed disc as in one- and four-term relations for knots from the beginning of Sec. 1. This does not restrict generality, because tangles are compact.

Definition. A *tangled chord diagram* (*T-chord diagram*) is a tangle T supplied with a set of horizontal chords considered up to a diffeomorphism of the tangle that preserves the horizontal fibration. The isotopy class of the embedding of the tangle into the slice of $\mathbf{R}^3$ is here irrelevant.

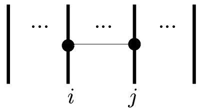
Multiplication of tangles induces a multiplication of tangled chord diagrams.

Definition. Let T be a tangle. The *space of T-chord diagrams $\mathcal{A}_T$* is a vector space generated by all chord diagrams on T modulo the set of *tangled one- and four-term relations.*

Tangled one-term relation: A tangled chord diagram with an isolated chord is equal to zero in $\mathcal{A}_T$:

 $= 0$ or  $= 0$.

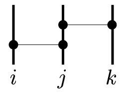
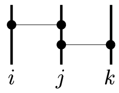
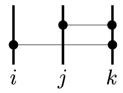
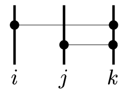
Tangled four-term relation: Denote by t_{ij} the following tangled chord diagram consisting of n parallel vertical strings and one chord between the i-th and j-th strings:

$t_{ij} = (-1)^{\downarrow}$  ,

where $\downarrow$ stands for the number of endpoints of the chords in which the orientation is directed downwards. Then the tangled four-term relation has the form

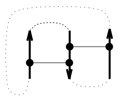
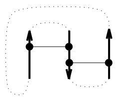
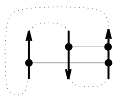
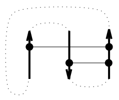
$$[t_{ij} + t_{ik}, t_{jk}] = 0$$

or, in pictures:

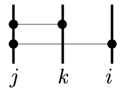
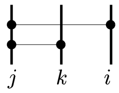
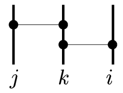
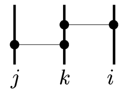
$(-1)^{\downarrow}$  $-$ $(-1)^{\downarrow}$  $+$ $(-1)^{\downarrow}$  $-$ $(-1)^{\downarrow}$  $= 0$. (1)

Exercise. If a tangle is closed into a circle in agreement with orientations of the strings, the tangled four-term relation carries over into the ordinary four-term relation of Sec. 1.2.

Here is an example:

 $-$  $-$  $+$  $= 0$.

Remark. Permuting the indices i, j, k we obtain another form of the tangled four-term relation:

$(-1)^{\downarrow}$  $-$ $(-1)^{\downarrow}$  $+$ $(-1)^{\downarrow}$  $-$ $(-1)^{\downarrow}$  $= 0$. (2)

If (top T_1) = (bottom T_2) then the multiplication of tangled chord diagrams descends to the multiplication of quotient spaces $\mathcal{A}_{T_1} \otimes \mathcal{A}_{T_2} \to \mathcal{A}_{T_1 \cdot T_2}$.

Now we can consider the Kontsevich integral for tangles. It is defined by the same formula only $t_{\min}$ must be the bottom and $t_{\max}$ the top of the tangle. The integral is multiplicative:

$$\boxed{Z(T_1) \cdot Z(T_2) = Z(T_1 \cdot T_2)}$$

whenever the product $T_1 \cdot T_2$ is defined. This important fact is a direct consequence of the Fubini theorem about multiple integrals.

Let us decompose the given knot into a product of tangles without critical points of the function t and very very thin tangles containing the critical levels (which we assume to be pointwise stable under the horizontal deformation). The contribution to the integral coming from these thin tangles is negligible as it tends to zero together with the thickness of the tangles. Thus, due to multiplicativity, it is enough to prove the invariance under horizontal deformations for a tangle without critical points. Such a tangle looks like a braid, the only difference being that the strings may have arbitrary orientations. Let T_0 be such a tangle and T_λ, a horizontal deformation of T_0 to T_1 (preserving the top and the bottom of the tangle).

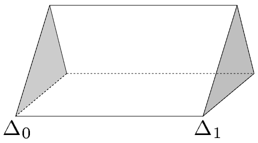
Theorem. $Z(T_0) = Z(T_1)$.

Proof. Denote by ω the integrand form in the m-th term of the Kontsevich integral:

$$\omega = \sum_{P=\{(z_j, z'_j)\}} (-1)^{\downarrow} D_P \bigwedge_{j=1}^{m} \frac{dz_j - dz'_j}{z_j - z'_j} \ .$$

Our tangles do not contain the critical values of t; therefore the integration domain for $Z(T_\lambda)$ is the entire m-dimensional simplex $\Delta_\lambda = \{t_{\min} < t_1 < \ldots < t_m < t_{\max}\}$, the same for all values of λ. Consider the product of this simplex with the unit interval of λ:

$$\Delta = \Delta_0 \times I_\lambda =$$



By Stokes theorem we have $\displaystyle\int_{\partial\Delta} \omega = \int_{\Delta} d\omega$.

The form ω is exact: $d\omega = 0$. The boundary of the integration domain is $\partial\Delta = \Delta_0 - \Delta_1 + \sum \{faces\}$. The theorem will follow from the fact that $\omega|_{\{face\}} = 0$. To show this consider two types of faces.

The first type corresponds to $t_j = t_{\min}$ or $t_j = t_{\max}$ for some j. In this situation $dz_j = dz'_j = 0$ because neither z_j nor z'_j depends on λ and on any other t_k, $k \neq j$, which form a coordinate system on the face.

The second type is that where we have $t_k = t_{k+1}$ for some k. In this case we have to choose the k-th and $(k+1)$-th chords on the same level $\{t = t_k\}$. In general, the endpoints of these chords may coincide and we don't get a chord diagram at all. Strictly speaking, we have to define the prolongation of ω (or, more precisely, D_P) to such a face more carefully. It is natural to prolong D_P to this face as a locally constant function. This means that in case when some endpoints of k-th and $(k+1)$-th chords belong to the same string (and therefore coincide) we place k-th chord a little lower than $(k+1)$-th chord, so that its endpoint differs from the endpoint of $(k+1)$-th chord. This trick yields a well-defined prolongation of D_P and ω to the face, and we use it here.

All summands of ω are divided into three parts:

1) k-th and $(k+1)$-th chords connect the same two strings;

2) k-th and $(k+1)$-th chords are chosen in such a way that their endpoints belong to four different strings;

3) k-th and $(k+1)$-th chords are chosen in such a way that there exist exactly three different strings containing their endpoints.

Consider all these cases one by one.

