Étale endomorphisms of algebraic surfaces with G_m-actions

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Abstract

Let X be an algebraic surface defined over the complex field $\mathbf{C}$ and endowed with an $\mathbf{A}^1_*$-fibration. As such a surface we have a Platonic $\mathbf{A}^1_*$-fiber space, a weighted hypersurface with its singular point deleted off and, more generally, an affine algebraic surface with an unmixed G_m-action and its fixpoint deleted off. We consider an étale endomorphism $\varphi : X \to X$ and show that φ is an automorphism in most cases. Of particular interest is the case of a Platonic $\mathbf{A}^1_*$-fiber space, for which φ being an automorphism is closely related to the Jacobian Problem for the affine plane $\mathbf{A}^2$. We observe also the automorphism group of such surfaces.

1 Introduction

Throughout the present article we work over the complex field $\mathbf{C}$. Let X be a smooth algebraic surface with a surjective morphism $\rho : X \to B$, where B is a smooth projective algebraic curve. We call a pair (X, ρ) (or simply X when the morphism ρ is tacitly given) an $\mathbf{A}^1_*$-fiber space (or a $\mathbf{C}^*$-fiber space) if the general fibers of ρ are isomorphic to $\mathbf{A}^1_*$, which is the affine line $\mathbf{A}^1$ with one point deleted off. We assume that there exists an open set U of B such that $\rho^{-1}(U)$ is a trivial $\mathbf{A}^1_*$-bundle over U. The morphism ρ is called an $\mathbf{A}^1_*$-fibration (or $\mathbf{C}^*$-fibration). As such surfaces, we consider in the present article the following three surfaces:

(1) Platonic $\mathbf{A}^1_*$-fiber spaces,

(2) A weighted hypersurface $x_1^{m_1} + x_2^{m_2} + x_3^{m_3} = 0$ in the affine 3-space $\mathbf{A}^3$ with its singular point $(0, 0, 0)$ deleted off,

(3) An algebraic surface with an unmixed G_m-action and its fixpoint deleted off.

The famous Jacobian problem asks whether or not an étale endomorphism φ of the affine plane $\mathbf{A}^2$ is an automorphism. We can generalize this problem as follows:

Generalized Jacobian Problem *Let X be a smooth algebraic surface and let $\varphi : X \to X$ be an étale endomorphism. Is φ an automorphism ?*

We consider the generalized Jacobian Problem for the above three surfaces. Of particular interest is the case of Platonic $\mathbf{A}^1_*$-fiber spaces. A Platonic $\mathbf{A}^1_*$-fiber space is written as $\mathbf{A}^2/G$ with the unique singular point deleted off, where G is a non-cyclic small finite subgroup of $GL(2, \mathbf{C})$, and the generalized Jacobian problem is equivalent to asking whether or not an étale endomorphism $\psi : \mathbf{A}^2 \to \mathbf{A}^2$ is an automorphism whenever ψ commutes with a linear action of the finite group G on $\mathbf{A}^2$ (cf. Lemma 2.2). Though we cannot solve the problem in the Platonic case, this became an introduction to consider the other surfaces with $\mathbf{A}^1_*$-fibrations. The generalized Jacobian problem does not necessarily have a positive answer (cf. Lemmas 5.4 and 5.8). The presentation of the present article is organized in such a way that we first treat the cases of Platonic $\mathbf{A}^1_*$-fiber spaces and weighted hypersurfaces as the motivations to the problem and then treat the general case of algebraic surfaces with G_m-actions. In the last section, we consider when a given $\mathbf{A}^1_*$-fibration comes from a G_m-action and also observe the automorphism group of algebraic surfaces with G_m-actions.

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2 Platonic $\mathbf{A}^1_*$-fiber spaces

Let $\rho : X \to B$ be an algebraic surface with an $\mathbf{A}^1_*$-fibration. We call X a *Platonic* $\mathbf{A}^1_*$-fiber space if the following conditions are satisfied:

(1) X is a smooth surface defined over $\mathbf{C}$ and the base curve B is isomorphic to the projective line $\mathbf{P}^1$.

(2) Every fiber of ρ is irreducible and isomorphic to $\mathbf{A}^1_*$ when taken with the reduced structure.

(3) There are three multiple fibers $m_i\Gamma_i$ $(i = 1, 2, 3)$ such that a triplet $\{m_1, m_2, m_3\}$ is a Platonic triplet. Namely, $\{m_1, m_2, m_3\}$ is one of the following triplets up to permutations:

$$\{2, 2, n\} \ (n \geq 2), \quad \{2, 3, 3\}, \quad \{2, 3, 4\}, \quad \{2, 3, 5\}$$

(4) There exists a smooth projective surface V with a $\mathbf{P}^1$-fibration $p : V \to B$ such that

  (i) V contains X as a Zariski open set and $V - X$ is a divisor with simple normal crossings. The restriction of p onto X coincides with the given $\mathbf{A}^1_*$-fibration ρ. We call V a *normal completion* of the Platonic $\mathbf{A}^1_*$-fiber space X.

  (ii) There exists two cross-sections S_0 and S_1 of p such that $S_0, S_1 \subset V - X$ and $S_0 \cap S_1 = \emptyset$, and other irreducible components of $V - X$ are contained in the fibers of p. Hence the $\mathbf{A}^1_*$-fibration ρ is untwisted.

It is known by [5] that a Platonic $\mathbf{A}^1_*$-fiber space X is isomorphic to $\mathbf{A}^2/G - (0)$, where G is a non-cyclic small finite subgroup of $GL(2, \mathbf{C})$ and (0) is the image of the point of origin on $\mathbf{A}^2$. The Platonic $\mathbf{A}^1_*$-fiber spaces represent an important class of almost minimal open algebraic surfaces with logarithmic Kodaira dimension $\overline{\kappa}(X) = -\infty$ which are not affine ruled (cf. [7]). Here we call a finite subgroup G of $GL(2, \mathbf{C})$ *small* if G contains no pseudo-reflections. Let us start with recalling the next result (cf. [5, Theorem 2]).

Lemma 2.1 (1) *Let X be a Platonic $\mathbf{A}^1_*$-fiber space. Then $\mathbf{A}^2 - (0)$ is the universal covering space of X, $G := \pi_1(X)$ is a small finite subgroup of $GL(2, \mathbf{C})$ which is not a cyclic group, and X is isomorphic to the quotient variety $\mathbf{A}^2/G$ with the unique singular point deleted off.*

(2) *Conversely, let G be a small finite subgroup of $GL(2,\mathbf{C})$ which is not a cyclic group. Then $X := \mathbf{A}^2/G - (0)$ is a Platonic $\mathbf{A}^1_*$-fiber space with respect to the G_m-action induced by the G_m-action on $\mathbf{A}^2$ via the center of $GL(2,\mathbf{C})$, where (0) represents the image of the point of origin of $\mathbf{A}^2$. Furthermore, the quotient morphism $\pi : \mathbf{A}^2 \to \mathbf{A}^2/G$ induces an étale finite Galois covering $\mathbf{A}^2 - (0) \to X$ with group G.*

Lemma 2.2 *Let X be a Platonic $\mathbf{A}^1_*$-fiber space and let $\varphi : X \to X$ be an étale endomorphism. Then there is an étale endomorphism $\widetilde{\varphi} : \mathbf{A}^2 \to \mathbf{A}^2$ such that $\varphi \circ \pi = \pi \circ \widetilde{\varphi}$, $\widetilde{\varphi}(0) = (0)$ and $\widetilde{\varphi}$ commutes with the $G := \pi_1(X)$-action on $\mathbf{A}^2$, where $\pi : \mathbf{A}^2 - (0) \to \mathbf{A}^2/G - (0)$ is the quotient morphism. Conversely, given an étale endomorphism $\widetilde{\varphi}$ of $\mathbf{A}^2$ such that $\widetilde{\varphi}(0) = (0)$ and $\widetilde{\varphi}$ commutes with an action of a small finite subgroup G of $GL(2,\mathbf{C})$, then the induced endomorphism $\overline{\varphi} : \mathbf{A}^2/G \to \mathbf{A}^2/G$ gives rise to an étale endomorphism $\varphi : X \to X$ such that $\varphi \circ \pi = \pi \circ \widetilde{\varphi}$.*

Proof. Since $\pi : \mathbf{A}^2 - (0) \to X$ is the universal covering, the endomorphism $\varphi : X \to X$ induces an étale endomorphism $\widehat{\varphi} : \mathbf{A}^2 - (0) \to \mathbf{A}^2 - (0)$ which commutes with the action of $\pi_1(X)$. Then $\widehat{\varphi}$ extends to an endomorphism $\widetilde{\varphi} : \mathbf{A}^2 \to \mathbf{A}^2$ such that $\widetilde{\varphi}(0) = (0)$. Note that $\mathbf{A}^2 - \widetilde{\varphi}(\mathbf{A}^2)$ is a finite set of points. For otherwise, there is an irreducible curve C defined by an equation $f = 0$ on $\mathbf{A}^2$ such that $C \subset \mathbf{A}^2 - \widetilde{\varphi}(\mathbf{A}^2)$. Then $\widetilde{\varphi}^*(f)$ is an invertible element on the upper $\mathbf{A}^2$, which is not the case. Write $\mathbf{A}^2 = \operatorname{Spec} k[x,y]$. Let $\widetilde{\varphi}$ is given by two polynomials $f := \widetilde{\varphi}^*(x)$ and $g := \widetilde{\varphi}^*(y)$. Then the Jacobian determinant $J = |\partial(f,g)/\partial(x,y)|$ is invertible on $\mathbf{A}^2 - (0)$ since $\widetilde{\varphi}$ is etale on $\mathbf{A}^2 - (0)$. Hence it is invertible on $\mathbf{A}^2$. This implies that $\widetilde{\varphi}$ is an étale endomorphism of $\mathbf{A}^2$. It is clear that $\widetilde{\varphi}$ commutes with the linear G-action. Conversely, given an endomorphism $\widetilde{\varphi}$ of $\mathbf{A}^2$ such that $\widetilde{\varphi}(0) = (0)$ and $\widetilde{\varphi}$ commutes with the G-action, $\widetilde{\varphi}$ induces an endomorphism $\overline{\varphi}$ on the quotient variety $\mathbf{A}^2/G$ such that $\overline{\varphi}(0) = (0)$. It is clear that $\varphi := \overline{\varphi}\mid_X$ is an étale endomorphism if so is $\widetilde{\varphi}$. Q.E.D.

