ON DYNAMICS OF MAPS CLOSE TO IDENTITY

DMITRY TURAEV

1 Introduction. Statement of results.

A long standing open problem in the theory of dynamical systems is to describe which kind of dynamical phenomena one can expect in close to identity maps. It started with a celebrated paper [1] where it was shown that any n-dimensional dynamics can be realized via C^{n+1}-small perturbations of the identity map of an n-dimensional torus. The paper seized a lot of attention by physicists, because it proposed a new view on the onset of hydrodynamical turbulence; at the same time it caused a lot of criticism. One of the reasons was that the C^{n+1}-small perturbations constructed in [1] were not small in C^{n+2}, which is quite unphysical. The controversy was resolved in [2] where it was shown that given any r, any C^r-diffeomorphism of a closed n-dimensional ball can be obtained as a restriction to an n-dimensional invariant manifold of some iteration of a C^r-close to identity map of the closed unit $(n+1)$-dimensional ball B^{n+1}. Thus, the restriction on smoothness of perturbations was removed by sacrificing one dimension of phase space; anyway, other scenarios of the transitions to turbulence had already been known.

From the purely mathematical point of view, the question still remained unsolved: can an arbitrary n-dimensional dynamics be obtained by iterations of a C^r-close to identity map of B^n, i.e. in the same dimension of phase space? The difficulty is that the straightforward construction proposed in [1] does not work for high r in principle. Indeed, given an orientation-preserving diffeomorphism $F : B^n \to R^n$, one can imbed it into a continuous family $\mathcal{F}_t$ of the diffeomorphisms such that $\mathcal{F}_1 = F$ and $\mathcal{F}_0 = id$. Then, given any N, the map F can be represented as a superposition of N maps

$$F = F_N \circ \ldots \circ F_1, \quad \text{where} \quad F_s = \mathcal{F}_{\frac{s}{N}} \circ \mathcal{F}^{-1}_{\frac{s-1}{N}}, \tag{1}$$

that are $O(1/N)$-close to identity. One can choose then N small balls $D_s \in B^n$ of radius $\rho \sim N^{-1/(n-1)}$ and define a map $\phi : B^n \to B^n$ such that

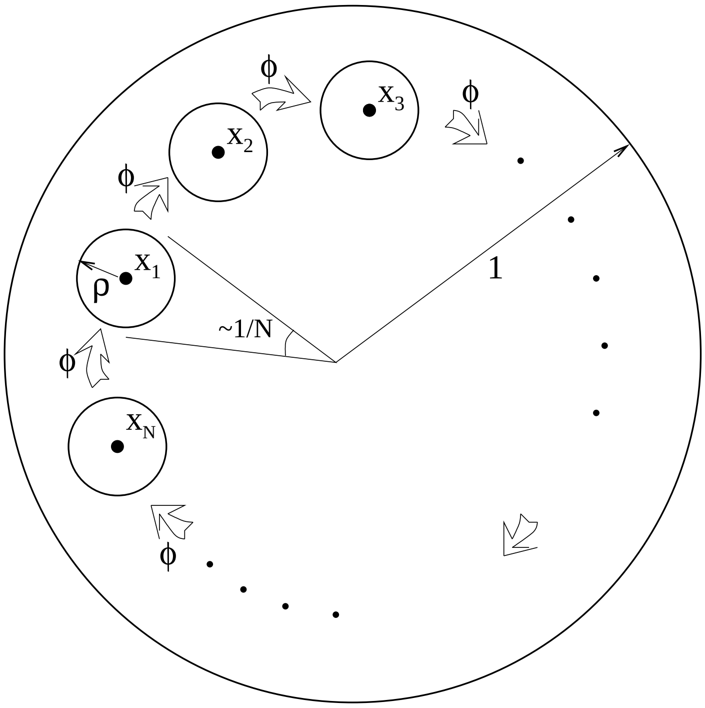

Figure 1: An illustration to Ruelle-Takens construction.

$\phi(x)|_{x\in D_s} \equiv x_{s+1} + \rho F_s(\frac{x-x_s}{\rho})$ where x_s is the center of D_s (Fig.1). Obviously, $\phi^N|_{D_0}$ is conjugate to F, i.e. the dynamics of $\phi^N|_{D_0}$ coincides with the dynamics of F. However, the derivatives of ϕ of order $n+1$ behave as $N^{-1}\rho^{-n}$, i.e. they do not, in general, tend to zero as $N \to +\infty$. Thus, an arbitrary n-dimensional dynamics can be realized by iterations of C^n-close to identity maps of B^n, but the construction gives no clue of whether the same can be said about the C^{n+1}-close to identity maps.

One could try to position the regions D_s differently, or make their radii vary, or change their shape. This, however, hardly can lead to an essential increase in the maximal order of the derivatives of ϕ which tend to zero as $N \to +\infty$. The reason lies in a well-known fact from the averaging theory that the $O(\delta)$-close to identity map

$$\bar{x} = x + \delta f(x)$$

can be approximated by a time shift of a certain autonomous flow with the accuracy $O(\delta^m)$ for an arbitrarily large m (if $f \in C^\infty$). That means that the number of iterations necessary in order to obtain a dynamics which is far from that of an autonomous flow, has to grow faster than $O(\delta^{-m})$, for every m. As we see, one has to control a very large number of iterations of close to identity maps, hence decompositions much longer than that is given by (1) have to be considered.

In this paper we propose such a decomposition (Theorem 3), using which we show that an arbitrary C^r-generic orientation-preserving n-dimensional dynamics can be obtained by iterations of C^r-close to identity maps of B^n, $n \geq 2$.

To make the formulations precise, we will borrow some definitions from [3]. Let g be a C^r-diffeomorphism of a certain closed n-dimensional ball D. Take any C^r-diffeomorphism ψ of R^n such that $\psi(B^n) \subseteq D$, and a positive integer m. The map $g_{m,\psi} = \psi^{-1} \circ g^m \circ \psi|_{B^n}$ is a C^r-diffeomorphism that maps B^n into R^n. We will call the maps $g_{m,\psi}$ obtained by this procedure *renormalized iterations of* g.

Theorem 1. *In space of C^r-smooth orientation-preserving diffeomorphisms of B^n into R^n ($n \geq 2$) there is a residual set $\mathcal{S}_r$ such that for every map $F \in \mathcal{S}_r$, for every $\delta > 0$ and for every n-dimensional ball D there exists a map $g : R^n \to R^n$, equal to identity outside D, such that $||g - id||_{C^r} < \delta$ and F is a renormalized iteration of g.*

This is the main result of the paper. The central part is to prove that for any $\delta > 0$ every orientation-preserving C^r-diffeomorphism $F : B^n \to R^n$ can be arbitrarily well approximated by renormalized iterations of δ-close to identity maps, equal to identity outside a given ball D (it is enough to prove this for one particular ball D, then for other balls the claim will remain true because there always exists an affine conjugacy that takes one ball to the other). In other words, we show that the set of all renormalized iterations of the maps $g : D \to D$ such that $||g - id||_{C^r} < \delta$ is dense in space of C^r-smooth orientation-preserving diffeomorphisms of B^n into R^n, for every $\delta > 0$. This set is also open (just by definition). Hence, the intersection $\mathcal{S}_r$ of these sets over all $\delta > 0$ is residual (and independent of the choice of D), which gives us the theorem.

We construct the approximations of the given map F by renormalized approximations of close to identity maps in Sections 2 and 3. As a first step, we represent F as a composition of a pair of certain special maps and some volume-preserving diffeomorphisms (Lemma 1). Each of the special maps can be realized as a flow map through a kind of saddle-node bifurcation (see Fig.2), reminiscent of the so-called "Iljashenko lips" (see [4]). For volume-preserving diffeomorphisms one may adjust the results of [3] for symplectic diffeomorphisms and prove (Lemma 2) the existence of an arbitrarily good, in the C^r-norm on any compact, polynomial approximation by a composition of the volume-preserving so-called Hénon-like maps. It is known that Hénon maps often appear as rescaled first-return maps near a homoclinic tangency (cf. [5, 6]). In this paper we find a kind of homoclinic tangency which does incorporate all the Hénon-like maps that appear in our volume-preserving polynomial approximations. Thus, we show that the map F can be approximated arbitrarily well by a composition of maps related to certain homoclinic bifurcations. The last step is to build a close to identity map which displays these bifurcations simultaneously. This is achieved by an arbitrarily small perturbation of the time-δ map of a certain C^∞ flow (the time δ map of a flow is, obviously, $O(\delta)$-close to identity).

Note that the approximation of any volume-preserving diffeomorphism of a unit ball into R^n by a polynomial volume-preserving diffeomorphism is not straightforward, because the Jacobian of the approximating diffeomorphism should be equal to 1 everywhere, and this constrain is quite strong for polynomial maps. Had the approximation result been true for all volume-preserving maps, i.e. not necessarily diffeomorphisms, it would produce a counterexample to a famous "Jacobian conjecture"; however, our approximation result uses in an essential way the injectivity of the map that has to be approximated (we represent the map as a shift by some smooth non-autonomous flow).

