Linear Forms in the p-adic Logarithms

by

Kunrui Yu

Max-Planck-Institut
für Mathematik
Gottfried-Claren-Str. 26
D-5300 Bonn 3

West Germany

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Introduction and results

The p-adic theory of transcendental numbers was initiated by Mahler in 1930s. Mahler[19],[20] obtained in 1932 and 1935 the p-adic analogues of both the Hermite-Lindemann and the Gelfond-Schneider theorems; and during the course he founded the p-adic theory of analytic functions.

In 1939, Gelfond [15] proved a quantitative result on linear forms in two p-adic logarithms; in 1967, Schinzel [26] improved Gelfond's result and computed explicitly all the constants. In 1975, Baker and Coates [8] established in the case $n = 2$ a p-adic analogue of a sharpened inequality of Baker [5].

Since Baker published in 1960s his first series of papers [3], [4] on linear forms in $n \geq 2$ logarithms of algebraic numbers, his method has been employed to the investigation on linear forms in $n \geq 2$ p-adic logarithms of algebraic numbers. In 1967, Brumer [11] proved that if $\alpha_1,\dots,\alpha_n$ are multiplicatively independent $\mathfrak{p}$-adic units then any nontrivial linear form in $\mathfrak{p}$-adic logarithms

$$\beta_1 \log \alpha_1 + \dots + \beta_n \log \alpha_n$$

does not vanish. Subsequently, Coates [12] proved a p-adic analogue of Baker's result [4]; Sprindžuk [28],[29] proved p-adic analogues of Baker's results [3],[4]; Kaufman [17] proved a $\mathfrak{p}$-adic analogue of Feldman's result [14]. In 1977, van der Poorten [25] published a paper, containing four theorems on linear forms in p-adic logarithms, with much more generality than the previous work and essentially the same degree of precision as Baker's result [6]. In order to state van der Poorten's resutls, we introduce notations. Denote by $\alpha_1,\dots,\alpha_n$ $(n \geq 2)$ non-zero algebraic numbers in an algebraic number field K of degree D over $\mathbb{Q}$, and of heights respectively not exceeding $A_1,\ldots,A_n$ (with $A_j \geq e^e$, $1 \leq j \leq n$). Write $\Omega' = \log A_1 \ldots \log A_{n-1}$, $\Omega = \Omega' \log A_n$. Denote by $b_1,\ldots,b_n$ ($b_n \neq 0$) rational integers with absolute values not exceeding B. Denote by $\mathfrak{p}$ a prime ideal in the ring of algebraic integers O_K in K, lying above the rational prime p; write $e_\mathfrak{p}$ for the ramification index of $\mathfrak{p}$ and $f_\mathfrak{p}$ for its residue class degree, so $N\mathfrak{p} = N_{K/\mathbb{Q}}\mathfrak{p} = p^{f_\mathfrak{p}}$. Let $g_\mathfrak{p} = [\frac{1}{2} + e_\mathfrak{p}/(p-1)]$, $G_\mathfrak{p} = N\mathfrak{p}^{g_\mathfrak{p}} \cdot (N\mathfrak{p} - 1)$. For $\alpha \in K$, $\alpha \neq 0$ denote by $\mathrm{ord}_\mathfrak{p}\alpha$ the order to which $\mathfrak{p}$ divides the fractional ideal (α) and put $\mathrm{ord}_\mathfrak{p} 0 = \infty$. Then van der Poorten's [25] Theorem 1 (the main theorem) and Theorem 2 are as follows.

<u>Theorem 1 VdP.</u> The inequalities

$$\infty > \mathrm{ord}_\mathfrak{p}(\alpha_1^{b_1}\ldots\alpha_n^{b_n} - 1) > (16(n+1)D)^{12(n+1)} G_\mathfrak{p} \Omega \log \Omega' \log B$$

have no solutions in rational integers $b_1,\ldots,b_n$; $b_n \not\equiv 0 \pmod p$, with absolute values at most B.

<u>Theorem 2 VdP.</u> The inequalities

$$\infty > \mathrm{ord}_\mathfrak{p}(\alpha_1^{b_1}\ldots\alpha_n^{b_n} - 1) > (16(n+1)D)^{12(n+1)} (G_\mathfrak{p}/\log p) \Omega (\log B)^2$$

have no solutions in rational integers $b_1,\ldots,b_n$ with absolute values at most B.

Unfortunately, the proof in van der Poorten [25] involves several errors and inaccuracies, which we should like to remark at the end of §.3.4 and in the Appendix, so that it seems to be necessary to restudy thoroughly the whole p-adic theory of linear forms in logarithms of algebraic numbers. In the present paper we prove two theorems, which imply the results we reported on in the Proceedings of the Durham Symposium on Transcendental Number Theory, July 1986. (See Yu [33]). Take now

$$K = \mathbb{Q}(\alpha_1, \ldots, \alpha_n)$$

and keep the notations $D, \mathfrak{p}, p, e_{\mathfrak{p}}, f_{\mathfrak{p}}, N\mathfrak{p} = N_{K/\mathbb{Q}}\mathfrak{p}$ and $\mathrm{ord}_{\mathfrak{p}}$ introduced above. Denote by $K_{\mathfrak{p}}$ the completion of K with respect to the (additive) valuation $\mathrm{ord}_{\mathfrak{p}}$, and the completion of $\mathrm{ord}_{\mathfrak{p}}$ will be denoted again by $\mathrm{ord}_{\mathfrak{p}}$. Now let Σ be an algebraic closure of $\mathbb{Q}_p$. Write $\mathbb{C}_p$ for the completion of Σ with respect to its valuation, which is the unique extension of the valuation $|\,|_p$ of $\mathbb{Q}_p$. Denote by ord_p the additive form of the valuation of $\mathbb{C}_p$. According to Hasse [16], pp. 298-302, we can embed $K_{\mathfrak{p}}$ into $\mathbb{C}_p$: there exists a $\mathbb{Q}$-isomorphism σ from K into Σ such that $K_{\mathfrak{p}}$ is value-isomorphic to $\mathbb{Q}_p(\sigma(K))$, whence we can identify $K_{\mathfrak{p}}$ with $\mathbb{Q}_p(\sigma(K))$. Obviously,

$$\mathrm{ord}_{\mathfrak{p}}\beta = e_{\mathfrak{p}}\ \mathrm{ord}_p\beta \quad \text{for all} \quad \beta \in K_{\mathfrak{p}} .$$

Further, for an algebraic number α , write $h(\alpha)$ for its logarithmic absolute height (see Chapter II). Let $b_1, \ldots, b_n$ be rational integers and q a rational prime such that

$$q \nmid p(p^{f_{\mathfrak{p}}} - 1). \tag{0.1}$$

Let $V_1, \ldots, V_n, V_{n-1}^{+}, B_0, B_n, B', B, W$ be real numbers satisfying the following conditions

$$\left.\begin{array}{l} V_j \geq \max(h(\alpha_j), \frac{f_{\mathfrak{p}} \log p}{D}) \quad (1 \leq j \leq n), \\ V_1 \leq \ldots \leq V_{n-1},\ V_{n-1}^{+} = \max(1, V_{n-1}), \end{array}\right\} \quad (0.2)$$

$$\left.\begin{array}{l} B_0 \geq \min_{1 \leq j \leq n,\ b_j \neq 0} |b_j|, \qquad B_n \geq |b_n|, \\ B' \geq \max_{1 \leq j < n} |b_j|, \quad B \geq \max\{|b_1|, \ldots, |b_n|, 2\}, \end{array}\right\} \quad (0.3)$$

$$W \geq \begin{cases} \max\left\{\log(1 + \frac{3}{8n} \frac{f_{\mathfrak{p}} \log p}{D} (\frac{B_n}{V_1} + \frac{B'}{V_n})), \log B_0, \frac{f_{\mathfrak{p}} \log p}{D}\right\}, & \text{if } \min_{1 \leq j \leq n} \operatorname{ord}_p b_j > 0, \\ \max\left\{\log(1 + \frac{3}{8n} \frac{f_{\mathfrak{p}} \log p}{D} (\frac{B_n}{V_1} + \frac{B'}{V_n})), \frac{f_{\mathfrak{p}} \log p}{D}\right\}, & \text{if } \min_{1 \leq j \leq n} \operatorname{ord}_p b_j = 0. \end{cases} \quad (0.4)$$

(It is easy to see, by (0.2), that (0.4) is implied by

$$W \geq \begin{cases} \max\left\{\log\left(1 + \frac{3}{4n} B\right), \log B_0, \frac{f_{\mathfrak{p}} \log p}{D}\right\}, & \text{if } \min_{1 \leq j \leq n} \operatorname{ord}_p b_j > 0, \\ \max\left\{\log\left(1 + \frac{3}{4n} B\right), \frac{f_{\mathfrak{p}} \log p}{D}\right\}, & \text{if } \min_{1 \leq j \leq n} \operatorname{ord}_p b_j = 0\ .) \end{cases}$$

Then we have

Theorem 1. Suppose that

$$\operatorname{ord}_{\mathfrak{p}} \alpha_j = 0 \quad (1 \leq j \leq n), \quad (0.5)$$

$$[K(\alpha_1^{\frac{1}{q}},\ldots,\alpha_n^{\frac{1}{q}}) : K] = q^n \quad , \tag{0.6}$$

$$\mathrm{ord}_p b_n \leq \mathrm{ord}_p b_j \quad (1 \leq j \leq n-1) \tag{0.7}$$

and

$$\alpha_1^{b_1}\ldots\alpha_n^{b_n} \neq 1 \ . \tag{0.8}$$

Then

$$\mathrm{ord}_{\mathfrak{p}}(\alpha_1^{b_1}\cdots\alpha_n^{b_n}-1) < C_1(p,n)a_1^n n^{n+\frac{5}{2}} q^{2n}(q-1)\log^2(nq)(p^{f_{\mathfrak{p}}}-1)\frac{(2+1/(p-1))^n}{(f_{\mathfrak{p}}\log p)^{n+2}} \cdot$$

$$\cdot\ D^{n+2}V_1\cdots V_n\cdot\left(\frac{W}{6n}+\log(4D)\right)\left(\log(4DV_{n-1}^+)+\frac{f_{\mathfrak{p}}\log p}{8n}\right),$$

where

$$a_1 = \begin{cases} \frac{56}{15}e, & 2 \le n \le 7, \\ \frac{8}{3}e, & n \ge 8 \end{cases}$$

and $C_1(p,n)$ is given by the following table with
$C_1(p,n) = C_1'(p,n)(2+\frac{1}{p-1})^2$ for $p \ge 5$.

n	2	3	4	5	6	7	n≥8
$C_1(2,n)$	768523	476217	373024	318871	284931	261379	2770008
$C_1(3,n)$	167881	104028	81486	69657	62243	57098	116055
$C_1'(p,n)$	87055	53944	42255	36121	32276	24584	311077

<u>Remark.</u> By a little computation it is easy to verify that

$$C_1(2,n)a_1^n \le 2770008\ (\tfrac{8}{3}e)^n \quad \text{for all } n \ge 2$$

and

$$C_1(p,n)a_1^n \le 311077(2+\frac{1}{p-1})^2(\tfrac{8}{3}e)^n \le 2770008(\tfrac{8}{3}e)^n$$

for all $p \ge 3$, $n \ge 2$.

Thus

$$C_1(p,n)a_1^n \leq 2770008 \ (\tfrac{8}{3}e)^n \quad \text{for all} \quad p \quad \text{and} \quad n \geq 2.$$

Therefore Theorem 1 implies Theorem 1 in Yu [33].

In the following Theorem 2, we assume, instead of (0.4),

$$W \geq \begin{cases} \max\{\log(1+\frac{2}{5n}\frac{f_\mathfrak{p}\log p}{D}(\frac{B_n}{V_1}+\frac{B'}{V_n})), \log B_0, \frac{f_\mathfrak{p}\log p}{D}\}, \text{if} \min\limits_{1\leq j\leq n} \operatorname{ord}_p b_j > 0, \\ \max\{\log(1+\frac{2}{5n}\frac{f_\mathfrak{p}\log p}{D}(\frac{B_n}{V_1}+\frac{B'}{V_n})), \frac{f_\mathfrak{p}\log p}{D}\}, \text{if} \min\limits_{1\leq j\leq n} \operatorname{ord}_p b_j = 0. \end{cases} \tag{0.9}$$

<u>Theorem 2.</u> Suppose that (0.5)-(0.8) hold. Then

$$\operatorname{ord}_\mathfrak{p}(\alpha_1^{b_1}\ldots\alpha_n^{b_n}-1) < C_2(p,n)a_2^n n^{n+\frac{7}{2}} q^{2n}(q-1)\log^2(nq)e_\mathfrak{p}(p^{f_\mathfrak{p}}-1)\frac{(2+1/(p-1))^n}{(f_\mathfrak{p}\log p)^{n+2}} \cdot$$

$$\cdot D^{n+2}V_1\ldots V_n(\frac{W}{6n}+\log(4D))^2,$$

where $a_2 = a_2(p,n)$ and $C_2(p,n)$ are given as follows

$$a_2(2,n) = \begin{cases} \frac{8}{3}e, & 2\leq n\leq 17, \\ \frac{5}{2}e, & n\geq 18 \end{cases}, \qquad a_2(3,n) = \begin{cases} \frac{8}{3}e, & 2\leq n\leq 7, \\ \frac{5}{2}e, & n\geq 8 \end{cases},$$

$$a_2(p,n) = \begin{cases} \frac{8}{3}e, & 2\leq n\leq 16, \\ \frac{5}{2}e, & n\geq 17 \end{cases}, \qquad (p\geq 5) \quad ;$$

n	2	3	4	5	6	7	8≤n≤17	n≥18
$C_2(2,n)$	338071	244589	202601	178202	161998	150321	141430	441432

n	2	3	4	5	6	7	$n \geq 8$
$C_2(3,n)$	61716	44650	36985	32531	29573	27442	24871

$$C_2(p,n) = C_2'(p,n)\left(2 + \frac{1}{p-1}\right)^3, \quad p \geq 5$$

n	2	3	4	5	6	7	$8 \leq n \leq 16$	$n \geq 17$
$C_2'(p,n)$	14491	10484	8685	7639	6944	6444	6063	17401

Remark. It is easy to verify, by a little computation, that Theorem 2 implies Theorem 2 in Yü [33].

Corollary of Theorem 2. One may remove in Theorem 2 the hypothesis (0.7), provided (0.9) is replaced by

$$W \geq \begin{cases} \max\{\log(1+\frac{4B}{5n}),\ \log B_0,\ \frac{f_p \log p}{D}\}, & \text{if } \min_{1\leq j\leq n} \operatorname{ord}_p b_j > 0, \\ \max\{\log(1+\frac{4B}{5n}),\ \frac{f_p \log p}{D}\}, & \text{if } \min_{1\leq j\leq n} \operatorname{ord}_p b_j = 0. \end{cases} \tag{0.10}$$

Deduction of the Corollary from Theorem 2. Choose j_0 with $1 \leq j_0 \leq n$ such that

$$\operatorname{ord}_p b_{j_0} \leq \operatorname{ord}_p b_j \quad (1 \leq j \leq n).$$

We reorder then $\alpha_1,\ldots,\alpha_n$, $b_1,\ldots,b_n$ as $\alpha_{i_1},\ldots,\alpha_{i_n}$, $b_{i_1},\ldots,b_{i_n}$ such that

$$b_{i_n} = b_{j_0}$$

and

$$V_{i_1} \leq \ldots \leq V_{i_{n-1}} \quad .$$

Then $\alpha_1',\ldots,\alpha_n', b_1',\ldots,b_n'$ (with $\alpha_j' = \alpha_{i_j}$, $b_j' = b_{i_j}$, $1 \leq j \leq n$) satisfy the conditions (0.5)-(0.8) of Theorem 2. On noting that

$$B \geq |b_n'| \quad , \quad B \geq \max_{1 \leq j < n} |b_j'|$$

and

$$\frac{f_p \log p}{DV_{i_1}} \leq 1, \quad \frac{f_p \log p}{DV_{i_n}} \leq 1, \quad \text{(see (0.2))}$$

we see that (0.10) implies (0.9) for $b_1',\ldots,b_n'$, $\alpha_1',\ldots,\alpha_n'$, whence the Corollary follows from Theorem 2 at once.

In a joint paper with G. Wüstholz, which is in preparation, we shall remove Kummer condition (0.6) and the appearance of V_{n-1}^+ in the bounds of Theorem 1. This is achieved by the recent work of Wüstholz concerning multiplicity estimates in connexion with Baker's theory of linear forms in logarithms of algebraic numbers. (See Wüstholz [32].) Furthermore in that joint paper we shall show how a combination of Kummer theory with multiplicity estimates will yield very sharp effective bounds.

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Chapter I. p-Adic analysis

In this chapter we work in $\mathbb{C}_p$ introduced in the Introduction. Thus $\mathbb{C}_p$ is a complete non-archimedean valued field of characteristic zero with residue class field of characteristic p, and $\mathrm{ord}_p z$ $(z \in \mathbb{C}_p)$ is the additive valuation of $\mathbb{C}_p$ such that

$$\mathrm{ord}_p\, p = 1.$$

Throughout this chapter, the variable z takes values from $\mathbb{C}_p$. If $\mathrm{ord}_p z \geq 0$, we say that z is integral.

1. p-Adic exponential and logarithmic functions in $\mathbb{C}_p$

We record the following facts, which can be found in Hasse [16], pp. 262-274.

(a) The exponential series

$$\exp(z) = \sum_{n=0}^{\infty} \frac{z^n}{n!}$$

has the region of convergence $\mathrm{ord}_p z > \frac{1}{p-1}$, where

$$\exp(z_1 + z_2) = \exp(z_1)\exp(z_2)$$

and

$$\mathrm{ord}_p(\exp(z) - 1) = \mathrm{ord}_p z \quad .$$

(b) The logarithmic series

$$\log(1+z) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}z}{n}$$

has the region of convergence $\mathrm{ord}_p z > 0$, where

$$\log((1+z_1)(1+z_2)) = \log(1+z_1) + \log(1+z_2).$$

In the subregion $\mathrm{ord}_p z > \frac{1}{p-1}$,

$$\mathrm{ord}_p \log(1+z) = \mathrm{ord}_p z.$$

(c) For $\mathrm{ord}_p z > \frac{1}{p-1}$, we have

$$\log \exp(z) = z$$

and

$$\exp(\log(1+z)) = 1 + z.$$

(d) For $\mathrm{ord}_p x > \frac{1}{p-1}$ and integral z, we define

$$(1+x)^z = \exp(z \log(1+x)).$$

(Note that, for $z \in \mathbb{Z}$, this definition coincides with the usual powers.) Thus, by (c), we have

$$\log(1+x)^z = z \log(1+x).$$

Furthermore for integral z, z' and x, x' with $\mathrm{ord}_p x > \frac{1}{p-1}$, $\mathrm{ord}_p x' > \frac{1}{p-1}$, we have

$$(1+x)^{z+z'} = (1+x)^z (1+x)^{z'},$$

$$(1+x)^{zz'} = ((1+x)^z)^{z'},$$

$$(1+x)^{z}(1+x')^{z} = ((1+x)(1+x'))^{z}.$$

Note that for $\beta_1,\ldots,\beta_m \in \mathbb{C}_p$ with

$$\operatorname{ord}_p(\beta_j - 1) > \frac{1}{p-1} \quad (1 \leq j \leq m) \tag{1.1}$$

and integral $z_1,\ldots,z_m \in \mathbb{C}_p$, we have

$$\operatorname{ord}_p(z_1 \log \beta_1 + \ldots + z_m \log \beta_m) = \operatorname{ord}_p(\beta_1^{z_1} \ldots \beta_m^{z_m} - 1). \tag{1.2}$$

This can be verified as follows. By (1.1), (d), (b), (a)

$$\operatorname{ord}_p(\beta_j^{z_j} - 1) = \operatorname{ord}_p(\exp(z_j \log \beta_j) - 1)$$

$$= \operatorname{ord}_p(z_j \log \beta_j) \geq \operatorname{ord}_p(\log \beta_j)$$

$$= \operatorname{ord}_p(\beta_j - 1) > \frac{1}{p-1}, \quad (1 \leq j \leq m). \tag{1.3}$$

By (1.3) and the identity

$$(1+x_1)\ldots(1+x_m) - 1 = x_1+\ldots+x_m+x_1x_2+\ldots+x_{m-1}x_m+\ldots+x_1\ldots x_m,$$

we get

$$\operatorname{ord}_p(\beta_1^{z_1}\ldots\beta_1^{z_m} - 1) > \frac{1}{p-1}. \tag{1.4}$$

On the other hand, by (b), (1.3), (d), (1.1),

$$\log(\beta_1^{z_1}\ldots\beta_m^{z_m}) = \log \beta_1^{z_1}+\ldots+\log\beta_m^{z_m}$$

$$= z_1\log \beta_1+\ldots+ z_m \log \beta_m .$$

On combining this and (b), (1.4), we see that

$$\mathrm{ord}_p(z_1\log\beta_1+\ldots+z_m\log\beta_m)=\mathrm{ord}_p\log(\beta_1^{z_1}\ldots\beta_m^{z_m})=\mathrm{ord}_p(\beta_1^{z_1}\ldots\beta_m^{z_m}-1).$$

This proves (1.2).

2. Normal series and functions

For the p-adic analytic parts of the proofs of our theorems, instead of using Schnirelman integral [27] (see also Adams [1]), which yields a p-adic analogue of the Cauchy integral formula, we introduce a kind of Hermite interpolation formula (see the Appendix, Theorem A); then we give, based on Mahler's [20] concept on normal functions, and similarly to the work of Schinzel [26] and van der Poorten [24], a lemma for the extrapolation procedure (see Section 4 of this chapter).

The following concepts of normal series and functions are due to Mahler [20]. We reintroduce them here, because of their importance for our work. A p-adic power series

$$f(z) = \sum_{h=0}^{\infty} f_h(z-z_0)^h, \quad f_h \in \mathbb{C}_p \quad (h = 0,1,\ldots),$$

where z_0 is an integral element of $\mathbb{C}_p$, is called a normal series, if

$$\mathrm{ord}_p\, f_h \geq 0 \quad (h = 0,1,\ldots)$$

and

$$\mathrm{ord}_p f_h \longrightarrow \infty \quad (h \longrightarrow \infty).$$

Clearly $f(z)$ converges for every integral z, because of

$$\mathrm{ord}_p(f_h(z-z_0)^h) \geq \mathrm{ord}_p f_h$$

and $\text{ord}_p f_h \longrightarrow \infty \quad (h \longrightarrow \infty)$. Let z_1 be an arbitrary integral element in $\mathbb{C}_p$. By the p-adic analogue of Taylor's theorem, we have

$$f(z) = \sum_{k=0}^{\infty} \frac{f^{(k)}(z_1)}{k!} (z - z_1)^k$$

where

$$f^{(k)}(z_1) = k! \sum_{h=k}^{\infty} \binom{h}{k} f_h (z_1 - z_0)^{h-k} \quad (k = 0,1,\ldots)$$

denotes the derivative at z_1 of order k. Obviously

$$\text{ord}_p \frac{f^{(k)}(z_1)}{k!} \geq 0 \quad (k = 0,1,\ldots)$$

and

$$\text{ord}_p \frac{f^{(k)}(z_1)}{k!} \longrightarrow \infty \quad (k \longrightarrow \infty).$$

So the new power series

$$f(z) = \sum_{k=0}^{\infty} \frac{f^{(k)}(z_1)}{k!} (z - z_1)^k$$

in $z - z_1$ is a normal series. Thus, if a p-adic function is representable by a normal series in a neighborhood of an integral point in $\mathbb{C}_p$, then so is it in a neighborhood of every integral point in $\mathbb{C}_p$. Therefore we may call a p-adic function, which is definable by a normal series in a neighborhood of an integral point in $\mathbb{C}_p$, a normal function.

The following lemma is fundamental.

<u>Lemma 1.1.</u> (Mahler [20]) If a normal function $f(z)$ has zeroes

at the distinct integral points $\beta_1,\ldots,\beta_h$ in $\mathbb{C}_p$ of multiplicities at least $m_1,\ldots,m_h$, respectively, then

$$f(z) = g(z) \prod_{j=1}^{h} (z-\beta_j)^{m_j} ,$$

where $g(z)$ is a normal function.

<u>Proof</u> Since $f(z)$ has zero at β_1 of multiplicity at least m_1, we have

$$f^{(k)}(\beta_1) = 0 \qquad (k = 0,1,\ldots,m_1-1).$$

Thus

$$f(z) = (z-\beta_1)^{m_1} g_1(z) ,$$

where

$$g_1(z) = \sum_{k=m_1}^{\infty} \frac{f^{(k)}(\beta_1)}{k!} (z-\beta_1)^{k-m_1} .$$

So $g_1(z)$ represents a normal function having zeros at $\beta_2,\ldots,\beta_h$ of multiplicities at least $m_2,\ldots,m_h$. On repeating this procedure, the lemma follows immediately.

<u>Remark.</u> If $\delta \in \mathbb{C}_p$ satisfies $\operatorname{ord}_p \delta > \frac{1}{p-1}$, then the p-adic series

$$\exp(\delta z) = \sum_{k=0}^{\infty} \frac{\delta^k}{k!} z^k$$

is a normal series, because of the well-known fact that $\operatorname{ord}_p k! \leq \frac{k}{p-1}$.

3. Supernormality

For $\theta = \frac{c}{d}$, where c,d are positive rational integers with $(c,d) = 1$, we define

$$p^{\theta} = \rho^{c},$$

where ρ is a fixed root of $x^d - p = 0$ in $\mathbb{C}_p$. Thus

$$\mathrm{ord}_p p^{\theta} = \theta .$$

If $\delta \in \mathbb{C}_p$ satisfies

$$\mathrm{ord}_p \delta > \theta + \frac{1}{p-1} \quad ,$$

then $\exp(\delta z)$ has supernormality in the sense that

$$\exp(\delta p^{-\theta} z) = \sum_{k=0}^{\infty} \frac{(\delta p^{-\theta})^k}{k!} z^k$$

is a normal function.

The following lemma shows that there exists an nonnegative integer κ bounded in terms of p and $e_{\mathfrak{p}}$ such that for every $\beta \in \mathbb{C}_p$ satisfying

$$\mathrm{ord}_p(\beta - 1) \geq \frac{1}{e_{\mathfrak{p}}}$$

the p-adic function

$$(\beta^{p^{\kappa}})^{z} = \exp(z \log \beta^{p^{\kappa}})$$

has supernormality required for our p-adic analytic part of the proofs of our theorems.

Lemma 1.2. Let κ be the rational integer satisfying

$$p^{\kappa-1}(p-1) \leq (1+\frac{p-1}{p})e_{\mathfrak{p}} < p^{\kappa}(p-1) \tag{1.5}$$

and

$$\theta = \begin{cases} 1, & \text{if } \kappa \geq 1 \text{ and } p^{\kappa-1}(p-1) > e_{\mathfrak{p}}; \\ \frac{p^{\kappa}}{(2+1/(p-1))e_{\mathfrak{p}}}, & \text{otherwise}. \end{cases} \tag{1.6}$$

If $\beta \in \mathbb{C}_p$ satisfies

$$\operatorname{ord}_p(\beta - 1) \geq \frac{1}{e_{\mathfrak{p}}},$$

then

$$\operatorname{ord}_p(\beta^{p^{\kappa}} - 1) > \theta + \frac{1}{p-1} .$$

Remark. By the remark at the end of the last section, $(\beta^{p^{\kappa}})^{p^{-\theta}z} = \exp(p^{-\theta}z \log \beta^{p^{\kappa}})$ is a normal function.

Proof. This is Lemma 2 in Yu [33]. For completeness of our exposition, we reintroduce the proof. It is easy to verify that for $\gamma \in \mathbb{C}_p$ and $h \in \mathbb{Z}$, $h > 0$, the condition

$$\operatorname{ord}_p(\gamma - 1) \geq \frac{h}{e_{\mathfrak{p}}}$$

implies

$$\operatorname{ord}_p(\gamma^p - 1) \geq \min(\frac{h}{e_{\mathfrak{p}}} p, \frac{h}{e_{\mathfrak{p}}} + 1). \tag{1.7}$$

Now we show that if $\kappa \geq 1$, then for $j = 0,1,\ldots,\kappa-1$

$$\mathrm{ord}_p(\beta^{p^j} - 1) \geq \frac{p^j}{e_{\mathfrak{p}}} \tag{1.8}$$

We may assume $\kappa \geq 2$, since (1.8) is obvious when $\kappa = 1$. Now (1.8) is valid for $j = 0$. Assuming (1.8) holds for some j with $0 \leq j \leq \kappa - 2$, we see by (1.8), (1.7) and (1.5) that

$$\mathrm{ord}_p(\beta^{p^{j+1}} - 1) = \mathrm{ord}_p((\beta^{p^j})^p - 1)$$

$$\geq \min(\frac{p^{j+1}}{e_{\mathfrak{p}}}, \frac{p^j}{e_{\mathfrak{p}}} + 1) = \frac{p^{j+1}}{e_{\mathfrak{p}}} .$$

This proves (1.8) for $\kappa \geq 1$ and $j = 0, 1, \ldots, \kappa - 1$.

The lemma is evidently true if $\kappa = 0$. If $\kappa \geq 1$, by (1.8) (with $j = \kappa - 1$) and (1.7), we obtain

$$\mathrm{ord}_p(\beta^{p^\kappa} - 1) = \mathrm{ord}_p((\beta^{p^{\kappa-1}})^p - 1)$$

$$\geq \min(\frac{p^\kappa}{e_{\mathfrak{p}}}, \frac{p^{\kappa-1}}{e_{\mathfrak{p}}} + 1) > \theta + \frac{1}{p-1} ,$$

where the last inequality follows from (1.5) and (1.6). This completes the proof of the lemma.

Let

$$G = N_{K/\mathbb{Q}}\mathfrak{p} - 1 = p^{f_{\mathfrak{p}}} - 1.$$

It is well-known (see Hasse [16], p. 220) that if m is a positive rational integer with $(p,m) = 1$, then $K_{\mathfrak{p}}$ contains the m-th roots of unity if and only if $m|G$. In particular, $K_{\mathfrak{p}}$ contains the G-th roots of unity. In the remaining part of this paper, let ζ be a fixed G-th primitive root of unity in $K_{\mathfrak{p}}$.

For any integral elements α, β in $K_{\mathfrak{p}}$ we write

$$\alpha \equiv \beta \pmod{\mathfrak{p}},$$

if $\mathrm{ord}_{\mathfrak{p}}(\alpha - \beta) \geq 1$. Obviously, this defines an equivalence relation on $O_{\mathfrak{p}} = \{\alpha \in K_{\mathfrak{p}} \mid \mathrm{ord}_{\mathfrak{p}}\alpha \geq 0\}$.

<u>Lemma 1.3.</u> For any $\alpha \in K_{\mathfrak{p}}$ with $\mathrm{ord}_{\mathfrak{p}}\alpha = 0$, there exists $r \in \mathbb{Z}$ with $0 \leq r < G$ such that

$$e_{\mathfrak{p}}\, \mathrm{ord}_p(\alpha\zeta^r - 1) = \mathrm{ord}_{\mathfrak{p}}(\alpha\zeta^r - 1) \geq 1.$$

<u>Proof.</u> By Hasse [16], p. 153, 155, 220, we see that the set

$$\{0, 1, \zeta, \zeta^2, \ldots, \zeta^{G-1}\}$$

is a complete residue system of $O_{\mathfrak{p}}$ mod $\mathfrak{p}$. Since $\mathrm{ord}_{\mathfrak{p}}\alpha = 0$, there exists $r' \in \mathbb{Z}$ with $0 \leq r' < G$ such that

$$\alpha \equiv \zeta^{r'} \pmod{\mathfrak{p}}.$$

Let $r \in \mathbb{Z}$ satisfy $r \equiv -r' \pmod{G}$ and $0 \leq r < G$. We get then

$$\alpha\zeta^r \equiv 1 \pmod{\mathfrak{p}},$$

and the lemma follows at once.

4. <u>A lemma for extrapolation</u>

<u>Lemma 1.4.</u> Suppose that $\theta > 0$ is a rational number, $q > 0$ is a rational prime with $q \neq p$, and $M > 0$, $R > 0$ are rational integers with $q \mid R$. Suppose further that $F(z)$ is a p-adic normal function and

$$\min_{\substack{1\le s\le R,(s,q)=1\\ t=0,\dots,M-1}} \left(\operatorname{ord}_p \frac{F^{(t)}(sp^\theta)}{t!} + t\theta\right)$$

$$\geq \left(1-\frac{1}{q}\right)RM\theta + M\operatorname{ord}_p R! + (M-1)\frac{\log R}{\log p}\ . \tag{1.9}$$

Then for all $\ell \in \mathbb{Z}$, we have

$$\operatorname{ord}_p F\left(\frac{\ell}{q}p^\theta\right) \geq \left(1-\frac{1}{q}\right)RM\theta.$$

Remark. Here $\log R$ and $\log p$ denote the usual logarithms for positive real numbers.