Lemma 2.3 *Let $\rho : X \to B$ be the $\mathbf{A}^1_*$-fibration of a Platonic $\mathbf{A}^1_*$-fiber space and let $\varphi : X \to X$ be an étale endomorphism. Suppose that φ preserves the $\mathbf{A}^1_*$-fibration ρ, i.e., there exists an endomorphism $\beta : B \to B$ such that $\beta \circ \rho = \rho \circ \varphi$. If β is an automorphism then φ is an automorphism.*

Proof. Write $X = \mathbf{A}^2/G - (0)$ as in Lemma 2.1. Let $\widetilde{\varphi} : \mathbf{A}^2 \to \mathbf{A}^2$ be an étale endomorphism which lifts φ. It suffices to show that $\widetilde{\varphi}$ is an automorphism of $\mathbf{A}^2$. Since β induces a permutation of the set of points $\{p_1, p_2, p_3\}$ with $p_i := \rho(\Gamma_i)$ we may assume by replacing φ by a suitable iteration of φ that β fixes these three points. Then β is the identity automorphism of $B := \mathbf{P}^1$. Note that each fiber of ρ complemented by the singular point on $\mathbf{A}^2/G$ is the image of a line on $\mathbf{A}^2$ passing through the point of origin. Furthermore, for each fiber of ρ there are finitely many such lines. For $i = 1, 2, 3$, let $\{\ell_i^{(j)}; 1 \le j \le r_i\}$ be the set of all such lines that $q(\ell_i^{(j)}) = \Gamma_i \cup (0)$. Then φ induces an endomorphism on the set $\{\ell_i^{(j)}; 1 \le j \le r_i\}$. Hence we may assume after replacing φ by a suitable iteration of φ that there are three lines ℓ_1, ℓ_2, ℓ_3 which are mapped by q to the fibers $\Gamma_1, \Gamma_2, \Gamma_3$ respectively and fixed by φ. We can choose a system of coordinates $\{x, y\}$ on $\mathbf{A}^2$ such that ℓ_1, ℓ_2, ℓ_3 are defined by $x = 0, y = 0, x = y$ respectively. Note that the set defined by the equation $\widetilde{\varphi}^*(x) = 0$ is a union of lines among $\{\ell_1^{(j)}; 1 \le j \le r_1\}$. Hence $\widetilde{\varphi}^*(x)$ is a homogeneous polynomial in x, y. Similarly, $\widetilde{\varphi}^*(y)$ is a homogeneous polynomial in x, y. Since $\widetilde{\varphi}$ is étale, the Jacobian determinant $J := |\partial(\widetilde{\varphi}^*(x), \widetilde{\varphi}^*(y))/\partial(x, y)|$ is a nonzero constant. Hence $\widetilde{\varphi}^*(x)$ and $\widetilde{\varphi}^*(y)$ are linear polynomials in x, y. Then φ is an automorphism. Q.E.D.

3 Étale endomorphisms preserving the $\mathbf{A}^1_*$-fibration

Let $\rho : X \to B$ be an $\mathbf{A}^1_*$-fibration as given in the introduction. In this section we assume that an étale endomorphism φ of X preserves the $\mathbf{A}^1_*$-fibration $\rho : X \to B$. Let $\beta : B \to B$ be the induced endomorphism of the base curve B. To avoid confusion of the notation, we write $\varphi : X_1 \to X_2$ to denote the endomorphism φ, where X_1, X_2 are isomorphic to X. We also denote the $\mathbf{A}^1_*$-fibrations for X_1, X_2 by $\rho_1 : X_1 \to B_1$ and $\rho_2 : X_2 \to B_2$ respectively. Hence we have $\rho_2 \circ \varphi = \beta \circ \rho_1$. We denote by g the genus of the curve B. The next result is useful.

Lemma 3.1 *With the above notation and assumptions, let $m\Gamma$ be an irreducible fiber of ρ_2 lying over a point $p \in B_2$ with $m \ge 1$ and Γ reduced, and let $q \in B_1$ be a point such that $\beta(q) = p$. Suppose $\rho_1^*(q) = \ell\Delta$, where Δ is* *reduced and irreducible and ℓ is its multiplicity. If the ramification index for β at q is e then $\ell e = m$. In particular, if $m = 1$ then $\ell = e = 1$.*

Proof. Let $Q \in \Delta$ and $P \in \Gamma$ be smooth points such that $\varphi(Q) = P$. Let $\eta = 0$ and $\xi = 0$ be the local equations of Δ and Γ at Q and P in X_1 and X_2 respectively. Choose local parameters u and t for $q \in B_1$ and $p \in B_2$ respectively. Then $\beta^*(t) \sim u^e$, where $\sim$ means that t and u^e differ by an invertible element at q. We have $\rho_1^*(u) \sim \eta^\ell$ near Q and $\rho_2^*(t) \sim \xi^m$ near P. Since φ is étale at Q, $\eta \sim \varphi^*(\xi)$. Hence we obtain

$$\eta^{e\ell} \sim \rho_1^*(u)^e = \rho_1^*(u^e) \sim \rho_1^*\beta^*(t) = \varphi^*\rho_2^*(t) \sim \varphi^*(\xi^m) \sim \eta^m .$$

So, the equality $\ell e = m$ holds. Q.E.D.

Let $m_1\Gamma_1, \ldots, m_r\Gamma_r$ exhaust all multiple fibers of ρ, where we assume without loss of generality that $2 \leq m_1 \leq \cdots \leq m_r$. We denote by $\{r; m_1, \ldots, m_r\}$ the number of multiple fibers and the multiplicities. Let $p_i = \rho(\Gamma_i)$ for $1 \leq i \leq r$.

Theorem 3.2 *With the above notation and assumptions, suppose $r \geq 3$. Then the induced endomorphism $\beta : B_1 \to B_2$ is an automorphism except when B is isomorphic to $\mathbf{P}^1$ and $\{r; m_1, \ldots, m_r\}$ is one of the following collections:*

$$\{4; 2, 2, 2, 2\}, \{3; 2, 3, 6\}, \{3; 2, 4, 4\}, \{3; 3, 3, 3\} .$$

The proof consists of the subsequent lemmas 3.3 $\sim$ 3.6. Note that $\{3; m_1, m_2, m_3\}$ is one of the above collections if and only if

$$1 = \frac{1}{m_1} + \frac{1}{m_2} + \frac{1}{m_3} .$$

Lemma 3.3 *Suppose that either $r \geq 5$ or $r > 0$ and $g > 0$. Then $\beta : B_1 \to B_2$ is an automorphism. If $r = 4$ and $g = 0$ then β is an automorphism except for the case $\{4; 2, 2, 2, 2\}$.*

Proof. Let $d = \deg \beta$. By Lemma 3.1 the endomorphism β induces an endomorphism of the set $\{p_1, \ldots, p_r\}$. Let $q_1, \ldots, q_s$ exhaust all points of B_1 which are contained in $B_1 - \{p_1, \ldots, p_r\}$ and mapped to the points in

$\{p_1, \ldots, p_r\}$ by $\beta : B_1 \to B_2$. In fact, we have s_i points of them mapped to the point p_i and $s = s_1 + \cdots + s_r$. First of all, since β induces an étale finite morphism

$$\beta : B_1 - \{p_1, \ldots, p_r, q_1, \ldots, q_s\} \longrightarrow B_2 - \{p_1, \ldots, p_r\} ,$$

the comparison of Euler numbers gives rise to an equality

$$2 - 2g - (r + s) = d(2 - 2g - r) . \tag{1}$$

Hence we have

$$s = (d - 1)\{2g - 2 + r\} .$$

Let e_j be the ramification index of β at the point q_j. Since $e_j \geq 2$ by Lemma 3.1 we have the following inequality

$$2s + r \leq dr . \tag{2}$$

By (1) and (2) we obtain an inequality

$$2(d - 1)\{2g - 2 + r\} \leq (d - 1)r . \tag{3}$$

Suppose $d > 1$. Then $4g - 4 + r \leq 0$ by (3). This is impossible if either $r \geq 5$ or $r > 0$ and $g > 0$.

Suppose $r = 4, g = 0$ and $d > 1$. Then $4g - 4 + r = 0$. Hence the equalities occur in (2) and (3). Hence the ramification index at the point $p_i \in B_1$ is one for $1 \leq i \leq 4$ and $e_j = 2$ for $1 \leq j \leq s$. Since $s = 2(d-1) > 0$ it follows from Lemma 3.1 that $m_1 = 2$. Since $d \geq 2$ by the hypothesis we have $2s > d$, which implies that one of the q_i is mapped to a point in $\{p_2, p_3, p_4\}$. Hence we may assume that $m_2 = 2$. Suppose $m_3 > 2$. Then $s_3 = s_4 = 0$, and $2d \geq 2s$ which implies that $d = 2$ and $s = 2$. This easily leads to a contradiction. Hence $m_3 = 2$. Suppose $m_4 > 2$. Then $3d \geq 2s$, whence $d \leq 4$. If $d = 2$ then $s = 2$. It is easily seen that this is impossible. If $d = 3$ then $s = 4$. In this case, no two of q_1, q_2, q_3, q_4 are mapped to one of $\{p_1, p_2, p_3, p_4\}$. But this leads to a contradiction becasuse $s_4 = 0$. Hence $d = 4$ and $s = 6$. Then the six points $q_1, \ldots, q_6$ form three couples which are mapped to p_1, p_2, p_3. Then p_1, p_2, p_3, p_4 are mapped to p_4 and $d \geq 2 + 2 + 2 + 1 = 7$. This is a contradiction. Thus we have shown that if $d > 1$ then $\{r; m_1, \ldots, m_r\} = \{4; 2, 2, 2, 2\}$. Q.E.D.

In the subsequent arguments we assume that $g = 0$ and $r = 3$.

Lemma 3.4 *Suppose that β induces a bijection on the set $\{p_1, p_2, p_3\}$. Then β is an automorphism except when $\{m_1, m_2, m_3\}$ is one of* $\{2,3,6\}, \{2,4,4\}, \{3,3,3\}$.

Proof. We consider only the case $\beta(p_i) = p_i$ for $1 \leq i \leq 3$. The other cases are treated in the same fashion. We have

$$d = m_1 s_1 + 1 = m_2 s_2 + 1 = m_3 s_3 + 1 \ .$$

By the Riemann-Hurwitz formula, we have

$$-2 = -2d + (m_1 - 1)s_1 + (m_2 - 1)s_2 + (m_3 - 1)s_3 \ .$$

By plugging into this equality the data

$$s_1 = \frac{d-1}{m_1}, \ s_2 = \frac{d-1}{m_2}, \ s_3 = \frac{d-1}{m_3},$$

we obtain

$$2(d-1) = (d-1)\left\{3 - \frac{1}{m_1} - \frac{1}{m_2} - \frac{1}{m_3}\right\}.$$

Hence if $d > 1$ we have

$$1 = \frac{1}{m_1} + \frac{1}{m_2} + \frac{1}{m_3}.$$

Q.E.D.