It should be mentioned that Theorem 1 does not hold true at $n = 1$. Indeed, if a map F on the interval B^1 has two fixed points (with the multipliers different from 1), then every close map $\hat{F}$ has a pair of fixed points $P_{1,2}$ as well. If such $\hat{F}$ is a renormalized iteration of a diffeomorphism g, then g will also have a pair of fixed points, $\psi(P_1)$ and $\psi(P_2)$ (at $n > 1$ this is not true). The interval between P_1 and P_2 will be invariant with respect to $\psi^{-1} \circ g \circ \psi$, hence $\psi^{-1} \circ g \circ \psi$ will be a root of degree $m > 1$ of the map $\hat{F}$ on this interval. Now note that the maps of the interval that have a root are not dense in C^2, according to [7], hence renormalized iterations are not dense either.

One can check through the proof of Theorem 1 that it holds true for finite-parameter families of orientation-preserving diffeomorphisms: in space of k-parameter families F_ε, $\varepsilon \in B^k$, of C^r-smooth orientation-preserving diffeomorphisms of B^n into R^n ($n \geq 2$) there is a residual set $\mathcal{S}_{kr}$ such that for every $F_\varepsilon \in \mathcal{S}_{kr}$ and for every $\delta > 0$ there exists $g_\varepsilon : D \times B^k \to D$ such that $||g_\varepsilon - id||_{C^r} < \delta$ and $F_\varepsilon = \psi_\varepsilon^{-1} \circ g_\varepsilon^m \circ \psi_\varepsilon|_{B^n}$ for some $m > 0$ and some family ψ_ε of C^r-diffeomorphisms of R^n.

Thus, any dynamical phenomenon which occurs generically in finite-parameter families of dynamical systems can be encountered in maps arbitrarily close to identity (with the same dimension of phase space).

To put the result into a general perspective, we recall that one of the main sources of difficulties in the theory of dynamical systems is that structurally stable systems are not dense in space of all systems [8, 9], moreover most natural examples of chaotic dynamics are indeed structurally unstable (like e.g. the famous Lorenz attractor). Understanding the dynamics of systems from the open regions of structural instability (i.e. the regions where arbitrarily close to every system there is a system which is not topologically conjugate to it) has been the subject of active research for the past four decades. It often happens, and helps a lot, that structurally unstable systems may possess certain robust properties, i.e. dynamical properties which are not destroyed by small perturbations. For example, systems with Lorenz attractor are pseudohyperbolic (or volume-hyperbolic) [10, 11, 12], and this is the property which, in fact, allowed for a very detailed description of them [13, 14, 15]. Another robust property is uniform partial hyperbolicity, a rich theory of systems possessing it is actively developing now [16, 17]. In fact, not so much of robust properties are known, it could even happen that beyond the mentioned partial hyperbolicity and volume-hyperbolicity no other robust dynamical properties exist. This claim can be demonstrated for various examples of homoclinic bifurcations (see [18]), and can be used as a guiding principle in the study of bifurcations of systems with a non-trivial dynamics:

given an n-dimensional system with a compact invariant set that is neither partially- nor volume-hyperbolic, every dynamics that is possible in B^n should be expected to occur at the bifurcations of this particular system.

The last statement is not a theorem and it might be not true in some situations, still it gives a useful view on global bifurcations. In particular, it was explicitly applied in [19] to Galerkin approximations of damped nonlinear wave equations in order to obtain estimates from below on the dimension of attractors in the situation where classical methods [20] do not work.

Theorem 1 gives one more example to the above stated principle: the identity map has no kind of hyperbolic structure, neither it contracts nor expands volumes, so it should not be surprising that its bifurcations provide an ultimately rich dynamics.

The same idea can be expressed in somewhat different terms. Let us call the set of all renormalized iterations of a map $g : D \to D$ its *dynamical conjugacy class*. The map will be called C^r*-universal* [3] if its dynamical conjugacy class is C^r-dense among all orientation-preserving C^r-diffeomorphisms of the closed unit ball B^n into R^n. By the definition, the dynamics of any single universal map is ultimately complicated and rich, and the detailed understanding of it is not simpler than understanding of all diffeomorphisms $B^n \to R^n$ *altogether*.

Theorem 2. *For every $r \geq 1$, C^r-universal diffeomorphisms of a given closed ball D exist arbitrarily close, in the C^r-metric, to the identity map.*

Proof. Take an arbitrary sequence of pairwise disjoint closed balls $D_j \subset D$, a sequence of maps F_j which is C^r-dense in space of orientation-preserving C^r-diffeomorphisms $B^n \to R^n$, and a sequence $\varepsilon_j \to +0$ as $j \to +\infty$. By Theorem 1, given any δ, there exist maps g_j such that g_j is identity outside D_j, some renormalized iteration of g_j is ε_j-close to F_j, and $\|g_j - id\|_{C^r} \leq \delta$. By construction, the map $g(x) \equiv g_j(x)$ at $x \in D_j$ $(j = 1, 2, \ldots)$ and $g(x) \equiv x$ at $x \in D \backslash \cup_{j=1}^{\infty} D_j$ is C^r-universal and δ-close to identity. $\square$

This work was done during the author's stay in MPI für Mathematik, Bonn. The author acknowledges the support by grant ISF 926/04.

2 An approximation theorem.

Let F be an orientation-preserving C^r-diffeomorphism $(r \geq 2)$ which maps the ball $B^n : \{\sum_{i=1}^n x_i^2 \leq 1\}$ into R^n. Without loss of generality we may assume that F is extended onto the whole R^n, i.e. it becomes a C^r-diffeomorphism $R^n \to R^n$, and it is identical (i.e. $F(x) = x$) at $\|x\|$ sufficiently large; such extension is always possible. Let K be a constant such that

$$\sup_{x \in R^n} \frac{\|\nabla J(x)\|}{J(x)} < K, \tag{2}$$

where $J(x)$ is the Jacobian of F. Denote $R^n_+ := \{x_n > 0\}$.

Lemma 1. *There exists a pair of volume-preserving, orientation-preserving C^r-diffeomorphisms $\Phi_1 : R^n_+ \to R^n_+$ and $\Phi_2 : R^n \to R^n$ such that*

$$F = \Phi_2 \circ \Psi_2 \circ \Phi_1 \circ \Psi_1, \tag{3}$$

where

$$\begin{array}{l} \Psi_1 := (x_1, \dots, x_{n-1}, x_n) \mapsto (x_1, \dots, x_{n-1}, e^{Kx_n}), \\ \Psi_2 := (x_1, \dots, x_{n-1}, x_n) \mapsto (x_1, \dots, x_{n-1}, \ln x_n). \end{array} \tag{4}$$

Proof. We need to construct a volume-preserving diffeomorphism $\Phi_1 : (x_1, \dots, x_n > 0) \mapsto (\bar{x}_1, \dots, \bar{x}_n > 0)$ in such a way that

$$\det \frac{\partial}{\partial x} \Psi_2 \circ \Phi_1 \circ \Psi_1(x) \equiv J(x) \tag{5}$$

(then the Jacobian of $\Phi_2 = F \circ (\Psi_2 \circ \Phi_1 \circ \Psi_1)^{-1}$ will be equal to 1 automatically). By (4), condition (5) is equivalent to

$$\bar{x}_n = \phi(x_1, \dots, x_n) \equiv \frac{Kx_n}{J(x_1, \dots, x_{n-1}, \frac{1}{K} \ln x_n)}.$$

It follows from (2) that $\partial\phi/\partial x_n > 0$ everywhere. Moreover, as F is the identity map outside a bounded region of R^n, we have that

$$\phi(x) = Kx_n \tag{6}$$

outside a compact subregion of R^n_+. Therefore, every trajectory of the vector field

$$\dot{x}_j = 0 \ \ (j \le n-2), \quad \dot{x}_{n-1} = \frac{\partial\phi}{\partial x_n}, \quad \dot{x}_n = -\frac{\partial\phi}{\partial x_{n-1}} \tag{7}$$

is extended for all $x_{n-1} \in (-\infty, +\infty)$, and the time $\tau(x)$ that the trajectory of the point x needs to get to $x_{n-1} = 0$ is a C^r-function of x, well defined everywhere in R^n_+. By (7),

$$\frac{\partial\tau}{\partial x_{n-1}} \frac{\partial\phi}{\partial x_n} - \frac{\partial\tau}{\partial x_n} \frac{\partial\phi}{\partial x_{n-1}} = -1. \tag{8}$$

Hence, the foliations of R_n^+ by level surfaces of the functions τ and ϕ are transverse, so (see (8),(6)) the map

$$\bar{x}_j = x_j \ \ (j \le n-2), \quad \bar{x}_{n-1} = -\tau(x), \quad \bar{x}_n = \phi(x) \tag{9}$$

is a volume-preserving, orientation-preserving C^r-diffeomorphism $R^n_+ \to R^n_+$, i.e. it is the sought map Φ_1. $\Box$

The maps $x \mapsto \bar{x}$ of the following form:

$$\bar{x}_1 = x_2, \; \dots, \; \bar{x}_{n-1} = x_n, \quad \bar{x}_n = (-1)^{n+1} x_1 + h(x_2, \dots, x_n), \tag{10}$$

will be called *Hénon-like volume-preserving maps.* Note that such maps are always injective, and the inverse map is also Hénon-like.