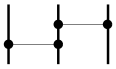
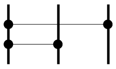
1) We have $z_k = z_{k+1}$ and $z'_k = z'_{k+1}$. So $d(z_k - z'_k) \wedge d(z_{k+1} - z'_{k+1}) = 0$ and therefore the restriction of this part of ω to the face is zero.

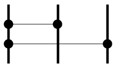
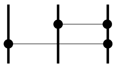
2) All choices of chords in this part of ω appear in mutually cancelling pairs. Fix four strings and number them by 1, 2, 3, 4. Suppose that for a certain choice of the pairing k-th chord connects the first two strings and $(k+1)$-th chord connects the last two strings. Then there exists another choice where on the contrary k-th chord connects the last two strings

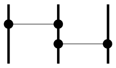
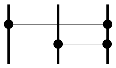
and $(k+1)$-th chord connects the first two strings. These two choices give two summands of ω differing by a sign:

$$\ldots d(z_k - z'_k) \wedge d(z_{k+1} - z'_{k+1}) \ldots + \ldots d(z_{k+1} - z'_{k+1}) \wedge d(z_k - z'_k) \ldots = 0.$$

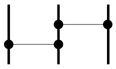
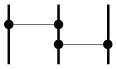
3) This is the most difficult case. There exist exactly one common string for the endpoints of k-th and $(k+1)$-th chords. Let the three strings be 1, 2, 3 and denote by ω_{ij} the form $\dfrac{dz_i - dz_j}{z_i - z_j}$. We have six possible summands of this part of ω:

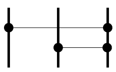
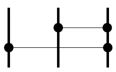
$(-1)^{\downarrow}$  $\omega_{12} \wedge \omega_{23}$ $+$ $(-1)^{\downarrow}$  $\omega_{12} \wedge \omega_{13}$

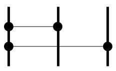
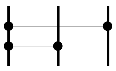
$+(-1)^{\downarrow}$  $\omega_{13} \wedge \omega_{12}$ $+$ $(-1)^{\downarrow}$  $\omega_{13} \wedge \omega_{23}$

$+(-1)^{\downarrow}$  $\omega_{23} \wedge \omega_{12}$ $+$ $(-1)^{\downarrow}$  $\omega_{23} \wedge \omega_{13}$.

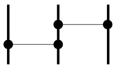
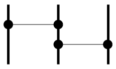
Using the fact that $\omega_{ij} = \omega_{ji}$, we can rewrite this as follows:

$\Big((-1)^{\downarrow}$  $-$ $(-1)^{\downarrow}$  $\Big)\ \omega_{12} \wedge \omega_{23}$

$+\Big((-1)^{\downarrow}$  $-$ $(-1)^{\downarrow}$  $\Big)\ \omega_{23} \wedge \omega_{31}$

$+\Big((-1)^{\downarrow}$  $-$ $(-1)^{\downarrow}$  $\Big)\ \omega_{31} \wedge \omega_{12}$.

The tangled four-term relations (1), (2) says that the expressions in parentheses are one and the same element of $\mathcal{A}_T$, hence the whole sum is equal to

$\Big((-1)^{\downarrow}$  $-$ $(-1)^{\downarrow}$  $\Big)\ (\omega_{12} \wedge \omega_{23} + \omega_{23} \wedge \omega_{31} + \omega_{31} \wedge \omega_{12}).$

The 2-form in parentheses is zero! This fact, known as *Arnold identity*, can be verified directly:

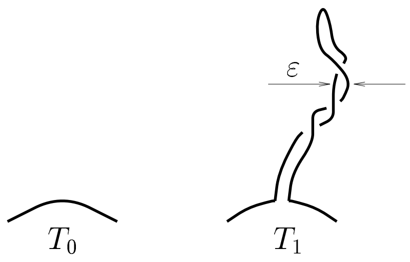
$$\frac{dz_1 - dz_2}{z_1 - z_2} \wedge \frac{dz_2 - dz_3}{z_2 - z_3} + \frac{dz_2 - dz_3}{z_2 - z_3} \wedge \frac{dz_3 - dz_1}{z_3 - z_1} + \frac{dz_3 - dz_1}{z_3 - z_1} \wedge \frac{dz_1 - dz_2}{z_1 - z_2}$$

$$\begin{aligned}
= \quad & \frac{dz_1 \wedge dz_2}{(z_1 - z_2)(z_2 - z_3)} - \frac{dz_1 \wedge dz_3}{(z_1 - z_2)(z_2 - z_3)} + \frac{dz_2 \wedge dz_3}{(z_1 - z_2)(z_2 - z_3)} \\
& + \frac{dz_2 \wedge dz_3}{(z_2 - z_3)(z_3 - z_1)} + \frac{dz_1 \wedge dz_2}{(z_2 - z_3)(z_3 - z_1)} - \frac{dz_1 \wedge dz_3}{(z_2 - z_3)(z_3 - z_1)} \\
& - \frac{dz_1 \wedge dz_3}{(z_3 - z_1)(z_1 - z_2)} + \frac{dz_2 \wedge dz_3}{(z_3 - z_1)(z_1 - z_2)} + \frac{dz_1 \wedge dz_2}{(z_3 - z_1)(z_1 - z_2)} \\
= \quad & \left(\frac{1}{(z_1 - z_2)(z_2 - z_3)} + \frac{1}{(z_2 - z_3)(z_3 - z_1)} + \frac{1}{(z_3 - z_1)(z_1 - z_2)} \right) \\
& \times (dz_1 \wedge dz_2 - dz_1 \wedge dz_3 + dz_2 \wedge dz_3) \quad = \quad 0 \, .
\end{aligned}$$

This proves the theorem and the horizontal invariance of the Kontsevich integral.

2.5 Moving the critical points

Let T_0 and T_1 be two tangles which are identical except a sharp tail of width ε, which may be twisted:



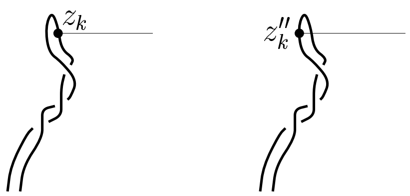
We assume that each level $\{t = const\}$ intersects the tail at most in two points and the distance between them is no greater than ε.

Theorem. $Z(T_0) = Z(T_1)$.

Proof. The difference between $Z(T_0)$ and $Z(T_1)$ can come only from terms with a chord ending on the tail.

If the highest of such chords connects the two sides of the tail then the corresponding tangled chord diagram is zero by one-term relation.

So we can assume that the highest, k-th, chord is a 'long' chord, which means that it connects the tail with another part of T_1. Suppose the endpoint of the chord belonging to the tail is (z_k, t_k). Then there exists another choice for k-th chord which is almost the same but touches the tail in another point (z_k'', t_k) of the same horizontal level:



The corresponding two terms appear in $Z(T_1)$ with the opposite signs due to $(-1)^{\downarrow}$.