Lemma 3.5 *Suppose that the image by β of $\{p_1, p_2, p_3\}$ consists of two points. Then β is an automorphism except when $\{m_1, m_2, m_3\}$ is one of* $\{2,3,6\}, \{2,4,4\}, \{3,3,3\}$.

Proof. We consider only the case $\beta(p_1) = \beta(p_2) = p_2$ and $\beta(p_3) = p_3$. The other cases are treated in the same fashion. The assumption implies that

$$d = m_1 s_1 = \frac{m_2}{m_1} + m_2 s_2 + 1 = m_3 s_3 + 1.$$

The Riemann-Hurwitz formula is then written as

$$-2 = -2d + (m_1 - 1)s_1 + \left(\frac{m_2}{m_1} - 1\right) + (m_2 - 1)s_2 + (m_3 - 1)s_3.$$

Since

$$s_1 = \frac{d}{m_1}, \; s_2 = \frac{d-1}{m_2} - \frac{1}{m_1}, \; s_3 = \frac{d-1}{m_3},$$

we obtain

$$2(d-1) = (d-1)\left\{3 - \frac{1}{m_1} - \frac{1}{m_2} - \frac{1}{m_3}\right\}.$$

Hence if $d > 1$ we have

$$1 = \frac{1}{m_1} + \frac{1}{m_2} + \frac{1}{m_3}.$$

Q.E.D.

Lemma 3.6 *Suppose that the image by β of $\{p_1, p_2, p_3\}$ consists of one point. Then β is an automorphism except when $\{m_1, m_2, m_3\}$ is one of $\{2,3,6\}, \{2,4,4\}, \{3,3,3\}$.*

Proof. Upto the renumbering of the points p_1, p_2, p_3 we may assume that $\beta(p_1) = \beta(p_2) = \beta(p_3) = p_3$. Then we have

$$d = m_1 s_1 = m_2 s_2 = \frac{m_3}{m_1} + \frac{m_3}{m_2} + m_3 s_3 + 1.$$

The Riemann-Hurwitz formula is written as

$$-2 = -2d + (m_1 - 1)s_1 + (m_2 - 1)s_2 + \left(\frac{m_3}{m_1} - 1\right) + \left(\frac{m_3}{m_2} - 1\right) + (m_3 - 1)s_3.$$

Then we obtain also an equality

$$2(d-1) = (d-1)\left\{3 - \frac{1}{m_1} - \frac{1}{m_2} - \frac{1}{m_3}\right\}.$$

Hence if $d > 1$ we have

$$1 = \frac{1}{m_1} + \frac{1}{m_2} + \frac{1}{m_3}.$$

Q.E.D.

Theorem 3.2 implies the next result.

Corollary 3.7 *Let $\rho : X \to B$ be a Platonic $\mathbf{A}^1_*$-fiber space. Suppose that an étale endomorphism φ of X preserves the $\mathbf{A}^1_*$-fibration ρ. Let $\beta : B \to B$ be the induced endomorphism. Then β is an automorphism. Hence φ is an automorphism by Lemma 2.3.*

4 Weighted hypersurfaces

In this section we consider a weighted hypersurface Σ in $\mathbf{A}^3$ defined by

$$x_1^{m_1} + x_2^{m_2} + x_3^{m_3} = 0,$$

where m_1, m_2, m_3 are integers larger than 1 and $\gcd(m_1, m_2, m_3) = 1$. Let d be the least common multiple of m_1, m_2, m_3 and let $q_i = d/m_i$ for $i = 1, 2, 3$. Then the one-dimensional algebraic torus G_m acts on X by

$$t \cdot (x_1, x_2, x_3) = (t^{q_1}x_1, t^{q_2}x_2, t^{q_3}x_3)$$

for $t \in G_m$. Let $X = \Sigma - (0, 0, 0)$, let $B = X/G_m$ and let $\pi : X \to B$ be the quotient morphism. The following result is known (cf. [10]), where we abbreviate $\gcd(m, n)$ as (m, n) :

Lemma 4.1 *The curve B is a nonsingular projective curve of genus given by*

$$g = \frac{d^2}{2q_1q_2q_3} - \frac{d}{2}\left\{\frac{(q_1, q_2)}{q_1q_2} + \frac{(q_2, q_3)}{q_2q_3} + \frac{(q_3, q_1)}{q_3q_1}\right\} + 1 \ .$$

The morphism $\pi : X \to B$ is an $\mathbf{A}^1_*$-fibration whose fibers are all isomorphic to $\mathbf{A}^1_*$ if taken with the reduced structure. If $(q_1, q_2) > 1$ (resp. $(q_2, q_3) > 1$, $(q_3, q_1) > 1$) there are $d(q_1, q_2)/q_1q_2$ (resp. $d(q_2, q_3)/q_2q_3, d(q_3, q_1)/q_3q_1$) multiple fibers of multiplicity (q_1, q_2) (resp. $(q_2, q_3), (q_3, q_1)$). Except for these fibers there are no multiple fibers.

We consider a special case where m_1, m_2, m_3 are pairwise coprime. Then $d = m_1m_2m_3, q_1 = m_2m_3, q_2 = m_1m_3, q_3 = m_1m_2$. Hence Lemma 4.1 implies the next result:

Corollary 4.2 *Suppose m_1, m_2, m_3 are pairwise coprime. Then the following assertions hold:*

(1) *The curve B is a rational curve.*

(2) *The $\mathbf{A}^1_*$-fibration $\pi : X \to B$ has three multiple fibers of respective multiplicities m_1, m_2, m_3.*

We consider the case where m_1, m_2, m_3 are pairwise coprime. The next result is due to Bundagaard-Nielsen [1] and Fox [2] (cf. [9, Theorem 1.2.15]).

Lemma 4.3 *Let B be a smooth projective curve of genus g. Let $r \geq 1$ and let $p_1, p_2, \ldots, p_r$ be distinct points in B and $m_1, m_2, \ldots, m_r$ arbitrary integers ≥ 2. Then there exists a finite Galois covering $\tau : \widetilde{B} \to B$ such that the ramification index at any point over p_i is m_i for $i = 1, 2, \ldots, r$ except when* (i) *$g = 0$ and $r = 1$ and* (ii) *$g = 0, r = 2$ and $m_1 \neq m_2$. If $B \cong \mathbf{P}^1, r = 2$ and* $\gcd(m_1, m_2) = d > 1$ *then we can construct a finite cyclic covering $\tau : \widetilde{B} \to \mathbf{P}^1$ with the ramification index d at p_1, p_2.*

Corollary 4.4 *Suppose $g = 0$ and $r = 3$. Then $\widetilde{B}$ in Lemma 4.3 has genus* 0 *(resp.* 1*) if and only if* $\{m_1, m_2, m_3\} = \{2, 2, n\}$ $(n \geq 2)$, $\{2, 3, 3\}$, $\{2, 3, 4\}$, $\{2, 3, 5\}$ *(resp.* $\{m_1, m_2, m_3\} = \{2, 3, 6\}, \{2, 4, 4\}, \{3, 3, 3\}$*), where we assume $m_1 \leq m_2 \leq m_3$. The genus of $\widetilde{B}$ is greater than* 1 *otherwise.*

Proof. Let d be the degree of $\tau : \widetilde{B} \to \mathbf{P}^1$ and let $\tilde{g}$ be the genus of $\widetilde{B}$. Let s_i be the number of points on $\widetilde{B}$ lying over the point p_i for $i = 1, 2, 3$. Then $d = m_1 s_1 = m_2 s_2 = m_3 s_3$. By the Riemann-Hurwitz formula, we have

$$2\tilde{g} - 2 = -2d + (m_1 - 1)s_1 + (m_2 - 1)s_2 + (m_3 - 1)s_3 \ .$$

Hence we have

$$\tilde{g} = \frac{d}{2}\left\{1 - \left(\frac{1}{m_1} + \frac{1}{m_2} + \frac{1}{m_3}\right)\right\} + 1 \ .$$

This implies $\tilde{g} = 0$ (resp. $1, > 1$) if and only if

$$1 - \left(\frac{1}{m_1} + \frac{1}{m_2} + \frac{1}{m_3}\right) < 0 \quad (\text{resp.} \ = 0, > 0) \ .$$

This yields the stated result. Q.E.D.

We consider an étale endomorphism $\varphi : X \to X$ when m_1, m_2, m_3 are pairwise coprime integers such that

$$\frac{1}{m_1} + \frac{1}{m_2} + \frac{1}{m_3} < 1 \tag{1}$$

Lemma 4.5 *Under the assumption* (1) *above, X has log Kodaira dimension one. If $\varphi : X \to X$ is an étale endomorphism then φ preserves the $\mathbf{A}^1_*$-fibration. Namely, there exists an endomorphism $\beta : \mathbf{P}^1 \to \mathbf{P}^1$ such that $\pi \circ \varphi = \beta \circ \pi$.*

Proof. Let $\widetilde{X}$ be the normalization of a fiber product $X \times_B \widetilde{B}$ and let $q : \widetilde{X} \to X$ be the natural finite morphism. Then the $\mathbf{A}^1_*$-fibration $\pi : X \to B$ induces an $\mathbf{A}^1_*$-fibration $\widetilde{\pi} : \widetilde{X} \to \widetilde{B}$. Since $\widetilde{B}$ has genus > 1 by the assumption (1), the addition formula of log Kodaira dimension yields

$$\overline{\kappa}(\widetilde{X}) \geq \overline{\kappa}(\widetilde{B}) + \overline{\kappa}(\mathbf{A}^1_*) = 1 \ .$$

On the other hand, since $q : \widetilde{X} \to X$ is an étale finite morphism (cf. Lemma 4.7 below), we have $\overline{\kappa}(\widetilde{X}) = \overline{\kappa}(X)$ and $\overline{\kappa}(X) \leq 1$ by the weak addition formula. Hence $\overline{\kappa}(\widetilde{X}) = \overline{\kappa}(X) = 1$.

The second assertion follows from [3, Lemma 2.4]. Q.E.D.