Proof. By Theorem A of the Appendix, the unique polynomial $Q(z)$ of degree at most $(1-\frac{1}{q})RM-1$ satisfying

$$Q^{(t-1)}(sp^\theta) = F^{(t-1)}(sp^\theta),\quad 1\le s\le R,\ (s,q)=1,\ 1\le t\le M$$

is given by the formula

$$Q(z) = \sum_{\substack{s=1\\(s,q)=1}}^{R} \sum_{t=1}^{M} \frac{F^{(t-1)}(sp^\theta)}{(t-1)!}(-1)^{M-t}(z-sp^\theta)^{t-1}\left\{\prod_{\substack{j=1\\(j,q)=1,j\neq s}}^{R}\left(\frac{z-jp^\theta}{(s-j)p^\theta}\right)^M\right\}\cdot$$

$$\cdot\sum_{h=1}^{M-t}(-1)^{h-1}\sum_{\substack{\lambda_1+\dots+\lambda_{M-t}=M-t\\ \lambda_i=0\ (i<h)\\ \lambda_i\ge 1\ (i\ge h)}}\prod_{i=1}^{M-t}\frac{1}{\lambda_i!}\left(\frac{\partial}{\partial\eta}\right)^{\lambda_i}\left\{(z-\eta)\prod_{\substack{k=1\\(k,q)=1,k\neq s}}^{R}\left(\frac{\eta-kp^\theta}{(s-k)p^\theta}\right)^M\right\}_{\eta=sp^\theta}, \tag{1.10}$$

where the second line of (1.10) reads as 1 when $t = M$. Let

$$E_s(z) = \prod_{\substack{k=1 \\ (k,q)=1, k\neq s}}^{R} \frac{z-kp^\theta}{(s-k)p^\theta},$$

$$A_{s,t}(z) = (z - sp^\theta)^{t-1}(E_s(z))^M,$$

$$B_{s,\lambda}(z) = \frac{1}{\lambda!}\left(\frac{\partial}{\partial\eta}\right)^\lambda \{(z-\eta)(E_s(\eta))^M\}_{\eta=sp^\theta}.$$

Then (1.10) can be written as

$$Q(z) = \sum_{\substack{s=1 \\ (s,q)=1}}^{R} \sum_{t=1}^{M} (-1)^{M-t} \frac{F^{(t-1)}(sp^\theta)}{(t-1)!} A_{s,t}(z) \sum_{h=1}^{M-t} (-1)^{h-1} \sum_{\substack{\lambda_1+\dots+\lambda_{M-t}=M-t \\ \lambda_i=0\,(i<h) \\ \lambda_i\geq 1\,(i\geq h)}} \prod_{i=1}^{M-t} B_{s,\lambda_i}(z).$$

(1.11)

We first show that for every $\ell \in \mathbf{Z}$,

$$\mathrm{ord}_p Q(\tfrac{\ell}{q}p^\theta) \geq \min_{\substack{1\leq s\leq R, (s,q)=1 \\ t=0,\dots,M-1}} (\mathrm{ord}_p \frac{F^{(t)}(sp^\theta)}{t!} + t\theta) - M\,\mathrm{ord}_p R! - (M-1)\frac{\log R}{\log p}.$$

(1.12)

Note that for every s with $1 \leq s \leq R$, $(s,q) = 1$, we have, by $(q,p) = 1$.

$$\mathrm{ord}_p E_s(\tfrac{\ell}{q}p^\theta) = \mathrm{ord}_p \prod_{\substack{k=1 \\ (k,q)=1, k\neq s}}^{R} \frac{\frac{\ell}{q} - k}{s-k} \geq -\mathrm{ord}_p \prod_{\substack{k=1 \\ k\neq s}}^{R} (s-k)$$

$$\geq -\mathrm{ord}_p (R-1)! \geq -\mathrm{ord}_p R!$$

Thus we get, by $(q,p) = 1$

$$\operatorname{ord}_p A_{s,t}\left(\frac{\ell}{q}\, p^{\theta}\right) \geq (t-1)\theta - M \operatorname{ord}_p R! \tag{1.13}$$

for $\ell \in \mathbb{Z}$, $1 \leq s \leq R$, $(s,q) = 1$, $1 \leq t \leq M$.

On noting that

$$E_s(sp^{\theta}) = 1 \tag{1.14}$$

and for every $\mu \in \mathbb{Z}$ with $1 \leq \mu \leq (1-\frac{1}{q})R - 1$

$$\frac{1}{\mu!}\left(\frac{d}{d\eta}\right)^{\mu} E_s(\eta) = E_s(\eta) \sum_{\substack{1 \leq k_1 < \ldots < k_\mu \leq R \\ (k_j,q)=1,\, k_j \neq s \\ (1 \leq j \leq \mu)}} \frac{1}{(\eta - k_1 p^{\theta}) \ldots (\eta - k_\mu p^{\theta})}\,, \tag{1.15}$$

we obtain

$$\frac{1}{\mu!}\left\{\left(\frac{d}{d\eta}\right)^{\mu} E_s(\eta)\right\}_{\eta = sp^{\theta}} = \sum_{\substack{1 \leq k_1 \ldots < k_\mu \leq R \\ (k_j,q)=1,\, k_j \neq s \\ (1 \leq j \leq \mu)}} \frac{1}{(s-k_1) \ldots (s-k_\mu) p^{\mu\theta}}\,.$$

Observing that

$$\operatorname{ord}_p (s - k_j) \leq \left[\frac{\log (R-1)}{\log p}\right] < \frac{\log R}{\log p}\,,$$

we get

$$\operatorname{ord}_p \frac{1}{\mu!}\left\{\left(\frac{d}{d\eta}\right)^{\mu} E_s(\eta)\right\}_{\eta = sp^{\theta}} \geq -\mu\left(\theta + \frac{\log R}{\log p}\right) \tag{1.16}$$

for $1 \leq \mu \leq (1-\frac{1}{q})R - 1$, $1 \leq s \leq R$, $(s,q) = 1$.

Note that (1.16) is also true for $\mu = 0$ and $\mu > (1-\frac{1}{q})R-1$, because of (1.14) and the fact that $E_s(z)$ is a polynomial in z of degree $(1-\frac{1}{q})R-1$. Now for positive $\lambda \in \mathbb{Z}$

$$\frac{1}{\lambda!}\left(\frac{d}{d\eta}\right)^{\lambda}(E_s(\eta))^M = \sum_{\substack{\mu_1+\dots+\mu_M=\lambda \\ \mu_j\geq 0\ (1\leq j\leq M)}} \prod_{j=1}^{M} \frac{1}{\mu_j!}\left(\frac{d}{d\eta}\right)^{\mu_j} E_s(\eta). \tag{1.17}$$

On combining (1.16) and (1.17), we get

$$\mathrm{ord}_p \frac{1}{\lambda!}\left\{\left(\frac{d}{d\eta}\right)^{\lambda}(E_s(\eta))^M\right\}_{\eta=sp^\theta} \geq -\lambda\left(\theta+\frac{\log R}{\log p}\right) \tag{1.18}$$

for $\lambda \geq 1$, $1 \leq s \leq R$, $(s,q) = 1$.

Note that (1.18) is also true for $\lambda = 0$, by (1.14). Now we estimate $\mathrm{ord}_p B_{s,\lambda_i}(\frac{\ell}{q}p^\theta)$. By the definition of $B_{s,\lambda}(z)$ we obtain for $\lambda \geq 1$

$$B_{s,\lambda}(z) = (z-sp^\theta)\frac{1}{\lambda!}\left\{\left(\frac{d}{d\eta}\right)^{\lambda}(E_s(\eta))^M\right\}_{\eta=sp^\theta} - \frac{1}{(\lambda-1)!}\left\{\left(\frac{d}{d\eta}\right)^{\lambda-1}(E_s(\eta))^M\right\}_{\eta=sp^\theta}. \tag{1.19}$$

So by (1.19) and (1.18) we get

$$\begin{aligned}\mathrm{ord}_p B_{s,\lambda}\left(\frac{\ell}{q}p^\theta\right) &\geq \min\left\{\theta-\lambda\left(\theta+\frac{\log R}{\log p}\right),\ -(\lambda-1)\left(\theta+\frac{\log R}{\log p}\right)\right\} \\ &= -(\lambda-1)\theta - \lambda\frac{\log R}{\log p}\end{aligned} \tag{1.20}$$

for $\lambda \geq 1$, $1 \leq s \leq R$, $(s,q) = 1$.

Note that (1.20) is also true for $\lambda = 0$. By (1.20) we see that

$$\operatorname{ord}_p \sum_{h=1}^{M-t} (-1)^{h-1} \sum_{\substack{\lambda_1+\dots+\lambda_{M-t}=M-t \\ \lambda_i=0\ (i<h) \\ \lambda_i \geq 1\ (i\geq h)}} \prod_{i=1}^{M-t} B_{s,\lambda_i}(\tfrac{\ell}{q}p^{\theta}) \geq -(M-t)\frac{\log R}{\log p} \tag{1.21}$$

for $\ell \in \mathbb{Z}$, $1 \leq s \leq R$, $(s,q) = 1$, $1 \leq t \leq M-1$.

Note that (1.21) is also valid for $t = M$. On combining (1.11), (1.13) and (1.21), we conclude

$$\operatorname{ord}_p Q(\tfrac{\ell}{q}\, p^{\theta}) \geq \min_{\substack{1\leq s\leq R,\,(s,q)=1 \\ t=1,\dots,M}} \{\frac{F^{(t-1)}(sp^{\theta})}{(t-1)!} + (t-1)\theta - M \operatorname{ord}_p R! - (M-t)\frac{\log R}{\log p}\},$$

which implies (1.12).

Now we proceed to prove that $Q(z)$ is a p-adic normal function, that is, to show that

$$\operatorname{ord}_p \frac{Q^{(m)}(0)}{m!} \geq 0 \tag{1.22}$$

for $0 \leq m \leq (1-\frac{1}{q})RM - 1$.

By (1.13) with $\ell = 0$ and (1.9), we see that (1.22) is true for $m = 0$. So we may assume $m \geq 1$ in the sequel. We assert that

$$\operatorname{ord}_p \frac{E_s^{(\mu)}(0)}{\mu!} \geq -\mu\theta - \operatorname{ord}_p R! \tag{1.23}$$

for $\mu \geq 0$ and $1 \leq s \leq R$, $(s,q) = 1$,

for by the definition of $E_s(z)$, (1.23) is true for $\mu = 0$; it is obvious for $\mu > (1-\frac{1}{q})R-1$; and for $1 \leq \mu \leq (1-\frac{1}{q})R-1$, it follows from (1.15) at once. Further (1.23) and (1.17) imply that

$$\operatorname{ord}_p \frac{1}{\lambda!} \{(\frac{d}{dz})^{\lambda}(E_s(z))^M\}_{z=0} \geq -\lambda\theta - M \operatorname{ord}_p R! \tag{1.24}$$

for $\lambda \geq 0$, $1 \leq s \leq R$, $(s,q) = 1$.

Now we show that

$$\operatorname{ord}_p \frac{1}{\mu!}\{(\frac{d}{dz})^{\mu} A_{s,t}(z)\}_{z=0} \geq (t-1-\mu)\theta - M \operatorname{ord}_p R! \tag{1.25}$$

for $\mu \geq 0$, $1 \leq s \leq R$, $(s,q) = 1$, $1 \leq t \leq M$.

By the definition of $A_{s,t}(z)$ and (1.24) with $\lambda = 0$, we see that (1.25) is true for $\mu = 0$. Assume $\mu \geq 1$. Then

$$\frac{1}{\mu!}(\frac{d}{dz})^{\mu} A_{s,t}(z) = \sum_{\substack{\lambda=0 \\ \lambda \geq \mu-t+1}}^{\mu} \{\frac{1}{\lambda!}(\frac{d}{dz})^{\lambda}(E_s(z))^M\}\binom{t-1}{\mu-\lambda}(z - sp^{\theta})^{t-1-(\mu-\lambda)} .$$

This and (1.24) imply (1.25) at once. Now we prove that if $1 \leq t \leq M-1$ and $\lambda_1, \ldots, \lambda_{M-t}$ are non-negative integers satisfying

$$\lambda_1 + \ldots + \lambda_{M-t} = M - t,$$

then

$$\operatorname{ord}_p \frac{1}{(m-\mu)!}\{(\frac{d}{dz})^{m-\mu}(\prod_{i=1}^{M-t} B_{s,\lambda_i}(z))\}_{z=0} \geq -(m-\mu)\theta - (M-t)\frac{\log R}{\log p} \tag{1.26}$$

for $1 \leq s \leq R$, $(s,q) = 1$, $0 \leq \mu \leq m$.

By (1.18) and (1.19), we have

$$B_{s,\lambda}(z) = a_{s,\lambda}(z - sp^{\theta}) + b_{s,\lambda} \tag{1.27}$$

for $\lambda \geq 0$, $1 \leq s \leq R$, $(s,q) = 1$,

where $a_{s,\lambda}$, $b_{s,\lambda} \in \mathbb{C}_p$ $(b_{s,0} = 0)$ satisfy

$$\left.\begin{aligned} \operatorname{ord}_p a_{s,\lambda} &\geq -\lambda\left(\theta + \frac{\log R}{\log p}\right), \\ \operatorname{ord}_p b_{s,\lambda} &\geq -(\lambda - 1)\left(\theta + \frac{\log R}{\log p}\right). \end{aligned}\right\} \tag{1.28}$$

(1.26) is obvious for μ with $m - \mu > M - t$, by (1.27). It is also true for $\mu = m$ by (1.20) with $\ell = 0$ and the fact that $\lambda_1 + \ldots + \lambda_{M-t} = M - t$. So we may assume $1 \leq m - \mu \leq M - t$.
Now

$$\frac{1}{(m-\mu)!}\left(\frac{d}{dz}\right)^{m-\mu}\left(\prod_{i=1}^{M-t} B_{s,\lambda_i}(z)\right) = \sum_{1 \leq i_1 < \ldots < i_{m-\mu} \leq M-t} \left(\prod_{j=1}^{m-\mu} a_{s,\lambda_{i_j}}\right) \prod_{\substack{1 \leq i \leq M-t \\ i \neq i_j (1 \leq j \leq m-\mu)}} B_{s,\lambda_i}(z$$

This together with (1.28), (1.20) with $\ell = 0$ and the fact that $\lambda_1 + \ldots + \lambda_{M-t} = M - t$ yields (1.26). Observing (1.11), (1.25) and (1.26), we obtain for $m = 0, 1, \ldots, (1 - \frac{1}{q})RM - 1$

$$\operatorname{ord}_p \frac{Q^{(m)}(0)}{m!} \geq \min_{\substack{1 \leq s \leq R, (s,q)=1 \\ t=1,\ldots,M \\ \mu=0,\ldots,m}} \left\{\frac{F^{(t-1)}(sp^{\theta})}{(t-1)!} + (t-1-\mu)\theta - (m-\mu)\theta - (M-t)\frac{\log R}{\log p}\right\} - M\operatorname{ord}_p R$$

$$\geq \min_{\substack{1 \leq s \leq R, (s,q)=1 \\ t=1,\ldots,M}} \left\{\frac{F^{(t-1)}(sp^{\theta})}{(t-1)!} + (t-1-m)\theta\right\} - M\operatorname{ord}_p R! - (M-1)\frac{\log R}{\log p}$$

$$\geq \min_{\substack{1 \leq s \leq R, (s,q)=1 \\ t=0,\ldots,M-1}} \left\{\frac{F^{(t)}(sp^{\theta})}{t!} + t\theta\right\} - \left(\left(1-\frac{1}{q}\right)RM-1\right)\theta - M\operatorname{ord}_p R! - (M-1)\frac{\log}{\log}$$

$$\geq \theta ,$$

where the last inequality follows from (1.9). This proves (1.22), i.e., $Q(z)$ is a normal function.

The normal function

$$F(z) - Q(z)$$

has zeroes at

$$sp^{\theta}, \qquad 1 \leq s \leq R, \ (s,q) = 1$$

of multiplicities at least M. By Lemma 1.1, there exists a normal function $g(z)$ such that

$$F(z) = Q(z) + g(z) \prod_{\substack{s=1 \\ (s,q)=1}}^{R} (z - sp^{\theta})^{M} .$$

Note that $\mathrm{ord}_p g(\frac{\ell}{q} p^{\theta}) \geq 0$, because $g(z)$ is normal and $(q,p) = 1$, whence $\mathrm{ord}_p (\frac{\ell}{q}p^{\theta}) \geq \theta > 0$. Thus for every $\ell \in \mathbb{Z}$, we have

$$\mathrm{ord}_p F(\frac{\ell}{q}p^{\theta}) \geq \min(\mathrm{ord}_p Q(\frac{\ell}{q}p^{\theta}), \ (1 - \frac{1}{q}) RM\theta).$$

This together with (1.12) and (1.9) implies

$$\mathrm{ord}_p F(\frac{\ell}{q}p^{\theta}) \geq (1 - \frac{1}{q}) RM\theta \ .$$

The proof of the lemma is thus complete.

Chapter II Arithmetic tools and estimates

We first introduce briefly the concept of logarithmic absolute height of an algebraic number α. Let α be of degree d, $a_0 > 0$ be the leading coefficient of its minimal polynomial f over $\mathbb{Z}$, $H_0(\alpha)$ be its usual height, i.e., the maximum of the absolute values of the coefficients of f, $\alpha_1,\ldots,\alpha_d$ be its conjugates over $\mathbb{Q}$. Write

$$M(\alpha) = a_0 \prod_{i=1}^{d} \max(1,|\alpha_i|).$$

Let E be a number field containing α. Write

$$H_E(\alpha) = \prod_{v} \max(1,|\alpha|_v) \;, \tag{2.1}$$

where v runs over all valuations of E normalized in the usual way to satisfy the product formula $\prod_v |\alpha|_v = 1$ for $\alpha \neq 0$. More precisely, for each embedding σ of E into $\mathbb{C}$ there is an archimedean valuation v defined by $|\alpha|_v = |\sigma(\alpha)|$; and for each prime ideal $\mathfrak{P}$ of O_E (the ring of algebraic integers in E) with absolute norm $N\mathfrak{P} = N_{E/\mathbb{Q}}\mathfrak{P}$ there is a non-archimedean valuation defined by

$$|\alpha|_v = (N\mathfrak{P})^{-\mathrm{ord}_{\mathfrak{P}}\alpha} \;,$$

where $\mathfrak{P}^{\mathrm{ord}_{\mathfrak{P}}\alpha}$ is the exact power of $\mathfrak{P}$ in the fractional principal ideal of E generated by α. The numbers

$$H(\alpha) = (H_E(\alpha))^{\frac{1}{[E:\mathbb{Q}]}}$$

and

$$h(\alpha) = \log H(\alpha)$$

are independent of E. We call $H(\alpha)$ and $h(\alpha)$ the absolute height and the logarithmic absolute height of α, respectively. The relation

$$H_{\mathbb{Q}(\alpha)}(\alpha) = M(\alpha)$$

(see, for example, Bertrand [10], Lemma 11) shows that

$$h(\alpha) = \frac{1}{d} \log M(\alpha) .$$

For any algebraic numbers $\alpha, \beta, \alpha_1, \ldots, \alpha_n$ and any $0 \neq m \in \mathbb{Z}$, we have

$$h(\alpha\beta) \leq h(\alpha) + h(\beta), \tag{2.2}$$

$$h(\alpha^m) = |m| h(\alpha), \tag{2.3}$$

$$h(\alpha_1 + \ldots + \alpha_n) \leq h(\alpha_1) + \ldots + h(\alpha_n) + \log n . \tag{2.4}$$

From the inequality

$$M(\alpha) \leq (d+1)^{\frac{1}{2}} H_0(\alpha)$$

(see Mahler [21]) it follows that

$$h(\alpha) \leq \frac{1}{d}(\log H_0(\alpha) + \log d),$$

since $h(\alpha) = \log H_0(\alpha)$ for $\alpha \in \mathbb{Q}$ and $x + 1 \leq x^2$ for $x \geq 2$. By (2.1) and the product formula, we have

$$H_E(\beta) = H_E(\tfrac{1}{\beta}) \quad \text{for} \quad \beta \in E,\ \beta \neq 0. \tag{2.5}$$

Now we give a p-adic analogue of the Liouville inequality.

For every prime ideal $\mathfrak{p}$ of O_E, let $e_\mathfrak{p}$ be its ramification index, $f_\mathfrak{p}$ its residue class degree, p the unique rational prime contained in $\mathfrak{p}$. Write

$$\mathrm{ord}_p = \frac{1}{e_\mathfrak{p}} \mathrm{ord}_\mathfrak{p} .$$

Denote by $|\ |_v$ the non-archimedean valuation determined by $\mathfrak{p}$. Then for every $\beta \in E$, we have

$$p^{-f_\mathfrak{p}\ \mathrm{ord}_\mathfrak{p}\beta} = (N\mathfrak{p})^{-\mathrm{ord}_\mathfrak{p}\beta} = |\beta|_v \leq H_E(\beta).$$

If $\beta \neq 0$, we can apply the above inequality to $\frac{1}{\beta}$ and obtain, by (2.5),

$$p^{f_\mathfrak{p}\mathrm{ord}_\mathfrak{p}\beta} \leq H_E(\beta) ,$$

whence

$$\mathrm{ord}_p\beta \leq \frac{\log H_E(\beta)}{e_\mathfrak{p}f_\mathfrak{p}\log p} = \frac{[E:\mathbb{Q}]}{e_\mathfrak{p}f_\mathfrak{p}\log p} h(\beta). \tag{2.6}$$

For a polynomial P denote by $L(P)$ its length, i.e., the sum of the absolute values of its coefficients.

<u>Lemma 2.1.</u> Suppose $P(x_1,\ldots,x_m) \in \mathbb{Z}[x_1,\ldots,x_m]$ satisfies

$$\deg_{x_k} P \leq N_k \ (\geq 1),\ 1 \leq k \leq m.$$

If $\beta_1,\ldots,\beta_m \in E$ and $P(\beta_1,\ldots,\beta_m) \neq 0$, then

$$\mathrm{ord}_p P(\beta_1,\ldots,\beta_m) \leq \frac{[E:\mathbb{Q}]}{e_\mathfrak{p}f_\mathfrak{p}\log p} \{\log L(P) + \sum_{k=1}^{m} N_k h(\beta_k)\}.$$

Proof. For each valuation v of E we have

$$\max(1,|P(\beta_1,\ldots,\beta_m)|_v) \le C_v \prod_{k=1}^{m} (\max(1,|\beta_k|_v))^{N_k}, \tag{2.7}$$

where $C_v = L(P)$ if v is archimedean and $C_v = 1$ otherwise. On multiplying (2.7) for all v and taking $[E:\mathbb{Q}]$-th root we obtain

$$H(P(\beta_1,\ldots,\beta_m)) \le L(P) \prod_{k=1}^{m} (H(\beta_k))^{N_k},$$

whence

$$h(P(\beta_1,\ldots,\beta_m)) \le \log L(P) + \sum_{k=1}^{m} N_k h(\beta_k).$$

This together with (2.6) proves the lemma.

We will deduce a version of Siegel's lemma (Lemma 2.2 below) from the following

Lemma (Anderson and Masser [2]) Let E be an algebraic number field of degree D. For each valuation v of E let μ_v be an element of E and let M_v be a non-negative real number such that $M_v = 1$ except for finitely many v. Put $M = \prod_v M_v$. Then there are at most $(2M^{\frac{1}{D}} + 1)^D$ elements ξ of E such that

$$|\xi - \mu_v|_v \le M_v$$

for all v.

Lemma 2.2. Let $\beta_1,\ldots,\beta_r$ be algebraic numbers in an algebraic number field E of degree D. Suppose that

$$P_{i,j} \in \mathbb{Z}[x_1,\ldots,x_r] \quad (1 \le i \le n,\ 1 \le j \le m) \quad \text{(not all zero)}$$

satisfy

$$\deg_{x_k} P_{i,j} \le N_{j,k} \quad (1 \le i \le n,\ 1 \le j \le m,\ 1 \le k \le r).$$

Write

$$X = \max_{1\le j\le m} \{ (\sum_{i=1}^{n} L(P_{i,j}))\exp(\sum_{k=1}^{r} N_{j,k} h(\beta_k)) \}$$

and

$$\gamma_{i,j} = P_{i,j}(\beta_1,\ldots,\beta_r) \quad (1 \le i \le n,\ 1 \le j \le m).$$

If $n > mD$, then there exist rational integers $y_1,\ldots,y_n$ with

$$0 < \max_{1\le i\le n} |y_i| \le X^{\frac{mD}{n-mD}}$$

such that

$$\sum_{i=1}^{n} \gamma_{i,j}\, y_i = 0 \quad (1 \le j \le m).$$

Remark. This is a slight refinement of Lemma 3 in Mignotte and Waldschmidt [22].

Proof. Let

$$A = [X^{\frac{mD}{n-mD}}]. \tag{2.8}$$

For each $\mathbf{y} = (y_1,\ldots,y_n) \in \mathbb{Z}^n$ with

$$0 \le y_i \le A \quad (1 \le i \le n)$$

we set $\boldsymbol{\lambda} = (\lambda_1,\ldots,\lambda_m)$ by

$$\lambda_j = \sum_{i=1}^{n} \gamma_{i,j}\, y_i \in E \quad (1 \le j \le m). \tag{2.9}$$

Further for each j with $1 \le j \le m$ and each valuation v of E, let

$$\mu_{v,j} = \begin{cases} \sum_{i=1}^{n} \gamma_{ij} \cdot \frac{1}{2}A, & \text{if } v \text{ is archimedean} \\ 0, & \text{otherwise,} \end{cases}$$

and

$$M_{v,j} = A_{v,j} \prod_{k=1}^{r} (\max(1, |\beta_k|_v))^{N_{j,k}}$$

where

$$A_{v,j} = \begin{cases} \frac{1}{2}A \sum_{i=1}^{n} L(P_{i,j}), & \text{if } v \text{ is archimedean} \\ 1, & \text{otherwise} \end{cases}.$$

Note that

$$M_j = \prod_v M_{v,j} = \{\tfrac{1}{2}A(\sum_{i=1}^{n} L(P_{i,j})) \prod_{k=1}^{r} (H(\beta_k))^{N_{j,k}}\}^D \le (\tfrac{1}{2} AX)^D. \tag{2.10}$$

Evidently $\mu_{v,j} \in E$ and for each j, $M_{v,j} = 1$ except for finitely many v. By (2.9), we have for archimedean v

$$|\lambda_j - \mu_{v,j}|_v = |\sum_{i=1}^{n} \gamma_{i,j}(y_i - \tfrac{1}{2}A)|_v \le \tfrac{1}{2} A \sum_{i=1}^{n} |\gamma_{i,j}|_v$$

$$\le \tfrac{1}{2}A(\sum_{i=1}^{n} L(P_{i,j})) \prod_{k=1}^{r} (\max(1, |\beta_k|_v))^{N_{j,k}} = M_{v,j} \qquad (1 \le j \le m),$$

and for non-archimdean v

$$|\lambda_j - \mu_{v,j}|_v = |\sum_{i=1}^{n} \gamma_{i,j}\, y_i|_v \leq \max_{1\leq i\leq n} |\gamma_{i,j}|_v$$

$$\leq \prod_{k=1}^{r} (\max(1, |\beta_k|_v))^{N_{j,k}} = M_{v,j} \quad (1 \leq j \leq m).$$

Thus all the $(A+1)^n$ $\boldsymbol{\lambda} = (\lambda_1, \ldots, \lambda_m)$, which correspond by (2.9) to the $(A+1)^n$ $\mathbf{y} = (y_1, \ldots, y_n) \in \mathbb{Z}^n$ with $0 \leq y_i \leq A$ $(1 \leq i \leq n)$, satisfy

$$|\lambda_j - \mu_{v,j}|_v \leq M_{v,j} \quad \text{for all } v \quad (1 \leq j \leq m).$$

On the other hand, by the Lemma of Anderson and Masser, and by (2.10), there exist at most

$$\prod_{j=1}^{m} (2M_j^{\frac{1}{D}} + 1)^D \leq (AX+1)^{mD}$$

$\boldsymbol{\xi} = (\xi_1, \ldots, \xi_m) \in E^m$ satisfying

$$|\xi_j - \mu_{v,j}|_v \leq M_{v,j} \quad \text{for all } v \quad (1 \leq j \leq m).$$

Now (2.8) and the fact that $X \geq 1$ imply

$$(AX+1)^{mD} \leq (X(A+1))^{mD} < (A+1)^n.$$

Thus by the box-principle, there exist two distinct integral points $\mathbf{y}' = (y_1', \ldots, y_n')$ and $\mathbf{y}'' = (y_1'', \ldots, y_n'')$ with

$$0 \leq y_i' \leq A, \quad 0 \leq y_i'' \leq A \quad (1 \leq i \leq n)$$

such that

$$\sum_{i=1}^{n} \gamma_{i,j} y_i' = \sum_{i=1}^{n} \gamma_{i,j} y_i'' \quad (1 \leq j \leq m).$$

Hence $\mathbf{y} = (y_1, \ldots, y_n) = (y_1' - y_1'', \ldots, y_n' - y_n'') \in \mathbb{Z}^n$ satisfies

$$\sum_{i=1}^{n} \gamma_{i,j} y_i = 0 \qquad (1 \le j \le m)$$

and

$$0 < \max_{1 \le i \le n} |y_i| \le A \le X^{\frac{mD}{n-mD}} .$$

This completes the proof of the lemma.

For every positive integer k, let $\nu(k)$ be the least common multiple of $1,2,\ldots,k$. Define for $z \in \mathbb{C}$.

$$\Delta(z;k) = (z+1)\ldots(z+k)/k! \quad (k \in \mathbb{Z}, k \geq 1) \quad \text{and} \quad \Delta(z;0) = 1, \tag{2.11}$$

and for ℓ, m non-negative integers

$$\Delta(z;k,\ell,m) = \frac{1}{m!} \left\{\frac{d^m}{dy^m} (\Delta(y;k))^{\ell}\right\}_{y=z} . \tag{2.12}$$

<u>Lemma 2.3.</u> (Waldschmidt [31], Lemma 2.4)
For any $z \in \mathbb{C}$ and any integers $k \geq 1$, $\ell \geq 1$, $m \geq 0$, we have

$$|\Delta(z;k,\ell,m)| \le (2e)^{k\ell} \left(\frac{|z| + k}{k}\right)^{k\ell} . \tag{2.13}$$

Let q be a positive integer, and let x be a rational number such that qx is a positive integer. Then

$$q^{2k\ell} (\nu(k))^m \Delta(x;k,\ell,m) \in \mathbb{Z}, \tag{2.14}$$

and we have

$$\nu(k) \le 3^k .$$

Finally, for any positive integers k, R and L with $k \geq R$, the polynomials $(\Delta(z+r;k))^{\ell}$ $(r=0,1,\ldots,R-1;\ell=1,\ldots,L)$ are linearly independent.

Remark 1. This is essentially Lemmas 3 and 4 of P.L. Cijsouw and M. Waldschmidt, Linear forms and simultaneous approximations, Compositio Math. 37 (1978), 21-50.

2. In Lemma 2.4 of Waldschmidt [31], the right-hand side of (2.13) is replaced by $(2e)^{k\ell}\left(\frac{|z|+k+1}{k}\right)^{k\ell}$.

3. (2.14) implies that its left-hand side is a positive integer when $m \leq k\ell$. Obviously, the left-hand side of (2.14) is zero when $m > k\ell$.

Proof. To prove (2.13), we may assume $m \leq k\ell$. Then

$$\Delta(y;k,\ell,m) = (\Delta(y;k))^{\ell} \sum((y+j_1)\ldots(y+j_m))^{-1}, \tag{2.15}$$

where the summation is over all selections $j_1,\ldots,j_m$ of m integers from the set $1,\ldots,k$ repeated ℓ times. Hence

$$|\Delta(z;k,\ell,m)| \leq \binom{k\ell}{m}(\Delta(|z|;k))^{\ell} \leq 2^{k\ell}(\Delta(|z|;k))^{\ell}.$$

This together with the fact that

$$|\Delta(z;k)| \leq \Delta(|z|;k) \leq \frac{(|z|+k)^k}{k!} \leq \left(\frac{|z|+k}{k}\right)^k e^k \tag{2.16}$$

implies (2.13) at once.

Note that (2.14) is just Lemma T1 of Tijdeman [30]. For the self-containess of our exposition, we reintroduce the proof here. Obviously, we may assume $m \leq k\ell$. Write $\Delta = \Delta(x;k,\ell,m)$. Then $q^{2k\ell}(\nu(k))^m\Delta$ is a positive rational number. Hence to prove (2.14) it suffices to show that

$$\mathrm{ord}_p(q^{2k\ell}(\nu(k))^m\Delta) \geq 0 \quad \text{for all rational primes } p. \tag{2.17}$$

By (2.15) with $y = x$, we see that

$$q^{k\ell-m}(k!)^{\ell}\,\Delta \in \mathbb{Z} .$$

If $p|q$, then $\mathrm{ord}_p q^k \geq k \geq \mathrm{ord}_p k!$. Hence

$$\mathrm{ord}_p(q^{2k\ell}\Delta) \geq \mathrm{ord}_p(q^{k\ell-m}(k!)^{\ell}\Delta) \geq 0 \quad \text{for } p \text{ with } p|q. \tag{2.18}$$

For p with $p\nmid q$, by a well-known counting argument, we have

$$\begin{aligned}\mathrm{ord}_p(q^k k!\ \Delta(x;k))^{\ell} &= \mathrm{ord}_p((qx+q1)^{\ell}\dots(qx+qk)^{\ell})\\ &\geq \ell([\tfrac{k}{p}] + [\tfrac{k}{p^2}] + \dots + [\tfrac{k}{p^t}])\\ &= \ell\ \mathrm{ord}_p\ k!\ ,\end{aligned} \tag{2.19}$$

where

$$t = [\frac{\log k}{\log p}] = \mathrm{ord}_p\ \nu(k). \tag{2.20}$$

Here we have counted at most t p-factors in each individual factor $qx + qj$. Hence by (2.19) and (2.20), we get

$$\begin{aligned}&\mathrm{ord}_p\{q^{k\ell-m}(k!)^{\ell}(\Delta(x;k))^{\ell}((x+j_1)\dots(x+j_m))^{-1}\}\\ &= \mathrm{ord}_p\{(q^k k!\Delta(x;k))^{\ell}(qx+qj_1)^{-1}\dots(qx+qj_m)^{-1}\}\\ &\geq \ell\ \mathrm{ord}_p k! - mt\\ &= \ell\ \mathrm{ord}_p k! - m\ \mathrm{ord}_p\ \nu(k).\end{aligned} \tag{2.21}$$

Recalling (2.15) with $y = x$, we obtain, by (2.21),

$$\mathrm{ord}_p\{q^{k\ell-m}(\nu(k))^m\Delta\} \geq 0 \quad \text{for all } p \text{ with } p\nmid q. \tag{2.22}$$

Now (2.18) and (2.22) implies (2.17), whence (2.14) follows at once.

To prove the final part of Lemma 2.3, we need the following

Lemma (Baker [7], p.26) If $P(x)$ is a polynomial with degree $n > 0$ and if K is a field containing its coefficients then, for any integer m with $0 \leq m \leq n$, the polynomials $P(x), P(x+1), \ldots, P(x+m)$ and $1, x, \ldots, x^{n-m-1}$ are linearly independent over K.

Now we prove the final part of Lemma 2.3 by an induction on L. The assertion is true for $L = 1$ by virtue of the lemma of Baker and the fact that $R \leq k$. Assume the assertion is valid for $L-1$, that is,

$$(\Delta(z+r;k))^{\ell}, \quad r = 0,1,\ldots,R-1, \quad \ell = 1,\ldots,L-1 \tag{2.23}$$

are linearly independent. Observe that by the lemma of Baker, the polynomials

$$(\Delta(z;k))^{L}, \ (\Delta(z+1;k))^{L}, \ldots, (\Delta(z+R-1;k))^{L}, 1, x, \ldots, x^{kL-R}$$

are linearly independent. But the polynomials in (2.23) are of degrees at most $k(L-1) \leq kL-R$. Hence the inductive hypothesis and the above observation imply that the polynomials

$$(\Delta(z+r;k))^{\ell}, \quad r = 0,1,\ldots,R-1, \quad \ell = 1,\ldots,L$$

are linearly independent. This proves the final part of Lemma 2.3. The proof of Lemma 2.3 is thus complete.

Let B', B_n be positive real numbers, $L_1, \ldots, L_n$ $(n \geq 2)$, T be positive integers. Put $L = \max_{1 \leq j \leq n-1} L_j$.