Let us estimate the difference of the integrals corresponding to such k-th chords:

$$\left| \int_{t_{k-1}}^{t_c} d(\ln(z_k - z_k')) \; - \int_{t_{k-1}}^{t_c} d(\ln(z_k'' - z_k')) \right| = \left| \ln\left(\frac{z_{k-1}'' - z_{k-1}'}{z_{k-1} - z_{k-1}'} \right) \right|$$

$$= \left| \ln\left(1 + \frac{z_{k-1}'' - z_{k-1}}{z_{k-1} - z_{k-1}'} \right) \right| \sim \left| z_{k-1}'' - z_{k-1} \right| \leq \varepsilon.$$

Now, if the next $(k-1)$-th chord is also long then analogously it can be paired with another one and they give a contribution to the integral proportional to $\left| z_{k-2}'' - z_{k-2} \right| \leq \varepsilon$.

In the case the $(k-1)$-th chord is short (i.e. connects two points z_{k-1}'', z_{k-1} of the tail) we have the following estimate for the double integral corresponding to k-th and $(k-1)$-th chords:

$$\left| \int\limits_{t_{k-2}}^{t_c} \left(\int\limits_{t_{k-1}}^{t_c} d(\ln(z_k - z'_k)) \;-\; \int\limits_{t_{k-1}}^{t_c} d(\ln(z''_k - z'_k)) \right) \frac{dz''_{k-1} - dz_{k-1}}{z''_{k-1} - z_{k-1}} \right|$$

$$\leq\; C \left| \int\limits_{t_{k-2}}^{t_c} \left| z''_{k-1} - z_{k-1} \right| \frac{dz''_{k-1} - dz_{k-1}}{\left| z''_{k-1} - z_{k-1} \right|} \right|$$

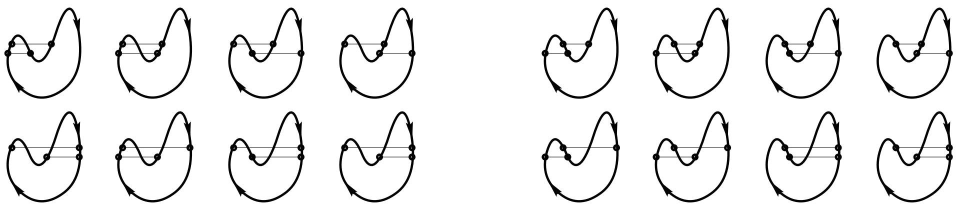
$$=\; C \left| \int\limits_{t_{k-2}}^{t_c} d(z''_{k-1} - z_{k-1}) \right| \sim \left| z''_{k-2} - z_{k-2} \right| \leq \varepsilon.$$

And again the integral is proportional to $\left| z''_{k-2} - z_{k-2} \right|$.

Continuing such considerations we obtain that the difference between $Z(T_0)$ and $Z(T_1)$ is proportional to ε. Now by horizontal deformation we can make ε tend to zero. This proves the theorem and completes the proof of the Kontsevich integral's invariance in the class of Morse knots.

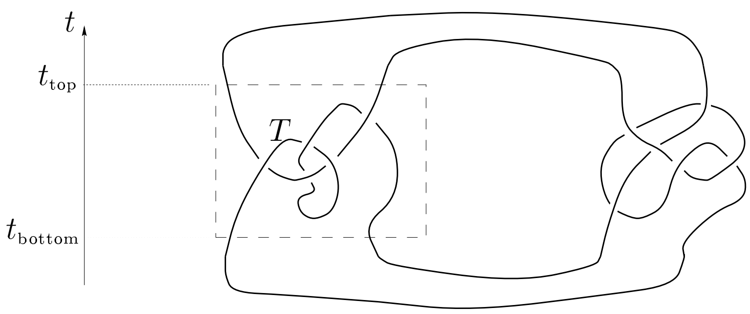
2.6 Reduction of the number of summands

We can use one-term relation to reduce the number of summands in the Kontsevich integral. Indeed, in the definition of the integral we can consider only those pairings $P = \{(z_j, z'_j)\}$ for which the corresponding chord diagram D_P contains no isolated chords. In particular, for the example of Sec. 2.1 this leaves only one component $t_{c_1} < t_1 < t_2 < t_{c_2}$ of the integration domain and, in this component, the following sixteen pairings out of the total number of thirty six:



The following useful observation belongs to folklore; we learnt it from T. Q. T. Le [Le]. It gives another way to reduce the number of summands and in fact is crucial for the construction of the universal Vassiliev invariant in the next subsection.

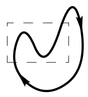
Suppose we have a Morse knot K with a distinguished tangle T.

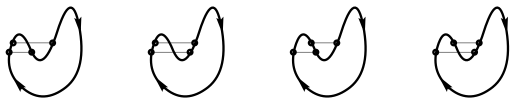


Let t_{top} and t_{bottom} be the maximal and minimal values of t on the tangle T. Consider an integral $Z_T(K)$ defined by the same formula as the Kontsevich integral $Z(K)$ but with a reduced number of summands in the integrand for those components of the integration domain whose projection on the coordinate axis t_j is entirely contained in the segment $[t_{\text{bottom}}, t_{\text{top}}]$. Namely, in $Z_T(K)$ we take into account only those pairings $P = \{(z_j, z'_j)\}$ on the level t_j ($t_{\text{bottom}} < t_j < t_{\text{top}}$) for which either both points (z_j, t_j) and (z'_j, t_j) do belong to the tangle T or neither of them does. This means that we omit the summands of $Z(K)$ which contain a 'long' chord (connecting T with the remaining part of K. We call $Z_T(K)$ the *Kontsevich integral reduced with respect to the tangle T.*

Lemma [Le]. *Reduction with respect to any tangle does not change the value of the Kontsevich integral: $Z_T(K) = Z(K)$.*

Warning. The Lemma does not say anything about the integrals over a particular component of the integration domain. It claims the coincidence of the total integrals.

Example. Consider the unknot from the example of Sec. 2.1 and the following tangle in it:  . Then the lemma leaves only following four summands for the component $t_{c_1} < t_1 < t_2 < t_{c_2}$ out of the initial (Sec. 2.1) 36 summands or out of 16 summands listed in the beginning of this subsection:



We shall explicitly compute the corresponding integral in Sec. 2.9.2, thus providing an example of how the Lemma can be used in practice.

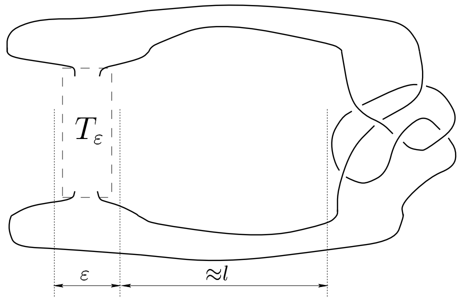
To prove the Lemma we need the following assertion which, as far as we know, is tacitly kept in mind in all texts about Kontsevich integral but is nowhere stated explicitly.