Theorem 3. *Every orientation-preserving C^r-diffeomorphism $F : B^n \to R^n$ can be arbitrarily closely approximated, in the C^r-norm on B^n, by a map of the following form:*

$$H_{2q_2} \circ \ldots \circ H_{21} \circ \Psi_2 \circ H_{1q_1} \circ \ldots \circ H_{11} \circ \Psi_1, \tag{11}$$

where the maps $\Psi_{1,2}$ are given by (4), *and H_{js} ($j = 1, 2; s = 1, \ldots q_j$) are certain polynomial Hénon-like volume-preserving maps.*

Proof. The map Φ_1 defined in Lemma 1 can be extended onto $x_n \leq 0$ by the rule $\bar{x}_n = K x_n$, $\bar{x}_{n-1} = x_{n-1}/K$, so it becomes a volume-preserving C^r-diffeomorphism $R^n \to R^n$. Then the theorem follows immediately from Lemma 1 and from Lemma 2 below. $\Box$

Lemma 2. *Every volume-preserving, orientation-preserving C^r-diffeomorphism $\Phi : R^n \to R^n$ can be arbitrarily closely approximated, in the C^r-norm on any given compact, by a composition of polynomial Hénon-like volume-preserving maps.*

Proof. It is well known that Φ can be imbedded in a smooth in t family $\mathcal{F}_t$ of volume-preserving C^r-diffeomorphisms $R^n \to R^n$ such that $\mathcal{F}_0 \equiv id$ and $\mathcal{F}_1 = \Phi$. The derivative $\frac{d}{dt}\mathcal{F}_t$ defines a divergence-free vector field $X(t, x)$, i.e. the diffeomorphism $\mathcal{F}_t$ is the time-t shift by the flow generated by the field X. One can approximate X arbitrarily closely on any given compact by a C^∞-smooth divergence-free vector field which is defined and bounded for all $(x, t) \in R^n \times [0, 1]$. Therefore, it is enough to prove the lemma only for those Φ which can be obtained as the time-1 shift by the flow generated by such a vector field, i.e. we may assume that $X \in C^\infty$ with no loss of generality.

Let $T_{\tau t} = \mathcal{F}_{t+\tau} \circ \mathcal{F}_t^{-1}$, i.e. it is the shift by the flow of X from the time t to $t + \tau$. This map is $O(\tau)$-close to identity, in the C^r-norm on any compact subset of $R^n \times [0,1]$. By construction, given any, arbitrarily large integer N,

$$\Phi = T_{\tau,(N-1)\tau} \circ \ldots \circ T_{\tau,m\tau} \circ \ldots \circ T_{\tau,0} \tag{12}$$

where $\tau = 1/N$, and $m = 0, \ldots, N-1$.

Note that the vector field X admits the following representation:

$$X = \sum_{i=1}^{n-1} X^{(i)} \tag{13}$$

where $X^{(i)}$ is a C^∞-smooth divergence-free vector field such that

$$\dot{x}_j \equiv 0 \quad \text{at} \quad j \neq i, i+1. \tag{14}$$

Indeed, if we write the field X as

$$\dot{x}_i = \xi_i(x,t), \quad i = 1, \ldots, n,$$

where $\sum_{i=1}^{n} \frac{\partial \xi_i}{\partial x_i} \equiv 0$, then the fields $X^{(i)}$ are defined as

$$\dot{x}_i = \eta_i(x,t), \quad \dot{x}_{i+1} = \zeta_i(x,t)$$

with

$$\eta_1 \equiv \xi_1, \quad \eta_i \equiv \xi_i - \zeta_{i-1} \ \ (i = 2, \ldots, n-1),$$

$$\zeta_i = -\int_0^{x_{i+1}} \frac{\partial}{\partial x_i} \eta_i(x_1, \ldots, x_i, s, x_{i+2}, \ldots, x_n, t) ds \ \ (i = 1, \ldots, n-2), \quad \zeta_{n-1} \equiv \xi_n.$$

By construction, the fields $X^{(1)}, \ldots, X^{(n-2)}$ are divergence-free, and $X^{(n-1)} = X - X^{(1)} - \ldots - X^{(n-2)}$, so $X^{(n-1)}$ is also divergence-free, as X is.

It follows from (13) that

$$T_{\tau t} = T_{\tau t}^{(n-1)} \circ \ldots \circ T_{\tau t}^{(1)} + O(\tau^2), \tag{15}$$

where $T_{\tau t}^{(i)}$ is the shift by the flow generated by the vector field $X^{(i)}$. Recall that the maps $T_{\tau,i\tau}$ in (12) are $O(1/N)$-close to identity. Therefore, it follows from (15),(12) that

$$\Phi = T^{(n-1)}_{\tau,(N-1)\tau} \circ \ldots \circ T^{(1)}_{\tau,(N-1)\tau} \circ \ldots \circ T^{(n-1)}_{\tau,m\tau} \circ \ldots \circ T^{(1)}_{\tau,m\tau} \circ \ldots \circ T^{(n-1)}_{\tau,0} \circ \ldots \circ T^{(1)}_{\tau,0} + O(\tau), \tag{16}$$

uniformly on compacta.

As τ can be taken arbitrarily small, it follows that in order to prove the lemma, it is enough to prove that every of the maps $T^{(i)}_{\tau t}$ in the right-hand side of (16) can be approximated arbitrarily well by a composition of Hénon-like volume-preserving maps. The maps $T^{(i)}_{\tau t}$ are volume-preserving and satisfy

$$\bar{x}_j = x_j \quad \text{at} \quad j \neq i, i+1 \tag{17}$$

(see (14)). Therefore, if we denote

$$\bar{x}_i = p(x), \quad \bar{x}_{i+1} = q(x), \tag{18}$$

then

$$\frac{\partial(p,q)}{\partial(x_i, x_{i+1})} = 1. \tag{19}$$

Thus, we can view (18) as an $(n-2)$-parameter family of symplectic two-dimensional maps $(x_i, x_{i+1}) \mapsto (\bar{x}_i, \bar{x}_{i+1})$ parametrized by $(x_1, \dots, x_{i-1}, x_{i+2}, \dots, x_n)$.

According to [3], every finite-parameter family of symplectic maps can be approximated (on any compact) by a composition of families of Hénon-like maps, i.e., in our case, maps of the form

$$\bar{x}_i = x_{i+1}, \quad \bar{x}_{i+1} = -x_i + h(x_{i+1}; x_1, \dots, x_{i-1}, x_{i+2}, \dots, x_n).$$

It follows that every map of the form (17),(18),(19) can be approximated arbitrarily closely by a composition of the maps of the form

$$\begin{aligned} &\bar{x}_j = x_j \quad \text{at} \quad j \neq i, i+1, \\ &\bar{x}_i = x_{i+1}, \\ &\bar{x}_{i+1} = -x_i + h(x_{i+1}; x_1, \dots, x_{i-1}; x_{i+2}, \dots, x_n). \end{aligned} \tag{20}$$

This proves the lemma if $n = 2$. In the case $n > 2$, it just remains to note that every map of form (20) is a composition of volume-preserving Hénon-like maps; namely, it equals to

$$S^{n-i-1} \circ H \circ S \circ Q^{n-1} \circ S^{i+1},$$

where

$$S := (x_1, \dots, x_n) \mapsto (x_2, \dots, x_n, (-1)^{n+1} x_1),$$

$$Q := (x_1, \dots, x_n) \mapsto (x_2, \dots, x_n, \sum_{j=1}^{n} (-1)^{n+j} x_j),$$

$$H := \{\bar{x}_1 = x_2, \dots, \bar{x}_{n-1} = x_n,\ \bar{x}_n =$$
$$= \sum_{j=1}^{n-1} (-1)^{n+j} x_j - x_n + h(x_n; x_{n-i+1}, \dots, x_{n-1}; (-1)^{n+1} x_2, \dots, (-1)^{n+1} x_{n-i})\}.$$

End of the proof. □

3 Proof of Theorem 1.

Given a diffeomorphism $F : B^n \to R^n$ we will take its sufficiently close approximation $\hat{F}$ in a form, similar to (11) (see (33) below). Then we construct a close to identity map whose some renormalized iteration is a close approximation of $\hat{F}$.