Corollary 4.6 *With the notations and assumptions in Lemma 4.5, the endomorphism β is the identity.*

Proof. We make use of Lemma 3.1. Let $m_i\Gamma_i$ $(i = 1, 2, 3)$ be the multiple fibers of $\pi : X \to B$, where $\Gamma_i \cong \mathbf{A}^1_*$. Since m_1, m_2, m_3 are pairwise coprime, $\varphi : X \to X$ maps the fiber $m_i\Gamma_i$ to itself. Furthermore, assume that there are s_i smooth fibers of π mapped to the fiber $m_i\Gamma_i$. Let those fibers lie over the points $q_i^{(j)}$ of B, where $1 \leq j \leq s_i$. Then $\beta(q_i^{(j)}) = p_i$. The ramification index of β at $q_i^{(j)}$ is m_i. By the Riemann-Hurwitz formula, we have

$$-2 = -2d + (m_1 - 1)s_1 + (m_2 - 1)s_2 + (m_3 - 1)s_3 \ ,$$

where $d := \deg \beta = 1 + m_1 s_1 = 1 + m_2 s_2 = 1 + m_3 s_3$. Since m_1, m_2, m_3 are pairwise coprime integers, we can write $s_1 = q_1 s, s_2 = q_2 s, s_3 = q_3 s$ for some integer s. From these relations one can deduce

$$m_1 m_2 m_3 s = (m_1 m_2 + m_2 m_3 + m_3 m_1)s \ ,$$

where $m_1 m_2 m_3 \neq m_1 m_2 + m_2 m_3 + m_3 m_1$. So, $s = 0$ and $d = 1$. Namely β is an automorphism. Meanwhile, since β fixes three points p_1, p_2, p_3 it is the identity. Q.E.D.

Before going to make further arguments we need the following result:

Lemma 4.7 *Let $\rho : X \to B$ be an $\mathbf{A}^1_*$-fibration as given in the introduction. Let $m_1\Gamma_1, \ldots, m_r\Gamma_r$ exhaust all multiple fibers of ρ and let $p_i = \rho(\Gamma_i)$. Suppose $r \geq 3$. Let $\tau : \widetilde{B} \to B$ be a finite Galois covering which ramifies over the* *points $p_1, \dots, p_r$ with respective multiplicities $m_1, \dots, m_r$ (cf. Lemma 4.3). Let $\widetilde{X}$ be the normalization of the fiber product $X \times_B \widetilde{B}$, let $\widetilde{\rho} : \widetilde{X} \to \widetilde{B}$ be the $\mathbf{A}^1_*$-fibration induced by ρ and let $\mu : \widetilde{X} \to X$ be a composite of the normalization morphism $\nu : \widetilde{X} \to X \times_B \widetilde{B}$ and the projection $X \times_B \widetilde{B} \to X$. Then the following assertions hold:*

(1) *The $\mathbf{A}^1_*$-fibration $\widetilde{\rho} : \widetilde{X} \to \widetilde{B}$ is an $\mathbf{A}^1_*$-fiber bundle over $\widetilde{B}$. In particular, there are no multiple fibers in $\widetilde{\rho}$.*

(2) *The morphism $\mu : \widetilde{X} \to X$ is a finite étale morphism.*

Proof. Let p be a point of B. Suppose first that p is none of $p_1, \dots, p_r$. Then τ is unramified over p. So, if q is a point of $\widetilde{B}$ lying over p then $\widetilde{B}$ and B are (analytically) locally isomorphic to each other near q and p. Hence the fibers $\widetilde{\rho}^{-1}(q)$ and $\rho^{-1}(p)$ are isomorphic to each other. Suppose p coincides with one of $p_1, \dots, p_r$, say $p = p_1$. Write $m = m_1$. Let t be a local parameter of B at p. Let P be a point on $\Gamma = \Gamma_1$. Then there exists a local system of parameters $\{\xi, \eta\}$ at P such that $\xi = 0$ defines locally the curve Γ, $t = a\xi^m$ and η is a fiber coordinate of Γ, i.e., a local parameter of the curve Γ. Here a is a regular function such that $a(P) \neq 0$. Then we can write analytically $a = b^m$ with b holomorphic and nonzero near P. Replacing ξ by $b\xi$, we can write $t = \xi^m$. Let q be a point of $\widetilde{B}$ lying above p and let u be a local parameter at q. Since the ramification index of τ is m at q, we may assume that $t = u^m$. Then the fiber product $X \times_B \widetilde{B}$ is identified near the point (q, P) with a hypersurface $u^m = \xi^m$ in the (u, ξ, η)-space. This hypersurface is singular because it is isomorphic to a union of m-planes meeting along the line $u = \xi = 0$. Taking the normalization of the fiber product $X \times_B \widetilde{B}$ is equivalent to separating these m-planes to disjoint m-planes. Recall here that we are looking an analytic local chart. Anyway, the fiber of $\widetilde{\rho}$ over q consists of m-copies of a curve locally given by $u = 0$ in the (u, η)-plane. Hence it is smooth and has multiplicity one. Furthermore, the morphism $\mu : \widetilde{X} \to X$ is étale, while μ is finite because the normalization morphism ν is finite and $\tau : \widetilde{B} \to B$ is finite. In order to show that $\widetilde{\rho} : \widetilde{X} \to \widetilde{B}$ is an $\mathbf{A}^1_*$-fiber bundle it suffices to show that the fiber $\widetilde{\rho}^{-1}(q)$ is irreducible.

For this purpose we embed the given $\mathbf{A}^1_*$-fibration $\rho : X \to B$ into a $\mathbf{P}^1$-fibration $\pi : V \to B$ in such a way that the restriction of π onto the open set X is ρ and that if G_i is the fiber of π containing the multiple fiber $m_i\Gamma_i$ the dual graph of G_i is a linear chain and the closure $\overline{\Gamma}$ of Γ in V is a unique

(-1) curve in G_i, where V is a smooth projective surface. Then there exist two cross-sections S_0, S_1 of π lying in $V - X$. Let $\widetilde{V}$ be the normalization of the fiber product $V \times_B \widetilde{B}$. Then $\widetilde{V}$ is a normal projective surface with the induced $\mathbf{P}^1$-fibration $\widetilde{\pi} : \widetilde{V} \to \widetilde{B}$ such that $\widetilde{X}$ is an open set of $\widetilde{V}$ and $\widetilde{\pi}|_{\widetilde{X}} = \widetilde{\rho}$. Let $h : \widehat{V} \to \widetilde{V}$ be a minimal resolution of singularities of $\widetilde{V}$. Then there is an induced $\mathbf{P}^1$-fibration $\widehat{\pi} : \widehat{V} \to B$ and every connected component of the exceptional locus of h is contained in a fiber of $\widehat{\pi}$. It is known that every singular point obtained by contracting a connected union of irreducible components of a reducible fiber of a $\mathbf{P}^1$-fibration is a rational singularity. The surface $\widetilde{X}$ is an open set of $\widehat{V}$ and the proper transforms $\widehat{S}_0, \widehat{S}_1$ of S_0, S_1 by h are the cross-sections of $\widehat{\pi}$.

Let $\widetilde{\Gamma} = \widetilde{\rho}^{-1}(q)$ with the above notation. Since $\mu : \widetilde{\Gamma} \to \Gamma$ is a finite étale morphism, $\widetilde{\Gamma}$ is a disjoint union of $\Delta_1, \ldots, \Delta_n$, where $\Delta_i \cong \mathbf{A}^1_*$ for each i. We have only to show that $n = 1$. Let $\overline{\Delta}_i$ be the closure of Δ_i in $\widehat{V}$. Then $\overline{\Delta}_1, \ldots, \overline{\Delta}_n$ are contained in the fiber $\overline{G} = \widehat{\pi}^{-1}(q)$. These components together with the components in $\overline{G} \cap (\widehat{V} - \widetilde{X})$ are linked to the cross-sections $\widehat{S}_0, \widehat{S}_1$. Then it is easy to see that the dual graph of $\overline{G}$ should contain loops provided $n > 1$. But this is impossible. Hence $n = 1$. Q.E.D.

Now we shall come back to the situation before Lemma 4.7 and look for a smooth completion of $\widetilde{X}$ which is the normalization of $X \times_B \widetilde{B}$. Note that every fiber of the $\mathbf{A}^1_*$-fibration $\widetilde{\pi} : \widetilde{X} \to \widetilde{B}$ is smooth and irreducible. Hence there is a $\mathbf{P}^1$-bundle $\rho : V \to \widetilde{B}$ such that $\widetilde{X} = V - (S_0 \cup S_1)$, where S_0, S_1 are the cross-sections of ρ and that $\widetilde{\pi}$ is the restriction of ρ onto $\widetilde{X}$. Since S_0, S_1 are disjoint from each other, the $\mathbf{P}^1$-bundle V is written as $\mathbf{P}(\mathcal{S}) := \mathrm{Proj}\,(\mathcal{S})$, where $\mathcal{S}$ is the symmetric $\mathcal{O}_{\widetilde{B}}$-algebra generated by a rank 2 locally free module $\mathcal{O}_{\widetilde{B}} \oplus \mathcal{L}$ with an invertible sheaf $\mathcal{L}$. Let $\varphi : X \to X$ be an étale endomorphism. Then φ is a B-morphism by Corollary 4.6. Hence φ lifts to an étale endomorphism $\widetilde{\varphi} : \widetilde{X} \to \widetilde{X}$ which is a $\widetilde{B}$-morphism. Let $\widetilde{K}$ be the function field of $\widetilde{B}$ and let $\widetilde{X}_{\widetilde{K}}$ be the generic fiber of $\widetilde{\pi}$. If K is the function field of B and X_K is the generic fiber of π then $\widetilde{X}_{\widetilde{K}} = X_K \otimes_K \widetilde{K}$. Hence we may write $\widetilde{X}_{\widetilde{K}} = X_{\widetilde{K}}$. By the above construction, $X_{\widetilde{K}} \cong \mathbf{A}^1_{*,\widetilde{K}} := \mathrm{Spec}\,\widetilde{K}[t, t^{-1}]$. Furthermore, $X_{\widetilde{K}} = \mathbf{P}^1_{\widetilde{K}} - \{S_{0,\widetilde{K}}, S_{1,\widetilde{K}}\}$, where $S_{0,\widetilde{K}}, S_{1,\widetilde{K}}$ are the generic points of S_0, S_1. The $\widetilde{B}$-morphism $\widetilde{\varphi}$ induces a $\widetilde{K}$-endomorphism $\widetilde{\varphi}_{\widetilde{K}} : X_{\widetilde{K}} \to X_{\widetilde{K}}$. Replacing $\widetilde{\varphi}$ by $\widetilde{\varphi}^2$ if necessary, we may assume that $\widetilde{\varphi}_{\widetilde{K}}$ extends to an endomorphism from $\mathbf{P}^1_{\widetilde{K}}$ into itself which ramifies totally over the points $S_{0,\widetilde{K}}, S_{1,\widetilde{K}}$. We denote this endomorphism by the same letter $\widetilde{\varphi}_{\widetilde{K}}$. This implies that there is a rational mapping $\widetilde{\varphi} : V \to V$ which is regular except for finitely many points lying on the cross-sections S_0, S_1. Then $\widetilde{\varphi}$ is an endomorphism, that is to say, $\widetilde{\varphi}$ is regular everywhere. Furthermore, $\widetilde{\varphi} : V \to V$ is a finite morphism and totally ramifying over S_0, S_1. It then follows that $\widetilde{\varphi}^*(S_0) = NS_0$, where N is the degree of $\widetilde{\varphi}$. We may assume that the cross-section S_0 contracts to the point $(0,0,0)$ on $\overline{\Sigma}$ by a morphism $V \to \overline{\Sigma}$ which is induced by the composite $\widetilde{X} \to X \times_B \widetilde{B} \to X \hookrightarrow \Sigma$, where $\overline{\Sigma}$ is a suitable projective completion of Σ. Hence $(S_0^2) < 0$. We have $N(S_0^2) = (S_0^2)$ by the projection formula. So, $N = 1$. Namely, $\widetilde{\varphi}$ is an automorphism. Hence $\varphi : X \to X$ is also an automorphism. If m_1, m_2, m_3 are pairwise coprime integers larger than one and if $\{m_1, m_2, m_3\} \neq \{2, 3, 5\}$ then

$$\frac{1}{m_1} + \frac{1}{m_2} + \frac{1}{m_3} < 1.$$

Hence, in view of Lemma 4.5 and Corollary 4.6, we have shown the following result.