Sublemma. *Let (K_s, T_s) be a one-parameter s deformation of the pair $(K = K_0, T = T_0)$ of a knot K with a distionguished tangle T in the class of Morse knots. Then the reduced integral is invariant: $Z_{T_s}(K_s) = Z_T(K)$.*

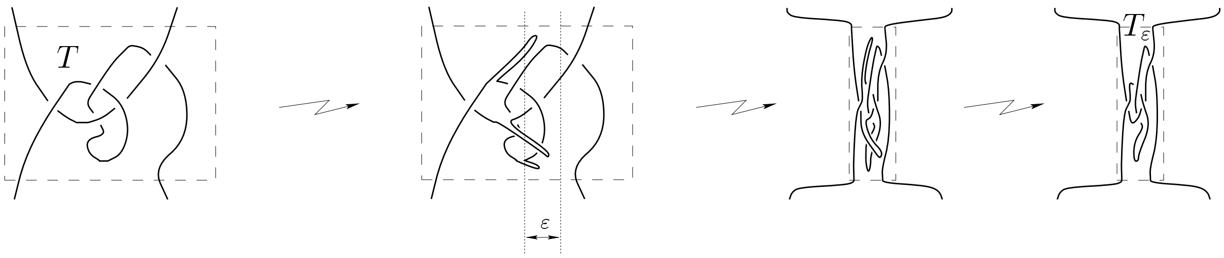
Proof of the sublemma. We can repeat step by step the proof of invariance of the ordinary Kontsevich integral given in Sec. 2.4 – 2.5. Invariance under the movement of a critical point proceeds exactly the same way, word by word. In the proof of invariance under a horizontal deformation there is, however, a minor difference. It appears when we try to repeat the proof of the theorem from Sec. 2.4 in case (3) (see page 11). In this case we use the four-term relation and it is assumed that we have enough pairings: all of the six tangled chord diagrams displayed on page 11 have to be at our disposal. But, *a priori*, it is a question whenever all of them participate in the reduced integral. An affirmative answer to this question follows from the next, rather obvious, fact. If one of the six tangled chord diagrams contains no 'long' chords and thus enters in the reduced integral, then all the remaining five diagrams have the same property, too. This proves the sublemma.

Proof of the lemma. We have to prove that the difference $Z(K) - Z_T(K)$ is equal to zero.

Let us deform the knot K shrinking the tangle T sideways into a narrow rectangle box of width ε:



We will prove that the difference $Z(K) - Z_{T_\varepsilon}(K_\varepsilon)$ tends to zero as $\varepsilon \to 0$. Since both integrals are invariant under the deformation, this immediately implies the lemma. Unfortunately this shrinking cannot be done by only horizontal deformations because we have to drag critical points into the narrow box. To perform the shrinking, we will first move all critical points of T into the box, using special movements of critical points. Then, by a horizontal deformation, we shrink the whole tangle to the box. After that return the critical points to their initial levels:



The process under study involves moving the critical points, hence there is no sense to consider the integral over one particular component. This is why the lemma is about the integrals over the whole integration domain.

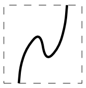
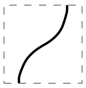
Now the difference $Z(K) - Z_{T_\varepsilon}(K_\varepsilon)$ consists of summands of $Z(K)$ with 'long' chords. So it is enough to prove that the integral corresponding to one 'long' chord tends to zero. For a long chord (z_j, z_j') connecting T_ε with the remaining part of K_ε the corresponding integral is equal to the logarithm of the quotient of the differences $z_j - z_j'$ at the upper and lower limits of integration. Both limits are such that the chord still connects T_ε with $K_\varepsilon \setminus T_e$ because we are interested only in such components of the integration domain. Assume that at the lower limit the distance is l. So l is approximately the distance between T_ε and the remaining part of the knot. Therefore at the upper limit the distance won't be more than $l + \varepsilon$. And the corresponding integral can be estimated by

$$\left|\ln\Big(\frac{l+\varepsilon}{l}\Big)\right| = \left|\ln\Big(1 + \frac{\varepsilon}{l}\Big)\right| \sim \varepsilon.$$

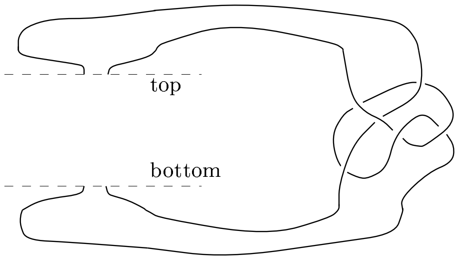
which tends to zero as $\varepsilon \to 0$. The lemma is proved.

2.7 Universal Vassiliev invariant

We have mentioned before that the Kontsevich integral is not invariant under deformations of the knot that change the number of critical points of the function t. An elementary perestroika of this kind is the *insertion of a hump.* The next theorem shows how the Kontsevich integral changes under this operation.

Theorem. $Z\Big($  $\Big) = Z\Big($  $\Big) \cdot Z\Big($  $\Big)$, *where the first and the third pictures depict an arbitrary knot, differing only in the shown fragment, while the second picture represents an unknot embedded in* $\mathbf{R}^3$ *in the specified way and the product is the product in the completed algebra* $\overline{\mathcal{A}}$ *of chord diagrams.*

To prove the theorem we will extend the construction of the Kontsevich integral for a tangles $Z(T) \in \overline{\mathcal{A}}_T$ to the complements of tangles. For example, the complement $K \setminus T$ for the pair K, T from Sec. 2.6 looks like



We consider chord diagrams on $K \setminus T$ made up of horizontal chords modulo one- and four-term relations which look exactly like tangled ones. The Kontsevich integral $Z(K \setminus T) \in \overline{\mathcal{A}}_{K\setminus T}$ is defined in the same way as for the tangles,

If T' be a tangle whose top and bottom coincide with top and bottom of $K \setminus T$ then one can obviously define their product $T' \cdot (K \setminus T)$ and the product of chord diagrams on them (*concatenation product*):

$$\overline{\mathcal{A}}_{T'} \times \overline{\mathcal{A}}_{T\setminus K} \to \overline{\mathcal{A}}.$$

By definition, for any tangle T in the knot K

$$Z_T(K) = Z(T) \cdot Z(K \setminus T),$$

and the lemma of Sec. 2.6 implies

Lemma 1. (*Multiplication formula for the Konstevich integral*)

$$Z(K) = Z(T) \cdot Z(K \setminus T).$$

We will investigate a little further the properties of the Kontsevich integral for *one-string* tangles and their complements and then use them to prove the theorem.