First, we define a certain C^∞ vector field Y (and the flow generated by it) in R^n by means of the following procedure: we give explicit formulas for the vector field inside certain blocks $U_{1\pm}, U_{2\pm}, V_{1,2}$ described below, while between the blocks we specify only the transition time from the boundary of one block to another and the corresponding Poincaré map. The existence of the C^∞ flow with arbitrary (of class C^∞) transition times and orientation-preserving Poincaré maps between block boundaries is a routine fact (at least for the given geometry of the blocks, see Fig.2).

Let $\Phi_{1,2}$ and $\Psi_{1,2}$ be the maps defined by (3). Let $I_{1\pm}$ and $I_{2\pm}$ be intervals of values of x_n such that $x_n \in I_{1+}$ at $x \in B^n$, $x_n \in I_{1-}$ at $x \in \Psi_1(B^n)$, $x_n \in I_{2+}$ at $x \in \Phi_1 \circ \Psi_1(B^n)$ and $x_n \in I_{2-}$ at $x \in \Psi_2 \circ \Phi_1 \circ \Psi_1(B^n)$. Let R be such that all the intervals $I_{j\pm}$ lie within $\{|x_n| \leq R\}$. Choose numbers $a_{1+} = a_{1-} + 3 = b_1 + 3 = a_{2+} + 3 = a_{2-} + 3 = b_2 + 3$. Let the vector field Y in the regions $U_{j\sigma} : \{|x_{n-1} - a_{j\sigma}\| \leq 1, |x_n| \leq R, |x_i| \leq 1\ (i \leq n-2)\}$, $j = 1, 2$, $\sigma = \pm 1$, be equal to

$$\begin{aligned} \dot{x}_{n-1} &= -\mu_j - (1 - \mu_j)(1 - \xi(x_{n-1} - a_{j\sigma})), \\ \dot{x}_i &= \sigma \gamma_{i\sigma}\, x_i\, \xi(x_{n-1} - a_{j\sigma}) \quad (i \neq n-1), \end{aligned} \tag{21}$$

where $\mu_{1,2} > 0$ are small (see (53),(54)), $\gamma_{i\pm} \in [0, 1]$ (see (54),(55)), and

$$0 \leq \xi \leq 1, \quad \xi(0) = 1, \quad \xi(z) \equiv 0 \ \text{ at } \ |z| \geq \frac{1}{2}. \tag{22}$$

In the regions $V_j : \{|x_{n-1} - b_j| \leq 1, |x_i| \leq 1\ \ (i \neq n-1)\}$ we denote $\hat{x}_i = x_i$ at $i \neq n-1$ and $\hat{x}_{n-1} = x_{n-1} - b_j$, and put Y to be equal to

$$\dot{x}_i = -\lambda_i \hat{x}_i \ \ (i = 1, \dots, n-1), \quad \dot{x}_n = x_n, \tag{23}$$

where $\lambda_i > 0$ are some numbers (see (35)).

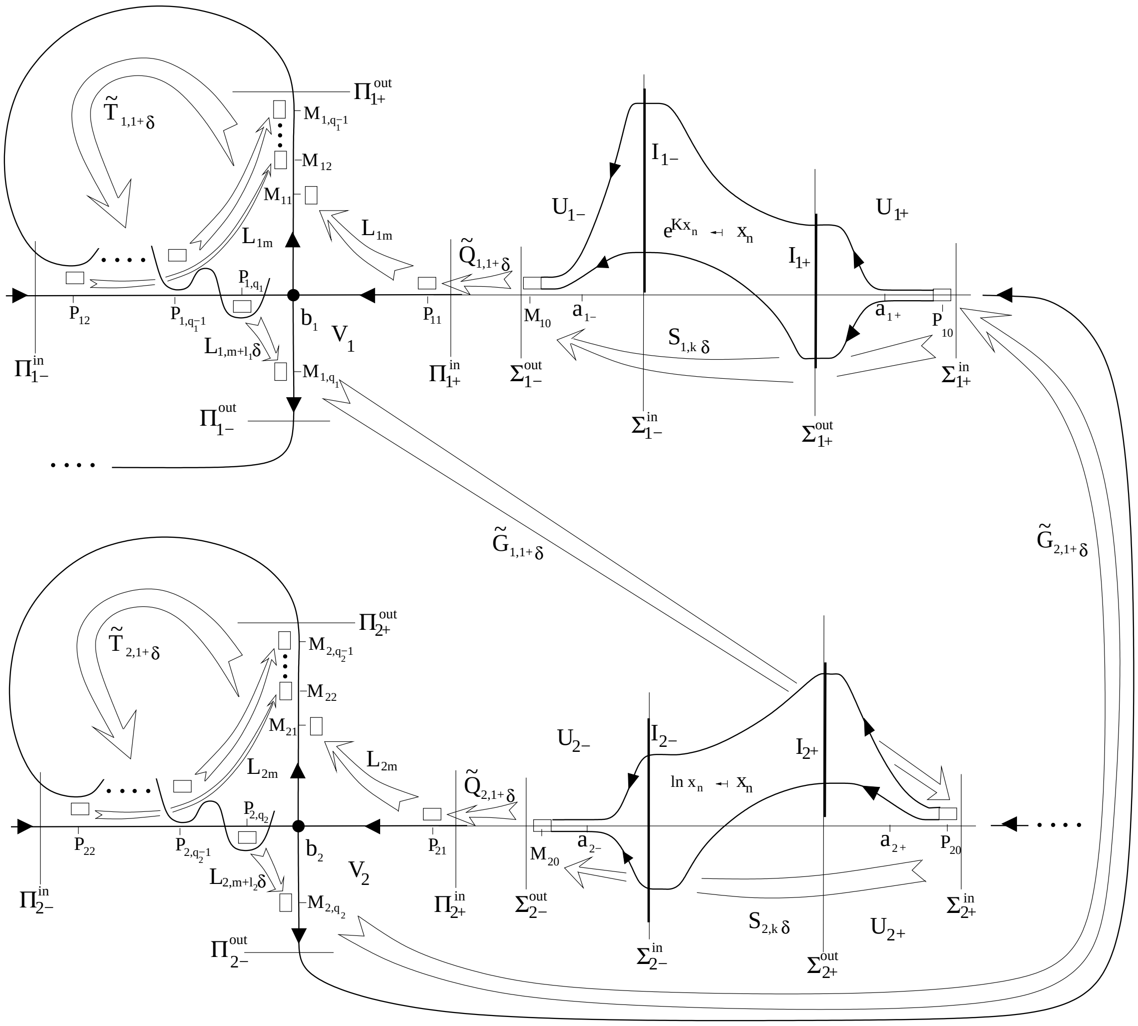

Figure 2: An illustration to the proof of Theorem 1.

As $\dot{x}_{n-1} < 0$ in $U_{j\sigma}$, every orbit of Y that starts in $U_{j\sigma}$ near $x_{n-1} = a_{j\sigma} + 1$ must come in the vicinity of $x_{n-1} = a_{j\sigma} - 1$ as time grows. For the corresponding time-t map, we have

$$x_i(t) = e^{\sigma\gamma_{i\sigma}\alpha(\mu_j)} x_i(0) \quad (i \neq n-1), \qquad x_n(t) = x_n(0) - t + \frac{1}{2}\beta(\mu_j), \quad (24)$$

where (see (21))

$$\alpha(\mu) = \int_{x_{n-1}(t)-a_{j\sigma}}^{x_{n-1}(0)-a_{j\sigma}} \frac{\xi(z)}{\mu + (1-\mu)(1-\xi(z))} dz = \int_{-1/2}^{1/2} \frac{\xi(z)}{\mu + (1-\mu)(1-\xi(z))} dz,$$

$$\beta(\mu) = 2\left(\int_{-1/2}^{1/2} \frac{1}{\mu + (1-\mu)(1-\xi(z))} dz - 1 \right). \tag{25}$$

Note that $\alpha(\mu) > 0$, it is independent of $x_{n-1}(0)$ and t (because we assume that the integration limits $(x_{n-1}(0) - a_{j\sigma})$ and $(x_{n-1}(t) - a_{j\sigma})$ are close to 1 in the absolute value, i.e. they fall in the region where $\xi(z) \equiv 0$; see (22)), and both $\alpha(\mu)$ and $\beta(\mu)$ tend to infinity as $\mu \to +0$ (the integrals diverge at $\mu = 0$ because $\xi(0) = 1$).