Theorem 4.8 *Let Σ be an affine hypersurface in $\mathbf{A}^3$ defined by $x_1^{m_1} + x_2^{m_2} + x_3^{m_3} = 0$, where m_1, m_2, m_3 are pairwise coprime positive integers larger than one. Let $X = \Sigma - (0,0,0)$. Then any étale endomorphism $\varphi : X \to X$ is an automorphism possibly except for the case $\{m_1, m_2, m_3\} = \{2, 3, 5\}$.*

5 Algebraic surfaces with G_m-action

Let Σ be anew an affine normal algebraic surface defined over $\mathbf{C}$. We suppose that Σ has an unmixed algebraic action of the one-dimensional torus G_m satisfying the following conditions:

(1) Σ has a unique fixpoint, say O. The point O is in the closure of every orbit.

(2) Let $X := \Sigma - \{O\}$. Then the algebraic quotient $B := X/G_m$ is a smooth projective curve. We denote its genus by g.

Let $\pi : X \to B$ be the quotient morphism. Then X is a smooth surface and π gives rise to an $\mathbf{A}^1_*$-fibration on X. By abuse of notations, we use π

in this section to denote the $\mathbf{A}^1_*$-fibration induced by the quotient morphism $X \to X/G_m$.

Lemma 5.1 *If $g > 0$ then an étale endmorphsim $\varphi : X \to X$ preserves the $\mathbf{A}^1_*$-fibration π. Hence there exists an endomorphism $\beta : B \to B$ such that $\pi \circ \varphi = \beta \circ \pi$. If either $g > 1$ or $g = 1$ and π has a multiple fiber then β is an automorphism.*

Proof. Let F be a fiber of π. Since $F_{\text{red}} \cong \mathbf{A}^1_*$, $(\pi \circ \varphi)(F_{\text{red}})$ is a point of B. Hence $\varphi(F_{red})$ is contained in a fiber of π. Namely φ preserves the $\mathbf{A}^1_*$-fibration. If $g > 1$ the Riemann-Hurwitz formula shows that β is an automorphism. Suppose $g = 1$. Let $m_1\Gamma_1, \ldots, m_r\Gamma_r$ exhaust all multiple fibers of π, where $\Gamma_i \cong \mathbf{A}^1_*$ and $m_1 \leq \cdots \leq m_r$. Let $p_i = \pi(\Gamma_i)$. By the hypothesis, $r \geq 1$. Suppose $d := \deg \beta > 1$. If $\beta(p) = p_1$, it follows from Lemma 3.1 that either $p = p_i$ with $m_i = m_1$ or p is a ramification point of index m_1. Since β is surjective, one can show easily that there exists a ramification point for β. Then the Riemann-Hurwitz formula gives a contradiction. Q.E.D.

Note that any automorphism β of a smooth projective curve of genus g is of finite order if either $g > 1$ or $g = 1$ and β fixes one point.

Lemma 5.2 *Let $\pi : X \to B$ be as in Lemma 5.1. Let $\varphi : X \to X$ be an étale endomorphism. If either $g > 1$ or $g = 1$ and π has a multiple fiber then φ is an automorphism.*

Proof. Let β be the automorphism of B induced by φ. Then β permutes the points over which lie the multiple fibers of π. Hence, replacing β by its iteration, we may assume that β fixes those points on B. Then β has finite order, say n. Now, replacing φ by φ^n, we may and shall assume that β is the identity. Now the argument proceeds as in the last part of section 4. Let $m_1\Gamma_1, \ldots, m_r\Gamma_r$ exhaust all multiple fibers of π and let $p_i = \pi(\Gamma_i)$. Let $\widetilde{B}$ be a finite Galois covering of B ramified over the points p_i with ramification indices m_i. Such a covering exists by Lemma 4.3. Let $\widetilde{X}$ be the normalization of $X \times_B \widetilde{B}$. Then $\widetilde{X}$ together with the induced $\mathbf{A}^1_*$-fibration $\widetilde{\pi} : \widetilde{X} \to \widetilde{B}$ is embedded into a $\mathbf{P}^1$-bundle $\rho : V \to \widetilde{B}$ in such a way that $\widetilde{X} = V - (S_0 \cup S_1)$ with disjoint cross-sections S_0, S_1 and $\widetilde{\pi} = \rho|_{\widetilde{X}}$. Here we may assume that $(S_0^2) < 0$. On the other hand, the given endomorphism $\varphi : X \to X$

extends to an étale endomorphism $\widetilde{\varphi} : \widetilde{X} \to \widetilde{X}$, which also extends to an endomorphism $\widetilde{\varphi} : V \to V$. Then $\widetilde{\varphi}$ is totally ramifying over S_0. Hence we have $(S_0^2) = N(S_0^2)$ with $N := \deg \widetilde{\varphi}$. Hence $N = 1$ and this shows that φ is an automorphism. Q.E.D.

Lemma 5.3 *Let $\pi : X \to B$ be as in Lemma 5.1. Assume that B is an elliptic curve and that π has no multiple fibers. Then there exists a nontrivial étale endomorphism $\varphi : X \to X$ if and only if there exist an endomorphism β of B, an invertible sheaf $\mathcal{L}$ on B of positive degree and a positive integer $d > 1$ satisfying the condition: $\mathcal{L}^{\otimes d} \cong \beta^*(\mathcal{L})$. We have necessarily $d = \deg \beta$ and $\deg \varphi = d^2$.*

Proof. There exists an invertible sheaf $\mathcal{L}$ such that $X = \mathbf{P}(\mathcal{O}_B \oplus \mathcal{L}) - (S_0 \cup S_1)$, where S_0, S_1 are the cross-sections of the $\mathbf{P}^1$-bundle. The invertible sheaf $\mathcal{L}$ is determined uniquely up to isomorphisms by the $\mathbf{A}^1_*$-bundle X. We may assume that $(S_0^2) < 0$. Then S_0 (resp. S_1) corresponds to the projection from $\mathcal{O}_B \oplus \mathcal{L}$ to $\mathcal{O}_B$ (resp. $\mathcal{L}$). Suppose there exists an étale endomorphism $\varphi : X \to X$. Then $\pi \circ \varphi = \beta \circ \pi$ for an endomorphism β of B. Let now Y be a fiber product $X \times_{(B,\beta)} B$ and $\pi_Y : Y \to B$ be the induced $\mathbf{A}^1_*$-fibration. Then $Y = \mathbf{P}(\mathcal{O}_B \oplus \beta^*(\mathcal{L})) - (S_{0,Y} \cup S_{1,Y})$, where $S_{0,Y} = S_0 \times_{(B,\beta)} B$ and $S_{1,Y} = S_1 \times_{(B,\beta)} B$. Let $\beta_Y : Y \to X$ be the base change of β. Then φ splits as a composite $\varphi = \beta_Y \circ \psi$, where $\psi : X \to Y$ is an étale B-morphism. Then ψ extends to a B-morphism of $\mathbf{P}^1$-bundles $\overline{\psi} : \mathbf{P}(\mathcal{O}_B \oplus \mathcal{L}) \to \mathbf{P}(\mathcal{O}_B \oplus \beta^*(\mathcal{L}))$ such that $\overline{\psi}^{-1}(S_{0,Y}) = S_0$ and $\overline{\psi}^{-1}(S_{1,Y}) = S_1$. Take an affine open covering $\mathcal{U} = \{U_i\}_{i \in I}$ of B such that $\mathcal{L} \mid_{U_i} = \mathcal{O}_{U_i} e_i$ for every $i \in I$. Write $\mathcal{O}_B = \mathcal{O}_B e$. Then $\pi^{-1}(U_i) = \mathrm{Spec}\, A_i[z_i, z_i^{-1}]$, where $z_i = e_i/e$ and $A_i = \Gamma(U_i, \mathcal{O}_B)$. Let $\{a_{ij}\}$ be the transition functions of $\mathcal{L}$ with respect to $\mathcal{U}$. Then $\psi \mid_{U_i} : \pi^{-1}(U_i) \to \pi_Y^{-1}(U_i)$ is given by $\beta^*(z_i) = \lambda_i z_i^d$ with $\lambda_i \in A_i^*$ and $d := \deg \psi$. Then it is easy to deduce a relation $\beta^*(a_{ji}) = a_{ji}^d \lambda_j \lambda_i^{-1}$ for $i, j \in I$. This implies that $\mathcal{L}^{\otimes d} \cong \beta^*(\mathcal{L})$. Since $\deg \mathcal{L}^{\otimes d} = d \deg \mathcal{L}$ and $\deg \beta^*(\mathcal{L}) = \deg \beta \deg \mathcal{L}$, we have necessarily $d = \deg \beta$.

Conversely, if we are given $\beta, \mathcal{L}, d$ as above, we can construct an endomorphism $\varphi : X \to X$ as a composite of β_Y and ψ, where ψ is a d-th power morphism on the generic fibers. This d-th power morphism is extended to a B-morphism $\psi : X \to Y$ because of the condition $\mathcal{L}^{\otimes d} \cong \beta^*(\mathcal{L})$. Since $\varphi = \beta_Y \circ \psi$, $\deg \beta_Y = \deg \beta$ and $\deg \psi = d$, we have $\deg \varphi = d^2$. Q.E.D.