A one-string tangle is a tangle that has only one string at the top and at the bottom. For such tangles, as well as for their complements, we have two naturally defined *closure* operations:

- For the tangle.

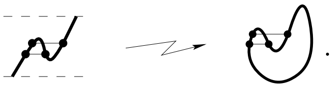
  1. Closure of the tangle $T \mapsto \overline{T}$. The tangle is converted into a knot by adding an arc with two nondegenerate critical points.

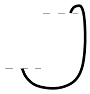
2. Closure of chord diagrams: the mapping $\overline{\mathcal{A}}_T \to \overline{\mathcal{A}}$ defined by the identification of the two endpoints of T. Let $\overline{Z}(T) \in \overline{\mathcal{A}}$ stand for the Kontsevich integral of the tangle closed in this way.

- For the complement.

1. Closure of the complement $K \setminus T \mapsto \overline{K \setminus T}$. The complement is converted into a knot by adding an arc without critical points.
2. Closure of chord diagrams: the mapping $\overline{\mathcal{A}}_{K\setminus T} \to \overline{\mathcal{A}}$ defined by the identification of the two endpoints of $K \setminus T$. Let $\overline{Z}(K \setminus T) \in \overline{\mathcal{A}}$ stand for the Kontsevich integral of the complement closed in this way.

Here is a figure illustrating the closure operation for tangles:

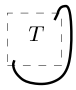
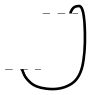
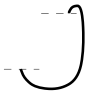


As you can see, the closure is carried out by adding the arc  .

Lemma 2. For a one-string tangle T, the following identities hold:

1. $\overline{Z}(T) = Z(\overline{T})$.
2. $\overline{Z}(K \setminus T) = Z(\overline{K \setminus T})$.

Proof.

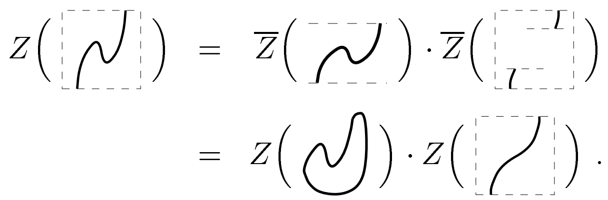
1. Indeed, by Lemma 1, $Z(\overline{T}) = Z\Big($  $\Big) = Z(T) \cdot Z\Big($  $\Big)$. But each chord diagram on  is equal to zero by one-term relation except for the diagram of order 0 (without any chords) which closes the chord diagrams on T.
2. The same argument holds for the complements, too, but it is even simpler, because in this case on the additional arc there is only one (empty) chord diagram.

Lemma 3. *For a one-string tangle T and its complement $K\setminus T$ the concatenation product $\overline{\mathcal{A}}_T \times \overline{\mathcal{A}}_{K\setminus T} \to \overline{\mathcal{A}}$ agrees with the product in the algebra $\overline{\mathcal{A}}$ in the sense that the following identity holds:*

$$Z(T) \cdot Z(K \setminus T) = \overline{Z}(T) \cdot \overline{Z}(K \setminus T).$$

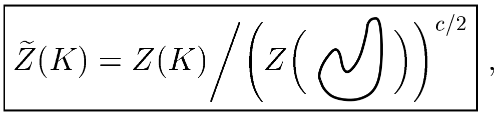
Proof. This follows directly from the definitions.

Proof of the theorem. By Lemmas 2 and 3 we have:



The theorem is proved.

Definition. The theorem allows to define the universal Vassiliev invariant by the formula



where c denotes the number of critical points of K and quotient means division in the algebra $\overline{\mathcal{A}}$: $(1+a)^{-1} = 1 - a + a^2 - a^3 + \ldots$. The invariance of $Z(K)$ in the class of Morse knots and the theorem above imply that $\tilde{Z}(K)$ is invariant under an arbitrary deformation of K.

2.8 Proof of the Kontsevich Theorem

First of all we reformulate the Kontsevich theorem as follows.

Theorem. *Let w be a weight system of order n (i. e. it vanishes on chord diagrams with the number of chords different from n). Then there exists a Vassiliev invariant of order $\leq n$ whose symbol is w.*

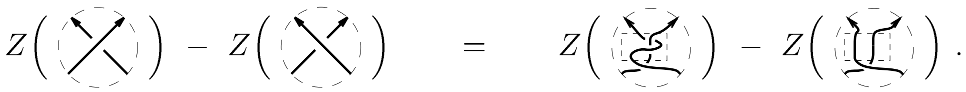
Proof. The desired knot invariant is given by the formula

$$K \longmapsto w(\tilde{Z}(K)).$$

Let D be a chord diagram of order n and K_D be a singular knot with chord diagram D. The theorem follows from the fact $\tilde{Z}(K_D) = D + (\text{terms of order} > n)$. Since the denominator of $\tilde{Z}(K)$ always starts with the unit of the algebra $\mathcal{A}$, it is sufficient to prove that

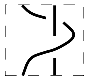
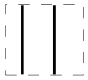
$$\boxed{Z(K_D) = D + (\text{terms of order} > n)}$$

To see what happens at a single double point let us consider the difference of Z on two knots K_+ and K_- where an overcrossing in K_+ is changed to an undercrossing in K_-. These knots can be deformed as follows:



The two knots differ only in that the first knot has one string that makes a small curl around another string. We assume that this curl is very thin and contained in a slice of thickness δ.

Enclose the curl in a box of height δ and consider it as a tangle T_+. Let T_- be the similar tangle for K_- consisting of two parallel vertical strings:

$T_+ =$  $T_- =$  .

We have

$$Z(K_+) = Z(T_+) \cdot Z(K_+ \setminus T_+) \qquad\qquad Z(K_-) = Z(T_-) \cdot Z(K_- \setminus T_-) \ .$$

But $K_+ \setminus T_+$ coincides with $K_- \setminus T_-$. By the multiplication formula for Kontsevich integral (Lemma 1 of Sec. 2.7),

$$Z(K_+) - Z(K_-) = (Z(T_+) - Z(T_-)) \cdot Z(K_+ \setminus T_+) \ .$$

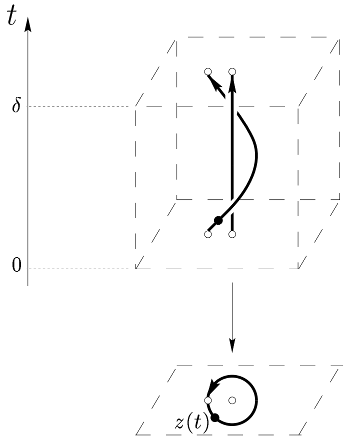
Now, for the tangle T_- both functions $z(t)$ and $z'(t)$ are constant, hence $Z(T_-)$ consists of only one summand: the trivial tangled chord diagram $D_-^{(0)}$ on T_- (here 'trivial' means with an empty set of chords) with coefficient one. The zero degree term $D_+^{(0)}$ of $Z(T_+)$ is also the trivial chord diagram on T_+ with coefficient one. But $D_-^{(0)} = D_+^{(0)}$ as chord diagrams. Thus the difference $Z(T_+) - Z(T_-)$, and therefore the difference $Z(K_+) - Z(K_-)$, starts with first degree term which comes from the chord connecting the two strings of T_+.