Denote $\Sigma_{j+}^{in} := \{x_{n-1} = a_{j+}+1, |x_n| \leq 1\}$, $\Sigma_{j+}^{out} := \{x_{n-1} = a_{j+}-1, |x_n| \leq R\}$, $\Sigma_{j-}^{in} := \{x_{n-1} = a_{j-} + 1, |x_n| \leq R\}$, $\Sigma_{j-}^{out} := \{x_{n-1} = a_{j-} - 1, |x_n| \leq 1\}$ (we also assume that $|x_i| \leq 1$ for $i \leq n-2$ on $\Sigma_{j\pm}^{in,out}$). Every orbit of Y that intersects Σ_{j+}^{in} at x_n, x_i $(i \leq n-2)$ sufficiently small leaves U_{j+} by crossing Σ_{j+}^{out}, and the orbits that intersect Σ_{j-}^{in} leave U_{j-} by crossing Σ_{j-}^{out} (see (24)). We define the vector field Y in the region between Σ_{j+}^{out} and Σ_{j-}^{in} in such a way that the orbits starting in Σ_{+}^{out} reach Σ_{-}^{in} at time 1, and the corresponding Poincaré map $\Sigma_{j+}^{out} \to \Sigma_{j-}^{in}$ is $(x_1, \ldots, x_{n-2}, x_n) \mapsto (x_1, \ldots, x_{n-2}, \psi_j(x_n))$, where we define the functions ψ_j in such a way that

$$\psi_1(x_n) = e^{Kx_n} \ \text{ at } \ x_n \in I_{1+}, \qquad \psi_2(x_n) = \ln x_n \ \text{ at } \ x_n \in I_{2+} \tag{26}$$

(see (4)). Then, the flow takes the points from the vicinity of $x_{n-1} = a_{j+}+1$ in U_{j+} into the vicinity of $x_{n-1} = a_{j-} - 1$ in U_{j-}. By (24), the corresponding time-t map S_{jt} is

$$x_i(t) = e^{(\gamma_{i+} - \gamma_{i-})\alpha(\mu_j)} x_i(0) \qquad (i \leq n-2),$$

$$x_{n-1}(t) = x_{n-1}(0) - t + \beta(\mu_j), \qquad x_n(t) = e^{-\gamma_{n-}\alpha(\mu_j)} \psi_j(e^{\gamma_{n+}\alpha(\mu_j)} x_n(0)). \tag{27}$$

In the region between Σ_{j-}^{out} and $\Pi_{j+}^{in} := \{x_{n-1} = b_j + 1, |x_i| \leq 1\}$, we define Y in such a way that all the orbits starting in a small neighborhood of $x_n = x_1 = \ldots = x_{n-2} = 0$ in Σ_{j-}^{out} intersect Π_{j+}^{in} at time 1, and the corresponding Poincaré map is the identity: $(x_1, \ldots, x_{n-2}, x_n) \mapsto (x_1, \ldots, x_{n-2}, x_n)$. Then the time-$t$ map Q_{jt} from a small neighborhood of $x_{n-1} = a_{j-} - 1, x_n = x_1 =$

$\ldots = x_{n-2} = 0$ in U_{j-} into a small neighborhood of $x_{n-1} = b_j + 1$ in V_j is given by

$$\begin{aligned} &x_i(t) = e^{-\lambda_i(t - x_{n-1}(0) - 2 + a_{j-})} x_i(0) \qquad (i \le n-2), \\ &x_{n-1}(t) - b_j = e^{-\lambda_{n-1}(t - x_{n-1}(0) - 2 + a_{j-})}, \;\; x_n(t) = e^{t - x_{n-1}(0) - 2 + a_{j-}} x_n(0) \end{aligned} \tag{28}$$

(see (23); the term $x_{n-1}(0) + 2 - a_{j-}$ in (28) is the time the orbit spends in order to get from $x(0)$ to Π^{in}_{j+}).

In V_j ($j = 1, 2$), the point $O_j : \{x_{n-1} = b_j, x_i = 0\ (i \neq n-1)\}$ is a linear saddle (see (23)). Its local stable manifold W^s_j is $x_n = 0$, and the local unstable manifold W^u_j is $x_{n-1} = b_j, x_1 = \ldots = x_{n-2} = 0$. The time-$t$ map L_{jt} within V_j is given by

$$\hat{x}_i(t) = e^{-\lambda_i t} \hat{x}_i(0) \quad (i \le n-1), \qquad x_n(t) = e^t x_n(0). \tag{29}$$

Every orbit that enters V_j at $x_n > 0$ leaves V_j by crossing the cross-section $\Pi^{out}_{j+} := \{x_n = 1, |\hat{x}_i| \le 1\ (i \le n-1)\}$, and every orbit that enters V_j at $x_n < 0$ leaves it by crossing the cross-section $\Pi^{out}_{j-} := \{x_n = -1, |\hat{x}_i| \le 1\ (i \le n-1)\}$. We assume that the orbits that start at Π^{out}_{j+} close to the point $W^u_j \cap \Pi^{out}_{j+} = (x_1 = \ldots = x_{n-2} = 0, x_{n-1} = b_j)$ return to W_j at time 1 and cross $\Pi^{in}_{j-} := \{x_{n-1} = b_j - 1, |x_i| \le 1 (i \neq n-1)\}$; we also assume that the corresponding Poincaré map $(x_1, \ldots, x_{n-1}) \mapsto (\bar{x}_1, \ldots, \bar{x}_{n-2}, \bar{x}_n)$ is given by

$$\bar{x}_i = \hat{x}_{i+1} \quad (i \le n-2), \qquad \bar{x}_n = (-1)^{n+1} \hat{x}_1$$

(the factor $(-1)^{n+1}$ stands to ensure the orientability). It follows that the time-t map T_{jt} from a small neighborhood of $W^u_j \cap \Pi^{out}_{j+}$ in V_j into a small neighborhood of $x_{n-1} = b_j - 1$ in V_j is given by

$$\begin{aligned} &x_i(t) = e^{-\lambda_i(t-1)} x_n(0)^{\lambda_{i+1} - \lambda_i} \hat{x}_{i+1}(0) \qquad (i \le n-2), \\ &x_{n-1}(t) = b_j - e^{-\lambda_{n-1}(t-1)} x_n(0)^{-\lambda_{n-1}}, \;\; x_n(t) = (-1)^{n+1} e^{t-1} x_n(0)^{1+\lambda_1} \hat{x}_1(0). \end{aligned} \tag{30}$$

For the orbits that leave V_j via Π^{out}_{j-}, we assume that the orbits that start at Π^{out}_{j-} close to the point $W^u_j \cap \Pi^{out}_{j-} = (x_1 = \ldots = x_{n-2} = 0, x_{n-1} = b_j)$ cross $\Sigma^{in}_{3-j,+}$ at time 1, and the corresponding Poincaré map $(x_1, \ldots, x_{n-1}) \mapsto (\bar{x}_1, \ldots, \bar{x}_{n-2}, \bar{x}_n)$ is given by $\bar{x}_i = x_i$ at $i = 1, \ldots, n-2$ and $\bar{x}_n = -(x_{n-1} - b_j)$.

Thus (see (21),(22),(23)), the time-t map G_{jt} from a small neighborhood of $W_j^u \cap \Pi_{j-}^{out}$ in V_j into a small neighborhood of $x_{n-1} = a_{3-j,+} + 1$ in $U_{3-j,+}$ is

$$\begin{aligned} x_i(t) &= |x_n(0)|^{\lambda_i} x_i(0) \qquad (i \le n-2), \\ x_{n-1}(t) &= a_{3-j,+} + 2 - t - \ln|x_n(0)|, \\ x_n(t) &= -|x_n(0)|^{\lambda_{n-1}} (x_{n-1}(0) - b_j). \end{aligned} \tag{31}$$

Every C^∞ flow Y, which satisfies (27),(28),(29),(30),(31), is good for our purposes. We may therefore assume that the vector field of Y is identically zero outside some sufficiently large ball D. For small δ, the time-δ map of the flow (which we denote as Y_δ) is $O(\delta)$-close to identity in the C^r-norm, for any given r. It also equals to identity outside D. Thus, if δ is small, then a small perturbation of Y_δ is a small perturbation of the identity map. Let us fix a certain r, and take a sufficiently small δ (for convenience, we assume that $N := \delta^{-1}$ is an integer). Below we construct an arbitrarily small (in the C^r-norm), localized in D perturbation of Y_δ as follows.