Lemma 2.4. Suppose that $b_1,\dots,b_n,\lambda_1,\dots,\lambda_n,\tau_1,\dots,\tau_{n-1}$ are rational integers satisfying

$$|b_j| \le B' \quad (1 \le j \le n-1), \quad |b_n| \le B_n ,$$

$$0 \le \lambda_j \le L_j \quad (1 \le j \le n),$$

$$\tau_j \ge 0 \quad (1 \le j \le n-1), \quad \tau_1+\dots+\tau_{n-1} \le T.$$

Then

$$\prod_{j=1}^{n-1} |\Delta(b_n\lambda_j - b_j\lambda_n;\tau_j)| \le e^T \left(1 + \frac{(n-1)(B_nL+B'L_n)}{T}\right)^T . \tag{2.24}$$

Remark. This is essentially an estimate in Loxton, Mignotte, van der Poorten and Waldschmidt [18], but we have modified their estimate

$$\prod_{j=1}^{n-1} |\Delta(b_n\lambda_j - b_j\lambda_n;\tau_j)| \le \{2e(1 + (n-1)\frac{B_nL+B'L_n+1}{T})\}^T$$

by (2.24).

Proof. Without loss of generality, we may assume $\tau_1 > 0,\dots,\tau_{n-1}>0$. By (2.16), we have

$$|\Delta(b_n\lambda_j - b_j\lambda_n;\tau_j)| \le e^{\tau_j}\left(\frac{B_nL+B'L_n+\tau_j}{\tau_j}\right)^{\tau_j} \tag{2.25}$$

From the convexity of the function $f(x) = x \log x$, we see that for any $a_i > 0$ and $x_i > 0$ $(i = 1,\dots,m)$

$$\sum_{i=1}^{m} \frac{a_i}{a_1+\dots+a_m} \cdot \frac{x_i}{a_i} \log \frac{x_i}{a_i} \geq \frac{x_1+\dots+x_m}{a_1+\dots+a_m} \log \frac{x_1+\dots+x_m}{a_1+\dots+a_m} ,$$

whence

$$\sum_{i=1}^{m} x_i \log \frac{a_i}{x_i} \leq (x_1+\dots+x_m) \log \frac{a_1+\dots+a_m}{x_1+\dots+x_m} .$$

Hence

$$\sum_{j=1}^{n-1} \tau_j \log \frac{B_n L+B'L_n+\tau_j}{\tau_j} \leq (\tau_1+\dots+\tau_{n-1}) \log\left(1 + \frac{(n-1)(B_n L+B'L_n)}{\tau_1+\dots+\tau_{n-1}}\right)$$

$$\leq T \log\left(1 + \frac{(n-1)(B_n L+B'L_n)}{T}\right), \tag{2.26}$$

where the last inequality follows from the fact that $g(x) = x \log (1+\frac{a}{x})$ $(a > 0)$ increases for $x > 0$. On multiplying (2.25) for $j = 1,\dots,n-1$ and using (2.26), the lemma follows at once.

By an integral valued polynomial we mean a polynomial $f(x) \in \mathbb{C}[x]$ such that

$$f(m) \in \mathbb{Z} \quad \text{for every} \quad m \in \mathbb{Z}.$$

Write $\delta f(x)$ for $f(x) - f(x-1)$. Then

$$\delta\Delta(x;0) = 0,$$

$$\delta\Delta(x;k) = \Delta(x;k-1) \quad (k \geq 1), \tag{2.27}$$

for if $k \geq 2$ then

$$\delta\Delta(x;k) = \frac{(x+1)\dots(x+k)}{k!} - \frac{x\dots(x+k-1)}{k!}$$

$$= \frac{(x+1)\dots(x+k-1)(x+k-x)}{k!} = \Delta(x;k-1) ,$$

and $\delta\Delta(x;0) = 0$, $\delta\Delta(x;1) = \Delta(x;0)$ are obvious. Let $\mathbb{N}=\{m\in\mathbb{Z} \mid m\geq 0\}$.

Lemma 2.5. Suppose $m \in \mathbb{N}$, $a \in \mathbb{C}$, $a \neq 0$. Then

$$\det(\Delta(aj;k))_{0\leq j,k\leq m} \neq 0 .$$

Proof. The case $m = 0$ is trivial. So we may assume $m \geq 1$. Suppose that the determinant equals to zero, we proceed to deduce a contradiction. Thus there exist complex numbers $\lambda_0, \lambda_1, \ldots, \lambda_m$, not all zero, such that

$$\sum_{k=0}^{m} \lambda_k \Delta(aj;k) = 0, \qquad j = 0,1,\ldots,m.$$

Hence the polynomial

$$\sum_{k=0}^{m} \lambda_k \Delta(x;k) \quad ,$$

being the degree at most m, has $m + 1$ zeroes at aj with $j = 0,1,\ldots,m$. So $\sum_{k=0}^{m} \lambda_k \Delta(x;k)$ is identically zero, a contradiction to the fact that $\Delta(x;0), \Delta(x;1), \ldots, \Delta(x;m)$ are linearly independent over $\mathbb{C}$. This prove the lemma.

Lemma 2.6. Every integral valued polynomial $f(x)$ of degree $k > 0$ can be expressed as

$$f(x) = a_k\Delta(x;k) + a_{k-1}\Delta(x;k-1)+\ldots+a_1\Delta(x;1) + a_0\Delta(x;0), \tag{2.28}$$

where $a_0, \ldots, a_k$ are rational integers.

Proof. By Lemma 2.5 with $a = 1$, there exists unique $(n+1)$-tuple $(a_0,\ldots,a_k) \in \mathbb{C}^{k+1}$ such that (2.28) holds. It remains only to show that $a_0,\ldots,a_k$ are rational integers. By (2.27), (2.28) we get

$$\delta f(x) = a_k \Delta(x;k-1) + a_{k-1}\Delta(x;k-2)+\ldots+a_1 .$$

Write

$$\delta^2 f(x) = \delta(\delta f(x)),\ldots,\delta^k f(x) = \delta(\delta^{k-1} f(x)).$$

Then

$$f(-1) = a_0,\quad (\delta f(x))_{x=-1} = a_1,\ldots,(\delta^k f(x))_{x=-1} = a_k .$$

Since $f(x)$ is integral valued, so are $\delta f(x)$, $\delta^2 f(x),\ldots,\delta^k f(x)$. Hence $a_0,a_1,\ldots,a_k$ are rational integers. This completes the proof of the lemma.

Lemma 2.7. For every positive integer n, we have

$$n! > \sqrt{2\pi}\, n^{n+\frac{1}{2}} e^{-n} .$$

Proof. Let

$$f(n) = \frac{n!e^n}{\sqrt{2\pi}\, n^{n+\frac{1}{2}}} .$$

The lemma is equivalent to the inequality

$$f(n) > 1, \quad n = 1,2,\ldots .$$

Since $f(n) \longrightarrow 1$ $(n \longrightarrow \infty)$ by the Stirling's formula, to prove the lemma it suffices to show that

$$f(n) > f(n+1), \; n = 1,2,\ldots,$$

i.e.

$$g(n) = \frac{f(n)}{f(n+1)} = \left(1+\frac{1}{n}\right)^{n+\frac{1}{2}} \frac{1}{e} > 1, \; n = 1,2,\ldots . \tag{2.29}$$

Now $g(n) \longrightarrow 1$ $(n \longrightarrow \infty)$. So it suffices to prove

$$h(x) = \left(x+\frac{1}{2}\right) \log \left(1+\frac{1}{x}\right)$$

decreases strictly for $x \geq 1$, since this implies that so does $g(n)$ for $n \geq 1$, whence (2.29) follows at once. Note that

$$h'(x) = \log\left(1+\frac{1}{x}\right) - \frac{x+\frac{1}{2}}{x(x+1)}$$

and

$$h'(1) = \log 2 - \frac{3}{4} < 0, \quad h'(x) \longrightarrow 0 \quad (x \longrightarrow \infty)$$

$$h''(x) = \frac{1}{2(x(x+1))^2} > 0 \qquad (x \geq 1).$$

Thus

$$h'(x) < 0 \quad (x \geq 1).$$

This completes the proof of the lemma.

Chapter III A proposition towards the proof of Theorem 1

In this chapter we prove a proposition towards the proof of Theorem 1. The proof follows the main lines of Baker [6] and Waldschmidt [31].

We use the notations introduced for Theorem 1 and let κ and θ be defined in Lemma 1.2. Put

$$G = N\mathfrak{p} - 1 = p^{f_{\mathfrak{p}}} - 1$$

and let $\zeta \in K_{\mathfrak{p}}$ be the G-th primitive root of unity fixed in § 1.3. By the fact that $\mathrm{ord}_{\mathfrak{p}}\alpha_j = 0$ $(1 \leq j \leq n)$ (see (0.5)) and Lemma 1.3, there exists $(r_1', \ldots, r_n') \in \mathbb{Z}^n$ with $0 \leq r_j' < G$ $(1 \leq j \leq n)$ such that

$$e_{\mathfrak{p}}\ \mathrm{ord}_p(\alpha_j \zeta^{r_j'} - 1) \geq 1 \qquad (1 \leq j \leq n).$$

Let $r_1, \ldots, r_n$ be the rational integers such that

$$r_j \equiv p^{\kappa} r_j' \pmod{G},\ 0 \leq r_j < G \quad (1 \leq j \leq n).$$

Then we see, by Lemma 1.2, that

$$\mathrm{ord}_p(\alpha_j^{p^{\kappa}} \zeta^{r_j} - 1) > \theta + \frac{1}{p-1} \quad (1 \leq j \leq n). \tag{3.1}$$

For later references, we give an expression (the following formula (3.3)) for

$$(\alpha_j^{p^{\kappa}} \zeta^{r_j})^{\frac{1}{q}} = \exp(\frac{1}{q} \log\ (\alpha_j^{p^{\kappa}} \zeta^{r_j})),$$

where the logarithmic and exponential functions are p-adic functions, which are well defined by (3.1) and the fact that $\mathrm{ord}_p q = 0$ (see (0.1)).

Since $(q,G) = 1$ by (0.1), we can choose $a,b \in \mathbb{Z}$ such that

$$aG + bq = 1.$$

Let $\zeta_q \in \mathbb{C}_p$ be a fixed q-th primitive root of unity and put

$$\xi = \zeta_q^a \zeta^b .$$

On noting $(q,p^\kappa) = 1$ and

$$((\alpha_j^{p^\kappa} \zeta^{r_j})^{\frac{1}{q}})^q = \alpha_j^{p^\kappa} \zeta^{r_j}$$

by § 1.1, (d), it is easy to see that there exists a q-th root $\alpha_j' \in \mathbb{C}_p$ of α_j such that

$$(\alpha_j^{p^\kappa} \zeta^{r_j})^{\frac{1}{q}} = \alpha_j'^{p^\kappa} \xi^{r_j} = \alpha_j'^{p^\kappa} \zeta_q^{ar_j} \zeta^{br_j} \quad (1 \leq j \leq n) . \tag{3.2}$$

By $(q,p^\kappa) = 1$, for each j with $1 \leq j \leq n$ there exists unique $k_j \in \mathbb{Z}$ such that

$$p^\kappa k_j \equiv ar_j \pmod q, \quad 0 \leq k_j < q.$$

Writing $\alpha_j^{\frac{1}{q}}$ for $\alpha_j' \zeta_q^{k_j}$, which is a q-th root of α_j in $\mathbb{C}_p$, we get, by (3.2),

$$(\alpha_j^{p^\kappa} \zeta^{r_j})^{\frac{1}{q}} = (\alpha_j^{\frac{1}{q}})^{p^\kappa} \zeta^{br_j} \qquad (1 \leq j \leq n). \tag{3.3}$$

1. Statement of the Proposition

We define $h_j = h_j(n,q;c_0,c_2)$ $(0 \leq j \leq 7)$, $h_8 = h_8(n,q;c_0,c_2,c_3)$,

$\varepsilon_j = \varepsilon_j(n,q;c_0,c_2)$ $(j = 1,2)$ by the following 11 formulas, which will be referred as (3.4):

$$
\begin{aligned}
h_0 &= n \log (2^{11}nq) , \\
h_1 &= 2^5 c_0 (2c_2q)^n (q-1) \frac{n^{2n+1}}{n!} h_0 , \\
h_2 &= 2^5 c_0 (2c_2q)^{n-1} (q-1) \frac{n^{2n-1}}{n!} , \\
1+\varepsilon_1 &= \left(1 - \frac{1}{h_2}\right)^{-n} , \\
h_3 &= \frac{h_1 - 1}{n^2} , \\
1+\varepsilon_2 &= e^{h_3^{-1}} , \\
h_4 &= \frac{h_1}{h_0+1} , \\
h_5 &= \frac{2^8 c_0 (1+\varepsilon_1)(1+\varepsilon_2)}{\sqrt{2\pi n}\,\left(1 - \frac{1}{32n}\right)} , \\
h_6 &= \frac{2^6 h_1}{n} , \\
\frac{1}{h_7} &= \frac{9 \times 10^{-15}}{h_0 h_1} + \frac{(n+1)\log(2^6 h_0 h_1)}{2^6 h_0 h_1} , \\
h_8 &= c_2 n (q-1)\left(1 - \frac{1}{c_3 n}\right)\left(1 - \frac{1}{h_1}\right) ,
\end{aligned}
\qquad (3.4)
$$

where $\log (2^{11}nq)$ and $\log (2^6 h_0 h_1)$ denote the usual logarithms. (In the sequel, it is easy to distinguish from the context what the symbol log (or exp) means: the usual or p-adic logarithmic (or exponential) function.)

In this chapter we suppose c_0, c_1, c_2, c_3, c_4 are real numbers satisfying the following conditions (3.5), (3.6) and (3.7).

$$2 \le c_0 \le 2^4,\ 2 \le c_1 \le \frac{7}{2},\ \frac{8}{3} \le c_2 \le 14,\ 2^5 \le c_3 \le 2^8,\ 2^5 \le c_4 \le 2^8 ; \tag{3.5}$$

$$\begin{aligned}
&\left(1-\frac{1}{c_3 n}\right)\left(1-\frac{1}{h_1}\right)\left(1-\frac{1}{q}\right)^2 \\
&\ge \left(\frac{1}{h_6}+\frac{1}{h_7}\right)\left(1+\frac{1}{c_0-1}\right)c_1 + \left(1+\frac{1}{c_0-1}\right)\frac{1}{c_2} \\
&+ \left\{1+\left(1+\frac{1}{h_0}\right)\log 3\right\}\left(\frac{1}{q}+\frac{1}{c_0-1}\right)\left(2+\frac{1}{p-1}\right)\frac{1}{c_3} \\
&+ \left(1+\frac{1}{h_4}\right)\left\{4+\frac{1}{2^{10}nq}+\frac{2\log h_5}{h_0}+\frac{1}{n}\left(1+\frac{1}{c_0-1}\right)+\left(1+\frac{1}{p-1}\right)\frac{1}{q^n}\right\}\frac{1}{c_4} ;
\end{aligned} \tag{3.6}$$

$$\begin{aligned}
c_1 &\ge \left(1+\frac{1}{h_8}\right)\left(2+\frac{1}{p-1}\right) \\
&+ \left\{2+\frac{1}{h_8}+\frac{\log h_0}{h_0}+\frac{1}{h_0}\cdot\frac{\log q}{q}+\frac{n\log(h_0+1)}{h_0}\right\}\cdot\frac{2+\frac{1}{p-1}}{nq^n}\cdot\frac{1}{c_3} .
\end{aligned} \tag{3.7}$$

The existence of such real numbers $c_0, \ldots, c_4$ will be proved in Chapter V.

Put

$$V_{n-1}^* = \max\left(p^{f_\mathfrak{p}},\ \left(2^{11} nq^{\frac{n+1}{n-1}} D^{\frac{n}{n-1}} V_{n-1}^+\right)^n\right) , \tag{3.8}$$

$$W^* = \max\left(W,\ n\log(2^{11}nqD)\right) . \tag{3.9}$$

Let U be a real number satisfying

$$U \ge (1+\varepsilon_1)(1+\varepsilon_2)c_0 c_1 c_2^n c_3 c_4 \frac{n^{2n+1}}{n!} q^{2n}(q-1)\frac{G\left(2+\frac{1}{p-1}\right)^n}{e_\mathfrak{p}(f_\mathfrak{p}\log p)^{n+2}} D^{n+2} V_1 \cdots V_n W^* \log V_{n-1}^* . \tag{3.10}$$

<u>Proposition 1</u>. Suppose that (0.5)-(0.8) hold. Then

$$\operatorname{ord}_p(\alpha_1^{b_1} \cdots \alpha_n^{b_n} - 1) < U .$$

2. Notations

The following 8 formulas will be referred as (3.11).

$$Y = \frac{e_{\mathfrak{p}} f_{\mathfrak{p}} \log p}{q^n D} : U ,$$

$$S = q\left[\frac{c_3 n D W^*}{f_{\mathfrak{p}} \log p}\right] ,$$

$$T = \left[\frac{U f_{\mathfrak{p}} \log p}{q^n D} \cdot \frac{1}{c_1 c_3 W^* \theta}\right] = \left[\frac{Y}{c_1 c_3 W^* e_{\mathfrak{p}} \theta}\right] ,$$

$$L_{-1} = [W^*] , \qquad (3.11)$$

$$L_0 = \left[\frac{U e_{\mathfrak{p}} f_{\mathfrak{p}} \log p}{q^n D} \cdot \frac{1}{c_1 c_4 (L_{-1}+1) \log V_{n-1}^*}\right] = \left[\frac{Y}{c_1 c_4 (L_{-1}+1) \log V_{n-1}^*}\right] ,$$

$$L_j = \left[\frac{U e_{\mathfrak{p}} f_{\mathfrak{p}} \log p}{q^n D} \cdot \frac{1}{c_1 c_2 n p^{\kappa} S V_j}\right] = \left[\frac{Y}{c_1 c_2 n p^{\kappa} S V_j}\right] (1 \le j \le n) ,$$

$$L = \max_{1 \le j < n} L_j = L_1 (\text{see } (0.2))$$

$$X_0 = \left\{D \prod_{j=-1}^{n} (L_j + 1)\right\} 3^{T(L_{-1}+1)} \left(2e\left(2 + \frac{S}{L_{-1}+1}\right)\right)^{(L_{-1}+1)(L_0+1)} \left(1 + \frac{(n-1)(B_n L_1 + B' L_n)}{T}\right)^T .$$

$$\cdot \exp\left\{p^{\kappa} S \sum_{j=1}^{n} L_j V_j + nD \max_{1 \le j \le n} V_j\right\} .$$

For later convenience we proceed to prove the following inequalities (3.12)-(3.27).

$$(L_{-1}+1)(L_0+1)\prod_{j=1}^{n}(L_j+1-G) \geq c_0G(1-\tfrac{1}{q})S\binom{T+n}{n}, \tag{3.12}$$

$$\tfrac{1}{n}q^{n-1}ST\theta > \left(1-\frac{1}{c_3n}\right)\left(1-\frac{1}{h_1}\right)\frac{1}{c_1}U, \tag{3.13}$$

$$p^{\kappa}S\sum_{j=1}^{n}L_jV_j \leq \frac{1}{c_1c_2}Y, \tag{3.14}$$

$$T(L_{-1}+1) \leq (1+\frac{1}{h_0})(2+\frac{1}{p-1})\frac{1}{c_1c_3}Y, \tag{3.15}$$

$$T\log\left(1+\frac{(n-1)q(B_nL_1+B'L_n)}{T}\right) \leq \left(2+\frac{1}{p-1}\right)\frac{1}{c_1c_3}Y, \tag{3.16}$$

$$(L_{-1}+1)(L_0+1)\left(\theta+\frac{1}{p-1}\right) \leq \left(1+\frac{1}{h_4}\right)\left(1+\frac{1}{p-1}\right)\frac{1}{q^n}\cdot\frac{1}{c_1c_4}U, \tag{3.17}$$

$$(L_{-1}+1)(L_0+1)\log\left(2e\left(2+\frac{S}{L_{-1}+1}\right)\right) \leq \left(1+\frac{1}{h_4}\right)\frac{1}{n}\cdot\frac{1}{c_1c_4}Y, \tag{3.18}$$

$$(L_{-1}+1)(L_0+1)\log(qL_n) \leq \left(1+\frac{1}{h_4}\right)\left(2+\frac{1}{2^{11}nq}+\frac{\log h_5}{h_0}\right)\frac{1}{c_1c_4}Y, \tag{3.19}$$

$$n\,D\max_{1\leq j\leq n}V_j \leq \frac{1}{h_6}Y, \tag{3.20}$$

$$\log(D(L_{-1}+1)\dots(L_n+1)) \leq \frac{1}{h_7}Y, \tag{3.21}$$

$$\frac{T\log(L_{-1}+1)}{\log p} \leq \frac{\log(h_0+1)}{h_0}\cdot\frac{2+\frac{1}{p-1}}{q^n}\cdot\frac{1}{c_1c_3}U. \tag{3.22}$$

In (3.23)-(3.25), J, k are integers with $0 \leq J \leq \left[\frac{\log L_n}{\log q}\right]$, $0 \leq k \leq n-1$.

$$\left(\left(1-\frac{1}{q}\right)\frac{1}{n}q^{-J}T+1\right)\mathrm{ord}_p b_n \leq \left(1+\frac{1}{h_8}\right)\frac{2+\frac{1}{p-1}}{nq^n}\cdot\frac{1}{c_1c_3}U, \tag{3.23}$$

$$\left(\left(1-\frac{1}{q}\right)\frac{1}{n}q^{-J}T+1\right)q^{J+k}S\left(\frac{1}{p-1}+\left(1-\frac{1}{q}\right)\theta\right) \leq \left(1+\frac{1}{h_8}\right)\left(2+\frac{1}{p-1}\right)\frac{1}{c_1}U, \tag{3.24}$$

$$\left(1-\frac{1}{q}\right)\frac{1}{n} q^{-J_T} \frac{\log(q^{J+k}S)}{\log p} \leq \left(1+\frac{\log h_0}{h_0} + \frac{1}{h_0} \cdot \frac{\log q}{q}\right)^{2+\frac{1}{p-1}} \frac{1}{nq^n} \cdot \frac{1}{c_1c_3} U , \tag{3.25}$$

$$L_1 + \ldots + L_{n-1} < \frac{1}{2} T , \tag{3.26}$$

$$(L_{-1}+1)(L_0+1) < \frac{1}{4} ST . \tag{3.27}$$

<u>Proof of (3.12)</u>. By the facts that $p^{\kappa} \leq \left(2+\frac{1}{p-1}\right)e_{\mathfrak{p}}$ (see (1.5)), $DV_j \geq f_{\mathfrak{p}} \log p$ $(1 \leq j \leq n)$ (see (0.2)), $\log V_{n-1}^{*} \geq f_{\mathfrak{p}} \log p$ (see (3.8)), $D \geq e_{\mathfrak{p}}$ and (3.10), we see that

$$\frac{U e_{\mathfrak{p}} f_{\mathfrak{p}} \log p}{q^n D c_1 c_2 n p^{\kappa} S V_j G} \geq h_2 \quad (1 \leq j \leq n) , \tag{3.28}$$

whence

$$\begin{aligned} L_j+1-G &> \frac{U e_{\mathfrak{p}} f_{\mathfrak{p}} \log p}{q^n D c_1 c_2 n p^{\kappa} S V_j} - G \\ &\geq \frac{U e_{\mathfrak{p}} f_{\mathfrak{p}} \log p}{q^n D c_1 c_2 n p^{\kappa} S V_j} \left(1-\frac{1}{h_2}\right) \quad (1 \leq j \leq n) . \end{aligned} \tag{3.29}$$

By (3.29), (3.11) and $1 + \varepsilon_1 = \left(1-\frac{1}{h_2}\right)^{-n}$ (see (3.4)) we get

$$\begin{aligned} &(L_{-1}+1)(L_0+1)\prod_{j=1}^{n}(L_j+1-G) \\ &> \left(\frac{U e_{\mathfrak{p}} f_{\mathfrak{p}} \log p}{q^n D}\right)^{n+1} \frac{1}{c_1 c_4 \log V_{n-1}^{*}} \cdot \frac{1}{(c_1 c_2 n p^{\kappa} S)^n V_1 \ldots V_n} (1+\varepsilon_1)^{-1} . \end{aligned} \tag{3.30}$$

Further, by (0.2) and by the facts that $\log V_{n-1}^{*} \geq h_0$ (see (3.8)), $G = p^{f_{\mathfrak{p}}}-1 \geq f_{\mathfrak{p}} \log p$, $D \geq e_{\mathfrak{p}}$, $\theta \leq 1$ (see (1.6)), we obtain

$$\frac{U\ f_{\mathfrak{p}} \log p}{q^n D c_1 c_3 W^* \theta} \geq h_1 \ . \tag{3.31}$$

This and (3.11) yield

$$\frac{n^2}{T} \leq \frac{n^2}{h_1 - 1} = \frac{1}{h_3} \ ,$$

whence by $1 + \varepsilon_2 = e^{h_3^{-1}}$ (see (3.4))

$$\binom{T+n}{n} \leq \left(1 + \frac{n}{T}\right)^n \frac{T^n}{n!} \leq \exp\left(\frac{n^2}{T}\right) \frac{T^n}{n!}$$

$$\leq e^{h_3^{-1}} \frac{T^n}{n!} = (1 + \varepsilon_2) \frac{T^n}{n!} \ .$$

Thus

$$c_0 G\left(1 - \frac{1}{q}\right) S \binom{T+n}{n} \leq (1+\varepsilon_2) c_0 G\left(1 - \frac{1}{q}\right) S \frac{T^n}{n!} \ . \tag{3.32}$$

By (3.11) we have

$$ST \leq \frac{n}{q^{n-1}\theta} \cdot \frac{1}{c_1} U \ , \quad S \leq \frac{c_3 n q D W^*}{f_{\mathfrak{p}} \log p} \ . \tag{3.33}$$

In virtue of (3.30), (3.32), (3.33), to prove (3.12) it suffices to show

$$U \geq (1+\varepsilon_1)(1+\varepsilon_2) c_0 c_1 c_2^n c_3 c_4 \frac{n^{2n+1}}{n!} q^{2n} (q-1) \frac{G}{e_{\mathfrak{p}} (f_{\mathfrak{p}} \log p)^{n+2}} \left(\frac{p^{\kappa}}{e_{\mathfrak{p}} \theta}\right)^n \cdot$$

$$\cdot D^{n+2} V_1 \ldots V_n W^* \log V_{n-1}^* \ . \tag{3.34}$$

On noting

$$\frac{p^{\kappa}}{e_{\mathfrak{p}}\theta} \le 2 + \frac{1}{p-1} \tag{3.35}$$

by (1.5) and (1.6), (3.34) follows from (3.10) at once. This proves (3.12).

Proof of (3.13). By (3.11), (0.4) and (3.9), we have

$$S > q\left(\frac{c_3 nDW^*}{f_{\mathfrak{p}}\log p} - 1\right) \ge \frac{c_3 nqDW^*}{f_{\mathfrak{p}}\log p}\left(1 - \frac{1}{c_3 n}\right) . \tag{3.36}$$

By (3.31) we get

$$T > \frac{U\ f_{\mathfrak{p}}\log p}{q^n Dc_1 c_3 W^* \theta} - 1 \ge \frac{U\ f_{\mathfrak{p}}\log p}{q^n Dc_1 c_3 W^* \theta}\left(1 - \frac{1}{h_1}\right) . \tag{3.37}$$

Now (3.36) and (3.37) imply (3.13) immediately.

(3.14) is a direct consequence of the definition of L_j $(1 \le j \le n)$ (see (3.11)).

Proof of (3.15). By (3.9), $W^* \ge h_0$. Hence we see, by (3.11) and (3.35), that

$$T(L_{-1}+1) \le \frac{Y(W^*+1)}{c_1 c_3 W^* e_{\mathfrak{p}}\theta} \le \left(1 + \frac{1}{h_0}\right)\left(2 + \frac{1}{p-1}\right)\frac{1}{c_1 c_3} Y .$$

Proof of (3.16). By (3.4), (3.5), we have $h_1 > 32n$, $c_3 \ge 32$. Hence

$$\left(1-\frac{1}{c_3 n}\right)\left(1-\frac{1}{h_1}\right) > 1-\frac{1}{c_3 n} - \frac{1}{h_1} > 1 - \frac{1}{n} .$$

By (1.6),

$$\frac{e_{\mathfrak{p}}\theta}{p^{\kappa}} < \frac{p-1}{p} < 1 .$$

So by (3.11) and (3.13) we see that

$$\frac{(n-1)qL_j}{T} \leq (n-1)q\,\frac{U\, e_{\mathfrak{p}} f_{\mathfrak{p}} \log p}{q^n D} \cdot \frac{1}{c_1 c_2 n p^{\kappa} STV_j}$$

$$\leq \frac{e_{\mathfrak{p}}\theta}{p^{\kappa}} \cdot \frac{f_{\mathfrak{p}}\log p}{DV_j} \cdot \frac{n-1}{c_2 n^2\left(1-\frac{1}{c_3 n}\right)\left(1-\frac{1}{h_1}\right)}$$

$$\leq \frac{1}{c_2 n} \cdot \frac{f_{\mathfrak{p}}\log p}{DV_j}. \tag{3.38}$$

Hence, on noting that $c_2 \geq \frac{8}{3}$ (see (3.5)), we get

$$\log\left(1+\frac{(n-1)q(B_n L_1 + B'L_n)}{T}\right) \leq \log\left(1+\frac{1}{c_2 n}\,\frac{f_{\mathfrak{p}}\log p}{D}\left(\frac{B_n}{V_1}+\frac{B'}{V_n}\right)\right)$$

$$\leq \log\left(1+\frac{3}{8n}\,\frac{f_{\mathfrak{p}}\log p}{D}\left(\frac{B_n}{V_1}+\frac{B'}{V_n}\right)\right) \leq W \leq W^* .$$

This together with (3.35) implies (3.16):

$$T\log\left(1+\frac{(n-1)q(B_n L_1 + B'L_n)}{T}\right) \leq TW^* \leq \frac{Y}{c_1 c_3 e_{\mathfrak{p}}\theta} \leq \left(2+\frac{1}{p-1}\right)\frac{1}{c_1 c_3} .$$

In order to prove (3.17)-(3.19), we first establish

$$(L_{-1}+1)(L_0+1) \leq \left(1+\frac{1}{h_4}\right)\frac{1}{\log V^*_{n-1}} \cdot \frac{Y}{c_1 c_4} . \tag{3.39}$$

By $DV_j \geq f_{\mathfrak{p}}\log p$ (see (0.2)), $W^* \geq h_0$ (see (3.9)) and $G = p^{f_{\mathfrak{p}}}-1 \geq f_{\mathfrak{p}}\log p$, we have

$$\frac{Y}{c_1 c_4 (L_{-1}+1)\log V^*_{n-1}} \geq h_4 ,$$

whence

$$L_0+1 \leqq \frac{Y}{c_1c_4(L_{-1}+1)\log V^*_{n-1}}\left(1+\frac{1}{h_4}\right)$$

and (3.39) follows at once.

Proof of (3.17). By $\log V^*_{n-1} \geq f_{\mathfrak{p}}\log p$ (see (3.8)) and $e_{\mathfrak{p}} \leqq D$, we have

$$\frac{Y}{\log V^*_{n-1}} = \frac{U\, e_{\mathfrak{p}}f_{\mathfrak{p}}\log p}{q^nD\log V^*_{n-1}} \leq \frac{U}{q^n} .$$

On noting the above inequality and the fact that $\theta \leqq 1$ (see (1.5), (1.6)), (3.39) implies (3.17) immediately.

Proof of (3.18). Note that by (3.11) and (3.5)

$$2e\left(2+\frac{S}{L_{-1}+1}\right) \leq 2e\left(2+\frac{S}{W^*}\right) \leq 2e\left(2+\frac{c_3nqD}{f_{\mathfrak{p}}\log p}\right)$$

$$\leq 2e\left(2+\frac{2^8nqD}{\log 2}\right) \leqq 2e\left(1+\frac{2^8}{\log 2}\right)nq\,D$$

$$\leq 2^{11}nq\,D \leq \left(V^*_{n-1}\right)^{\frac{1}{n}} ,$$

where the last inequality follows from (3.8). This and (3.39) imply (3.18).

Proof of (3.19). By (3.10), (3.11) and (3.36), we see that

$$qL_n \leq \frac{(1+\varepsilon_1)(1+\varepsilon_2)c_0c_4}{\left(1-\frac{1}{c_3n}\right)}\, c_2^{n-1}\frac{n^{2n-1}}{n!}\, q^n(q-1)\frac{G\left(2+\frac{1}{p-1}\right)^n}{p^{\kappa}(f_{\mathfrak{p}}\log p)^n}\, D^nV_1\cdots V_{n-1}\log V^*_{n-1} .$$

On noting the facts that $c_4 \leq 2^8$, $c_2 \leqq 14$ (see (3.5)),

$n! \geq \sqrt{2\pi n}\, n^n e^{-n}$ (Lemma 2.7) and $V_1 \leq \ldots \leq V_{n-1} \leq V_{n-1}^+$ (see (0.2)), the above inequality gives

$$qL_n \leq \frac{(1+\varepsilon_1)(1+\varepsilon_2)c_0 2^8}{\sqrt{2\pi n}\left(1 - \frac{1}{c_3 n}\right)} \left(14\left(\frac{3\,e}{\log\, 2}\right)^{\frac{n}{n-1}} nq^{\frac{n+1}{n-1}} D^{\frac{n}{n-1}}\, V_{n-1}^+\right)^{n-1} G \log V_{n-1}^* \,.$$

It is easy to check that

$$14\left(\frac{3e}{\log\, 2}\right)^{\frac{n}{n-1}} \leq 14\left(\frac{3e}{\log\, 2}\right)^2 < 2^{11} \,.$$

So by the definitions of V_{n-1}^* (see (3.8)) and h_5 (see (3.4)), we get

$$qL_n \leq h_5 (V_{n-1}^*)^{2-\frac{1}{n}} \log V_{n-1}^* \,. \tag{3.40}$$

Now we show that

$$\log V_{n-1}^* \leq (V_{n-1}^*)^{\frac{1+c}{n}} \quad \text{with} \quad c = \frac{1}{2^{11}q} \,. \tag{3.41}$$

Put

$$g(x) = 2^{11} q^{\frac{x+1}{x-1}} D^{\frac{x}{x-1}} V_{n-1}^+ \geq 2^{11} q \quad \text{for} \quad x \geq 2 \,.$$

By (3.8)

$$V_{n-1}^* \geq (ng(n))^n > n^n \,. \tag{3.42}$$

Note that

$$\left(x^{\frac{1+c}{n}} - \log x\right)' > 0 \quad \text{for} \quad x > n^n . \tag{3.43}$$

By (3.42) and (3.43) we see that, in order to prove (3.41), it suffices to show

$$\left(\left(ng(n)\right)^n\right)^{\frac{1+c}{n}} \geq \log(ng(n))^n \quad (n \geq 2)$$

or equivalently

$$n^c(g(n))^{1+c} \geq \log n + \log g(n) \quad (n \geq 2) . \tag{3.44}$$

Now by $g(x) \geqq 2^{11}q \ (x \geq 2)$ and recalling $c = \frac{1}{2^{11}q}$ (see (3.41)), we obtain

$$n^c(g(n))^{1+c} - \log g(n) \geq (n^c-1)(g(n))^{1+c}$$

$$\geq (n^c-1)2^{11}q \geq 2^{11}q \ c \log n = \log n .$$

This proves (3.44), whence (3.41) follows. On combining (3.39), (3.40), (3.41) and noting $\log V_{n-1}^* \geq h_0$ (see (3.8)), we obtain (3.19).