Now consider the n double points $p_1, \ldots, p_n$ of the knot K_D. Let $\varepsilon_1, \ldots, \varepsilon_n$ be a sequence of signs '+' or '−'. Denote by $K_{\varepsilon_1,\ldots,\varepsilon_n}$ the knot obtained from K_D by positive (like in K_+ above) resulution of p_j if $\varepsilon_j = +$ or by negative (like in K_- above) resolution of p_j if $\varepsilon_j = -$. Deforming all knots $K_{\varepsilon_1,\ldots,\varepsilon_n}$ near the points $p_1, \ldots, p_n$ in the same manner as above for K_+ and K_-, we get, for every $j = 1, \ldots, n$, two tangles $T_{j,+}$ and $T_{j,-}$ contained in a box of height δ near the point p_j. The tangle $T_{j,-}$ always consists of two vertical strings. Then we can write

$$\begin{aligned}
Z(K_D) &= \sum_{\varepsilon_1,\ldots,\varepsilon_n} \varepsilon_1 \cdot \ldots \cdot \varepsilon_n \cdot Z(K_{\varepsilon_1,\ldots,\varepsilon_n}) \\
&= \sum_{\varepsilon_1,\ldots,\varepsilon_{n-1}} \varepsilon_1 \cdot \ldots \cdot \varepsilon_{n-1} \cdot (Z(T_{n,+}) - Z(T_{n,-})) \cdot Z(K_{\varepsilon_1,\ldots,\varepsilon_{n-1},+} \setminus T_{n,+}) \\
&= \sum_{\varepsilon_1,\ldots,\varepsilon_{n-2}} \varepsilon_1 \cdot \ldots \cdot \varepsilon_{n-2} \cdot (Z(T_{n,+}) - Z(T_{n,-})) \cdot (Z(T_{n-1,+}) - Z(T_{n-1,-})) \\
&\qquad\qquad \cdot Z(K_{\varepsilon_1,\ldots,\varepsilon_{n-2},+,+} \setminus (T_{n,+} \cup T_{n-1,+})) \\
&\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\
&= (Z(T_{1,+}) - Z(T_{1,-})) \cdot \ldots \cdot (Z(T_{n,+}) - Z(T_{n,-})) \cdot Z(K_{+,\ldots,+} \setminus (\bigcup_j T_{j,+})) \ .
\end{aligned}$$

As above, each difference $Z(T_{j,+}) - Z(T_{j,-})$ starts with a term of degree 1. Hence the product $Z(K_D)$ starts from degree n. Moreover, the term of degree n is proportional to a chord diagram whose j-th chord connects the two strings of the tangle $T_{j,+}$. Since the two strings of $T_{j,+}$ correspond to the double point p_j in K_D, this chord diagram is precisely D. Now to calculate the order n part of $Z(K_D)$ we have to compute the coefficient of D. It is equal to the product of coefficients of one chord terms in $Z(T_{j,+})$. We will show that all of them are equal to 1. Indeed, it is possible to choose the coordinates z and t is such a way that $z'(t) \equiv 0$

and when t varies from 0 to δ the point $z(t)$ makes one complete turn around zero:



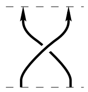
So we have

$$\frac{1}{2\pi i}\int_0^\delta \frac{dz-dz'}{z-z'}=\frac{1}{2\pi i}\oint_{|z|=1}\frac{dz}{z}=1$$

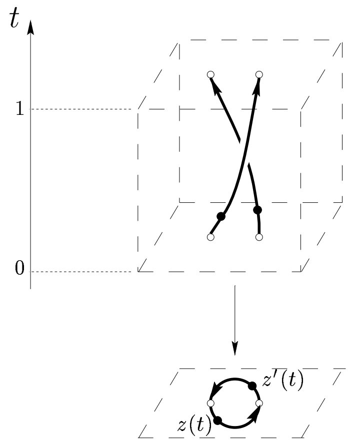
by the Cauchy theorem. The Kontsevich theorem is proved.

2.9 Examples of calculation

Explicite computation of the Kontsevich integral for a given knot is a difficult problem. Let us note that, so far, no closed form expression for the entire Kontsevich integral even for the trefoil is known! In the examples below, we consider a simple tangle with two strings and an unknot with four critical levels.

2.9.1. Here we compute the Kontsevich integral of the tangle  .

To start with introduce parametrizations $z(t)$ and $z'(t)$ of the two strings of the tangle according to the picture:



$$z(t)=e^{i(-\pi+\pi t)}\ ;$$

$$z'(t)=e^{i\pi t}\ ;$$

$$0\le t\le 1\ ;$$

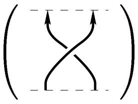
$$z(t)-z'(t)=e^{i\pi t}(e^{-i\pi}-1)\ ;$$

$$\ln(z(t)-z'(t))=i\pi t+\ln(e^{-i\pi}-1)\ ;$$

$$d(\ln(z(t)-z'(t)))=i\pi dt\ .$$

Now it is easy to compute the integral:

$$Z\Big(\ \Big) = \sum_{m=0}^{\infty} \frac{1}{(2\pi i)^m} \cdot D_m \int\limits_{0\le t_1<\ldots<t_m\le 1} \bigwedge_{j=1}^{m} d(\ln(z(t_j)-z'(t_j)))$$

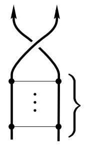


$$= \sum_{m=0}^{\infty} \frac{1}{2^m} \cdot D_m \int\limits_{0\le t_1<\ldots<t_m\le 1} dt_1\wedge\ldots\wedge dt_m$$

$$= \sum_{m=0}^{\infty} \frac{D_m}{2^m\cdot m!}\ ,$$

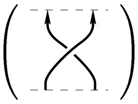
where D_m is the following tangled chord diagram.

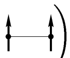
$$D_m =$$

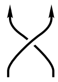


In terms of multiplication of tangled chord diagrams the answer can be written nicely as

$$Z\Big(\ \Big) = \exp\Big(\frac{1}{2}\cdot\ \Big)\cdot\ ,$$

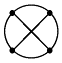
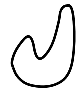


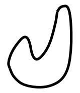
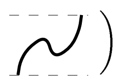
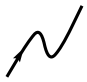


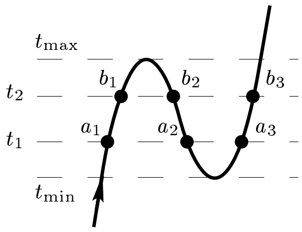


where the last factor means a tangled chord diagram without chords.