Note first, that the map

$$\Phi_0 := (x_1, \ldots, x_{n-1}, x_n) \mapsto (x_1, \ldots, x_n, -x_{n-1}) \tag{32}$$

is an orientation-preserving, volume-preserving diffeomorphism of R^n. Therefore, we may rewrite (3) as follows:

$$F = \Phi_0 \circ \tilde{\Phi}_2 \circ \Psi_2 \circ \Phi_0 \circ \tilde{\Phi}_1 \circ \Psi_1,$$

where $\tilde{\Phi}_{1,2}$ are orientation-preserving, volume-preserving C^r-diffeomorphisms ($\tilde{\Phi}_j = \Phi_0^{-1} \circ \Phi_j$; we assume that Φ_1 is extended onto the whole of R^n, like in Theorem 3). Now, by Lemma 2, we obtain the following, more convenient for us, analog of Theorem 3: the map F can be arbitrarily closely approximated by a map of the following form:

$$\Phi_0 \circ \tilde{H}_{2q_2} \circ \ldots \circ \tilde{H}_{21} \circ \Psi_2 \circ \Phi_0 \circ \tilde{H}_{1q_1} \circ \ldots \circ \tilde{H}_{11} \circ \Psi_1, \tag{33}$$

with polynomial Hénon-like volume-preserving maps $\tilde{H}_{js}$. Take a sufficiently close such approximation. There exists some finite $d \ge 1$ (common for all $\tilde{H}_{js}$) such that the maps $\tilde{H}_{js}$ are written as follows:

$$\bar{x}_i = x_{i+1} \quad (i = 1, \ldots, n-1), \qquad \bar{x}_n = (-1)^{n+1} x_1 + \sum_{\substack{\nu_2 \ge 0, \ldots, \nu_n \ge 0 \\ \nu_2 + \ldots + \nu_n \le d}} h_{js\nu} \prod_{2 \le p \le n} x_p^{\nu_p}. \tag{34}$$

In the segment $I_j^{out} := \{e^{-\delta} \le x_n < 1\}$ of W_j^u $(j = 1, 2)$, we choose $q_j - 1$ different points $M_{j1}, \dots, M_{j,q_j-1}$, and one point $M_{jq_j} \in W_j^u$ will be chosen in the segment $-e^{-\delta} \ge x_n > -1$. Let u_{js} denote the coordinate x_n of M_{js} $(s = 1, \dots, q_j)$. As $N\delta = 1$ =flight time from Π_{j+}^{out} to Π_{j-}^{in}, near the segment I_j^{out} the $(N+1)$-th iteration of the time-δ map Y_δ is the map $T_{j,1+\delta}$ from (30). We will fix

$$\lambda_i = \lambda \quad (i \ge 2), \qquad \lambda_1 = 1 - (n-2)\lambda, \tag{35}$$

where λ is a positive number such that

$$\lambda < \frac{1}{(n-1)d + r}. \tag{36}$$

Then the map Y_δ^{N+1} near I_j^{out} will be given by

$$\begin{aligned} &\bar{x}_1 = e^{-\lambda_1\delta} x_n^{(n-1)\lambda - 1} \hat{x}_2 \ \text{ at } \ n \ge 3, \qquad \bar{x}_i = e^{-\lambda\delta} \hat{x}_{i+1} \quad (2 \le i \le n-2), \\ &\bar{x}_{n-1} = b_j - e^{-\lambda_{n-1}\delta} x_n^{-\lambda_{n-1}}, \qquad \bar{x}_n = (-1)^{n+1} e^{\delta} x_n^{2-(n-2)\lambda} \hat{x}_1 \end{aligned} \tag{37}$$

(at $n = 2$ the first line is irrelevant). This map takes the segment I_j^{out} onto the segment $\{b_j - 1 < x_{n-1} < b_j - e^{-\lambda_{n-1}\delta}, x_1 = \ldots = x_{n-2} = x_n = 0\} \in W_j^s$. Let $P_{j,s+1} = T_{j,1+\delta} M_{js}$ $(s = 1, \dots, q_j - 1)$, and let P_{j1} be a point from $\{b_j + e^{-\lambda_{n-1}\delta} < x_{n-1} < b_j + 1, x_1 = \ldots = x_{n-2} = x_n = 0\} \in W_j^s$. By (37), the coordinate x_{n-1} of $P_{j,s+1}$ equals to $b_j - e^{-\lambda_{n-1}\delta} u_{js}^{-\lambda_{n-1}}$.

We take sufficiently large integer m and choose some points P'_{js} and M'_{js}, sufficiently close to P_{js} and M_{js} respectively $(j = 1, 2; s = 1, \dots, q_j)$, such that at $s \le q_j - 1$ we have $M'_{js} = L_{jm} P'_{js}$ (where L_{jt} is the map (29)). At $s = q_j$ we assume $M'_{jq_j} = L_{j,m+l_j\delta} P'_{jq_j}$ where l_j is an integer to be defined later (see (56); note that $l_j\delta$ is uniformly bounded). Denote the coordinates of P'_{js} and M'_{js} as $(z'_{js1}, \dots, z'_{js,n-2}, b_j + z'_{js,n-1}, z'_{jsn})$ and $(u'_{js1}, \dots, u'_{js,n-2}, b_j + u'_{js,n-1}, u'_{jsn})$ respectively. By (29),

$$u'_{jsi} = e^{-\lambda_i m} z'_{jsi} \ \ (i = 1, \dots, n-1), \ \ u'_{jsn} = e^{m} z'_{jsn} \tag{38}$$

at $s \le q_j - 1$. At $s = q_j$ we have

$$u'_{jq_ji} = e^{-\lambda_i \tau_j} z'_{jq_ji} \ \ (i = 1, \dots, n-1), \ \ u'_{jq_jn} = e^{m + l_j\delta} z'_{jq_jn}. \tag{39}$$

Note that u'_{jsi} are small at $i \le n-1$, as m is assumed to be large, and $l_j\delta$ is bounded. The values of z'_{jsi} with $i \ne n-1$ will be taken sufficiently small as well, and we will keep

$$u'_{jsn} = u_{js} \quad \text{and} \quad z'_{j,s+1,n-1} = -e^{-\lambda_{n-1}\delta} u_{js}^{-\lambda_{n-1}}, \tag{40}$$

in order to ensure the closeness of P'_{js} to P_{js} and M'_{js} to M_{js}.

We will add to the map Y_δ a small perturbation, which is localized in a small neighborhood of the points $Y_\delta^{-1}(P_{j,s+1})$ (so outside these small neighborhoods Y_δ remains unchanged). We require that these localized perturbations are such that in a sufficiently small neighborhood of $M_{js} = Y_\delta^{-N}(Y_\delta^{-1}P_{j,s+1})$ the map $\tilde{Y}_\delta^{N+1}$ (where $\tilde{Y}_\delta$ denotes the perturbed map) is given by (37) with the following correction term

$$\begin{aligned} z'_{j,s+1,n} - (-1)^{n+1} e^\delta x_n^{2-(n-2)\lambda} \left|\hat{x}_{n-1}/z'_{js,n-1}\right|^{\frac{1}{\lambda_{n-1}}-(n-1)} e^{-\lambda_{n-1}m} z'_{js1} \\ + \sum_{\substack{\nu_2\ge 0,\dots,\nu_n\ge 0 \\ \nu_2+\dots+\nu_n\le d}} \varepsilon_{js\nu} \prod_{2\le p\le n} (\hat{x}_p - u'_{jsp})^{\nu_p} \end{aligned} \tag{41}$$

added into the equation for $\bar{x}_n$, where $\varepsilon_{js\nu}$ are small coefficients to be determined later (see (46)). The first term in (41) is small as well (see (38),(39)); in the second term the values of x_n and $z'_{j,s,n-1}$ are bounded away from zero, and the exponent $\frac{1}{\lambda_{n-1}} - (n-1)$ is either zero (at $n = 2$) or larger than r (see (35),(36)), hence the second term is also small with the derivatives up to the order r at least. Note that the first two terms in (41) ensure, in particular, that at $\varepsilon = 0$ the coordinate x_n of $\tilde{Y}_\delta^{N+1}M'_{js}$ coincides with that of $P'_{j,s+1}$ (see (38),(40),(35)). We want $P'_{j,s+1} = \tilde{Y}_\delta^{N+1}M'_{js}$ at $\varepsilon = 0$, so we put

$$\begin{aligned} z'_{j,s+1,1} &= e^{-\lambda_1\delta-\lambda m} u_{js}^{(n-1)\lambda-1} z'_{js2} \quad \text{at} \quad n\ge 3, \\ z'_{j,s+1,i} &= e^{-\lambda(m+\delta)} z'_{js,i+1} \quad (2\le i\le n-2), \end{aligned} \tag{42}$$

(see (37),(38),(40)). At $s = 1$ we assume

$$z'_{j1i} = 0 \quad \text{at} \quad i \le n-2, \quad z'_{j1,n-1} = e^{-\lambda_{n-1}\delta/2}. \tag{43}$$

Now, the values of z'_{jsi}, u'_{jsi} are defined by (38),(39),(40),(42) for all j, s, i. As one can see, z'_{jsi} at $i \ne n-1$ and u'_{jsi} at $i \ne n$ tend to zero as $m \to +\infty$, i.e. $P'_{js} \to P_{js}$ and $M'_{js} \to M_{js}$ indeed.