<u>Proof of (3.20)</u>. By (3.8)-(3.11), (3.5), (3.4) and $G = p^{f_p}-1 \geq f_p \log p$, it is readily verified that

$$Y \geq 2^6 h_1 D \max(h_0, \max_{1 \leq j \leq n} V_j) . \tag{3.45}$$

This implies (3.20).

Proof of (3.21). Since $n \geq 2$, $q \geq 3$ (see (0.1)), we have, by (3.4), (3.5),

$$h_0 \geq 18.83, \ h_2 \geq 2^{13}, \ h_4 \geq 2^{19} \times \frac{18.83}{19.83} \ . \tag{3.46}$$

By (3.39), (3.46), (3.5) and $\log V^*_{n-1} \geq h_0$ (see (3.8)), we get

$$(L_{-1}+1)(L_0+1) \leq \frac{1}{2^6 \times 18.83} \times \left(1+\frac{19.83}{2^{19}\times 18.83}\right) Y \ . \tag{3.47}$$

By (3.36), (0.2), (3.5), (3.9) we see that

$$(c_1c_2np^{\kappa}S)^n V_1 \ldots V_n \geq (c_1c_2n)^n\left(1-\frac{1}{c_3n}\right)^n (c_3nqW^*)^n \frac{D^n V_1 \ldots V_n}{(f_p \log p)^n}$$

$$\geq (c_1c_2(c_3n^2-n)qW^*)^n \geq (2\times\tfrac{8}{3}\times(2^7-2)\times 3\times 18.83)^n$$

$$= (37961.28)^n \ . \tag{3.48}$$

Now (3.28) yields

$$L_j+1 \leq \frac{Y}{c_1c_2np^{\kappa}SV_j}\left(1+\frac{1}{h_2}\right) \ (1 \leq j \leq n) \ ,$$

whence on applying (3.46) and (3.48), we get

$$(L_1+1)\ldots(L_n+1) \leq \left(\frac{1+2^{-13}}{37961.28}\right)^n Y^n \leq \left(\frac{1+2^{-13}}{37961.28}\right)^2 Y^n \ . \tag{3.49}$$

(3.47) and (3.49) imply

$$D(L_{-1}+1)\ldots(L_n+1) \leq 5.76 \times 10^{-13} Y^{n+1} D \ .$$

This together with (3.45) implies

$$\frac{\log(D(L_{-1}+1)\ldots(L_n+1))}{Y} \le \frac{\log(5.76\times10^{-13}D)}{Y} + (n+1)\frac{\log Y}{Y}$$

$$\le \frac{5.76\times10^{-13}D}{2^6 h_0 h_1 D} + (n+1)\,\frac{\log(2^6 h_0 h_1)}{2^6 h_0 h_1} = \frac{1}{h_7}\ .$$

<u>Proof of (3.22)</u>. By the facts that

$$\left(\frac{\log(x+1)}{x}\right)' < 0 \quad \text{for} \quad x \ge 2$$

and $W^* \ge h_0$ (see (3.9)), and by (3.11), (3.35), we see that

$$\frac{T\log(L_{-1}+1)}{\log p} \le \frac{U}{q^n}\cdot\frac{f_p}{D\theta}\cdot\frac{\log(W^*+1)}{W^*}\cdot\frac{1}{c_1c_3}$$

$$\le \frac{U}{q^n}\cdot\frac{1}{e_p\theta}\cdot\frac{\log(h_0+1)}{h_0}\cdot\frac{1}{c_1c_3}$$

$$\le \frac{\log(h_0+1)}{h_0}\cdot\frac{2+\frac{1}{p-1}}{q^n}\cdot\frac{1}{c_1c_3}\,U\ .$$

<u>Proof of (3.23)</u>. We may assume $\operatorname{ord}_p b_n \ne 0$, since if $\operatorname{ord}_p b_n = 0$ (3.23) is trivial. By (0.7), we have

$$\operatorname{ord}_p b_n \le \frac{\log B_0}{\log p} \le \frac{W}{\log p} \le \frac{W^*}{\log p}\ .$$

By (0.2), (3.38) (using its second line) and the fact that $\frac{p^\kappa}{e_p\theta} > 1$, we see that

$$(1-\tfrac{1}{q})\tfrac{1}{n}\, q^{-J}T \ge \left(1-\tfrac{1}{q}\right)\frac{1}{n}\cdot\frac{T}{L_n} \ge h_8\ . \tag{3.50}$$

So by $e_{\mathfrak{p}}f_{\mathfrak{p}} \le D$, (3.35) and (3.50), we obtain

$$\left(\left(1-\frac{1}{q}\right)\frac{1}{n}\, q^{-J}T + 1\right)\mathrm{ord}_p b_n \le \left(1-\frac{1}{q}\right)\frac{1}{n}\, q^{-J}T\left(1+\frac{1}{h_8}\right)\frac{W^*}{\log p}$$

$$\le \left(1+\frac{1}{h_8}\right)\frac{2+\frac{1}{p-1}}{nq^n}\cdot\frac{1}{c_1c_3}\, U\ .$$

Proof of (3.24). By (1.5), (1.6) we have $\theta > \frac{1}{p}$, whence $\frac{1}{p-1} < \frac{p}{p-1}\theta$ and

$$\frac{1}{p-1} + \left(1-\frac{1}{q}\right)\theta < \left(2+\frac{1}{p-1} - \frac{1}{q}\right)\theta < \left(2+\frac{1}{p-1}\right)\theta\ .$$

By (3.50), (3.11) and the fact that $k \le n-1$, we see that

$$\left(\left(1-\frac{1}{q}\right)\frac{1}{n}\, q^{-J}T + 1\right)q^{J+k}S\left(\frac{1}{p-1} + \left(1-\frac{1}{q}\right)\theta\right)$$

$$< \left(1-\frac{1}{q}\right)\frac{1}{n}\, q^{-J}T\left(1+\frac{1}{h_8}\right)q^{J+n-1}S\left(2+\frac{1}{p-1}\right)\theta$$

$$< \left(1+\frac{1}{h_8}\right)\left(2+\frac{1}{p-1}\right)\frac{1}{n}\, q^{n-1}ST\theta$$

$$\le \left(1+\frac{1}{h_8}\right)\left(2+\frac{1}{p-1}\right)\frac{1}{c_1}\, U\ .$$

Proof of (3.25). By (3.9), $W^* \ge n\log(2^{11}nqD) \ge h_0$. So by (3.5) we get

$$\log\left(\frac{c_3 nqD}{f_{\mathfrak{p}}\log p}\right) \le \log\left(\frac{2^8}{\log 2}\, nqD\right) < \log\,(2^{11}nqD) \le \frac{1}{n}\, W^*$$

and

$$\frac{\log S}{W^*} \le \frac{1}{W^*}\left(\log\left(\frac{c_3 nqD}{f_{\mathfrak{p}}\log p}\right) + \log W^*\right) \le \frac{1}{n} + \frac{\log h_0}{h_0} \quad .$$

Hence, by $e_{\mathfrak{p}} f_{\mathfrak{p}} \le D$ and (3.35), we obtain

$$T\,\frac{\log S}{\log p} \le \frac{f_{\mathfrak{p}}}{D\theta}\cdot\frac{1}{q^n}\cdot\frac{U}{c_1c_3}\cdot\frac{\log S}{W^*} \le \left(\frac{1}{n}+\frac{\log h_0}{h_0}\right)\frac{2+\frac{1}{p-1}}{q^n}\cdot\frac{1}{c_1c_3}\,U\;. \tag{3.51}$$

Similarly, by the fact that $k \le n-1$,

$$T\,\frac{\log q^k}{\log p} \le \frac{(n-1)T\log q}{\log p} \le \frac{(n-1)T}{\log p}\cdot\frac{1}{n}\,W^* \le \left(1-\frac{1}{n}\right)\frac{U}{q^n c_1 c_3}\cdot\frac{f_{\mathfrak{p}}}{D\theta}$$

$$\le \left(1-\frac{1}{n}\right)\frac{2+\frac{1}{p-1}}{q^n}\cdot\frac{1}{c_1c_3}\,U\;. \tag{3.52}$$

On noting that

$$\frac{\log q^J}{q^J} \le \frac{\log q}{q} \quad \text{for} \quad J \ge 0\;,$$

we get (again by $e_{\mathfrak{p}} f_{\mathfrak{p}} \le D$, (3.35) and (3.9))

$$\frac{T}{\log p}\cdot\frac{\log q^J}{q^J} \le \frac{U f_{\mathfrak{p}}}{q^n D c_1 c_3 W^* \theta}\cdot\frac{\log q}{q} \le \frac{1}{h_0}\cdot\frac{\log q}{q}\cdot\frac{2+\frac{1}{p-1}}{q^n}\cdot\frac{1}{c_1c_3}\,U\;. \tag{3.53}$$

It follows from (3.51)-(3.53) that

$$\left(1-\frac{1}{q}\right)\frac{1}{n}\,q^{-J}T\,\frac{\log(q^{J+k}S)}{\log p}$$

$$\le \frac{1}{n}\left(T\,\frac{\log S}{\log p} + T\,\frac{\log q^k}{\log p} + \frac{T}{\log p}\cdot\frac{\log q^J}{q^J}\right)$$

$$\le \left(1+\frac{\log h_0}{h_0} + \frac{1}{h_0}\cdot\frac{\log q}{q}\right)\frac{2+\frac{1}{p-1}}{nq^n}\cdot\frac{1}{c_1c_3}\,U\;.$$

<u>Proof of (3.26)</u>. By (3.38), (0.2) and $c_2 \geq \frac{8}{3}$ (see (3.5)), $q \geq 3$ (see (0.1)), we have

$$\frac{L_j}{T} \leq \frac{1}{c_2 n(n-1)q} \leq \frac{1}{8n(n-1)} \quad (1 \leq j \leq n) \ .$$

Hence

$$\frac{L_1 + \dots + L_{n-1}}{T} \leq \frac{1}{8n} < \frac{1}{2} \ .$$

This proves (3.26).

<u>Proof of (3.27)</u>. By (3.13) and the facts that $\theta \leq 1$ (see (1.5), (1.6)), $c_3 \geq 2^5$, $h_1 > 2$ (see (3.4), (3.5)), we have

$$\begin{aligned} \frac{1}{4}\ ST &> \frac{1}{q^{n-1}\theta} \cdot \frac{1}{4} \cdot \left(n-\frac{1}{c_3}\right)\left(1-\frac{1}{h_1}\right) \frac{1}{c_1}\ U \\ &> \frac{1}{4} \times (2-1) \times \left(1-\frac{1}{2}\right) \cdot \frac{U}{q^{n-1}c_1} \\ &= \frac{U}{8q^{n-1}c_1} \ . \end{aligned}$$

On the other hand, by (3.39) and the facts that $h_4 \geq 1$ (see (3.46)), $\log V_{n-1}^* \geq f_p \log p$ (see (3.8)), $c_4 \geq 2^5$ (see (3.5)), $q \geq 3$ (see (0.1)), we obtain

$$\begin{aligned} (L_{-1}+1)(L_0+1) &\leq \frac{2}{c_1 c_4} \cdot \frac{1}{\log V_{n-1}^*} \cdot \frac{U\ e_p f_p \log p}{q^n D} \\ &\leq \frac{1}{16q} \cdot \frac{U}{q^{n-1}c_1} \\ &\leq \frac{U}{48\ q^{n-1}c_1} \ . \end{aligned}$$

Now (3.27) follows from the above two inequalities.

So far we have established the inequalities (3.12)-(3.27). Now we introduce two more notations. For $(J,\lambda_{-1},\dots,\lambda_n, \tau_0,\dots,\tau_{n-1}) \in \mathbb{N}^{2n+3}$ set

$$\Lambda_J(z,\tau) = \Delta(q^{-J}z+\lambda_{-1};L_{-1}+1,\lambda_0+1,\tau_0) \prod_{j=1}^{n-1} \Delta(b_n\lambda_j-b_j\lambda_n;\tau_j) , \qquad (3.54)$$

where $\Delta(z;k)$ and $\Delta(z;k,l,m)$ are defined by (2.11) and (2.12). In the sequel of this chapter, we abbreviate $(\lambda_{-1},\dots,\lambda_n)$ as λ, $(\tau_0,\dots,\tau_{n-1})$ as τ and write $|\tau| = \tau_0 + \dots + \tau_{n-1}$. Using a remark from Mignotte and Waldschmidt [22], § 4.2, we can fix a basis $\xi_1,\dots,\xi_D$ of $K = \mathbb{Q}(\alpha_1,\dots,\alpha_n)$ over $\mathbb{Q}$ of the shape

$$\xi_d = \alpha_1^{k_{1d}} \dots \alpha_n^{k_{nd}} \quad \text{with} \quad (k_{1d},\dots,k_{nd}) \in \mathbb{N}^n$$

$$\text{and} \quad \sum_{j=1}^{n} k_{jd} \leq D-1 \quad (1 \leq d \leq D) . \qquad (3.55)$$

3. Construction of the rational integers $p_d(\lambda)$

We recall that $r_1,\dots,r_n$ are the rational integers introduced in the beginning of this chapter, $G = p^{f_\mathfrak{p}}-1$, X_0 is defined in (3.11).

Lemma 3.1. For $d = 1,\dots,D$ and $\lambda = (\lambda_{-1},\dots,\lambda_n)$ in the range

$$0 \leq \lambda_j \leq L_j (-1 \leq j \leq n), \; r_1\lambda_1 + \dots + r_n\lambda_n \equiv 0 \pmod G) \qquad (3.56)$$

there exist rational integers $p_d(\lambda)$ with

$$0 < \max_{d,\lambda} |p_d(\lambda)| \leq X_0^{\frac{1}{c_0-1}}$$

such that

$$\sum_{\lambda} \sum_{d=1}^{D} p_d(\lambda)\xi_d \Lambda_0(s,\tau) \prod_{j=1}^{n} \left(\alpha_j^{p^\kappa} \zeta^{r_j}\right)^{\lambda_j s} = 0 \tag{3.57}$$

for all $(s,\tau_0,\ldots,\tau_{n-1}) \in \mathbb{N}^{n+1}$ satisfying

$$1 \leq s \leq S \ , \ (s,q) = 1 \ , \ |\tau| \leq T \ ,$$

where $\sum_{\lambda}$ ranges over (3.56).

Remark. In the rest of this chapter s always denotes a rational integer and τ a point $(\tau_0,\ldots,\tau_{n-1}) \in \mathbb{N}^n$. The expression "for $(s,\tau_0,\ldots,\tau_{n-1}) \in \mathbb{N}^{n+1}$" will be omitted.

Proof. Write

$$P_{d,\lambda;s,\tau}(x_1,\ldots,x_n) = (\nu(L_{-1}+1))^{\tau_0}\Lambda_0(s,\tau)x_1^{p^\kappa\lambda_1 s+k_{1d}} \ldots x_n^{p^\kappa\lambda_n s+k_{nd}}$$

$$= (\nu(L_{-1}+1))^{\tau_0}\Delta(s+\lambda_{-1};L_{-1}+1,\lambda_0+1,\tau_0)\prod_{j=1}^{n-1}\Delta(b_n\lambda_j-b_j\lambda_n;\tau_j)\cdot\prod_{j=1}^{n} x_j^{p^\kappa\lambda_j s+k_{jd}}$$

for d,λ,s,τ with $1 \leq d \leq D$, λ in the range (3.56), $1 \leq s \leq S$, $(s,q) = 1$ and $|\tau| \leq T$. By Lemmas 2.3 and 2.4 we see that each $P_{d,\lambda;s,\tau}$ is a monomial in $x_1,\ldots,x_n$ with rational integer coefficient, whose absolute value is at most

$$3^{(L_{-1}+1)\tau_0} e^{T-\tau_0}\left(1+\frac{(n-1)(B_nL_1+B'L_n)}{T-\tau_0}\right)^{T-\tau_0} \left(2e\left(2+\frac{S}{L_{-1}+1}\right)\right)^{(L_{-1}+1)(L_0+1)}$$

$$\leq 3^{(L_{-1}+1)T}\left(1+\frac{(n-1)(B_nL_1+B'L_n)}{T}\right)^{T} \left(2e\left(2+\frac{S}{L_{-1}+1}\right)\right)^{(L_{-1}+1)(L_0+1)} .$$

Further

$$\deg_{x_j} P_{d,\lambda;s,\tau} \le p^{\kappa} S L_j + D \quad (1 \le j \le n) .$$

On noting that

$$\zeta^{r_1\lambda_1 s+\dots+r_n\lambda_n s} = \zeta^{(r_1\lambda_1+\dots+r_n\lambda_n)s} = 1$$

for $\lambda_1,\dots,\lambda_n$ satisfying the congruence in (3.56), we see that (3.57) is equivalent to

$$\sum_{\lambda}\sum_{d} P_{d,\lambda;s,\tau}(\alpha_1,\dots,\alpha_n)p_d(\lambda) = 0$$

$$1 \le s \le S,\ (s,q) = 1,\ |\tau| \le T . \qquad (3.57)'$$

In (3.57)' there are $\left(1-\frac{1}{q}\right)S\binom{T+n}{n}$ equations and at least

$$D(L_{-1}+1)(L_0+1)\prod_{j=1}^{n}\left[\frac{L_j+1}{G}\right]\cdot G^{n-1}\text{g.c.d.}(r_1,\dots,r_n,G)$$

$$\ge \frac{1}{G}D(L_{-1}+1)(L_0+1)\prod_{j=1}^{n}(L_j+1-G)$$

unknowns $p_d(\lambda)$. By (3.12), we can apply Lemma 2.2 to $\alpha_1,\dots,\alpha_n$, the field $K = \mathbb{Q}(\alpha_1,\dots,\alpha_n)$ and the polynomials $P_{d,\lambda;s,\tau}$. Then the lemma follows at once.

4. The main inductive argument

For rational integers $r^{(J)}$, $L_j^{(J)}$ $(-1 \le j \le n)$ and

$p_d^{(J)}(\lambda) = p_d^{(J)}(\lambda_{-1},\ldots,\lambda_n)$, which will be constructed in the following "main inductive argument", set

$$\varphi_J(z,\tau) = \sum_\lambda^{(J)} \sum_{d=1}^{D} p_d^{(J)}(\lambda)\,\xi_d \Lambda_J(z,\tau) \prod_{j=1}^{n} \left(\alpha_j^{p^\kappa} \zeta^{r_j}\right)^{\lambda_j z} , \tag{3.58}$$

where $\sum_\lambda^{(J)}$ is taken over the range of $\lambda = (\lambda_{-1},\ldots,\lambda_n)$:

$$0 \le \lambda_j \le L_j^{(J)} (-1 \le j \le n),\ r_1\lambda_1+\ldots+r_n\lambda_n \equiv r^{(J)} (\mathrm{mod}\ G) . \tag{3.59}$$

Note that, by (3.1), the p-adic functions

$$\left(\alpha_j^{p^\kappa} \zeta^{r_j}\right)^{\lambda_j z} = \exp\left(\lambda_j z \log\left(\alpha_j^{p^\kappa} \zeta^{r_j}\right)\right) \ (1 \le j \le n)$$

are normal.

<u>The main inductive argument</u>. Suppose that there are algebraic numbers $\alpha_1,\ldots,\alpha_n$ and rational integers $b_1,\ldots,b_n$ satisfying (0.5)-(0.8), such that

$$\mathrm{ord}_p\left(\alpha_1^{b_1} \ldots \alpha_n^{b_n} - 1\right) \ge U . \tag{3.60}$$

Then for every rational integer J with

$$0 \le J \le \left[\frac{\log L_n}{\log q}\right] + 1 ,$$

there exist rational integers $r^{(J)}$, $L_j^{(J)} (-1 \le j \le n)$ with

$$0 \le r^{(J)} < G \ ,\ \text{g.c.d.}\ (r_1,\ldots,r_n,G) \,|\, r^{(J)} ,$$

$$L_{-1}^{(J)} = L_{-1} \ ,\ L_0^{(J)} = L_0 \ ,\ 0 \le L_j^{(J)} \le q^{-J} L_j \ (1 \le j \le n) ,$$

and rational integers

$$p_d^{(J)}(\lambda) \quad \text{for} \quad d=1,\ldots,D \quad \text{and} \quad \lambda \quad \text{in the range (3.59)},$$

not all zero, with absolute values not exceeding $X_0^{\frac{1}{c_0-1}}$, such that

$$\varphi_J(s,\tau) = 0 \quad \text{for} \quad 1 \leq s \leq q^J S \ , \ (s,q) = 1 \ , \ |\tau| \leq q^{-J}T \ .$$

The main inductive argument will be proved by an induction on J . On taking $r^{(0)} = 0$, $L_j^{(0)} = L_j (-1 \leq j \leq n)$, $p_d^{(0)}(\lambda) = p_d(\lambda)$, which are constructed in Lemma 3.1, we see, by Lemma 3.1, that the case $J = 0$ is true. In the rest of this section, we suppose the main inductive argument is valid for some J with $0 \leq J \leq \left[\frac{\log L_n}{\log q}\right]$, we are going to prove it for $J + 1$. So we always keep the hypothesis (3.60). We first prove the following Lemmas 3.2, 3.3, 3.4, then deduce from Lemma 3.4 the main inductive argument for $J + 1$.

Let

$$\gamma_j = \lambda_j - \frac{b_j}{b_n}\lambda_n \quad (1 \leq j \leq n-1)$$

and

$$p^{(J)}(\lambda) = \sum_{d=1}^{D} p_d^{(J)}(\lambda)\xi_d \ .$$

Set

$$f_J(z,\tau) = \sum_{\lambda}^{(J)} \sum_{d=1}^{D} p_d^{(J)}(\lambda)\xi_d \Lambda_J(z,\tau) \prod_{j=1}^{n-1} \left(\alpha_j^{p^\kappa} \zeta^{r_j}\right)^{\gamma_j z}$$

$$= \sum_{\lambda}^{(J)} p^{(J)}(\lambda)\Lambda_J(z,\tau) \prod_{j=1}^{n-1} \left(\alpha_j^{p^\kappa} \zeta^{r_j}\right)^{\gamma_j z} .$$

Note that, by (3.1) and (0.7), the p-adic functions

$$\left(\alpha_j^{p^\kappa} \zeta^{r_j}\right)^{\gamma_j p^{-\theta} z} = \exp\left(\gamma_j p^{-\theta} z \log\left(\alpha_j^{p^\kappa} \zeta^{r_j}\right)\right) \quad (1 \le j \le n-1)$$

are normal.

<u>Lemma 3.2</u>. For any τ with $|\tau| \le T$ and any rational number $y > 0$ with $\mathrm{ord}_p y \ge 0$, we have

$$\mathrm{ord}_p(\varphi_J(y,\tau) - f_J(y,\tau)) \ge U - \frac{T \log(L_{-1}+1)}{\log p} - \mathrm{ord}_p b_n .$$

<u>Proof</u>. We first show that

$$b_1 r_1 + \ldots + b_n r_n \equiv 0 \pmod{G} . \tag{3.61}$$

We use the concept of congruence mod $\mathfrak{p}$ (introduced in § 1.3) on $O_{\mathfrak{p}} = \{\alpha \in K_{\mathfrak{p}} | \mathrm{ord}_{\mathfrak{p}}\alpha \ge 0\}$. Note that if $\alpha, \beta, \gamma, \delta$ in $O_{\mathfrak{p}}$ satisfy $\alpha \equiv \beta \pmod{\mathfrak{p}}$, $\gamma \equiv \delta \pmod{\mathfrak{p}}$, then $\alpha\beta \equiv \gamma\delta \pmod{\mathfrak{p}}$; and if $\mathrm{ord}_{\mathfrak{p}}\alpha = \mathrm{ord}_{\mathfrak{p}}\beta = 0$, $\alpha \equiv \beta \pmod{\mathfrak{p}}$, then $\alpha^{-1} \equiv \beta^{-1} \pmod{\mathfrak{p}}$. Hence from the congruences

$$\alpha_j \zeta^{r_j'} \equiv 1 \pmod{\mathfrak{p}} \quad (1 \le j \le n)$$

(see the beginning of this chapter) and $\mathrm{ord}_{\mathfrak{p}}\alpha_j = 0$ $(1 \le j \le n)$ (see (0.5)) we get

$$\alpha_j^{b_j} \zeta^{b_j r_j'} \equiv 1 \pmod{\mathfrak{p}} \quad (1 \le j \le n) ,$$

whence

$$\zeta^{-b_j r_j'} \equiv \alpha_j^{b_j} \pmod{\mathfrak{p}} , \quad (1 \le j \le n) .$$

This together with (3.60) and the fact that $U \geqq 2$ implies

$$\zeta^{-(b_1 r_1' + \ldots + b_n r_n')} \equiv \alpha_1^{b_1} \ldots \alpha_n^{b_n} \equiv 1 \pmod{\mathfrak{p}} .$$

Since $\zeta \in K_{\mathfrak{p}}$ is a primitive G-th root of unity, we obtain, by Hasse [16], p. 153, 155, 220,

$$b_1 r_1' + \ldots + b_n r_n' \equiv 0 \pmod{G} .$$

On recalling $r_j \equiv p^{\kappa} r_j' \pmod{G}$, (3.61) follows at once.

Next we show that

$$\operatorname{ord}_p\left(\prod_{j=1}^{n}\left(\alpha_j^{p^{\kappa}} \zeta^{r_j}\right)^{-\frac{b_j}{b_n}\lambda_n y} - 1\right) \geqq U - \operatorname{ord}_p b_n . \tag{3.62}$$

By (0.7), (3.1), § 1.1 (b), we see that

$$\operatorname{ord}_p\left(-\frac{b_j}{b_n}\lambda_n y \log\left(\alpha_j^{p^{\kappa}} \zeta^{r_j}\right)\right) \geqq \operatorname{ord}_p \log\left(\alpha_j^{p^{\kappa}} \zeta^{r_j}\right)$$

$$= \operatorname{ord}_p\left(\alpha_j^{p^{\kappa}} \zeta^{r_j} - 1\right) > \theta + \frac{1}{p-1} .$$

From this inequality and by § 1.1 (a), (b), (d), (3.61), (3.60) and the fact that $U \geq 16\ W^* \geq 16$, we obtain

$$\prod_{j=1}^{n} \left(\alpha_j^{p^\kappa} \zeta^{r_j}\right)^{-\frac{b_j}{b_n}\lambda_n y} = \prod_{j=1}^{n} \exp\left(-\frac{b_j}{b_n}\lambda_n y \log\left(\alpha_j^{p^\kappa} \zeta^{r_j}\right)\right)$$

$$= \exp\left(-\frac{\lambda_n}{b_n} y \sum_{j=1}^{n} b_j \log\left(\alpha_j^{p^\kappa} \zeta^{r_j}\right)\right)$$

$$= \exp\left(-\frac{\lambda_n}{b_n} y \sum_{j=1}^{n} \log\left(\alpha_j^{p^\kappa} \zeta^{r_j}\right)^{b_j}\right)$$

$$= \exp\left(-\frac{\lambda_n}{b_n} y \log \prod_{j=1}^{n} \left(\alpha_j^{p^\kappa} \zeta^{r_j}\right)^{b_j}\right)$$

$$= \exp\left(-\frac{\lambda_n}{b_n} y \log\left(\alpha_1^{b_1} \cdots \alpha_n^{b_n}\right)^{p^\kappa}\right)$$

$$= \exp\left(-\frac{\lambda_n}{b_n} y\, p^\kappa \log\left(\alpha_1^{b_1} \cdots \alpha_n^{b_n}\right)\right) .$$

On noting that if $\operatorname{ord}_p b_n > 0$

$$U - \operatorname{ord}_p b_n \geq U - \frac{\log B_0}{\log p} \geq U - 2W^* \geq \frac{7}{8}\, U > \frac{1}{p-1}$$

and using (3.60), § 1.1 (b), we get

$$\operatorname{ord}_p\left(-\frac{\lambda_n}{b_n} y\, p^\kappa \log\left(\alpha_1^{b_1} \cdots \alpha_n^{b_n}\right)\right) \geq \operatorname{ord}_p \log\left(\alpha_1^{b_1} \cdots \alpha_n^{b_n}\right) - \operatorname{ord}_p b_n$$

$$= \operatorname{ord}_p\left(\alpha_1^{b_1} \cdots \alpha_n^{b_n} - 1\right) - \operatorname{ord}_p b_n \geq U - \operatorname{ord}_p b_n$$

$$> \frac{1}{p-1} .$$

Therefore by § 1.1 (a)

$$\mathrm{ord}_p\left(\prod_{j=1}^{n}\left(\alpha_j^{p^\kappa}\zeta^{r_j}\right)^{-\frac{b_j}{b_n}\lambda_n y} - 1\right)$$

$$= \mathrm{ord}_p\left(\exp\left(-\frac{\lambda_n}{b_n}y\ p^\kappa \log\left(\alpha_1^{b_1} \cdots \alpha_n^{b_n}\right)\right) - 1\right)$$

$$= \mathrm{ord}_p\left(-\frac{\lambda_n}{b_n}y\ p^\kappa \log\left(\alpha_1^{b_1} \cdots \alpha_n^{b_n}\right)\right)$$

$$\geqq U - \mathrm{ord}_p b_n \ .$$

This proves (3.62).

We assert that

$$\mathrm{ord}_p\left(\alpha_j^{p^\kappa}\zeta^{r_j}\right)^{\lambda_j y} = 0 \quad (1 \leq j \leq n) \ ,$$

for the inequality

$$\mathrm{ord}_p\left(\lambda_j y \log\left(\alpha_j^{p^\kappa}\zeta^{r_j}\right)\right) \geqq \mathrm{ord}_p \log\left(\alpha_j^{p^\kappa}\zeta^{r_j}\right) = \mathrm{ord}_p\left(\alpha_j^{p^\kappa}\zeta^{r_j} - 1\right)$$

$$> \theta + \frac{1}{p-1}$$

implies

$$\mathrm{ord}_p\left(\left(\alpha_j^{p^\kappa}\zeta^{r_j}\right)^{\lambda_j y} - 1\right) = \mathrm{ord}_p\left(\exp\left(\lambda_j y \log\left(\alpha_j^{p^\kappa}\zeta^{r_j}\right)\right) - 1\right)$$

$$= \mathrm{ord}_p\left(\lambda_j y \log\left(\alpha_j^{p^\kappa}\zeta^{r_j}\right)\right) > \theta + \frac{1}{p-1} \ ,$$

whence

$$\mathrm{ord}_p\left(\alpha_j^{p^\kappa}\zeta^{r_j}\right)^{\lambda_j y} = \min\left\{\mathrm{ord}_p\ 1,\ \mathrm{ord}_p\left(\left(\alpha_j^{p^\kappa}\zeta^{r_j}\right)^{\lambda_j y} - 1\right)\right\} = 0 \ .$$

On combining the above assertion and (3.62), and noting, by § 1.1 (d), that

$$\prod_{j=1}^{n-1}\left(\alpha_j^{p^\kappa}\zeta^{r_j}\right)^{\gamma_j y} - \prod_{j=1}^{n}\left(\alpha_j^{p^\kappa}\zeta^{r_j}\right)^{\lambda_j y}$$
$$= \left\{\prod_{j=1}^{n-1}\left(\alpha_j^{p^\kappa}\zeta^{r_j}\right)^{\lambda_j y}\right\}\left(\prod_{j=1}^{n}\left(\alpha_j^{p^\kappa}\zeta^{r_j}\right)^{-\frac{b_j}{b_n}\lambda_n y} - 1\right),$$

we obtain

$$\mathrm{ord}_p\left\{\prod_{j=1}^{n-1}\left(\alpha_j^{p^\kappa}\zeta^{r_j}\right)^{\gamma_j y} - \prod_{j=1}^{n}\left(\alpha_j^{p^\kappa}\zeta^{r_j}\right)^{\lambda_j y}\right\} \geq U - \mathrm{ord}_p b_n . \qquad (3.63)$$

Write $y = \frac{k}{h}$, where $h > 0$, $k > 0$ are coprime rational integers. Then $\mathrm{ord}_p h = 0$, since $\mathrm{ord}_p y \geq 0$. Note also that $\mathrm{ord}_p q = 0$ (see (0.1)). Now by Lemma 2.3 we have

$$(q^J h)^{2(L_{-1}+1)(L_0+1)}(\nu(L_{-1}+1))^T \Lambda_J(y,\tau) \in \mathbb{Z},$$

whence

$$\mathrm{ord}_p \Lambda_J(y,\tau) \geq -T\ \mathrm{ord}_p \nu(L_{-1}+1) \geq -T\frac{\log(L_{-1}+1)}{\log p} . \qquad (3.64)$$

Obviously for any d with $1 \leq d \leq D$ and λ in the range (3.59), we have, by (0.5),

$$\mathrm{ord}_p(p_d^{(J)}(\lambda)\xi_d) \geq 0 . \qquad (3.65)$$

Now on noting

$$f_J(y,\tau)-\varphi_J(y,\tau) = \sum_\lambda^{(J)}\sum_{d=1}^{D} p_d^{(J)}(\lambda)\xi_d\Lambda_J(y,\tau)\left(\prod_{j=1}^{n-1}\left(\alpha_j^{p^\kappa}\zeta^{r_j}\right)^{\gamma_j y} - \prod_{j=1}^{n}\left(\alpha_j^{p^\kappa}\zeta^{r_j}\right)^{\lambda_j y}\right),$$

the lemma follows from (3.63)-(3.65) immediately.

Lemma 3.3. For $k = 0,1,\ldots,n-1$, we have

$$\varphi_J(s,\tau) = 0 \tag{3.66}$$

for $1 \le s \le q^{J+k}S$, $(s,q) = 1$, $|\tau| \le \left(1-(1-\frac{1}{q})\frac{k}{n}\right)q^{-J}T$.

Proof. We argue by a further induction on k . By the main inductive hypothesis for J , (3.66) with $k = 0$ is true. We assume (3.66) is valid for some k with $0 \le k \le n-2$, and we prove it for $k+1$. Thus, we see, by Lemma 3.2, that

$$\operatorname{ord}_p f_J(s,\tau) \ge U - T\,\frac{\log(L_{-1}+1)}{\log p} - \operatorname{ord}_p b_n \tag{3.67}$$

for $1 \le s \le q^{J+k}S$, $(s,q) = 1$, $|\tau| \le \left(1-(1-\frac{1}{q})\frac{k}{n}\right)q^{-J}T$.