2.9.2. We borrowed this example from [Ar]. It provides a good exercise in undergraduate calculus.

Let us calculate the coefficient of the chord diagram  in $Z\Big(\ \Big)$. 

Since $Z\Big(\ \Big) = \overline{Z\Big(\ \Big)}$   it is sufficient to calculate the integral corresponding to the choice ot two levels t_1 and t_2 and pairs (z_1, z_1') and (z_2, z_2') such that the points of the first pair alternate with the points of the second one along the curve  . Choose an orientation on the curve as shown in the picture. Denote by a_1, a_2, a_3 (resp. b_1, b_2, b_3) the three points of intersection of the curve with the level $\{t=t_1\}$ (resp. $\{t=t_2\}$):



The alternating condition leaves four possibilities for chosing (z_1, z_1') and (z_2, z_2'):

(z_2, z_2')	(b_1, b_3)	(b_2, b_3)	(b_1, b_3)	(b_2, b_3)
(z_1, z_1')	(a_1, a_2)	(a_1, a_2)	(a_1, a_3)	(a_1, a_3)

Put $a_{ij} = a_j - a_i$ and $b_{ij} = b_j - b_i$. Then our integral has the form

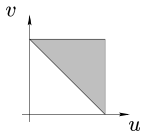
$$\frac{1}{(2\pi i)^2} \int\limits_{t_{\min}<t_1<t_2<t_{\max}} -d\ln a_{12} \wedge d\ln b_{13} + d\ln a_{12} \wedge d\ln b_{23} + d\ln a_{13} \wedge d\ln b_{13} - d\ln a_{13} \wedge d\ln b_{23}$$

$$= \frac{1}{-4\pi^2} \int\limits_{t_{\min}<t_1<t_2<t_{\max}} -d\ln\left(\frac{a_{12}}{a_{13}}\right) \wedge d\ln b_{13} + d\ln\left(\frac{a_{12}}{a_{13}}\right) \wedge d\ln b_{23}$$

$$= -\frac{1}{4\pi^2} \int\limits_{t_{\min}<t_1<t_2<t_{\max}} d\ln\left(\frac{a_{12}}{a_{13}}\right) \wedge d\ln\left(\frac{b_{23}}{b_{13}}\right) .$$

The mapping $u(t_1,t_2) = \frac{a_{12}}{a_{13}}$, $v(t_1,t_2) = \frac{b_{23}}{b_{13}}$ transforms the boundary of the integration domain $t_{\min} < t_1 < t_2 < t_{\max}$ to the standard triangle:

$$u \le 1; \quad v \le 1; \quad u + v \ge 1;$$



.

After the change of variables we obtain an integral over the standard triangle:

$$-\frac{1}{4\pi^2} \int\limits_{\substack{u\le 1;\ v\le 1;\\ u+v\ge 1}} d\ln u \wedge d\ln v = -\frac{1}{4\pi^2} \int\limits_0^1 \left(\int\limits_{1-u}^1 d\ln v \right) \frac{du}{u} = -\frac{1}{4\pi^2} \int\limits_0^1 \ln(1-u)\frac{du}{u} .$$

Taylorising the logarithm we get

$$-\frac{1}{4\pi^2} \sum_{k=1}^{\infty} \int\limits_0^1 \frac{u^k}{k}\frac{du}{u} = -\frac{1}{4\pi^2} \sum_{k=1}^{\infty} \frac{1}{k^2} = -\frac{1}{4\pi^2}\zeta(2) = -\frac{1}{24}.$$

Calculating more complicated terms of the Kontsevich integral leads to some values of D. Zagier's multivariate generalization of ζ-function:

$$\zeta(a_1, \ldots, a_n) = \sum_{0<k_1<k_2<\ldots<k_n} k_1^{-a_1} \ldots k_n^{-a_n}$$

(see [LM]).

3 Orientation and mutation

3.1 Orientation

Let S be the operation on knots which inverts their orientation. The same letter will also denote the analogous operation on chord diagrams (inverting the orientation of the outer circle or, which is the same thing, axial symmetry of the diagram).

Theorem (sf. [BN1]). *Kontsevich integral commutes with the operation S of inverting the orientation:*

$$Z(S(K)) = S(Z(K)).$$

where by $S(Z(K))$ we mean simultaneous application of S to all the chord diagrams participating in $Z(K)$.

Proof. The required identity follows directly from the definition of the Kontsevich integral in Sec.2. Indeed, the number $\downarrow$ for the knot $S(K)$ with inverse orientation is equal to the number $\uparrow$ for the initial knot K. Here by $\uparrow$ we of course mean the number of points (z_j, t_j) or (z'_j, t_j) in a pairing P where the coordinate t increases along the orientation of K. Since the number of points in a pairing is always even, $(-1)^{\downarrow} = (-1)^{\uparrow}$. Hence the corresponding chord diagrams appear in $Z(S(K))$ and in $S(Z(K))$ with the same sign. The theorem is proved.

Corollary. *Vassiliev invariants do not distinguish the orientation of knots if and only if all chord diagram are symmetric $D = S(D)$ modulo one and four-term relations.*

The caculations of [Knei] show that up to order 12 all chord diagrams are symmetric. For bigger orders the problem is still open.

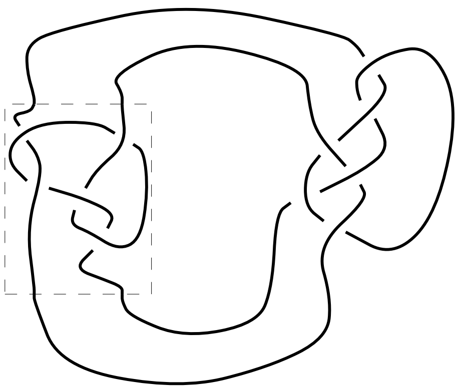
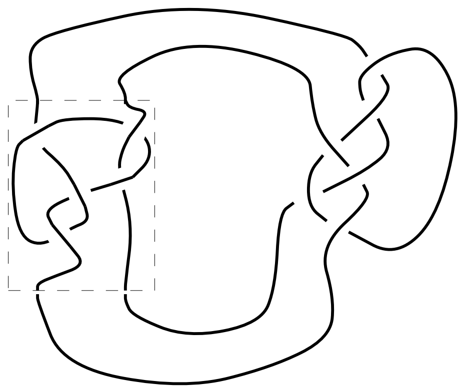
3.2 Mutation

The purpose of this subsection is to show that the Kontsevich integral commutes with the operation of mutation. We learned this fact from T. Q. T. Le [Le]. As an application, we will construct a counterexample to the intersection graph conjecture from [CDL1]. Finily we will formulate a conjecture about the combinatorial description of Vassiliev invariants which do not distingush mutant knots.