Lemma 5.4 *For any positive integer $n > 1$, let n_B denote the multiplication by n endomorphism of an elliptic curve B when B is viewed as an abelian variety. Then there exists an invertible sheaf $\mathcal{L}$ on B such that* $\deg \mathcal{L} > 0$ *and* $\mathcal{L}^{\otimes n^2} \cong n_B^*(\mathcal{L})$. *For this invertible sheaf $\mathcal{L}$, let $V := \mathbf{P}(\mathcal{O}_B \oplus \mathcal{L})$ and let S_0 (resp. S_1) be a cross-section such that $\mathcal{O}_{S_0}(S_0) \cong \mathcal{L}^{-1}$ (resp. $\mathcal{O}_{S_1}(S_1) \cong \mathcal{L}$). Then the cross-section S_0 contracts to an elliptic singularity O on a projective normal surface $\overline{V}$. Let $\Sigma := \overline{V} - S_1$. Then there exists a G_m-action on Σ such that the point O is a unique fixpoint, that every orbit is contained in a fiber of the $\mathbf{P}^1$-bundle projection $\rho : V \to B$ and that the closure of every orbit contains the point O.*

Proof. Take any invertible sheaf $\mathcal{L}$ on B with $\deg \mathcal{L} > 0$. Then, by [8, (iv),page 75], $n_B^*(\mathcal{L}) \cong \mathcal{L}^{\otimes n^2} \otimes \mathcal{M}$ with $\mathcal{M} \in \mathrm{Pic}^0(B)$. By [8, (iii), page 75], we have $n_B^*(\mathcal{N}) \cong \mathcal{N}^{\otimes n}$ for any $\mathcal{N} \in \mathrm{Pic}^0(B)$. Set $\mathcal{L}' := \mathcal{L} \otimes \mathcal{N}$. Then we compute

$$n_B^*(\mathcal{L}') = n_B^*(\mathcal{L} \otimes \mathcal{N}) \cong \mathcal{L}^{\otimes n^2} \otimes \mathcal{M} \otimes \mathcal{N}^{\otimes n} .$$

Hence, if we take $\mathcal{N}$ so that $\mathcal{M} \otimes \mathcal{N}^{\otimes n} \cong \mathcal{N}^{\otimes n^2}$, we have $n_B^*(\mathcal{L}') \cong \mathcal{L}'^{\otimes n^2}$. Meanwhile, since $\mathrm{Pic}^0(B)$ is a divisible group there exists $\mathcal{N} \in \mathrm{Pic}^0(B)$ such that $\mathcal{N}^{\otimes(n^2-n)} \cong \mathcal{M}$. Then we replace $\mathcal{L}$ by $\mathcal{L}'$. The second assertion is well-known. Q.E.D.

Now we shall consider the case where B is a smooth rational curve. Let $m_1\Gamma_1, \ldots, m_r\Gamma_r$ exhaust the multiple fibers of the $\mathbf{A}^1_*$-fibration $\pi : X \to B$. We denote by $\{r; m_1, m_2, \ldots, m_r\}$ the number of multiple fibers and the multiplicities with the condition $m_1 \le m_2 \le \ldots \le m_r$. Then we have the following:

Lemma 5.5 *With the above notations and assumptions, we have $\overline{\kappa}(X) = 1$ if $r \ge 3$ and except for the following cases :*

(i) $\{4; 2, 2, 2, 2\}$, $\{3; 2, 3, 6\}$, $\{3; 2, 4, 4\}$, $\{3; 3, 3, 3\}$

(ii) $\{3; 2, 2, n\}$ $(n \ge 2)$, $\{3; 2, 3, 3\}$, $\{3; 2, 3, 4\}$, $\{3; 2, 3, 5\}$.

Hence, in the case $\overline{\kappa}(X) = 1$, any étale endomorphism φ of X preserves the $\mathbf{A}^1_*$-fibration π.

Proof. By Lemma 4.3, there exists a cyclic Galois covering $\widetilde{B} \to B$ ramifying over the points $p_i := \pi(\Gamma_i)$ with index m_i. Let d be the degree of the covering. Then the genus g of $\widetilde{B}$ is given by

$$g = d\left\{(r-2) - \left(\frac{1}{m_1} + \ldots + \frac{1}{m_r}\right)\right\} + 1 \ .$$

Hence, if either $r \geq 5$ or except for the above-listed cases, $g > 1$. Let Y be the normalization of the fiber product $X \times_B \widetilde{B}$. Then, by argueing in the same fashion as in the proof of Lemma 4.5, we know that $\overline{\kappa}(Y) = \overline{\kappa}(X) = 1$. The preservation of the $\mathbf{A}^1_*$-fibration follows from [3, Lemma 2.4]. Q.E.D.

Lemma 5.6 *In the cases, which we will call the quasi-elliptic case,*

$$\{r; m_1, \ldots, m_r\} = \{4; 2,2,2,2\}, \{3; 2,3,6\}, \{3; 2,4,4\}, \{3; 3,3,3\},$$

we have $\overline{\kappa}(X) = 0$. In the cases, which we will call the Platonic case,

$$\{3; 2, 2, n\}\ (n \geq 2),\ \ \{3; 2, 3, 3\},\ \ \{3; 2, 3, 4\},\ \ \{3; 2, 3, 5\},$$

we have $\overline{\kappa}(X) = -\infty$.

Proof. In the first cases, there is a finite Galois covering $\widetilde{B} \to B = \mathbf{P}^1$ with ramification indices $m_1, \ldots, m_r$ such that the normalization $\widetilde{X}$ of the fiber product $X \times_B \widetilde{B}$ is an $\mathbf{A}^1_*$-bundle over an elliptic curve $\widetilde{B}$. It is easy to show that $\overline{\kappa}(\widetilde{X}) = 0$. Since the canonical morphism $\widetilde{X} \to X$ is a finite étale covering, we have $\overline{\kappa}(X) = 0$. In the second cases, X is a Platonic $\mathbf{A}^1_*$-fiber space. Hence $\overline{\kappa}(X) = -\infty$. Q.E.D.

Lemma 5.7 *In the quasi-elliptic case, any étale endomorphism φ of X preserves the $\mathbf{A}^1_*$-fibration $\pi : X \to B$.*

Proof. We denote $\varphi : X_1 \to X_2$ to distinguish the upper and lower X. Similarly, $\pi_i : X_i \to B_i$ $(1 = 1, 2)$ denotes the same $\pi : X \to B$, where $B_1 = B_2 = B$. Let (V_2, D_2) be a smooth completion of X_2 with a simple normal crossing divisor D_2 and let $\rho_2 : V_2 \to B_2$ be the $\mathbf{P}^1$-fibration on V_2 which extends π_2. Similarly, we take a smooth completion (V_1, D_1) of X_1 and a $\mathbf{P}^1$-fibration $\rho_1 : V_1 \to B_1$ extending π_1. Furthermore, we may assume that there exists a morphism $\Phi : V_1 \to V_2$ such that $\Phi\mid_{X_1} = \varphi$. Then the logarithmic ramification divisor formula implies that there exists an effective divisor R_Φ such that

$$D_1 + K_{V_1} = \Phi^*(D_2 + K_{V_2}) + R_\Phi$$

and that R_Φ contains the ramification locus of the morphism Φ.

Now let ℓ be a general fiber of the $\mathbf{P}^1$-fibration ρ_1. Then $(D_1 + K_{V_1} \cdot \ell) = 0$. Since $|\ell|$ is a linear pencil and since $\overline{\kappa}(X) = 0$ implies $|n(D_2 + K_{V_2})| \neq \emptyset$ for some $n > 0$, we have $(\Phi^*(D_2 + K_{V_2}) \cdot \ell) \geq 0$ and $(R_\Phi \cdot \ell) \geq 0$. Hence we must have

$$(\Phi^*(D_2 + K_{V_2}) \cdot \ell) = (R_\Phi \cdot \ell) = 0 \ .$$

This, in particular, implies that ℓ does not meet the ramification locus of the morphism Φ. It then follows that $\Phi(\ell)$ is a nonsingular rational curve with $(\Phi(\ell)^2) = 0$. Since $(D_2 + K_{V_2} \cdot \Phi(\ell)) = 0$ we have $(D_2 \cdot \Phi(\ell)) = 2$. Hence $\Phi(\ell) \cap X_2 = \varphi(F) \cong \mathbf{A}^1_*$, where $F = \ell \cap X_1$ is a general fiber of π_1.

Now consider $\widetilde{X}$ and the induced $\mathbf{A}^1_*$-fibration $\widetilde{\pi} : \widetilde{X} \to \widetilde{B}$. Let $\sigma : \widetilde{X} \to X$ be the canonical morphism. Then σ is a finite étale covering. Then each connected component of $\sigma^{-1}(\varphi(F))$ is a finite étale covering of $\mathbf{A}^1_*$. Hence each connected component must be isomorphic to $\mathbf{A}^1_*$. Then it is mapped to a point by $\widetilde{\pi}$ because $\widetilde{B}$ is an elliptic curve. Namely, each connected component of $\sigma^{-1}(\varphi(F))$ is a fiber of $\widetilde{\pi}$. Hence $\varphi(F)$ is a fiber of $\pi_2 : X_2 \to B_2$. So, φ preserves the $\mathbf{A}^1_*$-fibration. Q.E.D.

We shall give an example of a nontrivial étale endomorphism $\varphi : X \to X$ in the quasi-elliptic case $\{4; 2, 2, 2, 2\}$.

Lemma 5.8 *Let n be a positive integer > 1 such that $d := (n^2 - n)/2$ is odd. Let C be an elliptic curve which we view as an abelian variety by fixing one point as a point of origin. Let $\mathcal{L}$ be an invertible sheaf on C such that $\deg \mathcal{L} > 0$ and $n_C^*(\mathcal{L}) \cong \mathcal{L}^{\otimes n^2}$ (cf. Lemma 5.4). Let $\mathcal{M} := \mathcal{L}^{\otimes (n^2-n)/2}$. Let $(-1)_C : C \to C$ be the involution $p \mapsto -p$. Then the following assertions hold:*

(1) *$n_C^*(\mathcal{M}) \cong \mathcal{M}^{\otimes n^2}$ and $(-1)_C^*(\mathcal{M}) \cong \mathcal{M}$.*

(2) *Let $Y = \mathbf{P}(\mathcal{O}_C \oplus \mathcal{M}) - (S_0 \cup S_1)$, where S_0 (resp. S_1) is the cross-section corresponding to the projection $\mathcal{O}_C \oplus \mathcal{M} \to \mathcal{O}_C$ (resp. $\mathcal{O}_C \oplus \mathcal{M} \to \mathcal{M}$).*

Let $q : Y \to C$ be the canonical projection of $\mathbf{A}^1_*$-bundle. Then there exists an étale endomorphism $\psi : Y \to Y$ such that $n_C \circ q = q \circ \psi$ (cf. Lemmas 5.3 and 5.4).