Note that, by (3.1) and (0.7), the p-adic function

$$\prod_{j=1}^{n-1}\left(\alpha_j^{p^\kappa}\zeta^{r_j}\right)^{\gamma_j p^{-\theta}z}$$

is normal. Further by (2.15) and $\operatorname{ord}_p q = 0$ we see that

$$p^{(L_{-1}+1)(L_0+1)\theta}\left(\left(L_{-1}+1\right)!\right)^{L_0+1}\Lambda_J(p^{-\theta}z,\tau)$$

is a normal function, whence so is

$$p^{(L_{-1}+1)(L_0+1)\left(\theta+\frac{1}{p-1}\right)}\Lambda_J(p^{-\theta}z,\tau) \ .$$

Thus by the definition of $f_J(z,\tau)$,

$$F_J(z,\tau) = p^{(L_{-1}+1)(L_0+1)\left(\theta+\frac{1}{p-1}\right)} f_J\left(p^{-\theta}z,\tau\right) \tag{3.68}$$

for $|\tau| \le \left(1-\left(1-\frac{1}{q}\right)\frac{k+1}{n}\right) q^{-J}T$

are normal functions. We now apply Lemma 1.4 to each function $F_J(z,\tau)$ in (3.68), taking

$$R = q^{J+k}S \ , \ M = \left[\left(1-\frac{1}{q}\right)\frac{1}{n}\, q^{-J}T\right] + 1 \ . \tag{3.69}$$

Note that by (3.68)

$$\frac{1}{m!}\frac{d^m}{dz^m} F_J\left(sp^{\theta},\tau\right) = p^{(L_{-1}+1)(L_0+1)\left(\theta+\frac{1}{p-1}\right)-m\theta}\frac{1}{m!}\frac{d^m}{dz^m} f_J(s,\tau) \ . \tag{3.70}$$

It is also easy to verify that

$$\frac{1}{\mu_0!}\frac{d^{\mu_0}}{dz^{\mu_0}}\Delta\left(q^{-J}z+\lambda_{-1}\ ;\ L_{-1}+1,\ \lambda_0+1,\ \tau_0\right)$$

$$= q^{-J\mu_0}\binom{\tau_0+\mu_0}{\mu_0}\Delta\left(q^{-J}z+\lambda_{-1};\ L_{-1}+1,\ \lambda_0+1,\ \tau_0+\mu_0\right) \ . \tag{3.71}$$

Further we note that for any $t, m \in \mathbb{N}$, $\Delta(x;t)x^m$ is an integral valued polynomial of degree $t+m$, whence, by Lemma 2.6, there are $a_l^{(t,m)} \in \mathbb{Z}$ $(l = 0,1,\ldots,t+m)$, such that

$$\Delta(x;t)x^m = \sum_{l=0}^{t+m} a_l^{(t,m)} \Delta(x;l) \quad . \tag{3.72}$$

We abbreviate $(\mu_0,\ldots,\mu_{n-1}) \in \mathbb{N}^n$ to μ and write $|\mu|$ for $\mu_0 + \ldots + \mu_{n-1}$, and recall $p^{(J)}(\lambda) = \sum_{d=1}^{D} p_d^{(J)}(\lambda)\xi_d$, $\gamma_j = \lambda_j - \frac{b_j}{b_n}\lambda_n$. Now by (3.71), (3.72) we obtain

$$\frac{1}{m!}\frac{d^m}{dz^m} f_J(z,\tau)$$

$$= \sum_{|\mu|=m} \sum_{\lambda}^{(J)} p^{(J)}(\lambda)\left\{\frac{1}{\mu_0!}\frac{d^{\mu_0}}{dz^{\mu_0}} \Delta\left(q^{-J}z+\lambda_{-1};L_{-1}+1,\lambda_0+1,\tau_0\right)\right\}\prod_{j=1}^{n-1}\Delta(b_n\gamma_j;\tau_j)$$

$$\cdot \prod_{j=1}^{n-1}\frac{1}{\mu_j!}\frac{d^{\mu_j}}{dz^{\mu_j}}\left(\alpha_j^{p^\kappa}\ \xi^{r_j}\right)^{\gamma_j z}$$

$$= \sum_{|\mu|=m} q^{-J\mu_0}\binom{\tau_0+\mu_0}{\mu_0} b_n^{-(\mu_1+\ldots+\mu_{n-1})}\prod_{j=1}^{n-1}\frac{\left(\log\left(\alpha_j^{p^\kappa}\ \zeta^{r_j}\right)\right)^{\mu_j}}{\mu_j!} \cdot$$

$$\cdot \sum_{\lambda}^{(J)} p^{(J)}(\lambda)\Delta\left(q^{-J}z+\lambda_{-1};L_{-1}+1,\lambda_0+1,\tau_0+\mu_0\right)\left\{\prod_{j=1}^{n-1}\left(\Delta(b_n\gamma_j;\tau_j)(b_n\gamma_j)^{\mu_j}\right)\right\}\prod_{j=1}^{n-1}\left(\alpha_j^{p^\kappa}\ \zeta^{r_j}\right)^{\gamma_j z}$$

$$= \sum_{|\mu|=m} q^{-J\mu_0}\binom{\tau_0+\mu_0}{\mu_0} b_n^{-(m-\mu_0)}\left\{\prod_{j=1}^{n-1}\frac{\left(\log\left(\alpha_j^{p^\kappa}\ \zeta^{r_j}\right)\right)^{\mu_j}}{\mu_j!}\right\}\sum_{\sigma_1=0}^{\tau_1+\mu_1}\cdots\sum_{\sigma_{n-1}=0}^{\tau_{n-1}+\mu_{n-1}}\left\{\prod_{j=1}^{n-1} a_{\sigma_j}^{(\tau_j,\mu_j)}\right\} .$$

$$\cdot f_J(z,\tau_0+\mu_0,\sigma_1,\ldots,\sigma_{n-1}) \quad . \tag{3.73}$$

By (3.1) and § 1.1 (b),

$$\operatorname{ord}_p \prod_{j=1}^{n-1}\frac{\left(\log\left(\alpha_j^{p^\kappa}\ \zeta^{r_j}\right)\right)^{\mu_j}}{\mu_j!} \geq \sum_{j=1}^{n-1}\left\{\mu_j\left(\theta+\frac{1}{p-1}\right)-\frac{\mu_j}{p-1}\right\} \geq \theta(\mu_1+\ldots+\mu_{n-1}) \geq 0 .$$

For $|\tau| \leq \left(1-\left(1-\frac{1}{q}\right)\frac{k+1}{n}\right)q^{-J}T$, $|\mu| \leq m \leq M-1 = \left[\left(1-\frac{1}{q}\right)\frac{1}{n}\ q^{-J}T\right]$ and $(\sigma_1,\ldots,\sigma_{n-1}) \in \mathbb{N}^{n-1}$ with $\sigma_j \leq \tau_j + \mu_j$ $(1 \leq j \leq n-1)$, we have

$$\tau_0+\mu_0+\sigma_1+\ldots+\sigma_{n-1} \le \sum_{j=0}^{n-1}(\tau_j+\mu_j) = |\tau|+|\mu| \le \left(1-\left(1-\frac{1}{q}\right)\frac{k}{n}\right)q^{-J}T ,$$

whence by (3.67),

$$\mathrm{ord}_p\ f_J(s,\tau_0+\mu_0,\sigma_1,\ldots,\sigma_{n-1}) \ge U - \frac{T\log(L_{-1}+1)}{\log p} - \mathrm{ord}_p b_n$$

for all $f_J(z,\tau_0+\mu_0,\sigma_1,\ldots,\sigma_{n-1})$ appearing in (3.73) and $1 \le s \le q^{J+k}S$, $(s,q) = 1$. On combining the above observations, (3.73) yields

$$\mathrm{ord}_p\left(\frac{1}{m!}\frac{d^m}{dz^m}f_J(s,\tau)\right) \ge U - \frac{T\log(L_{-1}+1)}{\log p} - \left(\left(1-\frac{1}{q}\right)\frac{1}{n}q^{-J}T+1\right)\mathrm{ord}_p b_n$$

for $0 \le m \le M-1$, $1 \le s \le q^{J+k}S = R$, $(s,q) = 1$,
$|\tau| \le \left(1-\left(1-\frac{1}{q}\right)\frac{k+1}{n}\right)q^{-J}T$.

This together with (3.70), (3.22), (3.23) implies

$$\min_{\substack{1\le s\le R,\ (s,q)=1\\ 0\le t\le M-1}}\left\{\mathrm{ord}_p\left(\frac{1}{t!}\frac{d^t}{dz^t}F_J\left(sp^\theta,\tau\right)\right)+t\theta\right\}$$

$$\ge U + \left(L_{-1}+1\right)\left(L_0+1\right)\left(\theta+\frac{1}{p-1}\right) - \frac{T\log(L_{-1}+1)}{\log p} - \left(\left(1-\frac{1}{q}\right)\frac{1}{n}q^{-J}T+1\right)\mathrm{ord}_p b_n$$

$$\ge U - \left\{1+\frac{1}{h_8}+\frac{n\log(h_0+1)}{h_0}\right\}\frac{2+\frac{1}{p-1}}{nq^n}\cdot\frac{1}{c_1c_3}U \qquad (3.74)$$

for $|\tau| \le \left(1-\left(1-\frac{1}{q}\right)\frac{k+1}{n}\right)q^{-J}T$,

where

$$R = q^{J+k_S} , \quad M = \left[\left(1-\frac{1}{q}\right) \frac{1}{n}\, q^{-J_T}\right] + 1 , \quad \text{(see (3.69))}.$$

On the other hand, by (3.24), (3.25), we see that

$$\left(1-\frac{1}{q}\right)RM\theta + M \operatorname{ord}_p R! + (M-1) \frac{\log R}{\log p}$$

$$\le \left(\left(1-\frac{1}{q}\right)\frac{1}{n}q^{-J_T+1}\right)q^{J+k_S}\left(\left(1-\frac{1}{q}\right)\theta+\frac{1}{p-1}\right) + \left(1-\frac{1}{q}\right)\frac{1}{n}q^{-J_T} \frac{\log(q^{J+k_S})}{\log p}$$

$$\le \left(1+\frac{1}{h_8}\right)\left(2+\frac{1}{p-1}\right)\frac{1}{c_1}U + \left(1+\frac{\log h_0}{h_0} + \frac{1}{h_0} \cdot \frac{\log q}{q}\right)\frac{2+\frac{1}{p-1}}{nq^n} \cdot \frac{1}{c_1 c_3}U \ . \quad (3.75)$$

Now we see from (3.74), (3.75), (3.7) that each $F_J(z,\tau)$ in (3.68) satisfies the condition (1.9) with R,M given by (3.69). Thus by Lemma 1.4 and (3.68) we obtain

$$\operatorname{ord}_p f_J(s,\tau) \ge \operatorname{ord}_p F_J\left(sp^\theta,\tau\right) - \left(L_{-1}+1\right)\left(L_0+1\right)\left(\theta+\frac{1}{p-1}\right)$$

$$\ge \left(1-\frac{1}{q}\right) RM\,\theta - \left(L_{-1}+1\right)\left(L_0+1\right)\left(\theta+\frac{1}{p-1}\right)$$

$$> \left(1-\frac{1}{q}\right)^2 \frac{1}{n}\, q^{k_S T}\,\theta - \left(L_{-1}+1\right)\left(L_0+1\right)\left(\theta+\frac{1}{p-1}\right)$$

$$\text{for} \quad s \in \mathbb{Z}, \ |\tau| \le \left(1-\left(1-\frac{1}{q}\right)\frac{k+1}{n}\right)q^{-J_T} .$$

By Lemma 3.2 and again by (3.74), (3.75), (3.7) we get for $s \ge 1$

$$\operatorname{ord}_p\left(\varphi_J(s,\tau) - f_J(s,\tau)\right) \ge U - \frac{T\log(L_{-1}+1)}{\log p} - \operatorname{ord}_p b_n$$

$$> \left(1-\frac{1}{q}\right) RM\,\theta - \left(L_{-1}+1\right)\left(L_0+1\right)\left(\theta+\frac{1}{p-1}\right)$$

$$> \left(1-\frac{1}{q}\right)^2 \frac{1}{n}\, q^{k_S T}\,\theta - \left(L_{-1}+1\right)\left(L_0+1\right)\left(\theta+\frac{1}{p-1}\right) .$$

Hence

$$\begin{aligned}\mathrm{ord}_p\varphi_J(s,\tau) &\geq \min\Big(\mathrm{ord}_p f_J(s,\tau),\ \mathrm{ord}_p\Big(\varphi_J(s,\tau)-f_J(s,\tau)\Big)\Big)\\ &> \left(1-\frac{1}{q}\right)^2 \frac{1}{n}\, q^k ST\,\theta - \Big(L_{-1}+1\Big)\Big(L_0+1\Big)\Big(\theta+\frac{1}{p-1}\Big)\\ &\geq \frac{U}{c_1}\, q^{k+1-n}\left\{\left(1-\frac{1}{q}\right)^2\left(1-\frac{1}{c_3 n}\right)\left(1-\frac{1}{h_1}\right) - \frac{1}{q^{k+1}}\left(1+\frac{1}{h_4}\right)\left(1+\frac{1}{p-1}\right)\frac{1}{c_4}\right\}\end{aligned}$$

for $s \geq 1$, $|\tau| \leq \left(1-\left(1-\frac{1}{q}\right)\frac{k+1}{n}\right) q^{-J}T$, (3.76)

where the last inequality follows from (3.13) and (3.17).

On the other hand, by (3.59), we see that for $1 \leq s \leq q^{J+k+1}S$, $(s,q) = 1$, $|\tau| \leq \left(1-\left(1-\frac{1}{q}\right)\frac{k+1}{n}\right)\cdot q^{-J}T$,

$$\zeta^{-r(J)s}\, q^{J\cdot 2(L_{-1}+1)(L_0+1)}\Big(\nu(L_{-1}+1)\Big)^{\tau_0}\varphi_J(s,\tau)$$

$$= \sum_\lambda{}^{(J)} \sum_{d=1}^{D} p_d^{(J)}(\lambda)\, q^{2J(L_{-1}+1)(L_0+1)}\Big(\nu(L_{-1}+1)\Big)^{\tau_0}\Delta\Big(q^{-J}s+\lambda_{-1};L_{-1}+1,\lambda_0+1,\tau_0\Big)\cdot$$

$$\cdot\left\{\prod_{j=1}^{n-1}\Delta\Big(b_n\lambda_j - b_j\lambda_n;\tau_j\Big)\right\}\prod_{j=1}^{n}\alpha_j^{p^\kappa\lambda_j s+k_{jd}} ,$$

which is a polynomial (with rational integer coefficients) in $\alpha_1,\ldots,\alpha_n$, of degree at most

$$p^\kappa L_j^{(J)} q^{J+k+1}S + D \leq p^\kappa q^{k+1} SL_j + D$$

in α_j $(1 \leq j \leq n)$. Note that by the main inductive hypothesis for J and Lemmas 2.3, 2.4, for $1 \leq d \leq D$, λ satisfying (3.59), $1 \leq s \leq q^{J+k+1}S$, $(s,q) = 1$, $|\tau| \leq \left(1-\left(1-\frac{1}{q}\right)\frac{k+1}{n}\right)q^{-J}T$, we have

$$|p_d^{(J)}(\lambda)| \leq X_0^{\frac{1}{c_0-1}} ,$$

$$q^{2J(L_{-1}+1)(L_0+1)} \leq L_n^{2(L_{-1}+1)(L_0+1)} ,$$

$$\left|\Delta\left(q^{-J}s+\lambda_{-1};L_{-1}+1,\lambda_0+1,\tau_0\right)\right| \leq \left(2e\left(2+\frac{q^{k+1}S}{L_{-1}+1}\right)\right)^{(L_{-1}+1)(L_0+1)}$$
$$\leq \left(2e\left(2+\frac{S}{L_{-1}+1}\right)\right)^{q^{k+1}(L_{-1}+1)(L_0+1)} ,$$

$$\left(\nu(L_{-1}+1)\right)^{\tau_0} \prod_{j=1}^{n-1} \left| \Delta\left(b_n\lambda_j-b_j\lambda_n;\tau_j\right)\right|$$
$$\leq 3^{(L_{-1}+1)\tau_0} e^{T-\tau_0}\left(1+\frac{(n-1)(B_nL^{(J)}+B'L_n^{(J)})}{q^{-J}T}\right)^T$$
$$\leq 3^{(L_{-1}+1)T}\left(1+\frac{(n-1)(B_nL_1+B'L_n)}{T}\right)^T ,$$

where $L^{(J)} = \max_{1\leq j<n} L_j^{(J)}$.

So $\zeta^{-r^{(J)}s}\, q^{2J(L_{-1}+1)(L_0+1)}\left(\nu(L_{-1}+1)\right)^{\tau_0} \varphi_J(s,\tau)$ for $1 \leq s \leq q^{J+k+1}S$, $(s,q)=1$, $|\tau| \leq \left(1-\left(1-\frac{1}{q}\right)\frac{k+1}{n}\right) q^{-J}T$, as a polynomial in $\alpha_1,\ldots,\alpha_n$, has its length at most

$$\left(D\prod_{j=1}^{n}(L_j+1)\right)\cdot X_0^{\frac{1}{c_0-1}} L_n^{2(L_{-1}+1)(L_0+1)}\left(2e\left(2+\frac{S}{L_{-1}+1}\right)\right)^{q^{k+1}(L_{-1}+1)(L_0+1)} 3^{(L_{-1}+1)T} \cdot$$
$$\cdot \left(1+\frac{(n-1)(B_nL_1+B'L_n)}{T}\right)^T .$$

Now assume there exist s,τ with

$$1 \leq s \leq q^{J+k+1}S , \quad (s,q)=1 , \quad |\tau| \leq \left(1-\left(1-\frac{1}{q}\right)\frac{k+1}{n}\right) q^{-J}T$$

such that

$$\varphi_J(s,\tau) \neq 0 ,$$

and we proceed to deduce a contradiction. By Lemma 2.1 and the definition of X_0 (see (3.11)), and by (3.14)-(3.16), (3.18)-(3.21), the assumption $\varphi_J(s,\tau) \neq 0$ implies that

$$\mathrm{ord}_p \varphi_J(s,\tau) \leq \mathrm{ord}_p\left(\zeta^{-r^{(J)}} s_q^{2J(L_{-1}+1)(L_0+1)} \left(\nu\left(L_{-1}+1\right)\right)^{\tau_0} \varphi_J(s,\tau)\right)$$

$$\leq \frac{D}{e_{\mathfrak{p}} f_{\mathfrak{p}} \log p}\left\{\log\left(D\prod_{j=-1}^{n}(L_j+1)\right)+\frac{1}{c_0-1}\log X_0+\log 3\cdot T(L_{-1}+1)+T\log\left(1+\frac{(n-1)(B_n L_1+B' L_n)}{T}\right)\right.$$

$$+ 2(L_{-1}+1)(L_0+1)\log L_n + q^{k+1}(L_{-1}+1)(L_0+1)\log\left(2e\left(2+\frac{S}{L_{-1}+1}\right)\right)$$

$$\left.+ p^{\kappa} q^{k+1} S\sum_{j=1}^{n} L_j V_j + nD\max_{1\leq j\leq n} V_j\right\}$$

$$\leq q^{k+1-n}\frac{q^n D}{e_{\mathfrak{p}} f_{\mathfrak{p}} \log p}\left\{\frac{1}{q}\left(1+\frac{1}{c_0-1}\right)\left(\log\left(D\prod_{j=-1}^{n}(L_j+1)\right) + nD\max_{1\leq j\leq n} V_j\right)\right.$$

$$+ \left(1+\frac{1}{q(c_0-1)}\right)p^{\kappa} S\sum_{j=1}^{n} L_j V_j + \frac{1}{q}\left(1+\frac{1}{c_0-1}\right)\log 3 \cdot T(L_{-1}+1)$$

$$+ \frac{1}{q}\left(1+\frac{1}{c_0-1}\right)T\log\left(1+\frac{(n-1)(B_n L_1+B' L_n)}{T}\right)$$

$$+ \left(1+\frac{1}{q(c_0-1)}\right)(L_{-1}+1)(L_0+1)\log\left(2e\left(2+\frac{S}{L_{-1}+1}\right)\right)$$

$$\left.+ \frac{1}{q}\cdot 2(L_{-1}+1)(L_0+1)\log L_n\right\}$$

$$\leq \frac{U}{c_1} q^{k+1-n}\left\{\left(\frac{1}{h_6}+\frac{1}{h_7}\right)\left(1+\frac{1}{c_0-1}\right)c_1+\left(1+\frac{1}{c_0-1}\right)\frac{1}{c_2}\right.$$

$$+\left(1+\left(1+\frac{1}{h_0}\right)\log 3\right)\left(\frac{1}{q}+\frac{1}{c_0-1}\right)\left(2+\frac{1}{p-1}\right)\frac{1}{c_3}$$

$$+\left(1+\frac{1}{h_4}\right)\left(1+\frac{1}{c_0-1}\right)\frac{1}{n}\cdot\frac{1}{c_4}$$

$$\left.+\frac{1}{q}\left(1+\frac{1}{h_4}\right)\left(4+\frac{1}{2^{10}nq}+\frac{2\log h_5}{h_0}\right)\frac{1}{c_4}\right\} .$$

This together with (3.6) implies

$$\operatorname{ord}_p \varphi_J(s,\tau) \leq \frac{U}{c_1} q^{k+1-n}\left\{\left(1-\frac{1}{q}\right)^2\left(1-\frac{1}{c_3 n}\right)\left(1-\frac{1}{h_1}\right)\right.$$

$$-\left(1+\frac{1}{h_4}\right)\left(1+\frac{1}{p-1}\right)\frac{1}{q^n}\cdot\frac{1}{c_4}$$

$$\left.-\left(1-\frac{1}{q}\right)\left(1+\frac{1}{h_4}\right)\left(4+\frac{1}{2^{10}nq}+\frac{2\log h_5}{h_0}\right)\frac{1}{c_4}\right\} .$$

(3.77)

On noting that

$$\left(1+\frac{1}{p-1}\right)\frac{1}{q^n}+\left(1-\frac{1}{q}\right)\left(4+\frac{1}{2^{10}nq}+\frac{2\log h_5}{h_0}\right)$$

$$> \left(1+\frac{1}{p-1}\right)\frac{1}{q^n}+4\left(1-\frac{1}{q}\right) > \left(1+\frac{1}{p-1}\right)\frac{1}{q}$$

$$\geq \left(1+\frac{1}{p-1}\right)\frac{1}{q^{k+1}} ,$$

(3.77) yields

$$\operatorname{ord}_p \varphi_J(s,\tau) < \frac{U}{c_1} q^{k+1-n}\left\{\left(1-\frac{1}{q}\right)^2\left(1-\frac{1}{c_3 n}\right)\left(1-\frac{1}{h_1}\right)-\frac{1}{q^{k+1}}\left(1+\frac{1}{h_4}\right)\left(1+\frac{1}{p-1}\right)\frac{1}{c_4}\right\} ,$$

contradicting (3.76). This contradiction proves

$$\varphi_J(s,\tau) = 0$$

for $1 \leq s \leq q^{J+k+1}S$, $(s,q)=1$, $|\tau| \leq \left(1-\left(1-\frac{1}{q}\right)\frac{k+1}{n}\right) q^{-J}T$.

Thus the proof of the lemma is complete.

Lemma 3.4.

$$\varphi_J\left(\frac{s}{q},\tau\right) = 0$$

for $1 \leq s \leq q^{J+1}S$, $(s,q)=1$, $|\tau| \leq q^{-(J+1)}T$.

Proof. By Lemma 3.2 and Lemma 3.3 with $k = n-1$, we have

$$\text{ord}_p f_J(s,\tau) \geq U - \frac{T \log(L_{-1}+1)}{\log p} - \text{ord}_p b_n$$

for $1 \leq s \leq q^{J+n-1}S$, $(s,q)=1$, $|\tau| \leq \left(1-\left(1-\frac{1}{q}\right)\frac{n-1}{n}\right) q^{-J}T$.

Now we apply Lemma 1.4 to each function

$$F_J(z,\tau) = p^{(L_{-1}+1)(L_0+1)\left(\theta+\frac{1}{p-1}\right)} f_J(p^{-\theta}z,\tau)$$

with $|\tau| \leq q^{-(J+1)}T$,

taking

$$R = q^{J+n-1}S \ , \ M = \left[\left(1-\frac{1}{q}\right)\frac{1}{n}\, q^{-J}T\right] + 1 \ .$$

Then it is readily verified that the arguments from (3.67) to (3.75) in the proof of Lemma 3.3 work also for $k = n-1$. Hence we see that each p-adic normal function $F_J(z,\tau)$ with $|\tau| \le q^{-(J+1)}T$ satisfies the condition (1.9) of Lemma 1.4 with above specified R and M . So Lemma 1.4 implies

$$\begin{aligned}
\operatorname{ord}_p f_J\left(\frac{s}{q},\tau\right) &\ge \operatorname{ord}_p F_J\left(\frac{s}{q}p^{\theta},\tau\right) - (L_{-1}+1)(L_0+1)\left(\theta + \frac{1}{p-1}\right) \\
&\ge \left(1-\frac{1}{q}\right) RM\,\theta - (L_{-1}+1)(L_0+1)\left(\theta + \frac{1}{p-1}\right) \\
&> \left(1-\frac{1}{q}\right)^2 \frac{1}{n}\, q^{n-1} ST\,\theta - (L_{-1}+1)(L_0+1)\left(\theta + \frac{1}{p-1}\right)
\end{aligned}$$

for $s \in \mathbb{Z}$, $|\tau| \le q^{-(J+1)}T$.

By Lemma 3.2, (3.74), (3.75) (with $k = n-1$) and (3.7) we obtain for $s \ge 1$

$$\begin{aligned}
\operatorname{ord}_p\left(\varphi_J\left(\frac{s}{q},\tau\right) - f_J\left(\frac{s}{q},\tau\right)\right) &\ge U - \frac{T\log(L_{-1}+1)}{\log p} - \operatorname{ord}_p b_n \\
&> \left(1-\frac{1}{q}\right) RM\,\theta - (L_{-1}+1)(L_0+1)\left(\theta + \frac{1}{p-1}\right) \\
&> \left(1-\frac{1}{q}\right)^2 \frac{1}{n}\, q^{n-1} ST\,\theta - (L_{-1}+1)(L_0+1)\left(\theta + \frac{1}{p-1}\right).
\end{aligned}$$

Hence

$$\operatorname{ord}_p \varphi_J\left(\frac{s}{q},\tau\right) \geq \min\left(\operatorname{ord}_p f_J\left(\frac{s}{q},\tau\right),\ \operatorname{ord}_p\left(\varphi_J\left(\frac{s}{q},\tau\right) - f_J\left(\frac{s}{q},\tau\right)\right)\right)$$

$$> \left(1-\frac{1}{q}\right)^2 \frac{1}{n} q^{n-1} ST\,\theta - (L_{-1}+1)(L_0+1)\left(\theta + \frac{1}{p-1}\right)$$

$$\geq \frac{U}{c_1}\left\{\left(1-\frac{1}{q}\right)^2\left(1-\frac{1}{c_3 n}\right)\left(1-\frac{1}{h_1}\right)-\left(1+\frac{1}{h_4}\right)\left(1+\frac{1}{p-1}\right)\frac{1}{q^n}\cdot\frac{1}{c_4}\right\}$$

$$\text{for} \quad s \geq 1,\ |\tau| \leq q^{-(J+1)}T\ , \tag{3.78}$$

where the last inequality follows from (3.13) and (3.17).

On the other hand, on noting that, by § 1.1 (d) and (3.3), we have for $(\lambda_1,\ldots,\lambda_n) \in \mathbb{N}^n$ satisfying

$$r_1\lambda_1 + \ldots + r_n\lambda_n \equiv r^{(J)} \pmod G$$

$$\prod_{j=1}^{n}\left(\alpha_j^{p^\kappa}\ \zeta^{r_j}\right)^{\lambda_j \frac{s}{q}} = \prod_{j=1}^{n}\left(\left(\alpha_j^{p^\kappa}\ \zeta^{r_j}\right)^{\frac{1}{q}}\right)^{\lambda_j s}$$

$$= \prod_{j=1}^{n}\left\{\left(\alpha_j^{\frac{1}{q}}\right)^{p^\kappa}\zeta^{br_j}\right\}^{\lambda_j s} = \left\{\prod_{j=1}^{n}\left(\alpha_j^{\frac{1}{q}}\right)^{p^\kappa\lambda_j s}\right\}\cdot\zeta^{bs(r_1\lambda_1+\ldots+r_n\lambda_n)}$$

$$= \zeta^{bsr^{(J)}}\prod_{j=1}^{n}\left(\alpha_j^{\frac{1}{q}}\right)^{p^\kappa\lambda_j s}\ , \tag{3.79}$$

we see that for $1 \leq s \leq q^{J+1}S$, $(s,q)=1$, $|\tau| \leq q^{-(J+1)}T$

$$\zeta^{-bsr^{(J)}}\left(q^{J+1}\right)^{2(L_{-1}+1)(L_0+1)}\left(\nu(L_{-1}+1)\right)^{\tau_0}\varphi_J\left(\frac{s}{q},\tau\right)$$

$$= \sum_{\lambda}^{(J)}\sum_{d=1}^{D} p_d^{(J)}(\lambda)\, q^{2(J+1)(L_{-1}+1)(L_0+1)}\,\Delta\left(q^{-(J+1)}s+\lambda_{-1};L_{-1}+1,\lambda_0+1,\tau_0\right)\ .$$

$$\cdot\left(\nu(L_{-1}+1)\right)^{\tau_0}\prod_{j=1}^{n-1}\Delta(b_n\lambda_j-b_j\lambda_n;\tau_j)\cdot\prod_{j=1}^{n}\left(\alpha_j^{\frac{1}{q}}\right)^{p^\kappa\lambda_j s + qk_{jd}}$$

is a polynomial (with rational integer coefficients) in $\alpha_1^{\frac{1}{q}},\ldots,\alpha_n^{\frac{1}{q}}$ of degree at most

$$p^{\kappa}q^{J+1}SL_j^{(J)} + qD \leq p^{\kappa}qSL_j + qD$$

in $\alpha_j^{\frac{1}{q}}$ $(1 \leq j \leq n)$. By the main inductive hypothesis for J , Lemmas 2.3, 2.4 we have for $1 \leq s \leq q^{J+1}S$, $(s,q)=1$, $|\tau| \leq q^{-(J+1)}T$, $1 \leq d \leq D$, λ in the range (3.59),

$$\left|p_d^{(J)}(\lambda)\right| \leq X_0^{\frac{1}{c_0-1}} ,$$

$$q^{2(J+1)(L_{-1}+1)(L_0+1)} \leq (q\,L_n)^{2(L_{-1}+1)(L_0+1)} ,$$

$$\left|\Delta(q^{-(J+1)}s+\lambda_{-1};L_{-1}+1,\lambda_0+1,\tau_0)\right| \leq \left(2e\left(2+\frac{S}{L_{-1}+1}\right)\right)^{(L_{-1}+1)(L_0+1)}$$

$$\left(\nu(L_{-1}+1)\right)^{\tau_0} \prod_{j=1}^{n-1} \Delta(b_n\lambda_j-b_j\lambda_n;\tau_j)$$

$$\leq 3^{(L_{-1}+1)\tau_0}\, e^{\frac{1}{q}T-\tau_0} \left(1+\frac{(n-1)\left(B_nL^{(J)}+B'L_n^{(J)}\right)}{q^{-(J+1)}T}\right)^{q^{-(J+1)}T}$$

$$\leq 3^{\frac{1}{q}T(L_{-1}+1)}\left(1+\frac{(n-1)q(B_nL_1+B'L_n)}{T}\right)^{\frac{1}{q}T} .$$

Hence we see that

$$\zeta^{-bsr^{(J)}}q^{2(J+1)(L_{-1}+1)(L_0+1)}\left(\nu(L_{-1}+1)\right)^{\tau_0}\varphi_J\left(\frac{s}{q},\tau\right)$$

$$(1 \leq s \leq q^{J+1}S,\ (s,q)=1,\ |\tau| \leq q^{-(J+1)}T)$$

as a polynomial in $\alpha_1^{\frac{1}{q}},\ldots,\alpha_n^{\frac{1}{q}}$, has length not exceeding

$$\left\{D\prod_{j=-1}^{n}(L_j+1)\right\}X_0^{\frac{1}{c_0-1}}\ 3^{\frac{1}{q}T(L_{-1}+1)}\left(1+\frac{(n-1)q(B_nL_1+B'L_n)}{T}\right)^{\frac{1}{q}T} .$$

$$\cdot\left(2e\left(2+\frac{S}{L_{-1}+1}\right)\right)^{(L_{-1}+1)(L_0+1)}(qL_n)^{2(L_{-1}+1)(L_0+1)} .$$

No we assume that there exist s,τ satisfying

$$1\le s\le q^{J+1}S\ ,\ (s,q)=1\ ,\ |\tau|\le q^{-(J+1)}T$$

such that

$$\varphi_J\left(\frac{s}{q},\tau\right)\neq 0\ ,$$

and we proceed to deduce a contradiction. In Lemma 2.1, let $E=K\left(\alpha_1^{\frac{1}{q}},\ldots,\alpha_n^{\frac{1}{q}}\right)$, $\mathfrak{P}$ be a prime ideal of O_E lying above $\mathfrak{p}$. Thus

$$[E:\mathbb{Q}]=[E:K][K:\mathbb{Q}]=q^nD$$

(see (0.6)) and

$$e_{\mathfrak{P}}\ge e_{\mathfrak{p}}\ ,\ f_{\mathfrak{P}}\ge f_{\mathfrak{p}}\ .$$

Note that $h\left(\alpha_j^{\frac{1}{q}}\right)=\frac{1}{q}h(\alpha_j)$. Then by Lemma 2.1 and the definition of X_0 (see (3.11)), and by (3.14)-(3.16), (3.18)-(3.21), (3.6), we see that

$$\varphi_J\left(\frac{s}{q},\tau\right) \neq 0 \quad \text{with} \quad 1 \leq s \leq q^{J+1}S \ , \ (s,q)=1, \ |\tau| \leq q^{-(J+1)}T$$

implies

$$\operatorname{ord}_p \varphi_J\left(\frac{s}{q},\tau\right) \leq \operatorname{ord}_p\left\{\zeta^{-bsr^{(J)}} q^{2(J+1)(L_{-1}+1)(L_0+1)}\left(\nu(L_{-1}+1)\right)^{\tau_0} \varphi_J\left(\frac{s}{q},\tau\right)\right\}$$

$$\leq \frac{q^n D}{e_p f_p \log p}\left\{\log\left(D\prod_{j=1}^{n}(L_j+1)\right)+\frac{1}{c_0-1}\log X_0+p^{\kappa}S\sum_{j=1}^{n}L_jV_j+nD\max_{1\leq j\leq n}V_j\right.$$

$$+(\log 3)\frac{1}{q}T(L_{-1}+1)+\frac{1}{q}T\log\left(1+\frac{(n-1)q(B_nL_1+B'L_n')}{T}\right)$$

$$\left.+(L_{-1}+1)(L_0+1)\log\left(2e\left(2+\frac{1}{L_{-1}+1}\right)\right)+2(L_{-1}+1)(L_0+1)\log(qL_n)\right\}$$

$$\leq \frac{U}{c_1}\left\{\left(\frac{1}{h_6}+\frac{1}{h_7}\right)\left(1+\frac{1}{c_0-1}\right)c_1+\left(1+\frac{1}{c_0-1}\right)\frac{1}{c_2}\right.$$

$$+\left(1+\left(1+\frac{1}{h_0}\right)\log 3\right)\left(\frac{1}{q}+\frac{1}{c_0-1}\right)\left(2+\frac{1}{p-1}\right)\frac{1}{c_3}$$

$$\left.+\left(1+\frac{1}{h_4}\right)\left(4+\frac{1}{2^{10}nq}+\frac{2\log h_5}{h_0}+\frac{1}{n}\left(1+\frac{1}{c_0-1}\right)\right)\frac{1}{c_4}\right\}$$

$$\leq \frac{U}{c_1}\left\{\left(1-\frac{1}{q}\right)^2\left(1-\frac{1}{c_3 n}\right)\left(1-\frac{1}{h_1}\right)-\left(1+\frac{1}{h_4}\right)\left(1+\frac{1}{p-1}\right)\frac{1}{q^n}\cdot\frac{1}{c_4}\right\}$$

a contradiction to (3.78). This contradiction proves

$$\varphi_J\left(\frac{s}{q},\tau\right) = 0 \quad \text{for} \quad 1 \leq s \leq q^{J+1}S \ , \ (s,q)=1 \ , \ |\tau| \leq q^{-(J+1)}T \ .$$

The proof of the lemma is thus complete.