3.2.1 Mutation of knots

Definition. Suppose we have a knot K with a distinguished tangle T which has two strings at the bottom and two strings at the top. Let us cut out the tangle, rotate it through $180°$ and insert it back. This operation M is called a *mutation* and the obtained knot $M(K)$ is called a *mutant* of K.

Here is the widely known example of a mutant pair of knots, the *Conway* and *Kinoshita–Teresaka* knots:

$C =$  $KT =$ 

Theorem [MC]. *There exists a Vassiliev invariant v of order 11 such that $v(C) \neq v(KT)$.*

Morton and Cromwell produced the invariant v using the Lie algebra $\mathrm{gl}(N)$ with a non-standard representation (or, in other words, HOMFLY polynomial of certain cablings of the knots).

3.2.2 Mutation of Kontsevich integral

Let K be a knot with a distinguished tangle T which has two strings at the bottom and two strings at the top. Let $M(K)$ be the mutant of K obtained by the rotation of T. We shall define the mutation $M(Z(K))$ of the Kontsevich integral as the simultaneous mutation of all chord diagram participating in $Z(K)$. Then the following theorem holds.

Theorem [Le]. *Kontsevich integral commutes with the mutation operation M:*

$$Z(M(K)) = M(Z(K)).$$

By the lemma of Sec. 2.6 $Z(K)$ does not contain any chord diagram with a chord connecting T with the remaining part of K. This means that all chord diagrams which appear in $Z(K)$ are products (in sense of Sec. 2.7) of tangled chord diagrams on T and chord diagrams on $K \setminus T$. In [CDL2], we called a tangled chord diagram on T, considered as a part of the whole chord diagram for K, a *share*.

Definition. A *share* is a part of the chord diagram consisting of two arcs of the outer circle and all chords whose endpoints belong to these arcs. Moreover, we require that the whole chord diagram contains no chords that connect these two arcs with the remaining part of the circle.

Examples. In the left picture below the collection of four chords inside the dotted oval is a share. But the collection of three fat chords is not a share because there exists a fourth chord that separates their ends. In the right picture there are two shares shown by dotted arcs.

The choice of a tangle T in a knot K distinguishes the T-shares (which correspond to the tangled chord diagrams of $Z(T)$) in all chord diagrams that appear in $Z(K)$.

Definition. The *mutation* $M(Z(K))$ of $Z(K)$ is the simultaneous rotation of the T-shares in all diagrams of $Z(K)$.

Mutants of the chord diagrams of last two examples look as follows

Proof of the theorem. Let T_r be a tangle obtained from T by rotation under the mutation M. Then $Z(K) = Z(T) \cdot Z(K \setminus T)$ and $Z(M(K)) = Z(T_r) \cdot Z(K \setminus T)$. So

$$M(Z(K)) = M(Z(T) \cdot Z(K \setminus T)) = M(Z(T)) \cdot Z(K \setminus T) = Z(T_r) \cdot Z(K \setminus T) = Z(M(K)) \ .$$

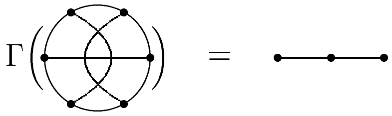
The theorem is proved.

3.2.3 Intersection graph conjecture

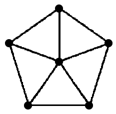
Here we introduce the notion of the intersection graph of a chord diagram and formulate the intersection graph conjecture [CDL2]. A variant of this notion was used in [BNG]. In the end we describe a counterexample to the intersection graph conjecture and formulate another conjecture instead.

Definition. *The intersection graph* $\Gamma(D)$ of a chord diagram D is a graph whose vertices correspond to the chords of D and two vertices are connected by an edge iff the corresponding chords intersect. (Two chords, a and b, are said to intersect if their endpoints a_1, a_2 and b_1, b_2 appear in the interchanging order a_1, b_1, a_2, b_2 along the circle.)

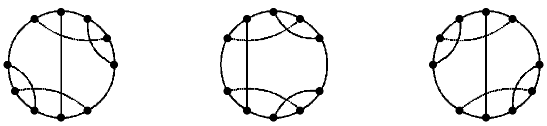
For example,



Note that not every graph can be obtained as the intersection graph of a chord diagram. The simplest example appears in degree 6 and is given by the graph



Different diagrams may have one and the same intersection graph. For example, there are three different diagrams



with the same intersection graph

Intersection graph conjecture. *If D_1 and D_2 are two chord diagrams whose intersection graphs are equal, $\Gamma(D_1) = \Gamma(D_2)$, then $D_1 = D_2$ as elements of $\mathcal{A}$ (i.e. modulo one- and four-term relations).*

This conjecture has been verified in the following situations:
1) for all diagrams D_1, D_2 with n chords up to $n = 8$ (we have checked this by computer);
2) for Vassiliev invariants v coming from the standard representations of Lie algebras $\mathrm{gl}(N)$ or

so(N) according to Bar-Natan's construction [BN1] (this means that $\Gamma(D_1) = \Gamma(D_2)$ implies $v(D_1) = v(D_2)$);
3) when $\Gamma(D_1) = \Gamma(D_2)$ is a tree (see [CDL2]) or, more generally, D_1, D_2 belong to the forest subalgebra (see [CDL3]).

3.2.4 Counterexample to the Intersection graph conjecture

It is easy to see that the mutation of chord diagrams does not change the intersection graph. So, if the intersection graph conjecture were true, the Kontsevich integrals of mutant knots would coincide. Hence all Vassiliev invariants would take the same value on mutant knots. But this contradicts the result of Morton and Cromwell [MC] from Sec. 3.2.1. Therefore the intersection graph conjecture is false.

By the way, it follows from these arguments and our computer calculations that Vassiliev invariants up to order 8 do not distinguish mutants.

It might be interesting to find an explicit combinatorial description of a weight system that distinguishes chord diagrams with the same intersection graph (like the symbol of the Morton–Cromwell invariant v from Sec. 3.2.1).

Note that the previous argument proves the following proposition.

Proposition. *Let w be a weight system whose value on each chord diagram D depends only on the intersection graph $\Gamma(D)$. Then the Vassiliev invariant $w(\widetilde{Z}(K))$ does not distinguish mutants.*

So now we can refine the intersection graph conjecture as follows.

Conjecture. *If a Vassiliev invariant $f \in \mathcal{V}_n$ does not distinguish mutants, then its symbol $[f]$ depends only on the intersection graph.*

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