(3) *Let $\{U_i\}_{i \in I}$ be an open covering of C such that $\mathcal{M} \mid_{U_i} = \mathcal{O}_{U_i} e_i$ for every $i \in I$. Then $q^{-1}(U_i) = \operatorname{Spec} \mathcal{O}_{U_i}[e_i, e_i^{-1}]$. Define an involution $\iota : Y \to Y$ locally by $\iota^*(e_i) = -e_i$ for $i \in I$. Then the involution ι is well-defined and satisfies the conditions $(-1)_C \circ q = q \circ \iota$ and $\psi \circ \iota = \iota \circ \psi$.*

(4) *Let $B := C/\langle (-1)_C \rangle$ and let $X := Y/\langle \iota \rangle$. Then the projection $q : Y \to C$ induces an $\mathbf{A}^1_*$-fibration $\pi : X \to B$, where $B \cong \mathbf{P}^1$ and π has four multiple fibers of multiplicity two. Furthermore, the multiplication by n endomorphism n_C of C induces an endomorphism $\beta : B \to B$.*

(5) *The étale endomorphism $\psi : Y \to Y$ induces an étale endomorphism $\varphi : X \to X$ such that $\pi \circ \varphi = \beta \circ \pi$. The endomorphism φ is not an automorphism.*

Proof. (1) By [8, Corollary 3, page 59], we have for any invertible sheaf $\mathcal{L}$

$$n_C^*(\mathcal{L}) \cong \mathcal{L}^{\left(\frac{n^2+n}{2}\right)} \otimes (-1)_C^* \mathcal{L}^{\left(\frac{n^2-n}{2}\right)} .$$

Since $n_C^*(\mathcal{L}) \cong \mathcal{L}^{\otimes n^2}$ we have

$$(-1)_C^* \mathcal{L}^{\left(\frac{n^2-n}{2}\right)} \cong \mathcal{L}^{\left(\frac{n^2-n}{2}\right)} .$$

Hence we have $n_C^*(\mathcal{M}) \cong \mathcal{M}^{\otimes n^2}$ and $(-1)_C^* \mathcal{M} \cong \mathcal{M}$.

(2) The assertion follows from Lemmas 5.3 and 5.4.

(3) Clear by the definition.

(4) The fixed points in C by $(-1)_C$ are the 2-torsion points, and there are four of them. The quotient morphism $C \to B$ ramifies at these four points, and B is therefore isomorphic to $\mathbf{P}^1$. Since $(-1)_C \circ n_C = n_C \circ (-1)_C$, n_C induces an endomorphism $\beta : B \to B$.

(5) The involution $\iota : Y \to Y$ has no fixed points by the definition. Hence the quotient morphism $Y \to X$ is a finite étale morphism. This implies that the fibers of $q : Y \to C$ lying over the four 2-torsion points of C produce the four multiple fibers of $\pi : X \to B$ with multiplicity 2. The rest of the assertion is now readily ascertained. Q.E.D.

The case (ii) of Lemma 5.5 is the case of Platonic $\mathbf{A}^1_*$-fiber spaces. In this case, we pose the following

Conjecture *Let $\pi : X \to B$ be a Platonic $\mathbf{A}^1_*$-fiber space and let $\varphi : X \to X$ be an étale endomorphism. Then there exists an endomorphism $\beta : B \to B$ such that $\beta \circ \pi = \pi \circ \varphi$.*

Let $\pi : X \to B$ be a Platonic fiber space. Then there exists a normal completion V of X as given at the beginning of the section 2. With the notations there we may further assume that the fibers G_i of the $\mathbf{P}^1$-fibration $p : V \to B$ containing the multiple fibers $m_i\Gamma_i$ have the linear chains as the dual graphes. The end components of the fibers G_i meet the cross-sections S_0 and S_1 of p. The components of the G_i other than $m_i\Gamma_i$ contract to the cyclic quotient singular points lying on the sections S_0 and S_1. Let $\widehat{V}$ be a normal projective surface obtained by contracting all the components of G_1, G_2, G_3 except for $m_1\Gamma_1, m_2\Gamma_2, m_3\Gamma_3$. Then $\widehat{V}$ inherits the $\mathbf{P}^1$-fibration $\widehat{p} : \widehat{V} \to B$. The images $\widehat{S}_0, \widehat{S}_1$ of S_0, S_1 are the cross-sections of $\widehat{p}$. Let $\varphi : X \to X$ be an étale endomorphism. By Lemma 2.2, φ extends to an étale endomorphism $\widetilde{\varphi}$ on $\mathbf{A}^2$ which fixes the point of origin O and commutes with the group $G := \pi_1(X)$. In particular, $\widetilde{\varphi}$ induces a local analytic isomorphism between $\mathbf{A}^2/G$ and its copy near the point O. This implies that φ extends to a regular endomorphism $\widehat{\varphi} : \widehat{V} - \widehat{S}_1 \to \widehat{V} - \widehat{S}_1$.

Theorem 5.9 *Suppose that $\widehat{\varphi}$ is also regular on the cross-section $\widehat{S}_1$. Then φ preserves the $\mathbf{A}^1_*$-fibration $\rho : X \to B$.*

Proof. Under the hypothesis, we know by modifying the logarithmic ramification formula that the following equality among the $\mathbf{Q}$-Caritier divisors holds:

$$\widehat{D_1} + K_{\widehat{V_1}} = \widehat{\varphi}^*(\widehat{D_2} + K_{\widehat{V_2}}) + R_{\widehat{\varphi}} \ ,$$

where $\widehat{V}_1$ and $\widehat{V}_2$ are copies of $\widehat{V}$, which are numbered to clarify the source and the target of the endomorphism $\widehat{\varphi}$, and similarly $\widehat{D_1} = \widehat{D_2} = \widehat{S}_0 + \widehat{S}_1$, and where the divisor $R_{\widehat{\varphi}}$ is an effective divisor. On the other hand, we can show that $N(\widehat{D} + K_{\widehat{V}})$ is a Cartier divisor for some integer $N > 0$ and $|-N(\widehat{D}+K_{\widehat{V}})| = |\ell|$, where ℓ is a general fiber of the $\mathbf{P}^1$-fibration $\widehat{p} : \widehat{V} \to B$ (for the computation, see [4, p. 81]). Namely we have the equality

$$-N(\widehat{D_1} + K_{\widehat{V_1}}) + R_{\widehat{\varphi}} = \widehat{\varphi}^*(-N(\widehat{D_1} + K_{\widehat{V_1}})) \ .$$

Then it follows that

$$(-N(\widehat{D}_1 + K_{\widehat{V}_1}) \cdot \widehat{\varphi}^*(\ell)) = (R_{\widehat{\varphi}} \cdot \widehat{\varphi}^*(\ell)) = 0 \ .$$

This implies that $\widehat{\varphi}^*(\ell)$ consists of several fibers of the $\mathbf{P}^1$-fibration $\widehat{p}$. So, φ preserves the $\mathbf{A}^1_*$-fibration φ. Q.E.D.

6 $\mathbf{A}^1_*$-fibrations given by G_m-actions

In this section we consider when a given $\mathbf{A}^1_*$-fibration comes from a G_m-action. Theorem 6.1 below asserts that in most cases the untwisted $\mathbf{A}^1_*$-fibrations are given by G_m-actions provided one of the two connected components in the boundary is contractible.

Let Σ be an affine normal algebraic surface with a unique singular point O and let $X := \Sigma - \{O\}$. Suppose that there exists a surjective morphism $\pi : X \to B$ from X to a nonsingular projective curve B of genus g such that every fiber of π is irreducible and is isomorphic to $\mathbf{A}^1_*$ if taken with reduced structure. Let $m_1\Gamma_1, \ldots, m_r\Gamma_r$ be the multiple fibers of π. Then our result is stated as follows.

Theorem 6.1 *Except for the cases* (i) $g = 0, r = 1$ *and* (ii) $g = 0, r = 2, m_1 \neq m_2$*, there exists a* G_m*-action on the surface* Σ *such that :*

(1) Σ *has a unique fixpoint* O*, and the point* O *is in the closure of every orbit.*

(2) $B \cong X/G_m$ *and the quotient morphism coincides with the given* $\mathbf{A}^1_*$*-fibration* π*.*

Proof. Our proof consists of several steps.

(I) Let $p_i = \pi(\Gamma_i)$ for $1 \leq i \leq r$. By Lemma 4.3, except for the cases (i) and (ii) above, there exists a Galois covering $\tau : C \to B$ with group G which ramifies over $p_1, \ldots, p_r$ with respective ramification indices $m_1, \ldots, m_r$. Let Y be the normalization of the fiber product $X \times_B C$ and let $\rho : Y \to C$ be the composition of the normalization morphism $Y \to X \times_B C$ and the canonical projection $X \times_B C \to C$. By Lemma 4.7 it then follows that $\rho : Y \to C$ *is an* $\mathbf{A}^1_*$*-fiber bundle*, i.e., there are no singular fibers.

The Galois group G acts on C. Hence G acts on the fiber product $X \times_B C$ via the factor C. Then the G-action extends to a G-action on the normalization Y and its quotient Y/G is isomorphic to X. Let $p : Y \to X$ be the quotient morphism.

(II) We can write Y as $Y = \mathbf{P}(\mathcal{O}_C \oplus \mathcal{L}) - (S_0 \cup S_1)$, where $\mathcal{L}$ is an invertible sheaf over the curve C and S_0, S_1 are two cross-sections of the $\mathbf{P}^1$-fibration $\mathbf{P}(\mathcal{O}_C \oplus \mathcal{L}) \to C$. We assume that $({S_0}^2) < 0$. Now note that X is an open set of the normal affine surface Σ. Let $\widetilde{\Sigma}$ be the normalization of Σ in the function field of $X \times_B C$ (or equivalently in the function field of Y). Then $\widetilde{\Sigma}$ is a normal affine surface such that the group G acts on $\widetilde{\Sigma}$ and Y is an open set of $\widetilde{\Sigma}$. In fact, $\widetilde{\Sigma} - Y$ consists of a single point $\widetilde{O}$. To see this, let $Z = \mathbf{P}(\mathcal{O} \oplus \mathcal{L}) - S_1$. Then the morphism $p : Y \to X$ extends to a morphism $\widehat{p} : Z \to \Sigma$. Then the morphism $\widehat{p}$ factors as

$$\widehat{p} : Z \xrightarrow{\sigma} \widetilde{\Sigma} \xrightarrow{\nu} \Sigma \ ,$$

where $\sigma : Z \to \widetilde{\Sigma}$ is the contraction of the section S_0 and $\nu : \widetilde{\Sigma} \to \Sigma$ is the normalization morphism. Actually, Z is a minimal resolution of the singular point $\widetilde{O}$. Then the G-action on $\widetilde{\Sigma}$ lifts to a G-action on Z. In particular, the conormal bundle $\mathcal{I}/\mathcal{I}^2$ of the section S_0 is G-stable, where $\mathcal{I}$ is the defining sheaf of ideals of S_0 in Z. We know that the conormal bundle $\mathcal{I}/\mathcal{I}^2$ is isomorphic to the bundle $\mathcal{L}$. Thus we have shown that $\mathcal{L}$ is a G-bundle in the sense that G acts on $\mathcal{L}$ and the canonical projection $\mathcal{L} \to C$ is a G-equivariant morphism, where we identify intentionally an invertible sheaf and the associated bundle.