Lemma 3.5. The main inductive argument is true for $J+1$.

Proof. Similarly to (3.79) we have for

$(\mu_1,\ldots,\mu_n,r) \in \mathbb{N}^{n+1}$ satisfying $r_1\mu_1+\ldots+r_n\mu_n \equiv r \pmod{G}$

the equality

$$\prod_{j=1}^{n}\left(\alpha_j^{p^\kappa}\zeta^{r_j}\right)^{\mu_j\frac{s}{q}} = \zeta^{bsr}\prod_{j=1}^{n}\left(\alpha_j^{\frac{1}{q}}\right)^{p^\kappa\mu_j s} . \tag{3.80}$$

Writing

$$\mu_j = \lambda_j^0 + q\lambda_j \ , \ 0 \le \lambda_j^0 < q \ (1 \le j \le n) \ ,$$

we see that

$$\left(\alpha_j^{\frac{1}{q}}\right)^{p^\kappa\mu_j s} = \alpha_j^{p^\kappa\lambda_j s}\left(\alpha_j^{\frac{1}{q}}\right)^{p^\kappa\lambda_j^0 s} \ (1 \le j \le n) \ . \tag{3.81}$$

By Lemma 3.4, (3.80), (3.81), we obtain

$$\sum_{\lambda_1^0=0}^{q-1}\cdots\sum_{\lambda_n^0=0}^{q-1}\prod_{j=1}^{n}\left(\alpha_j^{\frac{1}{q}}\right)^{p^\kappa\lambda_j^0 s}\sum_{\lambda_{-1}=0}^{L_{-1}^{(J)}}\sum_{\lambda_0=0}^{L_0^{(J)}}\sum_{\lambda_1,\ldots,\lambda_n}\sum_{d=1}^{D}p_d^{(J)}\left(\lambda_{-1},\lambda_0,\lambda_1^0+q\lambda_1,\ldots,\lambda_n^0+q\lambda_n\right)\xi_d \cdot$$

$$\cdot\,\Delta\!\left(q^{-(J+1)}s+\lambda_{-1};L_{-1}+1,\lambda_0+1,\tau_0\right)\prod_{j=1}^{n-1}\Delta\!\left(q(b_n\lambda_j-b_j\lambda_n)+(b_n\lambda_j^0-b_j\lambda_n^0);\tau_j\right)\cdot\prod_{j=1}^{n}\alpha_j^{p^\kappa\lambda_j s}$$

$$= 0 \tag{3.82}$$

for $1 \le s \le q^{J+1}S$, $(s,q)=1$, $|\tau| \le q^{-(J+1)}T$,

where $\sum\limits_{\lambda_1,\ldots,\lambda_n}$ ranges over the rational integers $\lambda_1,\ldots,\lambda_n$ satisfying

$$0 \leq \lambda_j \leq L_j^{(J+1)}(\lambda_1^0,\ldots,\lambda_n^0) = \left[\frac{L_j^{(J)}-\lambda_j^0}{q}\right] (1 \leq j \leq n) \qquad (3.83)$$

and

$$\sum_{j=1}^{n} r_j(\lambda_j^0 + q\lambda_j) \equiv r^{(J)} (\text{mod } G) \ . \qquad (3.84)$$

We emphasize that, by (0.1)

$$(q,G) = 1 \ ,$$

hence (3.84) is equivalent to

$$r_1\lambda_1+\ldots+r_n\lambda_n \equiv r^{(J+1)}(\lambda_1^0,\ldots,\lambda_n^0) \ (\text{mod } G) \ , \qquad (3.84)'$$

where $r^{(J+1)}(\lambda_1^0,\ldots,\lambda_n^0)$ is the unique solution of the congruence

$$qx \equiv r^{(J)} - (r_1\lambda_1^0+\ldots+r_n\lambda_n^0) \ (\text{mod } G)$$

in the range $0 \leq x < G$. Now by the mian inductive hypothesis for J , there exists a n-tuple $\lambda_1^0,\ldots,\lambda_n^0$ with $0 \leq \lambda_j^0 < q \ (1 \leq j \leq n)$, such that the rational integers

$$p_d^{(J)}(\lambda_{-1},\lambda_0,\lambda_1^0 + q\lambda_1,\ldots,\lambda_n^0 + q\lambda_n)$$

for $1 \leq d \leq D$, $0 \leq \lambda_j \leq L_j^{(J)}$ $(j=-1,0)$, $\lambda_1,\ldots,\lambda_n$ satisfying (3.83), (3.84)', are not all zero. Fix this n-tuple $\lambda_1^0,\ldots,\lambda_n^0$, take

$$r^{(J+1)} = r^{(J+1)}(\lambda_1^0,\ldots,\lambda_n^0) \ ,$$

which is obviously divisible by g.c.d. $(r_1,\ldots,r_n,G)$, and set

$$L_j^{(J+1)}=L_j^{(J)}=L_j\,(j=-1,0),\ L_j^{(J+1)}=L_j^{(J+1)}(\lambda_1^0,\ldots,\lambda_n^0)\ \ (1 \le j \le n)\ ,$$

$$p_d^{(J+1)}(\lambda_{-1},\lambda_0,\lambda_1,\ldots,\lambda_n) = p_d^{(J)}(\lambda_{-1},\lambda_0,\lambda_1^0+q\lambda_1,\ldots,\lambda_n^0+q\lambda_n)$$

for

$$1 \le d \le D,\ 0 \le \lambda_j \le L_j^{(J+1)}\,(-1 \le j \le n),\ r_1\lambda_1+\ldots+r_n\lambda_n \equiv r^{(J+1)}\,(\mathrm{mod}\ G)\ . \tag{3.85}$$

By the condition (0.6) and the fact that

$$(p^{\kappa}s,q) = 1 \ ,$$

we obtain from (3.82) that

$$\sum_{\lambda}^{(J+1)}\sum_{d=1}^{D} p_d^{(J+1)}(\lambda)\,\xi_d\,\Delta\Big(q^{-(J+1)}s+\lambda_{-1};\ L_{-1}+1,\ \lambda_0+1,\ \tau_0\Big)\ \cdot$$
$$\cdot\ \prod_{j=1}^{n-1}\Delta\Big(q(b_n\lambda_j-b_j\lambda_n)+(b_n\lambda_j^0-b_j\lambda_n^0);\tau_j\Big)\cdot\prod_{j=1}^{n}\alpha_j^{p^{\kappa}\lambda_j s} = 0 \tag{3.86}$$

for $1 \le s \le q^{J+1}S$, $(s,q)=1$, $|\tau| \le q^{-(J+1)}T$, where $\sum_{\lambda}^{(J+1)}$ denotes the summation over the λ's in (3.85). By Lemma 2.6 for each j with $1 \le j \le n-1$ and $0 \le k \le \tau_j$ $\Delta(q(b_n\lambda_j-b_j\lambda_n) + (b_n\lambda_j^0-b_j\lambda_n^0);k)$ is a linear combination of the $k+1$ numbers

$$\Delta(b_n\lambda_j - b_j\lambda_n; t) \ , \ t = 0,1,\dots,k \ ,$$

with coefficients independent of $\lambda_1,\dots,\lambda_n$, where the coefficient of $\Delta(b_n\lambda_j-b_j\lambda_n;k)$ is non-zero. Hence for each j with $1 \leq j \leq n-1$, $\Delta(b_n\lambda_j-b_j\lambda_n;\tau_j)$ is a linear combination of the τ_j+1 numbers

$$\Delta\left(q(b_n\lambda_j-b_j\lambda_n)+(b_n\lambda_j^0-b_j\lambda_n^0);k\right) \ , \ k = 0,1,\dots,\tau_j \ ,$$

with coefficients independent of $\lambda_1,\dots,\lambda_n$. By this observation and by (3.80), we see that (3.86) implies

$$\zeta^{-bsr^{(J+1)}} \varphi_{J+1}(s,\tau) = 0$$

for $1 \leq s \leq q^{J+1}S$, $(s,q) = 1$, $|\tau| \leq q^{-(J+1)}T$.

This completes the proof of the lemma.

Thus we have established the main inductive argument for $J = 0,1,\dots,\left[\frac{\log L_n}{\log q}\right] + 1$.

We should like to make some remarks on van der Poorten [25]. Recall

$$g_{\mathfrak{p}} = \left[\frac{1}{2}+\frac{e_{\mathfrak{p}}}{p-1}\right] \ , \ G_{\mathfrak{p}} = N\mathfrak{p}^{g_{\mathfrak{p}}} \cdot (N\mathfrak{p}-1)$$

and let ζ' be a $G_{\mathfrak{p}}$-th primitive root of unity in $\mathbb{C}_p$. It is asserted in [25], p. 35 that for $\alpha \in K$ with $\mathrm{ord}_{\mathfrak{p}}\alpha = 0$ there is an integer r , $0 \leq r < G_{\mathfrak{p}}$ such that

$$\mathrm{ord}_{\mathfrak{p}}\left(\alpha\ \zeta'^{r}-1\right) \geq g_{\mathfrak{p}} + 1 . \tag{3.87}$$

Note that this is false. A simple counter-example is the following. Take $K = \mathbb{Q}$, $\mathfrak{p} = 3\,\mathbb{Z}$, then $e_{\mathfrak{p}} = g_{\mathfrak{p}} = 1$, $G_{\mathfrak{p}} = 6$. Let ζ' be a 6-th primitive root of unity. Take $\alpha = \frac{2}{5}$, then $\mathrm{ord}_{\mathfrak{p}}\alpha = 0$ and it is readily verified that

$$\mathrm{ord}_{\mathfrak{p}}\left(\alpha\ \zeta'^{r}-1\right) \leq 1 < g_{\mathfrak{p}} + 1 \quad \text{for} \quad r = 0,1,\dots,G_{\mathfrak{p}}-1 .$$

We should also point out that the assertion (3.87) does hold for the special case where $g_{\mathfrak{p}}=0$, by virtue of our Lemma 1.3; but even in this special case, there are still some inaccuracies in [25]. For instance, in the proof of Lemma 7 in [25], p. 46, p. 47, which corresponds to our Lemma 3.5, the author of [25] does not put an additional restriction on q that

$$(q,G_{\mathfrak{p}}) = 1 , \tag{3.88}$$

which seems to be essential to make his proof work. On the other hand, if one does assume (3.88), then by Hasse [16], p. 220, $K_{\mathfrak{p}}$, whence K , does not contain the q-th primitive roots of unity, and we can not understand the arguments related to Kummer theory in Section 5 of [25], pp. 49-51. The same remark extends to the proofs of Theorems 2, 3, 4 of [25].

5. The completion of the proof of Proposition 1

We suppose that Proposition 1 is false, that is, there exist algebraic numbers $\alpha_1,\dots,\alpha_n$ and rational integers $b_1,\dots,b_n$

satisfying (0.5)-(0.6) such that

$$\operatorname{ord}_p(\alpha_1^{b_1}\cdots\alpha_n^{b_n} - 1) \geq U ,$$

then we proceed to deduce a contradiction. By the main inductive argument for

$$J = J_0 = \left[\frac{\log L_n}{\log q}\right] + 1 ,$$

we have

$$\varphi_{J_0}(s,\tau) = 0 \quad \text{for} \quad 1 \leq s \leq q^{J_0}S , \ (s,q)=1 , \ |\tau| \leq q^{-J_0}T . \quad (3.89)$$

Since $0 \leq L_n^{(J_0)} \leq q^{-J_0}L_n$, we see that $L_n^{(J_0)} = 0$. Further if $\tau = (\tau_0,\ldots,\tau_{n-1})$ satisfies

$$0 \leq \tau_0 \leq \frac{1}{2}\, q^{-J_0}T , \ 0 \leq \tau_j \leq L_j^{(J_0)} \quad (1 \leq j \leq n-1) ,$$

then we see, by (3.26), that

$$|\tau| = \tau_0+\ldots+\tau_{n-1} \leq \frac{1}{2}\, q^{-J_0}T + L_1^{(J_0)} + \ldots + L_{n-1}^{(J_0)}$$

$$\leq \frac{1}{2}\, q^{-J_0}T + q^{-J_0}(L_1 + \ldots + L_{n-1})$$

$$\leq q^{-J_0}T .$$

By these observations, (3.89) implies (writing again $p^{(J_0)}(\lambda) = \sum_{d=1}^{D} p_d^{(J_0)}(\lambda)\xi_d$)

$$\sum_{\lambda_{n-1}=0}^{L_{n-1}^{(J_0)}} \left\{ \sum_{\lambda_{-1}=0}^{L_{-1}^{(J_0)}} \cdots \sum_{\lambda_{n-2}=0}^{L_{n-2}^{(J_0)}} p^{(J_0)}(\lambda_{-1},\ldots,\lambda_{n-1},0)\Delta(q^{-J_0}s+\lambda_{-1};L_{-1}+1,\lambda_0+1,\tau_0)\cdot\right.$$

$$\left.\cdot\left(\prod_{j=1}^{n-2}\Delta(b_n\lambda_j;\tau_j)\right)\cdot\prod_{j=1}^{n-1}\left(\alpha_j^{p^\kappa}\zeta^{r_j}\right)^{\lambda_j s}\right\}\Delta(b_n\lambda_{n-1};\tau_{n-1}) = 0 \qquad (3.90)$$

for $1 \le s \le q^{J_0}S$, $(s,q)=1$, $0 \le \tau_0 \le \frac{1}{2}q^{-J_0}T$, $0 \le \tau_j \le L_j^{(J_0)}$ $(1 \le j \le n-1)$, where we have set

$$p^{(J_0)}(\lambda_{-1},\ldots,\lambda_{n-1},0) = 0$$

for $\lambda_{-1},\ldots,\lambda_{n-1}$ satisfying

$0 \le \lambda_j \le L_j^{(J_0)}$ $(-1 \le j \le n-1)$ and $r_1\lambda_1 + \ldots + r_{n-1}\lambda_{n-1} \not\equiv r^{(J_0)}$ (mod G).
By Lemma 2.5 we have

$$\det\Big(\Delta\big(b_n\lambda_{n-1};\tau_{n-1}\big)\Big)_{0 \le \lambda_{n-1},\tau_{n-1} \le L_{n-1}^{(J_0)}} \ne 0 .$$

So (3.90) implies that for each λ_{n-1} with $0 \le \lambda_{n-1} \le L_{n-1}^{(J_0)}$

$$\sum_{\lambda_{n-2}=0}^{L_{n-2}^{(J_0)}} \left\{ \sum_{\lambda_{-1}=0}^{L_{-1}^{(J_0)}} \cdots \sum_{\lambda_{n-3}=0}^{L_{n-3}^{(J_0)}} p^{(J_0)}(\lambda_{-1},\ldots,\lambda_{n-1},0)\Delta\left(q^{-J_0}s+\lambda_{-1};L_{-1}+1,\lambda_0+1,\tau_0\right)\cdot\right.$$

$$\left.\cdot\left(\prod_{j=1}^{n-3}\Delta(b_n\lambda_j;\tau_j)\right)\prod_{j=1}^{n-2}\left(\alpha_j^{p^\kappa}\zeta^{r_j}\right)^{\lambda_j s}\right\}\Delta(b_n\lambda_{n-2};\tau_{n-2}) = 0$$

for $1 \le s \le q^{J_0}S$, $(s,q)=1$, $0 \le \tau_0 \le \frac{1}{2}q^{-J_0}T$, $0 \le \tau_j \le L_j^{(J_0)}$ $(1 \le j \le n-2)$.
On repeating this argument $n-1$ times and noting $L_j^{(J_0)} = L_j$ $(j=-1,0)$, we obtain

$$\sum_{\lambda_{-1}=0}^{L_{-1}} \sum_{\lambda_0=0}^{L_0} p^{(J_0)}(\lambda_{-1},\lambda_0,\lambda_1,\ldots,\lambda_{n-1},0)\Delta\left(q^{-J_0}s+\lambda_{-1};L_{-1}+1,\lambda_0+1,\tau_0\right) = 0$$

$$\text{for} \quad 0 \le \lambda_j \le L_j^{(J_0)} \quad (1 \le j \le n-1) \quad \text{and}$$

$$1 \le s \le q^{J_0} S, \ (s,q) = 1, \ 0 \le \tau_0 \le \tfrac{1}{2} q^{-J_0} T \ .$$

This implies that each polynomial

$$Q_{\lambda_1,\ldots,\lambda_{n-1}}(x) = \sum_{\lambda_{-1}=0}^{L_{-1}} \sum_{\lambda_0=0}^{L_0} p^{(J_0)}(\lambda_{-1},\lambda_0,\lambda_1,\ldots,\lambda_{n-1},0)\,\Delta(x+\lambda_{-1};L_{-1}+1,\lambda_0+1,0) \tag{3.91}$$

with $0 \le \lambda_j \le L_j^{(J_0)}$ $(1 \le j \le n-1)$ has at least

$$\left(1-\frac{1}{q}\right) q^{J_0} S\left(\left[\frac{1}{2} q^{-J_0} T\right] + 1\right) > \frac{1}{2}\left(1-\frac{1}{q}\right) ST > \frac{1}{4} ST$$

zeros. But (3.27) yields

$$\frac{1}{4} ST > (L_{-1}+1)(L_0+1) \ge \deg Q_{\lambda_1,\ldots,\lambda_{n-1}}(x) \ .$$

So

$$Q_{\lambda_1,\ldots,\lambda_{n-1}}(x) = 0 \quad \text{for} \quad 0 \le \lambda_j \le L_j^{(J_0)} \ (1 \le j \le n-1) \ . \tag{3.92}$$

According to Lemma 2.3, the polynomials

$$\Delta(x+\lambda_{-1};L_{-1}+1,\lambda_0+1,0) = \Big(\Delta(x+\lambda_{-1};L_{-1}+1)\Big)^{\lambda_0+1}, \quad 0 \le \lambda_{-1} \le L_{-1} \ ,$$

$$0 \le \lambda_0 \le L_0$$

are linearly independent. Thus (3.91) and (3.92) imply

$$p^{(J_0)}(\lambda_{-1},\lambda_0,\lambda_1,\ldots,\lambda_{n-1},0) = 0 \quad \text{for} \quad 0 \le \lambda_j \le L_j^{(J_0)} \ (-1 \le j \le n-1) \ ,$$

that is,

$$p_d^{(J_0)}(\lambda_{-1},\ldots,\lambda_{n-1},0) = 0 \quad \text{for} \quad 1 \le d \le D,\ 0 \le \lambda_j \le L_j^{(J_0)}\ (-1 \le j \le n-1)\ ,$$

contradicting the construction in the main inductive argument. This contradiction proves Proposition 1.

Chapter IV A proposition towards the proof of Theorem 2

In this chapter we prove a proposition towards the proof of Theorem 2. The proof goes along the same line as in Chapter III. Since we do not introduce the polynomials $\Delta(x;k,l,m)$ in our auxiliary functions, we have some simplification. We use the notations introduced for Theorem 2 and those introduced at the beginning of Chapter III.

1. Statement of the proposition

We define $h_j = h_j(n,q;c_0,c_2)$ $(0 \le j \le 5)$, $h_6 = h_6(n,q;c_0,c_2,c_3)$, $\varepsilon_j = \varepsilon_j(n,q;c_0,c_2)$ $(j = 1,2)$ by the following 9 formulas, which will be referred as (4.1):

$$h_0 = n \log(2^{11} nq) ,$$

$$h_1 = 16\ c_0 (2\ c_2 q)^n (q-1) \frac{n^{2n+2}}{n!} h_0 ,$$

$$h_2 = 16\ c_0 (2\ c_2 q)^{n-1} (q-1) \frac{n^{2n}}{n!} ,\quad 1 + \varepsilon_1 = \left(1-\frac{1}{h_2}\right)^{-n} ,$$

$$h_3 = \frac{h_1 - 1}{(n-1)^2} ,\quad 1 + \varepsilon_2 = e^{h_3^{-1}} , \tag{4.1}$$

$$h_4 = \frac{2^5 h_1}{n} ,$$

$$h_5^{-1} = \frac{1.02\times 10^{-10}}{h_0 h_1} + \frac{n \log(2^5 h_0 h_1)}{2^5 h_0 h_1} ,$$

$$h_6 = c_2 n (q-1)\left(1-\frac{1}{c_3 n}\right)\left(1-\frac{1}{h_1}\right) .$$

In this chapter we suppose c_0, c_1, c_2, c_3 are real numbers satisfying the following conditions (4.2), (4.3), (4.4):

$$2 \le c_0 \le 2^4,\ 2 \le c_1 \le \tfrac{7}{2},\ c_2 \ge \tfrac{5}{2},\ 2^4 \le c_3 \le 2^8 \ ; \qquad (4.2)$$

$$\left(1-\frac{1}{c_3 n}\right)\left(1-\frac{1}{h_1}\right)\left(1-\frac{1}{q}\right)^2 \ge \left(\frac{1}{h_4}+\frac{1}{h_5}\right)\left(1+\frac{1}{c_0-1}\right)c_1 + \left(1+\frac{1}{c_0-1}\right)\frac{1}{c_2}$$
$$+ \left(\frac{1}{q}+\frac{1}{c_0-1}\right)\left(1+\frac{1}{h_0}\right)\left(2+\frac{1}{p-1}\right)\frac{1}{c_3} \ ; \qquad (4.3)$$

$$c_1 \ge \left(1+\frac{1}{h_6}\right)\left(2+\frac{1}{p-1}\right)+\left\{2+\frac{1}{h_6}+\frac{\log h_0}{h_0}+\frac{1}{h_0}\cdot\frac{\log q}{q}\right\}\cdot\frac{2+\frac{1}{p-1}}{nq^n}\cdot\frac{1}{c_3} \ . \qquad (4.4)$$

The existence of such real numbers c_0, c_1, c_2, c_3 will be proved in Chapter V. Let

$$W^* = \max(W, n\log(2^{11}nqD)) \ , \qquad (4.5)$$

where W is a real number satisfying (0.9), and let U be a real number satisfying

$$U \ge (1+\varepsilon_1)(1+\varepsilon_2)c_0c_1c_2^nc_3^2\,\frac{n^{2n+2}}{n!}\,q^{2n}(q-1)\,\frac{G\left(2+\frac{1}{p-1}\right)^n}{(f_p\log p)^{n+2}}\,D^{n+2}V_1\ldots V_n(W^*)^2 \ . \qquad (4.6)$$

Proposition 2. Suppose that (0.5)-(0.8) hold. Then

$$\operatorname{ord}_p\left(\alpha_1^{b_1}\ \ldots\ \alpha_n^{b_n}-1\right) < U \ .$$

2. Notations

The following 6 formulas will be referred as (4.7).

$$Y = \frac{e_p f_p \log p}{q^n D} U ,$$

$$S = q\left[\frac{c_3 n D W^*}{f_p \log p}\right] ;$$

$$T = \left[\frac{U f_p \log p}{q^n D} \cdot \frac{1}{c_1 c_3 W^* \theta}\right] = \left[\frac{Y}{c_1 c_3 W^* e_p \theta}\right] , \tag{4.7}$$

$$L_j = \left[\frac{U e_p f_p \log p}{q^n D} \cdot \frac{1}{c_1 c_2 n p^\kappa S V_j}\right] = \left[\frac{Y}{c_1 c_2 n p^\kappa S V_j}\right] \ (1 \le j \le n) ,$$

$$L = \max_{1 \le j < n} L_j = L_1 \ \text{(see (0.2))} ,$$

$$X_0 = \left(D \prod_{j=1}^{n} (L_j + 1)\right) e^T \left(1 + \frac{(n-1)(B_n L_1 + B' L_n)}{T}\right)^T \exp\left(p^\kappa S \sum_{j=1}^{n} L_j V_j + nD \max_{1 \le j \le n} V_j\right) .$$

The following 11 inequalities (4.8)-(4.18), which can be established in almost the same way as in § 3.2, will be required later. We give only the proofs of (4.11) and (4.14), and omit the proof of the rest.

$$\left(L_1 + 1 - G\right) \ldots \left(L_n + 1 - G\right) \ge c_0 G \left(1 - \frac{1}{q}\right) S \binom{T+n-1}{n-1} , \tag{4.8}$$

$$\frac{1}{n} q^{n-1} S T \theta > \left(1 - \frac{1}{c_3 n}\right)\left(1 - \frac{1}{h_1}\right) \frac{1}{c_1} U , \tag{4.9}$$

$$p^\kappa S \sum_{j=1}^{n} L_j V_j \le \frac{1}{c_1 c_2} Y , \tag{4.10}$$

$$T \le \frac{1}{h_0} \left(2 + \frac{1}{p-1}\right) \frac{1}{c_1 c_3} Y , \tag{4.11}$$

$$T \log\left(1 + \frac{(n-1) q (B_n L_1 + B' L_n)}{T}\right) \le \left(2 + \frac{1}{p-1}\right) \frac{1}{c_1 c_3} Y , \tag{4.12}$$

$$n D \max_{1 \le j \le n} V_j \le \frac{1}{h_4} Y , \tag{4.13}$$

$$\log\Big(D(L_1+1)\ldots(L_n+1)\Big) \le \frac{1}{h_5} Y , \tag{4.14}$$

$$\left(\left(1-\frac{1}{q}\right)\frac{1}{n}q^{-J}T+1\right)\operatorname{ord}_p b_n \le \left(1+\frac{1}{h_6}\right)\frac{2+\frac{1}{p-1}}{nq^n} \cdot \frac{1}{c_1 c_3} U , \tag{4.15}$$

$$\left(\left(1-\frac{1}{q}\right)\frac{1}{n}q^{-J}T+1\right)q^{J+k}S\left(\frac{1}{p-1}+\left(1-\frac{1}{q}\right)\theta\right) \le \left(1+\frac{1}{h_6}\right)\left(2+\frac{1}{p-1}\right)\frac{1}{c_1} U , \tag{4.16}$$

$$\left(1-\frac{1}{q}\right)\frac{1}{n}q^{-J}T \frac{\log(q^{J+k}S)}{\log p} \le \left(1+\frac{\log h_0}{h_0} + \frac{1}{h_0} \cdot \frac{\log q}{q}\right) \frac{2+\frac{1}{p-1}}{nq^n} \cdot \frac{1}{c_1 c_3} U , \tag{4.17}$$

(In (4.15)-(4-17), J,k are integers with $0 \le J \le \frac{\log L_n}{\log q}$, $0 \le k \le n-1$.)

$$L_1 + \ldots + L_{n-1} \le T . \tag{4.18}$$

Proof of (4.11). By (4.5), $W^* \ge n \log(2^{11}nqD) \ge h_0$. Hence the definition of T in (4.7) and (3.35) imply

$$T \le \frac{Y}{c_1 c_3 W^* e_p \theta} \le \frac{1}{h_0}\left(2+\frac{1}{p-1}\right)\frac{1}{c_1 c_3} Y .$$

Proof of (4.14). By (0.1), (0.2), (0.9), (4.1), (4.2), (4.5)-(4.7), we have

$$q \ge 3,\ W^* \ge h_0 \ge 2 \log(2^{11}\times 2\times 3) \ge 18.832,\ h_2 \ge 2^9 \times 15 ,$$

$$Y \ge 2^5 h_0 h_1 D ,$$

$$\frac{Y}{c_1 c_2 n p^{\kappa} S V_j} \ge h_0 h_2 ,$$

$$c_1 c_2 n p^{\kappa} S V_j \ge c_1 c_2 n (c_3 n-1) q W^* \ge 17513.76 .$$

Thus we see that

$$\prod_{j=1}^{n}(L_j+1) \le \prod_{j=1}^{n}\left(\frac{Y}{\tilde{c}_1 c_2 n p^{\kappa} S V_j} + 1\right)$$

$$\le \prod_{j=1}^{n}\left\{\frac{Y}{c_1 c_2 n p^{\kappa} S V_j}\left(1+\frac{1}{h_0 h_2}\right)\right\}$$

$$\le Y^n \left(\frac{1+6.9143\times 10^{-6}}{17513.76}\right)^2$$

$$\le 3.2603 \times 10^{-9}\, Y^n .$$

So

$$\frac{\log\left(D\prod_{j=1}^{n}(L_j+1)\right)}{Y} \le \frac{1}{Y}\left(\log\left(3.2603 \times 10^{-9} D\right) + n \log Y\right)$$

$$\le \frac{1.02\times 10^{-10}}{h_0 h_1} + \frac{n \log(2^5 h_0 h_1)}{2^5 h_0 h_1} = h_5^{-1} .$$

This proves (4.14).

In the sequel we abbreviate $(\lambda_1,\ldots,\lambda_n) \in \mathbb{N}^n$ as λ, $(\tau_1,\ldots,\tau_{n-1}) \in \mathbb{N}^{n-1}$ as τ, and write

$$|\tau| = \tau_1 + \ldots + \tau_{n-1} ,$$

$$\Lambda(\tau) = \prod_{j=1}^{n-1} \Delta(b_n\lambda_j - b_j\lambda_n;\, \tau_j) .$$

We also use the basis $\xi_1,\ldots,\xi_D$ of $K = \mathbb{Q}(\alpha_1,\ldots,\alpha_n)$ over $\mathbb{Q}$ to the shape (3.55).

3. Construction of the rational integers $p_d(\lambda)$

Lemma 4.1. For $d = 1,\dots,D$ and $\lambda = (\lambda_1,\dots,\lambda_n)$ satisfying

$$0 \le \lambda_j \le L_j \ (1 \le j \le n),\ r_1\lambda_1 + \dots + r_n\lambda_n \equiv 0 \pmod G) \tag{4.19}$$

there exist rational integers $p_d(\lambda)$ with

$$0 < \max_{d,\lambda} |p_d(\lambda)| \le X_0^{\frac{1}{c_0-1}}$$

such that

$$\sum_{\lambda} \sum_{d=1}^{D} p_d(\lambda)\xi_d \Lambda(\tau) \prod_{j=1}^{n} \left(\alpha_j^{p^\kappa} \zeta^{r_j}\right)^{\lambda_j s} = 0$$

for $1 \le s \le S$, $(s,q)=1$, $|\tau| \le T$, where $\sum_{\lambda}$ denotes the summation over the range (4.19).

Proof. Similar to the proof of Lemma 3.1.

4. The main inductive argument

For rational integers $r^{(J)}$, $L_j^{(J)}$ $(1 \le j \le n)$ and $p_d^{(J)}(\lambda) = p_d^{(J)}(\lambda_1,\dots,\lambda_n)$, which will be constructed in the following "main inductive argument", we set

$$\varphi_J(z,\tau) = \sum_{\lambda}^{(J)} \sum_{d=1}^{D} p_d^{(J)}(\lambda)\xi_d \Lambda(\tau) \prod_{j=1}^{n} \left(\alpha_j^{p^\kappa} \zeta^{r_j}\right)^{\lambda_j z}, \tag{4.20}$$

where $\sum_{\lambda}^{(J)}$ denotes the summation over the range of $\lambda = (\lambda_1,\dots,\lambda_n)$:

$$0 \leq \lambda_j \leq L_j^{(J)} \quad (1 \leq j \leq n), \quad r_1\lambda_1+\ldots+r_n\lambda_n \equiv r^{(J)} (\mathrm{mod}\ G) \ . \tag{4.21}$$

The main inductive argument. Suppose that there are algebraic numbers $\alpha_1,\ldots,\alpha_n$ and rational integers $b_1,\ldots,b_n$, satisfying (0.5)-(0.8), such that

$$\mathrm{ord}_p\left(\alpha_1^{b_1}\ldots\alpha_n^{b_n} - 1\right) \geq U \ . \tag{4.22}$$

Then for every rational integer J with

$$0 \leq J \leq \left[\frac{\log L_n}{\log q}\right] + 1$$

there exist rational integers $r^{(J)}$, $L_j^{(J)}$ $(1 \leq j \leq n)$ with

$$0 \leq r^{(J)} < G, \ \text{g.c.d.}\ (r_1,\ldots,r_n,G)\,|\,r^{(J)} \ ,$$

$$0 \leq L_j^{(J)} \leq q^{-J}L_j \quad (1 \leq j \leq n) \ ,$$

and rational integers

$$p_d^{(J)}(\lambda) \quad \text{for} \quad d = 1,\ldots,D \quad \text{and} \quad \lambda \quad \text{satisfying (4.21)},$$

not all zero, with absolute values not exceeding $X_0^{\frac{1}{c_0-1}}$, such that

$$\varphi_J(s,\tau) = 0 \quad \text{for} \quad 1 \leq s \leq q^J S, \ (s,q)=1, \ |\tau| \leq q^{-J}T \ .$$

The proof of the mainductive argument is similar to that in § 3.4. So we only give a detailed sketch. We prove it by an induction on J. On taking $r^{(0)} = 0$, $L_j^{(0)} = L_j$ $(1 \le j \le n)$, $p_d^{(0)}(\lambda) = p_d(\lambda)$ $(1 \le d \le D$, λ satisfying (4.21)), we see, by Lemma 4.1, that the case $J = 0$ is true. In the remaining part of this section, we assume the main inductive argument is valid for some J with

$$0 \le J \le \left[\frac{\log L_n}{\log q}\right],$$

and we shall prove it for $J + 1$. So we always keep the hypothesis (4.22). We first show the following Lemmas 4.2, 4.3, 4.4, then deduce the main inductive argument for $J + 1$.