At $\varepsilon = 0$ (hence at all small ε) the map $\tilde{T}_{j,1+\delta} \circ L_{jm} \equiv \tilde{Y}_\delta^{mN+N+1}$ (where $\tilde{T}$ stands for the perturbed map T) takes a small neighborhood of P'_{js} into a small neighborhood of $P'_{j,s+1}$. We choose some $\eta(m)$ that tends to zero as $m \to +\infty$ and some, independent of m, coefficients $C_{jsi} > 0$, and introduce rescaled coordinates $v_1, \ldots, v_n$ near P'_{js} by the rule

$$\hat{x}_1 |b_j - x_{n-1}|^{n-1-1/\lambda_{n-1}} = z'_{js1} |z'_{js,n-1}|^{n-1-1/\lambda_{n-1}} + C_{js1}\, \eta\, e^{-\lambda m(n-2)} v_1$$

$$\hat{x}_i = z'_{jsi} + C_{jsi}\, \eta\, e^{-\lambda m(n-i-1)} v_i \quad (2 \le i \le n-1), \qquad \hat{x}_n = z'_{jsn} + C_{jsn}\, \eta\, e^{-m} v_n. \tag{44}$$

Note that $\lambda_1 = 1$ at $n = 2$ (see (35)), so the exponent $n - 1 - 1/\lambda_{n-1}$ in the first line is non-zero only at $n \ge 3$ (recall that $|b_j - x_{n-1}|$ is close to 1 near P_{js}, hence (44) is a smooth coordinate transformation in any case). Since η tends to zero as $m \to +\infty$, any bounded region of values of v corresponds to a small neighborhood of P'_{js}.

After the rescaling, the map $\tilde{T}_{j,1+\delta} \circ L_{jm} \equiv \tilde{Y}_\delta^{mN+N+1}$ from a small neighborhood of P'_{js} into a small neighborhood of $P'_{j,s+1}$ takes the following form (see (37),(29),(35),(38),(40),(42),(44)):

$$C_{j,s+1,i} \bar{v}_i = e^{-\lambda\delta} C_{j,s,i+1} v_{i+1} \quad (i \le n-2),$$

$$C_{j,s+1,n-1} \bar{v}_{n-1} = e^{-\lambda_{n-1}\delta} \left(u_{js}^{-\lambda_{n-1}} - (u_{js} + C_{jsn}\, \eta\, v_n)^{-\lambda_{n-1}}\right)/\eta,$$

$$C_{j,s+1,n} \bar{v}_n = (-1)^{n+1} \phi_{js} C_{js1} v_1 + \sum_{\substack{\nu_2 \ge 0, \ldots, \nu_n \ge 0 \\ \nu_2 + \ldots + \nu_n \le d}} \varepsilon_{js\nu} E_{js\nu} \prod_{2 \le p \le n} v_p^{\nu_p}$$

where we denote

$$\phi_{js} = e^\delta (u_{js} + \eta C_{jsn} v_n)^{2-(n-2)\lambda} |z'_{js,n-1} + \eta C_{j,s,n-1} v_{n-1}|^{\frac{1}{\lambda_{n-1}} - (n-1)},$$

$$E_{js\nu} = e^{m(1 - \lambda \sum_{2 \le p \le n-1} (n-p)\nu_p)}\, \eta^{(-1 + \sum_{2 \le p \le n} \nu_p)} \prod_{2 \le p \le n} C_{jsp}^{\nu_p}.$$

Note that $\sum_{2 \le p \le n-1} (n-p)\nu_p \le (n-2)d$, hence $1 - \lambda \sum_{2 \le p \le n-1} (n-p)\nu_p > 0$ (see (36)). Therefore, all the coefficients $E_{js\nu}$ tend to infinity as $m \to +\infty$ (provided η tends to zero sufficiently slowly).

As we see, by putting

$$C_{j,s+1,i} = e^{-\lambda\delta} C_{j,s,i+1} \quad (i \le n-2), \qquad C_{j,s+1,n-1} = \lambda_{n-1} e^{-\lambda_{n-1}\delta} u_{js}^{-\lambda_{n-1}-1} C_{jsn},$$

$$C_{j,s+1,n} = e^{\delta} u_{js}^{2-(n-2)\lambda} |z'_{js,n-1}|^{\frac{1}{\lambda_{n-1}}-(n-1)} C_{js1}, \tag{45}$$

and

$$\varepsilon_{js\nu} = h_{jsi} \frac{C_{j,s+1,n}}{E_{js\nu}}, \tag{46}$$

the map $\tilde{T}_{j,1+\delta} \circ L_{jm}$ near P'_{js} takes the form

$$\bar{v}_i = v_{i+1} \quad (i \le n-2), \qquad \bar{v}_{n-1} = v_n + O(\eta),$$
$$\bar{v}_n = (-1)^{n+1} v_1 + O(\eta) + \sum_{\substack{\nu_2 \ge 0, \dots, \nu_n \ge 0 \\ \nu_2 + \dots + \nu_n \le d}} h_{js\nu} \prod_{2 \le p \le n} v_p^{\nu_p}, \tag{47}$$

i.e. it can be made as close as we want to the map $\tilde{H}_{js}$, provided m is taken large enough (recall that $\eta \to 0$ as $m \to +\infty$). We take η tending to zero sufficiently slowly, so, as we mentioned, $E_{ij\nu} \to \infty$ as $m \to +\infty$, which implies that all $\varepsilon_{js\nu} \to 0$ (see (46)), i.e. our perturbation to Y_δ is arbitrarily small indeed.

It follows that in the rescaled coordinates the map $\left(\tilde{T}_{j,1+\delta} \circ L_{jm}\right)^{q_j} \equiv \tilde{Y}_\delta^{q_j(mN+N+1)}$ from a small neighborhood of P'_{j1} into a small neighborhood of P'_{jq_j} can be made as close as we want to the map $\tilde{H}_{jq_j} \circ \dots \circ \tilde{H}_{j1}$, provided m is large enough (the rescaled coordinates near P'_{j1} and P'_{jq_j} are given by formulas (44), where the coefficients $C_{j1i} > 0$ are taken arbitrary, and the coefficients C_{jq_ji} are then recovered from the recursive formula (45); note that m does not enter (45), hence C_{jq_ji} stay bounded away from zero and infinity as $m \to +\infty$).

Now, from (29) we obtain that the same holds true for the map $L_{j,m+l_j\delta} \circ \left(\tilde{T}_{j,1+\delta} \circ L_{jm}\right)^{q_j} \equiv \tilde{Y}_\delta^{l_j+mN+q_j(mN+N+1)}$ from a small neighborhood of P'_{j1} into a small neighborhood of M'_{jq_j}, where the rescaled coordinates $(v_1, \dots, v_n)$ are introduced as follows:

$$\hat{x}_1 |\hat{x}_{n-1}|^{n-1-1/\lambda_{n-1}} = u'_{jq_j1} |u'_{jq_j,n-1}|^{n-1-1/\lambda_{n-1}} + C_{jq_j1}\, \eta\, e^{-\lambda_{n-1}(n-1)m-\lambda_{n-1}l_j\delta} v_1,$$

$$\hat{x}_i = u'_{jq_ji} + C_{jq_ji}\, \eta\, e^{-\lambda m(n-i)-\lambda l_j\delta} v_i \;\; (2 \le i \le n-1), \qquad x_n = u_{jq_j} + C_{jq_jn}\, \eta\, e^{l_j\delta} v_n, \tag{48}$$

with the same constants C_{jq_ji} as above.

Recall that, by construction, the point $Y_\delta^{N+1}M'_{jq_j}$ lies in $U_{3-j,+}$ in the region $a_{3-j,+}+1-\delta < x_{n-1} < a_{3-j,+}+1$. We add to the map Y_δ a perturbation, localized near the point $Y_\delta^N M'_{jq_j}$, such that the corresponding map $\tilde{G}_{j,1+\delta} \equiv \tilde{Y}_\delta^{N+1}$ will have the following form near M'_{jq_j}:

$$
\begin{aligned}
&\bar{x}_1 = |x_n|^{\lambda_1}(x_1 - e^{-\lambda(m+l_j\delta)} z'_{jq_j1} \left|(x_{n-1}-b_j)/z'_{jq_j,n-1}\right|^{\frac{1}{\lambda}-(n-1)}) \quad \text{at} \quad n \geq 3,\\
&\bar{x}_i = |x_n|^{\lambda_i}(x_i - u'_{jq_ji}) \ \ (2 \leq i \leq n-2),\\
&\bar{x}_{n-1} = a_{3-j,+} + 1 - \delta - \ln|x_n|, \qquad \bar{x}_n = -|x_n|^{\lambda_{n-1}}(x_{n-1} - b_j - u'_{jq_j,n-1}).
\end{aligned}
\tag{49}
$$

Note that u'_{jq_ji} at $i \leq n-1$ tend to zero as $m \to +\infty$ (see (39)), while the values of x_n near M'_{jq_j} and $z'_{j,q_j,n-1}$ are bounded away from zero; the exponent $\frac{1}{\lambda} - (n-1)$ in the first line is larger than r (see (36)). Thus, for sufficiently large m, map (49) is indeed a small perturbation of the map $G_{j,1+\delta}$ given by (31).