(III) Since $\mathcal{L}$ is a G-line bundle, $\mathcal{O}_C \oplus \mathcal{L}$ is a G-vector bundle over C. Hence the $\mathbf{P}^1$-bundle $V = \mathbf{P}(\mathcal{O}_C \oplus \mathcal{L})$ has a G-action which induces the G-actions on Y and Z. Now choose an affine open covering $\mathcal{U} = \{U_i\}_{i \in I}$ such that $\mathcal{L} \mid_{U_i} = \mathcal{O}_{U_i} e_i$ for every $i \in I$. Write $\mathcal{O}_C = \mathcal{O}_C e$, where one can take $e = 1$. Then $\pi^{-1}(U_i) \cong \text{Spec}\ A_i[z_i, z_i^{-1}]$, where $U_i = \text{Spec}\ A_i$ and $z_i = e_i/e$. In $U_i \times \mathbf{P}^1$, the cross-section S_0 is defined by $z_i = 0$. Now consider a G_m-action locally on $\pi^{-1}(U_i)$ by $t \cdot z_i = t z_i$ for $t \in G_m$. Then the local G_m-actions on the $\pi^{-1}(U_i)$ glue together to give a G_m-action on Y which commutes with the G-action. Hence the G_m-action on Y descends on the quotient $X := Y/G$. Since Σ is a normal surface and $X = \Sigma - \{O\}$, the G_m-action on X extends to Σ. In fact, let σ_t be the automorphism of X induced by an action of

$t \in G_m$. Then $\sigma_t : \Sigma \to \Sigma$ is a birational mapping which is regular except for the point O. Then σ_t is regular everywhere on Σ because Σ is a normal surface. By the above construction, one can readily show that the G_m-action on Σ satisfies two conditions (1) and (2) in the statement of the theorem.

Q.E.D.

Theorem 6.2 *Let Σ be an affine normal algebraic surface as in Theorem 6.1, let $X := \Sigma - \{O\}$ and let $\pi : X \to B$ be the given $\mathbf{A}^1_*$-fibration. Suppose that one of the following cases occurs:*

(1) $g > 1$.

(2) $g = 1$ *and there is at least one multiple fiber.*

(3) $g = 0, r \geq 3$ *and* $\{r; m_1, \ldots, m_r\}$ *is none of the cases listed as the exceptional cases in Lemma 5.5.*

Then Aut $(\Sigma) =$ Aut (X) *satisfies an exact sequence*

$$0 \to G_m \to \mathrm{Aut}\,(X) \to \mathrm{Aut}_*(B),$$

where $\mathrm{Aut}_*(B)$ *is a subgroup of* $\mathrm{Aut}(B)$ *consisting of elements which permute the points of B over which lie the multiple fibers of π. In any one of the above cases,* $\mathrm{Aut}_*(B)$ *is a finite group. Hence the connected component* $\mathrm{Aut}^0(X)$ *of* Aut (X) *is isomorphic to the one-dimensional algebraic torus G_m.*

Proof. We employ the notations in the proof of Theorem 6.1. Since O is the unique singular point of Σ, every automorphism φ fixes the point O. Conversely, any automorphism ψ of X is regular at O as a birational mapping from Σ into itself. Hence Aut $(\Sigma) =$ Aut (X) holds.

Note that $\overline{\kappa}(X) = 1$ in each of the above three cases. Hence every automorphism φ preserves the $\mathbf{A}^1_*$-fibration π. Namely, $\pi \circ \varphi = \beta \circ \varphi$. Hence we have a canonical group automorphism Aut $(X) \to$ Aut (B), whose kernel we denote by $\mathrm{Aut}_\pi(X)$. In fact, the image is contained in $\mathrm{Aut}_*(B)$ by the observations in the previous sections. In what follows, we shall show that $\mathrm{Aut}_\pi(X) = G_m$.

Let φ be an element of $\mathrm{Aut}_\pi(X)$. As in the proof of Theorem 6.1, let $\tau : C \to B$ be a finite Galois covering which ramifies over the points p_i

with ramification index m_i if there are multiple fibers of π and let Y be the normalization of the fiber product $X \times_B C$. If there are no multiple fibers we take B as C and X as Y. Then the induced $\mathbf{A}^1_*$-fibration on Y over C has no multiple fibers. The automorphism φ extends naturally to a C-automorphism of Y, which we denote by the same symbol φ. Then $Y = \mathbf{P}(\mathcal{O}_C \oplus \mathcal{L}) - (S_0 \cup S_1)$ as in the step (III) of the proof of Theorem 6.1. With the notations there, φ induces an A_i-algebra automorphism $\alpha = \varphi^*$ of $A_i[z_i, z_i^{-1}]$.Then $\alpha(z_i) = a_i z_i^{\pm}$ with $a_i \in A_i^*$ for every $i \in I$. Hence we know that φ extends to a C-automorphism of $V = \mathbf{P}(\mathcal{O}_C \oplus \mathcal{L})$ which fixes the cross-sections S_0 and S_1 or exchanges S_0 and S_1. This implies that if $\alpha(z_i) = a_i z_i$ (resp. $\alpha(z_i) = a_i z_i^{-1}$) for some $i \in I$ then $\alpha(z_j) = a_j z_j$ (resp. $\alpha(z_j) = a_j z_j^{-1}$) for every $j \in I$. Suppose that $\alpha(z_i) = a_i z_i^{-1}$ for every $i \in I$. Then we have

$$a_j a_{ji}^{-1} z_i^{-1} = a_j z_j^{-1} = \alpha(z_j) = \alpha(a_{ji} z_i) = a_{ji}\alpha(z_i) = a_{ji} a_i z_i^{-1}$$

over $U_{ij} = U_i \cap U_j$, where $\{a_{ji}\}$ is a family of transition functions of $\mathcal{L}$ with respect to the affine open covering $\mathcal{U} = \{U_i\}_{i \in I}$. Then $a_{ji}^2 = a_j a_i^{-1}$. This implies that $\mathcal{L}^2 \cong \mathcal{O}_C$. But this is impossible because $\deg \mathcal{L} = -(S_0^2) > 0$. So, $\alpha(z_i) = a_i z_i$ for every $i \in I$. Then $a_i = a_j$ for all $i, j \in I$ by a calculation similar to the above. Hence $a_i = a \in \Gamma(C, \mathcal{O}_C)^* = \mathbf{C}^*$. Then it is clear that φ comes from the G_m-action on X whose existence is shown in Theorem 6.1. Q.E.D.

Next we consider the case of Platonic $\mathbf{A}^1_*$-fiber spaces.

Theorem 6.3 *Let $\pi : X \to B$ be a Platonic $\mathbf{A}^1_*$-fiber space and let p_1, p_2, p_3 be the points of B such that $\pi^*(p_i)$ is a multiple fiber of π for $i = 1, 2, 3$. Then there exists an exact sequence*

$$0 \to G_m \to \mathrm{Aut}\,(X) \to \mathrm{Aut}_*(B),$$

where $\mathrm{Aut}_*(B)$ consists of all automorphisms of B permuting the set $\{p_1, p_2, p_3\}$.

Proof. Write $X = \mathbf{A}^2/G - \{0\}$ as in section 2, where G is a non-cyclic small finite subgroup of $GL(2, \mathbf{C})$. Let φ be an automorphism of X. Then φ lifts to an automorphism ψ of $\mathbf{A}^2$ such that ψ commutes with the linear G-action and $\psi(O) = O$ (cf. section 2). Then ψ induces an automorphism on

$\Sigma := \mathbf{A}^2/G$ whose restriction on X is the given automorphism φ. We denote by the same symbol φ the induced automorphism of Σ. We have $\varphi(O) = O$. Let $h : \widehat{\Sigma} \to \Sigma$ be the minimal resolution of the singularity O. Then the exceptional locus $h^{-1}(O)$ is an admissible rational fork $F = B_0 + T_1 + T_2 + T_3$ (see [6] for the definitions), where B_0 is the central component and T_1, T_2, T_3 are three maximal rational twigs meeting B_0. We can identify B_0 with the base curve B so that $B_0 \cap T_i = \{p_i\}$ for $i = 1, 2, 3$. Then φ lifts to an automorphism $\widehat{\varphi}$ of $\widehat{\Sigma}$ such that $h \circ \widehat{\varphi} = \varphi \circ h, \widehat{\varphi}(B_0) = B_0$ and $\varphi\mid_{B_0}$ induces a permutation on the set $\{p_1, p_2, p_3\}$. Thus there exists a group homomorphism

$$\mathrm{Aut}\,(X) \to \mathrm{Aut}_*(B),$$

whose kernel we denote by $\mathrm{Aut}_0(X)$.

Let φ be anew an element of $\mathrm{Aut}_0(X)$. Let C be an irreducible component of the twig T_1 such that $B_0 \cap C_1 = \{p_1\}$. Then $\widehat{\varphi}(C_1) = C_1$. If $\widehat{\varphi}\mid_{C_1}$ is the identity then $\widehat{\varphi}$ (and hence φ) is the identity. In general, there are two points of C_1 which are fixed by $\widehat{\varphi}\mid_{C_1}$. Namely they are the point $B_0 \cap C_1$ and the point where C_1 meets the adjacent component of the fiber of π containing C_1. By the restriction $\varphi \mapsto \widehat{\varphi} \mapsto \widehat{\varphi}\mid_{C_1}$, we have an injective group homomorphism

$$\mathrm{Aut}_0(X) \to G_m.$$

In particular, $\mathrm{Aut}_0(X)$ is an abelian group.

On the other hand, there exists a G_m-action by Theorem 6.1 which gives rise to the $\mathbf{A}^1_*$-fibration π. We can find an element $t \in G_m$ such that $\widehat{\varphi}\mid_{C_1}$ is the multiplication by t. Then $t^{-1} \circ \widehat{\varphi}\mid_{C_1}$ is the identity. Hence $\widehat{\varphi} = t$. It then follows that $\mathrm{Aut}_0(X) = G_m$. Q.E.D.

If we state Theorem 6.3 in terms of the automorphisms on $\mathbf{A}^2$ we obtain the following result.

Corollary 6.4 *Let G be a non-cyclic small finite subgroup of $GL(2, \mathbf{C})$. Let* $\mathrm{Aut}_G(\mathbf{A}^2)$ *be the subgroup of* $\mathrm{Aut}\,(\mathbf{A}^2)$ *consisting of automorphisms which commutes with the linear G-action. Then* $\mathrm{Aut}_G(\mathbf{A}^2)$ *is an algebraic group such that its connected component containing the identity element is isomorphic to G_m.*

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