Let

$$\gamma_j = \lambda_j - \frac{b_j}{b_n}\lambda_n \quad (1 \le j \le n-1)$$

and put

$$f_J(z,\tau) = \sum_{\lambda}{}^{(J)} \sum_{d=1}^{D} p_d^{(J)}(\lambda)\xi_d \Lambda(\tau) \prod_{j=1}^{n-1} \left(\alpha_j^{p^\kappa} \zeta^{r_j}\right)^{\gamma_j z}.$$

We write $p^{(J)}(\lambda)$ for $\sum_{d=1}^{D} p_d^{(J)}(\lambda)\xi_d$.

<u>Lemma 4.2</u>. For any $\tau = (\tau_1, \ldots, \tau_{n-1})$ with $|\tau| \le T$ and any $y \in \mathbb{Q}$, $y > 0$, with $\mathrm{ord}_p y \ge 0$, we have

$$\mathrm{ord}_p(\varphi_J(y,\tau) - f_J(y,\tau)) \ge U - \mathrm{ord}_p b_n.$$

Proof. By the definitions of $\varphi_J(z,\tau)$ and $f_J(z,\tau)$, we have

$$\varphi_J(y,\tau)-f_J(y,\tau)=\sum_{\lambda}{}^{(J)} p^{(J)}(\lambda)\Lambda(\tau)\left\{\prod_{j=1}^{n}\left(\alpha_j^{p^\kappa}\zeta^{r_j}\right)^{\lambda_j y}-\prod_{j=1}^{n-1}\left(\alpha_j^{p^\kappa}\zeta^{r_j}\right)^{\gamma_j y}\right\} .$$

It is easy to see $\mathrm{ord}_p\Lambda(\tau) \geq 0$ (since $\Lambda(\tau) \in \mathbb{Z}$) and $\mathrm{ord}_p p^{(J)}(\lambda) \geq 0$ by (0.5). Similarly to the proof of (3.63), we can readily show that

$$\mathrm{ord}_p\left\{\prod_{j=1}^{n}\left(\alpha_j^{p^\kappa}\zeta^{r_j}\right)^{\lambda_j y}-\prod_{j=1}^{n-1}\left(\alpha_j^{p^\kappa}\zeta^{r_j}\right)^{\gamma_j y}\right\} \geq U - \mathrm{ord}_p b_n .$$

Now the lemma follows from the above observations at once.

Lemma 4.3. For $k = 0,1,\ldots,n-1$, we have

$$\varphi_J(s,\tau) = 0 \tag{4.23}$$

for $1 \leq s \leq q^{J+k}S$, $(s,q)=1$, $|\tau| \leq \left(1-\left(1-\frac{1}{q}\right)\frac{k}{n}\right)q^{-J}T$.

Proof. We argue by an induction on k. By the main inductive hypothesis for J, (4.23) with $k = 0$ is true. Assuming (4.23) is valid for some k with $0 \leq k \leq n-2$, we prove it for $k+1$. Thus we see, by Lemma 4.2, that

$$\mathrm{ord}_p f_J(s,\tau) \geq U - \mathrm{ord}_p b \tag{4.24}$$

for $1 \leq s \leq q^{J+k}S$, $(s,q)=1$, $|\tau| \leq \left(1-\left(1-\frac{1}{q}\right)\frac{k}{n}\right) q^{-J}T$. By (0.7), (3.1) and the remark below the proof of Lemma 1.1,

$$\prod_{j=1}^{n-1}\left(\alpha_j^{p^\kappa}\zeta^{r_j}\right)^{\gamma_j p^{-\theta}z}$$

is a p-adic normal function, whence so are

$$F_J(z,\tau) = f_J(p^{-\theta}z,\tau) \quad \text{for} \quad |\tau| \leq \left(1-\left(1-\frac{1}{q}\right)\frac{k+1}{n}\right)q^{-J}T \ . \tag{4.25}$$

We now apply Lemma 1.4 to each $F_J(z,\tau)$ in (4.25), taking

$$R = q^{J+k}S, \ M = \left[\left(1-\frac{1}{q}\right)\frac{1}{n}\ q^{-J}T\right] + 1 \ . \tag{4.26}$$

Similarly to the proof of Lemma 3.3, we see, by (4.24), (4.25) and (4.15), that

$$\min_{\substack{1\leq s\leq R,\ (s,q)=1\\ 0\leq t\leq M-1}} \left\{\mathrm{ord}_p\left(\frac{1}{t!}\frac{d^t}{dz^t}F_J(sp^{\theta},\tau)\right)+t\theta\right\} \geq U - \left(\left(1-\frac{1}{q}\right)\frac{1}{n}q^{-J}T+1\right)\mathrm{ord}_p b_n$$

$$\geq U - \left(1+\frac{1}{h_6}\right)\frac{2+\frac{1}{p-1}}{nq^n}\cdot\frac{1}{c_1c_3}\,U \tag{4.27}$$

for $|\tau| \leq \left(1-\left(1-\frac{1}{q}\right)\frac{k+1}{n}\right)q^{-J}T$, where R, M are given by (4.26). On the other hand, by (4.16) and (4.17)

$$\left(1-\frac{1}{q}\right)RM\ \theta + M\ \mathrm{ord}_p R! + (M-1)\ \frac{\log R}{\log p}$$

$$\leq \left(1+\frac{1}{h_6}\right)\left(2+\frac{1}{p-1}\right)\frac{1}{c_1}U+\left(1+\frac{\log h_0}{h_0}+\frac{1}{h_0}\cdot\frac{\log q}{q}\right)\frac{2+\frac{1}{p-1}}{nq^n}\cdot\frac{1}{c_1c_3}\,U \ . \tag{4.28}$$

By (4.27), (4.28), (4.4), we see that each $F_J(z,\tau)$ in (4.25) satisfies the condition (1.9) of Lemma 1.4 with R, M given by (4.26). So Lemma 1.4 and (4.25) imply

$$\mathrm{ord}_p f_J(s,\tau) \geq \left(1-\frac{1}{q}\right)RM\ \theta > \left(1-\frac{1}{q}\right)^2\frac{1}{n}\ q^kST\ \theta$$

for $s \in \mathbb{Z}$, $|\tau| \leq \left(1-\left(1-\frac{1}{q}\right)\frac{k+1}{n}\right) q^{-J}T$. By Lemma 4.2 and again by (4.27), (4.28), we get for $s \geq 1$

$$\mathrm{ord}_p\Big(\varphi_J(s,\tau)-f_J(s,\tau)\Big) \geq U - \mathrm{ord}_p b_n > \left(1-\frac{1}{q}\right) RM\ \theta$$

$$> \left(1-\frac{1}{q}\right)^2 \frac{1}{n}\, q^k ST\ \theta\ .$$

Hence

$$\begin{aligned}\mathrm{ord}_p\varphi_J(s,\tau) &\geq \min\Big(\mathrm{ord}_p f_J(s,\tau),\ \mathrm{ord}_p(\varphi_J(s,\tau)-f_J(s,\tau))\Big)\\ &> \left(1-\frac{1}{q}\right)^2 \frac{1}{n}\, q^k ST\ \theta\\ &> \frac{U}{c_1}\, q^{k+1-n}\left(1-\frac{1}{q}\right)^2\left(1-\frac{1}{c_3 n}\right)\left(1-\frac{1}{h_1}\right) \qquad (4.29)\end{aligned}$$

for $s \geq 1$, $|\tau| \leq \left(1-\left(1-\frac{1}{q}\right)\frac{k+1}{n}\right) q^{-J}T$, where the last inequality follows from (4.9).

Now assuming there exist s, τ with

$$1 \leq s \leq q^{J+k+1}S,\ (s,q)=1,\ |\tau| \leq \left(1-\left(1-\frac{1}{q}\right)\frac{k+1}{n}\right) q^{-J}T$$

such that

$$\varphi_J(s,\tau) \neq 0\ ,$$

we proceed to deduce a contradiction. On applying Lemma 2.1 to $\zeta^{-r^{(J)}s}\varphi_J(s,\tau)$, which is a polynomial in $\alpha_1,\ldots,\alpha_n$ with rational integer coefficients, we see, by (4.10)-(4.14),

$|p_d^{(J)}(\lambda)| \leq X_0^{\frac{1}{c_0-1}}$ (from the main inductive argument for J)

and the definition of X_0 in (4.7), that

$$\mathrm{ord}_p \varphi_J(s,\tau) = \mathrm{ord}_p\left(\zeta^{-r^{(J)}s}\varphi_J(s,\tau)\right)$$

$$\leq \frac{D}{e_{\mathfrak{p}} f_{\mathfrak{p}} \log p}\left\{\log(D(L_{-1}+1)\dots(L_n+1)) + \frac{1}{c_0-1}\log X_0 + T + T\log\left(1+\frac{(n-1)(B_nL_1+B'L_n)}{T}\right)\right.$$

$$\left. + p^{\kappa}q^{k+1}S\sum_{j=1}^{n}L_jV_j + nD\max_{1\leq j\leq n}V_j\right\}$$

$$\leq q^{k+1-n}\frac{q^nD}{e_{\mathfrak{p}} f_{\mathfrak{p}} \log p}\left\{\frac{1}{q}\left(1+\frac{1}{c_0-1}\right)\left(\log(D(L_1+1)\dots(L_n+1)) + nD\max_{1\leq j\leq n}V_j\right)\right.$$

$$+\left(1+\frac{1}{q(c_0-1)}\right)p^{\kappa}S\sum_{j=1}^{n}L_jV_j$$

$$\left.+\frac{1}{q}\left(1+\frac{1}{c_0-1}\right)\left(T+T\log\left(1+\frac{(n-1)(B_nL_1+B'L_n)}{T}\right)\right)\right\}$$

$$\leq \frac{U}{c_1}q^{k+1-n}\left\{\left(\frac{1}{h_4}+\frac{1}{h_5}\right)\left(1+\frac{1}{c_0-1}\right)c_1 + \left(1+\frac{1}{c_0-1}\right)\frac{1}{c_2}\right.$$

$$\left.+\left(\frac{1}{q}+\frac{1}{c_0-1}\right)\left(1+\frac{1}{h_0}\right)\left(2+\frac{1}{p-1}\right)\frac{1}{c_3}\right\}$$

$$\leq \frac{U}{c_1}q^{k+1-n}\left(1-\frac{1}{c_3n}\right)\left(1-\frac{1}{h_1}\right)\left(1-\frac{1}{q}\right)^2$$

(where the last inequality follows from (4.3)), contrary to (4.29). This contradiction proves

$$\varphi_J(s,\tau)=0 \quad \text{for} \quad 1 \leq s \leq q^{J+k+1}S,\ (s,q)=1,\ |\tau| \leq \left(1-\left(1-\frac{1}{q}\right)\frac{k+1}{n}\right)q^{-J}T\ ,$$

thereby establishes the lemma.

Lemma 4.4.

$$\varphi_J\left(\frac{s}{q},\tau\right) = 0$$

for $1 \leq s \leq q^{J+1}S$, $(s,q)=1$, $|\tau| \leq q^{-(J+1)}T$.

Proof. By Lemma 4.2 and Lemma 4.3 with $k = n-1$, we have

$$\operatorname{ord}_p f_J(s,\tau) \geq U - \operatorname{ord}_p b_n$$

for $1 \leq s \leq q^{J+n-1}S$, $(s,q)=1$, $|\tau| \leq \left(1-\left(1-\frac{1}{q}\right)\frac{n-1}{n}\right) q^{-J}T$. On applying Lemma 1.4 to each of the functions

$$F_J(z,\tau) = f_J\left(p^{-\theta}z,\tau\right) \quad \text{for} \quad |\tau| \leq q^{-(J+1)}T ,$$

taking

$$R = q^{J+n-1}S \ , \ M = \left[\left(1-\frac{1}{q}\right)\frac{1}{n}q^{-J}T\right] + 1 \ ,$$

we can, similarly to the proof of Lemma 3.4, obtain

$$\operatorname{ord}_p\varphi_J\left(\frac{s}{q},\tau\right) \geq \frac{U}{c_1}\left(1-\frac{1}{q}\right)^2\left(1-\frac{1}{c_3n}\right)\left(1-\frac{1}{h_1}\right) \tag{4.30}$$

for $s \geq 1$, $|\tau| \leq q^{-(J+1)}T$.

Assuming that there exist s, τ with

$$1 \leq s \leq q^{J+1}S, \ (s,q)=1, \ |\tau| \leq q^{-(J+1)}T$$

such that

$$\varphi_J\left(\frac{s}{q},\tau\right) \neq 0 \ ,$$

we proceed to deduce a contradiction. On applying Lemma 2.1 to $\zeta^{-bsr^{(J)}}\varphi_J\left(\frac{s}{q},\tau\right)$ (recalling that b is introduced in the beginning of Chapter III and appears in (3.3)), which is a polynomial in $\alpha_j^{\frac{1}{q}}$ $(1 \le j \le n)$ of degree at most

$$p^{\kappa}L_j^{(J)}q^{J+1}s + qD \le q(p^{\kappa}SL_j + D) \quad (1 \le j \le n)$$

with rational integer coefficients, and on utilizing (4.10)-(4.14), $|p_d^{(J)}(\lambda)| \le X_0^{\frac{1}{c_0-1}}$ (from the main inductive argument for J) and the definition of X_0 in (4.7), we obtain

$$\begin{aligned}
\operatorname{ord}_p\varphi_J\left(\frac{s}{q},\tau\right) \le \frac{q^nD}{e_{\mathfrak{p}}f_{\mathfrak{p}}\log p}\Bigg\{&\left(1+\frac{1}{c_0-1}\right)\left(\log(D(L_1+1)\ldots(L_n+1))+nD\max_{1\le j\le n}V_j\right)\\
&+\left(1+\frac{1}{c_0-1}\right)p^{\kappa}S\sum_{j=1}^{n}L_jV_j\\
&+\left(\frac{1}{q}+\frac{1}{c_0-1}\right)\left(T+T'\log\left(1+\frac{(n-1)q(B_nL_1+B'L_n)}{T}\right)\right)\Bigg\}\\
\le \frac{U}{c_1}\Bigg\{&\left(\frac{1}{h_4}+\frac{1}{h_5}\right)\left(1+\frac{1}{c_0-1}\right)c_1+\left(1+\frac{1}{c_0-1}\right)\frac{1}{c_2}\\
&+\left(\frac{1}{q}+\frac{1}{c_0-1}\right)\left(1+\frac{1}{h_0}\right)\left(2+\frac{1}{p-1}\right)\frac{1}{c_3}\\
\le \frac{U}{c_1}&\left(1-\frac{1}{c_3n}\right)\left(1-\frac{1}{h_1}\right)\left(1-\frac{1}{q}\right)^2
\end{aligned}$$

(where the last inequality follows from (4.3)), contrary to (4.30). This contradiction proves

$$\varphi_J\left(\frac{s}{q},\tau\right) = 0 \quad \text{for} \quad 1 \leq s \leq q^{J+1}S, \ (s,q) = 1, \ |\tau| \leq q^{-(J+1)}T ,$$

thereby establishes the lemma.

Lemma 4.5. The main inductive argument is true for $J + 1$.

Proof. Similar to the proof of Lemma 3.5.

Thus we have established the main inductive argument for $J = 0,1,\ldots,\left[\frac{\log L_n}{\log q}\right] + 1$.

5. The completion of the proof of Proposition 2

We assume that Proposition 2 is false, that is, there exist algebraic numbers $\alpha_1,\ldots,\alpha_n$ and rational integers $b_1,\ldots,b_n$ satisfying (0.5)-(0.8), such that

$$\mathrm{ord}_p\left(\alpha_1^{b_1} \ldots \alpha_n^{b_n} - 1\right) \geq U ,$$

and we proceed the deduce a contradiction. By the main inductive argument for

$$J = J_0 = \left[\frac{\log L_n}{\log q}\right] + 1 ,$$

we have

$$\varphi_{J_0}(s,\tau)=0 \quad \text{for} \quad 1 \leq s \leq q^{J_0}S, \ (s,q)=1, \ |\tau| \leq q^{-J_0}T . \tag{4.31}$$

Since $0 \le L_n^{(J_0)} \le q^{-J_0} L_n$, we see that $L_n^{(J_0)} = 0$. Further if $\tau = (\tau_1, \ldots, \tau_{n-1})$ satisfies

$$0 \le \tau_j \le L_j^{(J_0)} \quad (1 \le j \le n-1) ,$$

then by (4.18)

$$|\tau| = \tau_1 + \ldots + \tau_{n-1} \le q^{-J_0}(L_1 + \ldots + L_{n-1}) \le q^{-J_0} T .$$

Thus (4.31) implies (writing $p^{(J)}(\lambda) = \sum_{d=1}^{D} p_d^{(J)}(\lambda)\xi_d$)

$$\sum_{\lambda_{n-1}=0}^{L_{n-1}^{(J_0)}} \left\{ \sum_{\lambda_1=0}^{L_1^{(J_0)}} \cdots \sum_{\lambda_{n-2}=0}^{L_{n-2}^{(J_0)}} p^{(J)}(\lambda_1,\ldots,\lambda_{n-1},0) \prod_{j=1}^{n-2} \Delta(b_n\lambda_j;\tau_j) \cdot \prod_{j=1}^{n-1} \left(\alpha_j^{p^{\kappa}} \zeta^{r_j}\right)^{\lambda_j s} \right\} \Delta(b_n\lambda_{n-1};\tau_{n-1}) = 0 \tag{4.32}$$

for $1 \le s \le q^{J_0} S$, $(s,q)=1$, $0 \le \tau_j \le L_j^{(J_0)}$ $(1 \le j \le n-1)$, where we have set

$$p^{(J_0)}(\lambda_1,\ldots,\lambda_{n-1},0) = 0$$

for $\lambda_1,\ldots,\lambda_{n-1}$ satisfying $0 \le \lambda_j \le L_j^{(J_0)}$ $(1 \le j \le n-1)$ and $r_1\lambda_1 + \ldots + r_{n-1}\lambda_{n-1} \not\equiv r^{(J_0)} \pmod G$. By Lemma 2.5 we have

$$\det(\Delta(b_n\lambda_{n-1};\tau_{n-1}))_{0 \le \lambda_{n-1},\tau_{n-1} \le L_{n-1}^{(J_0)}} \neq 0 .$$

So (4.32) implies that for each λ_{n-1} with $0 \le \lambda_{n-1} \le L_{n-1}^{(J_0)}$

$$\sum_{\lambda_{n-2}=0}^{L_{n-2}^{(J_0)}} \left\{ \sum_{\lambda_1=0}^{L_1^{(J_0)}} \cdots \sum_{\lambda_{n-3}=0}^{L_{n-3}^{(J_0)}} p^{(J_0)}(\lambda_1,\ldots,\lambda_{n-1},0) \prod_{j=1}^{n-3} \Delta(b_n\lambda_j;\tau_j) \cdot \prod_{j=1}^{n-2} \left(\alpha_j^{p^{\kappa}} \zeta^{r_j}\right)^{\lambda_j s} \right\} \Delta(b_n\lambda_{n-2};\tau_{n-2}) = 0$$

for $1 \leqq s \leqq q^{J_0}S$, $(s,q)=1$, $0 \leq \tau_j \leq L_j^{(J_0)}$ $(1 \leqq j \leqq n-2)$.
Repeating this argument $n-1$ times, we obtain

$$p^{(J_0)}(\lambda_1,\ldots,\lambda_{n-1},0) = 0 \quad \text{for} \quad 0 \leqq \lambda_j \leq L_j^{(J_0)} \; (1 \leq j \leqq n-1) ,$$

contrary to the construction in the main inductive argument. This contradiction proves Proposition 2.

Chapter V Completion of the proofs of Theorems 1 and 2

1. Solving the system of inequalities (3.5)-(3.7)

We solve the system of inequalities (3.5)-(3.7) in the following cases respectively:

(1.a) $p = 2 \; , \; 2 \leq n \leq 7 \; ;$

(1.b) $p = 2 \; , \; n \geq 8 \; ;$

(2.a) $p = 3 \; , \; 2 \leq n \leq 7 \; ;$

(2.b) $p = 3 \; , \; n \geq 8 \; ;$

(3.a) $p \geq 5 \; , \; 2 \leq n \leq 6 \; ;$

(3.b) $p \geq 5 \; , \; n = 7 \; ;$

(3.c) $p \geq 5 \; , \; n \geq 8 \; .$

We abbreviate $h_i(n,q;c_0,c_2)$ as $h_i (0 \leq i \leq 7)$, $h_8(n,q;c_0,c_2,c_3)$ as h_8 , $\varepsilon_i(n,q;c_0,c_2)$ as $\varepsilon_i (i = 1,2)$.

We first deal with the cases (1.a), (2.a), (3.a), (3.b). In these cases

$$n \geq 2 \; , \; q \geq 3$$

and we fix

$$c_0 = 8 \ , \ c_2 = \frac{56}{15} \ .$$

Then we have the following inequalities:

$$h_0 \geq h_0(2,3) \geq 18.832756 \ , \ \frac{1}{h_0} \leq 5.3099 \times 10^{-2} \ , \ h_0(2,3) \leq 18.832758$$

$$\frac{\log h_0}{h_0} \leq 1.5587732 \times 10^{-1} \ , \ \frac{\log(h_0+1)}{h_0} \leq 0.1586245 \ ,$$

$$h_1 \geq h_1(2,3;8,\tfrac{56}{15}) \geq 7.74103 \times 10^7 \ , \ \frac{1}{h_1} \leq 1.291818 \times 10^{-8} \ ,$$

$$h_2(2,3;8,\tfrac{56}{15}) \geq \frac{7}{5} \times 2^{15} \ , \ (h_2(2,3;8,\tfrac{56}{15}))^{-1} \leq 2.17983 \times 10^{-5} \ ,$$

$$1 + \varepsilon_1 \leq 1 + \varepsilon_1(2,3;8,\tfrac{56}{15}) \leq (1 - 2.17983 \times 10^{-5})^{-2} \leq 1 + 4.35986 \times 10^{-5} \ ,$$

$$(h_3(2,3;8,\tfrac{56}{15}))^{-1} \leq \frac{4}{7.74103 \times 10^7 - 1} \leq 5.167273 \times 10^{-8} \ ,$$

$$1 + \varepsilon_2 \leq 1 + \varepsilon_2(2,3;8,\tfrac{56}{15}) \leq 1 + 5.167274 \times 10^{-8} \ ,$$

$$(1 + \varepsilon_1)(1 + \varepsilon_2) \leq 1 + 4.366 \times 10^{-5} \ ,$$

$$\frac{1}{h_4} \leq (h_4(2,3;8,\tfrac{56}{15}))^{-1} \leq 1.291818 \times 10^{-8} \times 19.832758 \leq 2.5620315 \times 10^{-7} \ ,$$

$$\log h_5 \leq \log h_5(2,3;8,\tfrac{56}{15}) \leq 6.3749002 \ ,$$

$$\frac{1}{h_6} \leq (h_6(2,3;8,\tfrac{56}{15}))^{-1} \leq 4.03694 \times 10^{-10} \ ,$$

$$\frac{1}{h_7} \leq (h_7(2,3;8,\tfrac{56}{15}))^{-1} \leq 8.1217 \times 10^{-10} \ .$$

The above inequalities will be repeatedly used in the cases (1.a), (2.a), (3.a), (3.b).

Case (1.a) : $P = 2,\ 2 \leq n \leq 7$

It is easy to verify the following inequalities

$$\left(1-\frac{1}{c_3 n}\right)\left(1-\frac{1}{h_1}\right)\left(1-\frac{1}{q}\right)^2 \geq \frac{4}{9} \times (1 - 1.291818 \times 10^{-8}) \times \left(1-\frac{1}{2c_3}\right), \quad (5.1)$$

$$\left(\frac{1}{h_6} + \frac{1}{h_7}\right)\left(1+\frac{1}{c_0-1}\right)c_1 \leq 1.215864 \times 10^{-9} \times \left(1+\frac{1}{7}\right) \times \frac{7}{2} \leq 4.863456 \times 10^{-9}, \quad (5.2)$$

provided $c_1 \leq \frac{7}{2}$,

$$\left(1+\frac{1}{c_0-1}\right)\frac{1}{c_2} = \frac{15}{49}, \quad (5.3)$$

$$\left\{1+\left(1+\frac{1}{h_0}\right)\log 3\right\}\left(\frac{1}{q} + \frac{1}{c_0-1}\right)\left(2 + \frac{1}{p-1}\right)$$

$$\leq \left(1 + 1.053099 \times \log 3\right)\left(\frac{1}{3} + \frac{1}{7}\right) \times 3 \leq 3.081354, \quad (5.4)$$

$$\left(1+\frac{1}{h_4}\right)\left\{4 + \frac{1}{2^{10}nq} + \frac{2\log h_5}{h_0} + \frac{1}{n}\left(1 + \frac{1}{c_0-1}\right) + \left(1 + \frac{1}{p-1}\right)\frac{1}{q^n}\right\}$$

$$\leq \left(1 + 2.5620315 \times 10^{-7}\right)\left(4 + \frac{1}{2^{11}\times 3} + 6.7700168 \times 10^{-1} + \frac{4}{7} + \frac{2}{9}\right)$$

$$\leq 5.4708156. \quad (5.5)$$

On combining (5.1)-(5.5) we see that if $c_1 \leq \frac{7}{2}$, then the inequality

$$\frac{4}{9} \times \left(1 - 1.291818 \times 10^{-8}\right) - \frac{15}{49} - 4.863456 \times 10^{-9}$$

$$\geq \left(3.081354 + \frac{2}{9} \times \left(1 - 1.291818 \times 10^{-8}\right)\right) \frac{1}{c_3} + 5.4708156 \frac{1}{c_4} \quad (5.6)$$

implies (3.6). Letting the two terms on the right-hand side of (5.6) be equal, we see that

$$c_3 = 47.766502 \ , \ c_4 = 79.102681$$

satisfy (5.6).

Further, on substituting c_3 by 47.766502 , we see that

$$\frac{1}{h_8} \leq \left(h_8(2,3;8,\frac{56}{15},47.766502)\right)^{-1} \leq 6.76727 \times 10^{-2}$$

and

$$\left\{2 + \frac{1}{h_8} + \frac{\log h_0}{h_0} + \frac{1}{h_0} \cdot \frac{\log q}{q} + \frac{n \log(h_0+1)}{h_0}\right\} \frac{2+\frac{1}{p-1}}{nq^n} \cdot \frac{1}{c_3}$$

$$\leq \left\{2 + 6.76727 \times 10^{-2} + 1.5587732 \times 10^{-1} + 5.3099 \times 10^{-2} \times \frac{\log 3}{3} + \right.$$

$$\left. + 2 \times 0.1586245\right\} \times \frac{3}{2\times 3^2} \times \frac{1}{47.766502}$$

$$\leq 8.9332 \times 10^{-3} \ .$$

Thus

$$c_1 = (1 + 6.76727 \times 10^{-2}) \times 3 + 8.9332 \times 10^{-3} = 3.2119513 \ (< \tfrac{7}{2})$$

$$c_3 = 47.766502$$

satisfy (3.7).

From the above discussion, we conclude that

$$c_0 = 8,\ c_1 = 3.2119513,\ c_2 = \frac{56}{15},\ c_3 = 47.766502,\ c_4 = 79.102681$$

satisfy the system of inequalities (3.5)-(3.7).

<u>Case (2.a): $p = 3,\ 2 \le n \le 7$</u>

By (0.1), we have

$$q \ge 5 .$$

On noting $p = 3,\ n \geqq 2,\ q \ge 5$, it is easy to see that the following inequalities hold.

$$\left(1-\frac{1}{c_3 n}\right)\left(1-\frac{1}{h_1}\right)\left(1-\frac{1}{q}\right)^2 \ge \frac{16}{25} \times (1 - 1.291818 \times 10^{-8})\left(1-\frac{1}{2c_3}\right), \quad (5.7)$$

$$\left(\frac{1}{h_6}+\frac{1}{h_7}\right)\left(1+\frac{1}{c_0-1}\right)c_1 \le 4.863456 \times 10^{-9}, \text{ provided } c_1 \leqq \frac{7}{2}, \quad (5.8)$$

$$\left(1+\frac{1}{c_0-1}\right)\frac{1}{c_2} = \frac{15}{49}, \quad (5.9)$$

$$\left\{1+\left(1+\frac{1}{h_0}\right)\log 3\right\}\left(\frac{1}{q}+\frac{1}{c_0-1}\right)\left(2+\frac{1}{p-1}\right)$$

$$\leqq \left\{1+1.053099\times\log 3\right\}\left(\frac{1}{5}+\frac{1}{7}\right)\left(2+\frac{1}{p-1}\right) \le 0.739525\left(2+\frac{1}{p-1}\right), \quad (5.10)$$

$$\left(1+\frac{1}{h_4}\right)\left\{4+\frac{1}{2^{10}nq} + \frac{2\log h_5}{h_0} + \frac{1}{n}\left(1+\frac{1}{c_0-1}\right) + \left(1+\frac{1}{p-1}\right)\frac{1}{q^n}\right\}$$

$$\leq \left(1+2.5620315\times10^{-7}\right)\left\{4+\frac{1}{2^{11}\times5}+6.7700168\times10^{-1}+\frac{4}{7}+\frac{3}{2}\times\frac{1}{5^2}\right\}$$

$$\leq 5.3085281\ . \tag{5.11}$$

On combining (5.7)-(5.11), we see that the inequality

$$\frac{16}{25}\times(1-1.291818\times10^{-8})-\frac{15}{49}-4.863456\times10^{-9}$$

$$\geq \left(0.739525+\frac{8}{25}\times(1-1.291818\times10^{-8})\times\frac{2}{5}\right)\left(2+\frac{1}{p-1}\right)\frac{1}{c_3}+5.3085281\frac{1}{c_4} \tag{5.12}$$

implies (3.6), provided $c_1 \leq \frac{7}{2}$. Evidently

$$c_3 = c_4 = 32$$

satisfy (5.12).

Further, on substituting c_3 by 32 , we get

$$\frac{1}{h_8} \leq \left(h_8\left(2,5;8,\frac{56}{15},32\right)\right)^{-1} \leq 3.40137\times10^{-2}$$

and

$$\left\{2+\frac{1}{h_8}+\frac{\log h_0}{h_0}+\frac{1}{h_0}\cdot\frac{\log q}{q}+\frac{n\log(h_0+1)}{h_0}\right\}\frac{1}{nq^n}\cdot\frac{1}{c_3}$$

$$\leq \left\{2+0.0340137+1.5587732\times10^{-1}+5.3099\times10^{-2}\times\frac{\log 5}{5}+2\times0.1586245\right\}$$

$$\times\frac{1}{2\times5^2}\times\frac{1}{32}$$

$$\leq 0.0015777\ .$$

Thus

$$c_1 = \left(1+3.40137\times10^{-2}+0.0015777\right)\left(2+\frac{1}{p-1}\right) = 2.5889785 \ , \ (< \tfrac{7}{2})$$

$$c_3 = 32$$

satisfy (3.7).

From the above discussion we conclude that

$$c_0 = 8, \ c_1 = 2.5889785, \ c_2 = \frac{56}{15}, \ c_3 = c_4 = 32$$

satisfy the system of inequalities (3.5)-(3.7).

Case (3.a): $p \geq 5, \ 2 \leq n \leq 6$

On noting $p \geq 5, \ n \geq 2, \ q \geq 3$, it is readily verified that the inequality

$$\frac{4}{9} \times \left(1 - 1.291818 \times 10^{-8}\right) - \frac{15}{49} - 4.863456 \times 10^{-9}$$

$$\geq \left(1.027118+\frac{2}{9}\times\left(1-1.291818\times10^{-8}\right)\times\frac{1}{2}\right)\left(2+\frac{1}{p-1}\right)\frac{1}{c_3}+5.3874822\frac{1}{c_4} \qquad (5.13)$$

implies (3.6), provided $c_1 \leq \frac{7}{2}$. Evidently

$$c_3 = 16.457689 \left(2+\frac{1}{p-1}\right) , \ c_4 = 77.89776$$

satisfy (5.13).

Further on substituting c_3 by $16.457689\left(2+\frac{1}{p-1}\right)\left(> 32.915378\right)$ we see that

$$\frac{1}{h_8} \leq \left(h_8(2,3;8,\frac{56}{15},32.915378)\right)^{-1} \leq 6.79973 \times 10^{-2}$$

and

$$\left\{2 + \frac{1}{h_8} + \frac{\log h_0}{h_0} + \frac{1}{h_0} \cdot \frac{\log q}{q} + \frac{n \log(h_0+1)}{h_0}\right\} \frac{1}{nq^n} \cdot \frac{1}{c_3}$$

$$\leq \left\{2+6.79973\times10^{-2}+1.5587732\times10^{-1}+5.3099\times10^{-2}\times\frac{\log 3}{3}+2\times0.1586245\right\} \times$$

$$\times \frac{1}{2\times3^2} \times \frac{1}{16.457689\times2} \leq 4.3219 \times 10^{-3} .$$

Thus

$$c_1 = (1+6.79973\times10^{-2}+4.3219\times10^{-3})\left(2+\frac{1}{p-1}\right) = 1.0723192 \left(2+\frac{1}{p-1}\right)$$

$$c_3 = 16.457689 \left(2+\frac{1}{p-1}\right)$$

satisfy (3.7).

From the above discussion we conclude that

$$c_0=8,\ c_1=1.0723192\left(2+\frac{1}{p-1}\right),\ c_2=\frac{56}{15},\ c_3=16.457689\left(2+\frac{1}{p-1}\right),\ c_4=77.89776$$

satisfy the system of inequalities (3.5)-(3.7).

<u>Case (3.b): $p \geqq 5,\ n = 7$</u>

On noting $p \geqq 5$, $q \geqq 3$ and $n = 7$, we have

$$\left(1-\frac{1}{c_3 n}\right)\left(1-\frac{1}{h_1}\right)\left(1-\frac{1}{q}\right)^2 \geq \frac{4}{9} \times (1 - 1.291818 \times 10^{-8})\left(1-\frac{1}{7c_3}\right) ,$$

$$\left(1+\frac{1}{h_4}\right)\left\{4 + \frac{1}{2^{10}nq} + \frac{2 \log h_5}{h_0} + \frac{1}{n}\left(1+\frac{1}{c_0-1}\right) + \left(1+\frac{1}{p-1}\right) \frac{1}{q^n}\right\}$$

$$\leq (1+2.5620315\times 10^{-7})\left\{4+\frac{1}{2^{10}\times 7\times 3} + 6.7700168 \times 10^{-1} + \frac{1}{7} \times \frac{8}{7} + \left(1+\frac{1}{4}\right) \times \frac{1}{3^7}\right\}$$

$$\leq 4.8408865 .$$

Therefore the inequality

$$\frac{4}{9} \times \left(1 - 1.291818 \times 10^{-8}\right) - \frac{15}{49} - 4.863456 \times 10^{-9}$$

$$\geq \left(1.027118+\frac{4}{63}\times(1-1.291818\times 10^{-8}) \times\frac{1}{2}\right)\left(2+\frac{1}{p-1}\right)\frac{1}{c_3}+4.8408865\frac{1}{c_4} \qquad (5.14)$$

implies (3.6), provided $c_1 \leq \frac{7}{2}$. It is easy to verify that

$$c_3 = 16\left(2+\frac{1}{p-1}\right) , \quad c_4 = 69.994513$$

satisfy (5.14).