Denote $P'_{3-j,0} = \tilde{G}_{j,1+\delta}M'_{jq_j}$. By (49), this is a point with the coordinates $x_i = 0$ at $i \neq n-1$, and $x_{n-1} = a_{3-j,+} + 1 + (\kappa_j - 1)\delta$ (we assume that the coordinate x_n of M'_{jq_j} is $u'_{jq_jn} = u_{jq_j} = -e^{-\kappa_j\delta}$ for some $0 < \kappa_j \leq 1$). Introduce rescaled coordinates near $P'_{3-j,0}$ by the rule

$$
\begin{aligned}
&x_i = e^{-(l_j+\kappa_j)\delta\lambda_i} C_{jq_ji}\, \eta\, e^{-\lambda m(n-i)} v_i \ \ (i \leq n-2),\\
&x_{n-1} = a_{3-j,+} + 1 + (1-\kappa_j)\delta + e^{(l_j+\kappa_j)\delta} C_{jq_jn}\, \eta\, v_{n-1},\\
&x_n = e^{-(l_j+\kappa_j)\delta\lambda_{n-1}} C_{j,q_j,n-1}\, \eta\, e^{-\lambda_{n-1}m} v_n
\end{aligned}
\tag{50}
$$

(with the same constants C_{jq_ji} as above). In coordinates (48),(50), map (49) takes the form $(v_1, \ldots, v_{n-1}, v_n) \mapsto (v_1, \ldots, v_n, -v_{n-1}) + O(\eta)$, i.e. it becomes arbitrarily close to the map Φ_0 (see (32)) as $m_{\to} + \infty$. Thus, in the rescaled coordinates, the map $\tilde{G}_{j,1+\delta} \circ L_{j,m+l_j\delta} \circ \left(\tilde{T}_{j,1+\delta} \circ L_{jm}\right)^{q_j} \equiv \tilde{Y}_\delta^{l_j+(1+q_j)(mN+N+1)}$ from a small neighborhood of P'_{j1} into a small neighborhood of $P'_{3-j,0}$, can be made as close as we want to the map $\Phi_0 \circ \tilde{H}_{jq_j} \circ \ldots \circ \tilde{H}_{j1}$ as m grows.

Analogously, we take the point $M'_{j0} : \{x_{n-1} = a_{j-} - 1 + \delta/2, x_i = 0 \ (i \neq n-1)\} \in U_{j-}$, and perturb the map Y_δ near $Y_\delta^N P'_{j0}$ in such a way that the map $\tilde{Q}_{j,1+\delta} \equiv \tilde{Y}_\delta^{N+1}$ near M'_{j0} will be given by

$$\begin{aligned}
&\bar{x}_i = e^{-\lambda_i(\delta - x_{n-1} - 1 + a_{j-})} x_i \qquad (i \le n-2), \\
&\bar{x}_{n-1} - b_j = e^{-\lambda_{n-1}(\delta - x_{n-1} - 1 + a_{j-})}, \;\; \bar{x}_n = e^{\delta - x_{n-1} - 1 + a_{j-}} x_n + e^{-m} u_{j1}.
\end{aligned} \tag{51}$$

It is a small perturbation of the map $Q_{j,1+\delta}$ from (28), and it takes M'_{j0} to P'_{j1} (see (38),(40)). When we introduce rescaled variables near M'_{j0} by the rule

$$\begin{aligned}
&x_i = e^{\delta\lambda_i/2} C_{j1i}\, \eta\, e^{-\lambda m(n-i-1)} v_i \;\; (i \le n-2), \\
&x_{n-1} = a_{j-} - 1 + \delta/2 + \frac{1}{\lambda_{n-1}} e^{\lambda_{n-1}\delta/2} C_{j,1,n-1}\, \eta\, v_{n-1}, \\
&x_n = e^{-\delta/2} C_{j1n}\, \eta\, e^{-m} v_n,
\end{aligned} \tag{52}$$

map (51) will take the form $\bar{v} = v + O(\eta)$, i.e. it is close to the identity map. Thus, in the rescaled coordinates given by (52),(50), the map $\tilde{G}_{j,1+\delta} \circ L_{j,m+l_j\delta} \circ \left(\tilde{T}_{j,1+\delta} \circ L_{jm}\right)^{q_j} \circ \tilde{Q}_{j,1+\delta} \equiv \tilde{Y}_\delta^{l_j+(1+q_j)(mN+N+1)+N+1}$ from a small neighborhood of M'_{j0} into a small neighborhood of $P'_{3-j,0}$, is as close as we want to the map $\Phi_0 \circ \tilde{H}_{jq_j} \circ \ldots \circ \tilde{H}_{j1}$ at m large enough, i.e. it is a close approximation of the map Φ_j.

Let us now determine the form of the map $S_{jt} : v \mapsto \bar{v}$ from a small neighborhood of P'_{j0} into a small neighborhood of M'_{j0} in the rescaled coordinates (52),(50). By (27), for an integer $k > 0$, the map $S_{j,k\delta} \equiv Y_\delta^k$ takes the point P'_{j0} into M'_{j0} if

$$\beta(\mu_j) = (k + \kappa_{3-j} - 1/2)\delta - 5 \tag{53}$$

(see (52),(50)). Since $\beta \to +\infty$ as $\mu \to +0$ (see (25)), for every sufficiently large k equation (53) has a solution $\mu_j(k)$, and $\mu_j(k) \to +0$ as $k \to +\infty$. It follows that $\alpha(\mu_j(k)) \to +\infty$. Thus, for any sufficiently large m we can find $\gamma_{n\pm} \in (0,1]$ and k such that

$$\begin{aligned}
&e^{-\gamma_{n+}\alpha(\mu_j(k))} = e^{-(l_{3-j}+\kappa_{3-j})\delta\lambda_{n-1}} C_{3-j,q_{3-j},n-1}\, \eta\, e^{-\lambda_{n-1} m}, \\
&e^{-\gamma_{n-}\alpha(\mu_j(k))} = e^{-\delta/2} C_{j1n}\, \eta\, e^{-m}.
\end{aligned} \tag{54}$$

This guarantees that $\bar{v}_n = \psi_j(v_n)$ (see (52),(50),(27)).

We also obtain $\bar{v}_i = v_i$ at $i \le n-2$ by choosing $\gamma_{i\pm} \in (0,1]$ such that

$$\begin{aligned}
&e^{-\gamma_{i+}\alpha(\mu_j(k))} = e^{\delta\lambda_i/2} C_{j1i}\, \eta\, e^{-\lambda m}, \\
&e^{-\gamma_{i-}\alpha(\mu_j(k))} = e^{-(l_j+\kappa_j)\delta\lambda_i} C_{3-j,q_{3-j},i}\eta.
\end{aligned} \tag{55}$$

Finally, we fix the choice of the integer l_j and $\kappa_j \in (0,1]$ as follows:

$$e^{(l_j+\kappa_j)\delta} = \frac{1}{\lambda_{n-1}} e^{\lambda_{n-1}\delta/2} C_{3-j,1,n-1}/C_{jq_jn}. \tag{56}$$

This (along with (53)) gives us $\bar{v}_{n-1} = v_{n-1}$ for the map $S_{j,k\delta}$ in the coordinates (52),(50). As wee see, the map $S_{j,k\delta}$ in the rescaled coordinates coincides with the map Ψ_j for v from some open neighborhood of D (if $j=1$) or of $\Phi_1 \circ \Psi_1(D)$ (if $j=2$).

Summarizing, we obtain that the map

$$\begin{aligned} &\tilde{G}_{2,1+\delta} \circ L_{2,m+l_2\delta} \circ \left(\tilde{T}_{2,1+\delta} \circ L_{2m}\right)^{q_2} \circ \tilde{Q}_{2,1+\delta} \circ S_{2,k\delta} \\ &\quad \circ \tilde{G}_{1,1+\delta} \circ L_{1,m+l_1\delta} \circ \left(\tilde{T}_{1,1+\delta} \circ L_{1m}\right)^{q_1} \circ \tilde{Q}_{1,1+\delta} \circ S_{1,k\delta} \\ &\equiv \tilde{Y}_\delta^{2k+l_1+l_2+(2+q_1+q_2)(mN+N+1)+2(N+1)} \end{aligned}$$

is a close approximation to the map $\Phi_2 \circ \Psi_2 \circ \Phi_1 \circ \Psi_1$ (i.e. to the original map F), provided $\tilde{H}_{jq_j} \circ \ldots \circ \tilde{H}_{j1}$ are sufficiently close approximations to $\tilde{\Phi}_j$ ($j=1,2$) and m is large enough. $\Box$

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