Further, on substituting c_3 by $16\left(2+\frac{1}{p-1}\right)\left(> 32\right)$, we see that

$$\frac{1}{h_8} \leq \left(h_8(7,3;8,\frac{56}{15},32)\right)^{-1} \leq 1.92185 \times 10^{-2}$$

and

$$\left(2 + \frac{1}{h_8} + \frac{\log h_0}{h_0} + \frac{1}{h_0} \cdot \frac{\log q}{q} + \frac{n \log(h_0+1)}{h_0}\right) \frac{1}{nq^n} \cdot \frac{1}{c_3}$$

$$\leq \left(2+1.92185\times10^{-2}+1.5587732\times10^{-1}+5.3099\times10^{-2}\times\frac{\log 3}{3}+7\times 0.1586245\right) \times \frac{1}{7\times3^7} \times \frac{1}{32}$$

$$\leq 6.8 \times 10^{-6} .$$

Thus

$$c_1 = \left(1+1.92185\times10^{-2}+6.8\times10^{-6}\right)\left(2+\frac{1}{p-1}\right) = 1.0192253 \left(2+\frac{1}{p-1}\right) ,$$

$$c_3 = 16 \left(2+\frac{1}{p-1}\right)$$

satisfy (3.7).

We conclude from the above discussion that

$$c_0=8, \ c_1=1.0192253\left(2+\frac{1}{p-1}\right), \ c_2=\frac{56}{15}, \ c_3=16\left(2+\frac{1}{p-1}\right), \ c_4=69.994513$$

satisfy the system of inequalities (3.5)-(3.7).

Now we treat the cases (1.b), (2.b), (3.c). In these cases

$$n \geq 8 \ , \ q \geq 3$$

and we fix

$$c_0 = 16 \ , \ c_2 = \frac{8}{3} .$$

Then it is easy to establish the following inequalities:

$$h_0 \geq h_0(8,3) \geq 86.42138,\ \frac{1}{h_0} \leq 1.157122 \times 10^{-2},\ h_0(8,3) \leq 86.421384$$

$$\frac{\log h_0}{h_0} \leq 5.1598793 \times 10^{-2},\ \frac{\log(h_0+1)}{h_0} \leq 5.1731917 \times 10^{-2}\ ,$$

$$h_1 \geq h_1(8,3;16,\tfrac{8}{3}) \geq 2.1226 \times 10^{25}\ ,\ \frac{1}{h_1} \leq 4.711204 \times 10^{-26}\ ,$$

$$h_2(8,3;16,\tfrac{8}{3}) = \frac{2^{76}}{5\times7\times9}\ ,\ \left(h_2(8,3;16,\tfrac{8}{3})\right)^{-1} \leq 4.1689994 \times 10^{-21}\ ,$$

$$1+\varepsilon_1 \leq 1+\varepsilon_1(8,3;16,\tfrac{8}{3}) \leq (1-4.1689994\times10^{-21})^{-8} \leq 1+3.3352 \times 10^{-20}\ ,$$

$$\left(h_3(8,3;16,\tfrac{8}{3})\right)^{-1} \leq \frac{8^2}{2.1226\times10^{25}-1} \leq 3.0151703 \times 10^{-24}\ ,$$

$$1+\varepsilon_2 \leq 1+\varepsilon_2(8,3;16,\tfrac{8}{3}) \leq 1 + 3.0151704 \times 10^{-24}\ ,$$

$$(1+\varepsilon_1)(1+\varepsilon_2) \leq 1 + 4 \times 10^{-20}\ ,$$

$$\frac{1}{h_4} \leq \left(h_4(8,3;16,\tfrac{8}{3})\right)^{-1} \leq \frac{87.421384}{2.1226\times10^{25}} \leq 4.1185992 \times 10^{-24}\ ,$$

$$\log h_5 \leq \log h_5(8,3;16,\tfrac{8}{3}) \leq 6.3630211\ ,$$

$$\frac{1}{h_6} \leq \left(h_6(8,3;16,\tfrac{8}{3})\right)^{-1} \leq 5.889006 \times 10^{-27}\ ,$$

$$\frac{1}{h_7} \leq \left(h_7(8,3;16,\tfrac{8}{3})\right)^{-1} \leq 5.13132 \times 10^{-27}\ .$$

The above inequalities will be repeatedly used in the cases (1.b), (2.b), (3.c).

<u>Case (1.b): $p = 2,\ n \geq 8$</u>

On noting $p = 2,\ q \geq 3,\ n \geq 8$, we have

$$\left(1-\frac{1}{c_3 n}\right)\left(1-\frac{1}{h_1}\right)\left(1-\frac{1}{q}\right)^2 \geq \frac{4}{9} \times (1 - 4.711204 \times 10^{-26})\left(1-\frac{1}{8c_3}\right),$$

$$\left(\frac{1}{h_6} + \frac{1}{h_7}\right)\left(1+\frac{1}{c_0-1}\right) c_1 \leq 4.2 \times 10^{-26}, \text{ provided } c_1 \leq \frac{7}{2},$$

$$\left(1+\frac{1}{c_0-1}\right)\frac{1}{c_2} = \frac{2}{5},$$

$$\left\{1+\left(1+\frac{1}{h_0}\right)\log 3\right\}\left(\frac{1}{q} + \frac{1}{c_0-1}\right)\left(2+\frac{1}{p-1}\right) \leq 2.5335894,$$

$$\left(1+\frac{1}{h_4}\right)\left\{4+\frac{1}{2^{10}nq}+\frac{2 \log h_5}{h_0}+\frac{1}{n}\left(1+\frac{1}{c_0-1}\right)+\left(1+\frac{1}{p-1}\right)\frac{1}{q^n}\right\} \leq 4.2809348 .$$

So the inequality

$$\frac{4}{9} \times (1 - 4.711204 \times 10^{-26}) - \frac{2}{5} - 4.2 \times 10^{-26}$$

$$\geq (2.5335894 + \frac{1}{18}) \frac{1}{c_3} + 4.2809348 \frac{1}{c_4} \tag{5.15}$$

implies (3.6), provided $c_1 \leq \frac{7}{2}$. It is easy to check that

$$c_3 = 116.51153,\ c_4 = 192.64207$$

satisfy (5.15).

Further on substituting c_3 by 116.51153, we see that

$$\frac{1}{h_8} \leq 2.34627 \times 10^{-2}$$

and

$$\left(2+\frac{1}{h_8}+\frac{\log h_0}{h_0}+\frac{1}{h_0}\cdot\frac{\log q}{q}+\frac{n\log(h_0+1)}{h_0}\right)\frac{2+\frac{1}{p-1}}{nq^n}\cdot\frac{1}{c_3} \leq 1.3\times 10^{-6} .$$

So

$$c_1 = (1 + 2.34627\times 10^{-2})\times 3 + 1.3\times 10^{-6} = 3.0703894 ,$$

$$c_3 = 116.51153$$

satisfy (3.7).

We conclude from the above discussion that

$$c_0=16,\ c_1=3.0703894,\ c_2=\frac{8}{3},\ c_3=116.51153,\ c_4=192.64207$$

satisfy the system of the inequalities (3.5)-(3.7).

<u>Case (2.b): $p = 3,\ n \geq 8$</u>

By (0.1), we have $q \geq 5$. On noting $p = 3,\ q \geq 5,\ n \geq 8$, we see that the inequality

$$\frac{16}{25}\times\left(1 - 4.711204\times 10^{-26}\right) - \frac{2}{5} - 4.2\times 10^{-26}$$

$$\geq \left(5.6301992\times 10^{-1}+\frac{2}{25}\times\frac{2}{5}\right)\left(2+\frac{1}{p-1}\right)\frac{1}{c_3} + 4.2806175\ \frac{1}{c_4} \qquad (5.16)$$

implies (3.6), provided $c_1 \leq \frac{7}{2}$. Obviously

$$c_3 = 32, \; c_4 = 35.671814$$

satisfy (5.16).

Further, on substituting c_3 by 32 and noting that $n \geq 8$, $q \geq 5$ we see that

$$\frac{1}{h_8} \leq 1.17648 \times 10^{-2}$$

and

$$\left(2+\frac{1}{h_8}+\frac{\log h_0}{h_0}+\frac{1}{h_0} \cdot \frac{\log q}{q} + \frac{n \log(h_0+1)}{h_0}\right) \frac{1}{nq^n} \leq \frac{1}{c_3} \leq 3 \times 10^{-8} .$$

So

$$c_1 = \left(1+1.17648 \times 10^{-2}+10^{-7}\right)\left(2+\frac{1}{p-1}\right) = 1.0117649 \times \frac{5}{2} = 2.52941225 ,$$

$$c_3 = 32$$

satisfy (3.7).

From the above discussion we conclude that

$$c_0=16 , \; c_1=2.52941225 , \; c_2=\frac{8}{3} , \; c_3=32 , \; c_4=35.671814$$

satisfy the system of inequalities (3.5)-(3.7).

<u>Case (3.c): $p \geq 5$, $n \geq 8$</u>

It is easy to verify that the inequality

$$\frac{4}{9} \times (1 - 4.711204 \times 10^{-26}) - \frac{2}{5} - 4.2 \times 10^{-26}$$

$$\geq \left(0.8445298 + \frac{1}{18} \times \frac{1}{2}\right)\left(2+\frac{1}{p-1}\right) \frac{1}{c_3} + 4.2808204 \frac{1}{c_4} \qquad (5.17)$$

implies (3.6), provided $c_1 \leq \frac{7}{2}$. Evidently

$$c_3 = 39.253842 \left(2+\frac{1}{p-1}\right), \; c_4 = 192.63692$$

satisfy (5.17).

Further, on substituting c_3 by $39.253842 \left(2+\frac{1}{p-1}\right)$ (> 78.507684) we see that

$$\frac{1}{h_8} \leq 2.34749 \times 10^{-2}$$

and

$$\left(2+\frac{1}{h_8}+\frac{\log h_0}{h_0}+\frac{1}{h_0}\cdot\frac{\log q}{q}+\frac{n \log(h_0+1)}{h_0}\right) \frac{1}{nq^n} \cdot \frac{1}{c_3} \leq 7 \times 10^{-7} .$$

So

$$c_1 = (1+2.34749\times10^{-2}+7\times10^{-7})\left(2+\frac{1}{p-1}\right) = 1.0234756 \left(2+\frac{1}{p-1}\right),$$

$$c_3 = 39.253842 \left(2+\frac{1}{p-1}\right)$$

satisfy (3.7).

From the above discussion we conclude that

$$c_0=16, \; c_1=1.0234756\left(2+\frac{1}{p-1}\right), \; c_2=\frac{8}{3}, \; c_3=39.253842\left(2+\frac{1}{p-1}\right), \; c_4=192.63692$$

satisfy the system of inequalities (3.5)-(3.7).

On summing up all the cases (1.a)-(3.c) and applying Proposition 1, we obtain the following

<u>Proposition 3</u>. Let

$$\varepsilon = \varepsilon(n) = \begin{cases} 4.366 \times 10^{-5}, & 2 \le n \le 7 \\ 4 \times 10^{-20}, & n \geq 8 \end{cases}$$

and c_0, c_1, c_2, c_3, c_4 be positive numbers given by the following two tables.

Case		c_0	c_1	c_2	c_3	c_4
$p = 2$	$2 \le n \le 7$	8	3.2119513	$\frac{56}{15}$	47.766502	79.102681
	$n \ge 8$	16	3.0703894	$\frac{8}{3}$	116.51153	192.64207
$p = 3$	$2 \le n \le 7$	8	2.5889785	$\frac{56}{15}$	32	32
	$n \ge 8$	16	2.52941225	$\frac{8}{3}$	32	35.671814

Case		c_0	$c_1/\left(2+\frac{1}{p-1}\right)$	c_2	$c_3/\left(2+\frac{1}{p-1}\right)$	c_4
$p \ge 5$	$2 \le n \le 6$	8	1.0723192	$\frac{56}{15}$	16.457689	77.89776
	$n=7$	8	1.0192253	$\frac{56}{15}$	16	69.994513
	$n \ge 8$	16	1.0234756	$\frac{8}{3}$	39.253842	192.63692

Let

$$U = (1+\varepsilon)c_0c_1c_2^nc_3c_4\frac{n^{2n+1}}{n!}q^{2n}(q-1)\frac{G\left(2+\frac{1}{p-1}\right)^n}{e_{\mathfrak{p}}(f_{\mathfrak{p}}\log p)^{n+2}}D^{n+2}V_1\ldots V_nW^*\log V_{n-1}^* .$$

Suppose that (0.5)-(0.8) hold. Then

$$\mathrm{ord}_p\left(\alpha_1^{b_1}\cdots\alpha_n^{b_n} - 1\right) < U \ .$$

2. Solving the system of inequalities (4.2)-(4.4)

We solve the system of inequalities (4.2)-(4.4) in the following cases respectively

(1.a) $p = 2 \ , \ 2 \le n \le 7 \ ,$

(1.b) $p = 2 \ , \ n \ge 8 \ ,$

(2.a) $p = 3 \ , \ 2 \le n \le 7 \ ,$

(2.b) $p = 3 \ , \ n \ge 8 \ ,$

(3.a) $p \ge 5 \ , \ 2 \le n \le 7 \ ,$

(3.b) $p \ge 5 \ , \ n \ge 8 \ .$

We abbreviate $h_i(n,q;c_0,c_2)\,(0 \le i \le 5)$ as h_i, $h_6(n,q;c_0,c_2,c_3)$ as h_6 , $\varepsilon_i(n,q;c_0,c_2)\,(i = 1,2)$ as ε_i .

We first deal with the cases (1.a), (2.a), (3.a). In these cases

$$n \ge 2 \ , \ q \ge 3 \ ,$$

and we fix

$$c_0 = 16 \ , \ c_2 = \frac{8}{3} \ .$$

Then we have the following inequalities

$$h_0 \ge h_0(2,3) \ge 18.832756, \ \frac{1}{h_0} \le 5.3099 \times 10^{-2}, \ \frac{\log h_0}{h_0} \le 1.5587732 \times 10^{-1},$$

$$h_1 \ge h_1(2,3;16,\tfrac{8}{3}) \ge 78990303, \ \frac{1}{h_1} \le 1.26598 \times 10^{-8}$$

$$\left(h_2(2,3;16,\tfrac{8}{3})\right)^{-1} = 2^{-16} \leq 1.52588 \times 10^{-5} ,$$

$$1+\varepsilon_1 \leq 1+\varepsilon_1(2,3;16,\tfrac{8}{3}) \leq 1 + 3.05192 \times 10^{-5} ,$$

$$\left(h_3(2,3;16,\tfrac{8}{3})\right)^{-1} \leq 1.26598 \times 10^{-8} ,$$

$$1+\varepsilon_2 \leq 1+\varepsilon_2(2,3;16,\tfrac{8}{3}) \leq 1 + 1.266 \times 10^{-8} ,$$

$$(1+\varepsilon_1)(1+\varepsilon_2) \leq 1 + 3.0532 \times 10^{-5} ,$$

$$\frac{1}{h_4} \leq \left(h_4(2,3;16,\tfrac{8}{3})\right)^{-1} \leq 7.91238 \times 10^{-10} ,$$

$$\frac{1}{h_5} \leq \left(h_5(2,3;16,\tfrac{8}{3})\right)^{-1} \leq 1.03297 \times 10^{-9} .$$

The above inequalities will be repeated used in the cases (1.a), (2.a), (3.a).

Case (1.a): $p = 2,\ 2 \leq n \leq 7$

It is readily verified that the inequality

$$\frac{4}{9} \times (1 - 1.26598 \times 10^{-8}) - \frac{2}{5} - 6.81038 \times 10^{-9}$$

$$\geq \left(1.2637188 + \frac{2}{9} \times (1 - 1.26598 \times 10^{-8})\right) \frac{1}{c_3} \tag{5.18}$$

implies (4.3), provided $c_1 \leq \frac{7}{2}$. Obviously

$$c_3 = 33.433683$$

satisfies (5.18). On substituting c_3 by 33.433683 , we obtain

$$\frac{1}{h_6} \leq 9.51734 \times 10^{-2}$$

$$\left(2+\frac{1}{h_6}+\frac{\log h_0}{h_0}+\frac{1}{h_0}\cdot\frac{\log q}{q}\right)\frac{2+\frac{1}{p-1}}{nq^n}\cdot\frac{1}{c_3} \leq 1.13185 \times 10^{-2} .$$

So

$$c_1 = (1+9.51734\times 10^{-2})\times 3 + 1.13185\times 10^{-2} = 3.2968387$$

$$c_3 = 33.433683$$

satisfy (4.4).

From the above discussion we conclude that

$$c_0 = 16,\ c_1 = 3.2968387,\ c_2 = \frac{8}{3},\ c_3 = 33.433683$$

satisfy the system of inequalities (4.2)-(4.4).

<u>Case (2.a): $p = 3,\ 2 \leq n \leq 7$</u>

By (0.1) we have

$$q \geq 5 .$$

It is easy to verify that the inequality

$$\frac{16}{25} \times (1 - 1.26598 \times 10^{-8}) - \frac{2}{5} - 6.81038 \times 10^{-9}$$

$$\geq \left(0.2808265+\frac{8}{25}\times(1-1.26598\times10^{-8})\times\frac{2}{5}\right)\left(2+\frac{1}{p-1}\right)\frac{1}{c_3} \qquad (5.19)$$

implies (4.3), provided $c_1 \leq \frac{7}{2}$. Evidently

$$c_3 = 16$$

satisfies (5.19). On substituting c_3 by 16 , we obtain

$$\frac{1}{h_6} \leq 0.048388$$

$$\left(2+\frac{1}{h_6}+\frac{\log h_0}{h_0}+\frac{1}{h_0}\cdot\frac{\log q}{q}\right)\frac{1}{nq^n}\cdot\frac{1}{c_3} \leq 2.7767 \times 10^{-3} .$$

So

$$c_1 = (1 + 0.048388 + 2.7767 \times 10^{-3})\left(2+\frac{1}{p-1}\right) = 2.62791175 ,$$

$$c_3 = 16$$

satisfy (4.4).

From the above discussion we conclude that

$$c_0 = 16,\ c_1 = 2.62791175,\ c_2 = \frac{8}{3},\ c_3 = 16$$

satisfy the system of inequalities (4.2)-(4.4).

<u>Case (3.a): $p \geq 5$, $2 \leq n \leq 7$</u>

It is easy to verify that the inequality

$$\frac{4}{9} \times (1 - 1.26598 \times 10^{-8}) - \frac{2}{5} - 6.81038 \times 10^{-9}$$

$$\leq \left(0.4212396 + \frac{2}{9} \times \left(1 - 1.26598 \times 10^{-8}\right) \times \frac{1}{2}\right)\left(2+\frac{1}{p-1}\right) \frac{1}{c_3} \tag{5.20}$$

implies (4.3), provided $c_1 \leq \frac{7}{2}$. Evidently

$$c_3 = 11.977897 \left(2+\frac{1}{p-1}\right)$$

satisfies (4.3). On substituting c_3 by $11.977897 \left(2+\frac{1}{p-1}\right)$ (> 23.955794) , we get

$$\frac{1}{h_6} \leq 0.0957485$$

and

$$\left(2+\frac{1}{h_6}+\frac{\log h_0}{h_0}+\frac{1}{h_0}\cdot\frac{\log q}{q}\right) \frac{1}{nq^n} \cdot \frac{1}{c_3} \leq 5.267 \times 10^{-3} .$$

So

$$c_1 = (1 + 0.0957485 + 5.267 \times 10^{-3})\left(2+\frac{1}{p-1}\right) = 1.1010155\left(2+\frac{1}{p-1}\right) ,$$

$$c_3 = 11.977897\left(2+\frac{1}{p-1}\right)$$

satisfy (4.4).

We conclude that

$$c_0 = 16,\ c_1 = 1.1010155\left(2+\frac{1}{p-1}\right),\ c_2 = \frac{8}{3},\ c_3 = 11.977897\left(2+\frac{1}{p-1}\right)$$

satisfy the system of inequalities (4.2)-(4.4).

Remark. Note that the inequalities for $h_0,\ldots,h_5$, h_6, ε_1, ε_2 we used in the cases (1.a), (2.a), (3.a) depend on the fact that $n \geq 2$, but not on $n \leq 7$. Hence the solutions c_0, c_1, c_2, c_3 of the system of inequalities (4.2)-(4.4), which we obtained in the cases (1.a), (2.a), (3.a), are also the solutions of the system (4.2)-(4.4) for the cases (1.b), (2.b), (3.b).

Now we treat the cases (1.b), (2.b), (3.b). In these cases

$$n \geq 8\ ,\ q \geq 3$$

and we fix

$$c_0 = 16\ ,\ c_2 = \frac{5}{2}\ .$$

Then we have the following inequalities

$$h_0 \geq h_0(8,3) \geq 86.42138,\ \frac{1}{h_0} \leq 1.157122 \times 10^{-2},\ \frac{\log h_0}{h_0} \leq 5.1598793 \times 10^{-2}\ ,$$

$$h_1 \geq h_1(8,3;16,\tfrac{5}{2}) \geq 5.06661 \times 10^{25},\ \frac{1}{h_1} \leq 1.974 \times 10^{-26},$$

$$h_2(8,3;16,\tfrac{5}{2}) \geq 6.1068935 \times 10^{20},\ \left(h_2(8,3;16,\tfrac{5}{2})\right)^{-1} \leq 1.637494 \times 10^{-21}\ ,$$

$$1+\varepsilon_1 \leq 1+\varepsilon_1(8,3;16,\tfrac{5}{2}) \leq 1 + 1.31 \times 10^{-20}\ ,$$

$$h_3(8,3;16,\tfrac{5}{2}) \geq \frac{1}{49} \times 5.0666 \times 10^{25}, \quad \left(h_3(8,3;16,\tfrac{5}{2})\right)^{-1} \leq 9.67112 \times 10^{-25} ,$$

$$1+\bar{\varepsilon}_2 \leq 1+\varepsilon_2(8,3;16,\tfrac{5}{2}) \leq 1 + 9.68 \times 10^{-25} ,$$

$$(1+\varepsilon_1)(1+\varepsilon_2) \leq 1 + 1.4 \times 10^{-20} ,$$

$$\frac{1}{h_4} \leq \left(h_4(8,3;16,\tfrac{5}{2})\right)^{-1} \leq 4.935 \times 10^{-27} ,$$

$$\frac{1}{h_5} \leq \left(h_5(8,3;16,\tfrac{5}{2})\right)^{-1} \leq 3.835 \times 10^{-27} .$$

The above inequalities will be repeatedly used in the cases (1.b), (2.b), (3.b).

Case (1.b): $p = 2,\ n \geqq 8$

It is easy to verify that the inequality

$$\frac{4}{9} \times (1 - 1.974 \times 10^{-26}) - \frac{32}{75} - 3.275 \times 10^{-26}$$

$$\geqq \left(\frac{6}{5} \times (1 + 1.157122 \times 10^{-2}) + \frac{1}{18}\right) \frac{1}{c_3} \tag{5.21}$$

implies (4.3), provided $c_1 \leq \frac{7}{2}$. Clearly

$$c_3 = 71.406058$$

satisfies (5.21). On substituting c_3 by 71.406058 , we obtain

$$\frac{1}{h_6} \leq 2.50439 \times 10^{-2}$$

and

$$\left(2+\frac{1}{h_6}+\frac{\log h_0}{h_0}+\frac{1}{h_0}\cdot\frac{\log q}{q}\right)\frac{2+\frac{1}{p-1}}{nq^n}\cdot\frac{1}{c_3}\leq 1.7\times 10^{-6} .$$

So

$$c_1 = (1+2.50439\times 10^{-2})\times 3+1.7\times 10^{-6} = 3.0751334 ,$$

$$c_3 = 71.406058$$

satisfy (4.4).

We conclude that

$$c_0 = 16,\ c_1 = 3.0751334,\ c_2 = \frac{5}{2} ,\ c_3 = 71.406058$$

satisfy the system of inequalities (4.2)-(4.4).

Case (2.b): $p = 3,\ n \geq 8$

By (0.1) we have

$$q \geq 5 .$$

It is easy to verify that the inequality

$$\frac{16}{25}\times(1-1.974\times 10^{-26})-\frac{32}{75}-3.275\times 10^{-26}$$

$$\geq\left(0.2697524+\frac{2}{25}\times\frac{2}{5}\right)\left(2+\frac{1}{p-1}\right)\frac{1}{c_3} \qquad (5.22)$$

implies (4.3), provided $c_1 \leq \frac{7}{2}$. Obviously

$$c_3 = 16$$

satisfies (5.22). On substituting c_3 by 16 we obtain

$$\frac{1}{h_6} \leq 0.0125985$$

and

$$\left(2 + \frac{1}{h_6} + \frac{\log h_0}{h_0} + \frac{1}{h_0} \cdot \frac{\log q}{q}\right) \frac{1}{nq^n} \cdot \frac{1}{c_3} \leq 10^{-7} .$$

So

$$c_1 = (1 + 0.0125985 + 10^{-7})\left(2+\frac{1}{p-1}\right) = 2.5314965$$

$$c_3 = 16$$

satisfy (4.4).

We conclude that

$$c_0 = 16 , \ c_1 = 2.5314965 , \ c_2 = \frac{5}{2} , \ c_3 = 16$$

satisfy the system of inequalities (4.2)-(4.4).

Case (3.b): $p \geq 5$, $n \geq 8$

It is to verify that the inequality

$$\frac{4}{9} \times (1 - 1.974 \times 10^{-26}) - \frac{32}{75} - 3.275 \times 10^{-26}$$

$$\geq (0.4046285 + \frac{1}{18} \times \frac{1}{2})\left(2+\frac{1}{p-1}\right) \frac{1}{c_3} \qquad (5.23)$$

implies (4.3), provided $c_1 \leq \frac{7}{2}$. Obviously

$$c_3 = 24.322856 \left(2+\frac{1}{p-1}\right)$$

satisfies (4.3). On substituting c_3 by $24.322856 \left(2+\frac{1}{p-1}\right)$ (> 48.645712) we obtain

$$\frac{1}{h_6} \leq 0.0250645$$

and

$$\left(2+\frac{1}{h_6}+\frac{\log h_0}{h_0}+\frac{1}{h_0}\cdot\frac{\log q}{q}\right) \frac{1}{nq^n} \cdot \frac{1}{c_3} \leq 9 \times 10^{-7} .$$

So

$$c_1 = \left(1+0.0250645+9\times10^{-7}\right)\left(2+\frac{1}{p-1}\right) = 1.0250654 \left(2+\frac{1}{p-1}\right)$$

$$c_3 = 24.322856 \left(2+\frac{1}{p-1}\right)$$

satisfy (4.4).

We conclude that

$$c_0=16 \ , \ c_1=1.0250654\left(2+\frac{1}{p-1}\right) \ , \ c_2=\frac{5}{2} \ , \ c_3=24.322856\left(2+\frac{1}{p-1}\right)$$

satisfy the system of inequalities (4.2)-(4.4).

On summing up all the cases (1.a)-(3,b) and the remark at the end of the discussion of the case (3.a), and applying Proposition 2, we obtain the following

Proposition 4.

(i) Let

$$\varepsilon = \varepsilon(n) = \begin{cases} 1 + 3.0532 \times 10^{-5}, & 2 \leq n \leq 7 \\ 1 + 1.4 \times 10^{-20}, & n \geq 8 \end{cases}$$

and c_0, c_1, c_2, c_3 be positive numbers given by the following two tables.

Case		c_0	c_1	c_2	c_3
p = 2	$2 \leq n \leq 7$	16	3.2968387	$\frac{8}{3}$	33.433683
	$n \geq 8$	16	3.0751334	$\frac{5}{2}$	71.406058
p = 3	$2 \leq n \leq 7$	16	2.62791175	$\frac{8}{3}$	16
	$n \geq 8$	16	2.5314965	$\frac{5}{2}$	16

Case		c_0	$c_1/\left(2+\frac{1}{p-1}\right)$	c_2	$c_3/\left(2+\frac{1}{p-1}\right)$
p = 5	$2 \leq n \leq 7$	16	1.1010155	$\frac{8}{3}$	11.977897
	$n \geq 8$	16	1.0250654	$\frac{5}{2}$	24.322856

Let

$$U=(1+\varepsilon)c_0c_1c_2^nc_3^2\frac{n^{2n+2}}{n!}q^{2n}(q-1)\frac{G\left(2+\frac{1}{p-1}\right)^n}{(f_p\log p)^{n+2}}D^{n+2}V_1\ldots V_n(W^*)^2 \quad . \quad (5.24)$$

Suppose that (0.5)-(0.8) hold. Then

$$\mathrm{ord}_p\left(\alpha_1^{b_1}\ldots\alpha_n^{b_n}-1\right) < U \ . \qquad (5.25)$$

(ii) Suppose that (0.5)-(0.8) hold. If in (5.24), ε , c_0 , c_1 , c_2 , c_3 take the values (given in the above two tables) for the cases $p=2,\ 2 \leq n \leq 7$; $p=3,\ 2 \leq n \leq 7$; $p=5,\ 2 \leq n \leq 7$, respectively, then (5.25) holds also for the cases $p=2,\ n \geq 8$; $p=3,\ n \geq 8$; $p \geq 5,\ n \geq 8$, respectively.

3. Estimates for $\log V_{n-1}^*$ and W^*

Lemma 5.1. Let

$v_2 = 5.2336533,\ v_3 = 3.81275,\ v_4 = 3.2814667,\ v_5 = 2.9909667$,

$v_6 = 2.8030858,\ v_7 = 2.66939,\ v_n = 2.5681639\ (n \geq 8)$;

$w_2 = 3.7909562,\ w_3 = 3.2245056,\ w_4 = 2.9347108,\ w_5 = 2.7523294$,

$w_6 = 2.6242173,\ w_7 = 2.5278708,\ w_n = 2.4519668\ (n \geq 8)$.

Then for $n \geq 2$ we have

$$\log V_{n-1}^{*} \leq v_n n \log(nq) \cdot \left(\log(4DV_{n-1}^{+}) + \frac{f_p \log p}{8n}\right), \tag{5.26}$$

$$W^{*} \leqq w(n) n \log(nq) \cdot \left(\frac{W}{6n} + \log(4D)\right), \text{ where}$$

$$w(n) = \frac{\log(2^{11} \times 3n)}{\log 4 \cdot \log(3n)} \tag{5.27}$$

and

$$W^{*} \leq w_n n \log(nq) \cdot \left(\frac{W}{6n} + \log(4D)\right). \tag{5.28}$$

Proof. Note that by $q \geq 3$ we have

$$\log\left(2^{11} nq^{\frac{n+1}{n-1}} D^{\frac{n}{n-1}} V_{n-1}^{+}\right) = \log\left(2^{9} nq^{\frac{n+1}{n-1}}\right) + \log\left(4D^{\frac{n}{n-1}} V_{n-1}^{+}\right)$$

$$= \frac{n+1}{n-1} \log\left(2^{\frac{9(n-1)}{n+1}} n^{-\frac{2}{n+1}} nq\right) + \frac{n}{n-1} \log\left(\left(4V_{n-1}^{+}\right)^{-\frac{1}{n}} 4DV_{n-1}^{+}\right)$$

$$\leq \log(nq) \cdot \log(4DV_{n-1}^{+}) \left\{\frac{n+1}{n-1}\left(\frac{\log\left(2^{\frac{9(n-1)}{n+1}} n^{-\frac{2}{n+1}}\right)}{\log 4 \cdot \log(nq)} + \frac{1}{\log 4}\right) + \frac{n}{n-1} \cdot \frac{1}{\log(nq)}\right\}$$

$$\leq \log(nq) \cdot \log(4DV_{n-1}^{+}) \frac{\log(2^{9(n-1)} n^{-2}) + (n+1)\log(3n) + n \log 4}{\log 4 \cdot (n-1)\log(3n)}$$

$$= \log(nq) \cdot \log(4DV_{n-1}^{+}) \frac{(n-1)\log n + \log(2^{9(n-1)}) + (n+1)\log 3 + n \log 4}{\log 4 \cdot (n-1)\log(3n)}$$

$$= \log(nq) \cdot \log(4DV_{n-1}^{+}) v(n) \quad \text{(say)}. \tag{5.29}$$

It is easy to verify that $v(n)$ decreases monotonely and by a direct computation we see that

$$v(n) \leq v_n \quad (n \geq 2) \ . \tag{5.30}$$

Now by the definition of V^*_{n-1} (see (3.8)) and by (5.29), (5.30), we have

$$\log V^*_{n-1} \leq n \log \left(2^{11} nq^{\frac{n+1}{n-1}} D^{\frac{n}{n-1}} V^+_{n-1}\right) + f_p \log p$$

$$\leq v_n n \log(nq) \left(\log(4DV^+_{n-1}) + \frac{f_p \log p}{n\ v_n \log(3n)}\right) .$$

This together with the fact that $v_n \log(3n) \geq 8 \ (n \geq 2)$, which can be verified by a direct calculation, yields (5.26) at once.

Further, we have

$$\log(2^{11} nqD) = \log(nq) \cdot \log(4D) \cdot \frac{\log 2^9 + \log(nq) + \log(4D)}{\log(nq) \cdot \log(4D)}$$

$$\leq \log(nq) \cdot \log(4D) \left\{\frac{\log 2^9}{\log 4 \cdot \log(3n)} + \frac{1}{\log 4} + \frac{1}{\log(3n)}\right\}$$

$$= \log(nq) \cdot \log(4D) \frac{\log(2^{11} \times 3n)}{\log 4 \cdot \log(3n)}$$

$$= w(n) \log(nq) \cdot \log(4D) \ . \tag{5.31}$$

Obviously $w(n)$ decreases monotonely and by a direct calculation we see that

$$w(n) \leq w_n \quad (n \geq 2) \ . \tag{5.32}$$

Now by the definition of W^* (see (3.9) and (4.5)) and by (5.31), we get

$$W^{*} \leq W + n \log(2^{11} nqD) \leq w(n) n \log(nq)\left(\frac{W}{nw(n)\log(3n)} + \log(4D)\right) .$$

This together with the fact that

$$w(n)\log(3n) = \frac{\log(2^{11} \times 3n)}{\log 4} > 6 \quad (n \geq 2)$$

implies (5.27) immediately. Now (5.28) follows from (5.27) and (5.32). The proof of the lemma is thus complete.

4. Completion of the proofs of Theorems 1 and 2

Completion of the proof of Theorem 1. By Proposition 3, Lemma 5.1 and Lemma 2.7, we see that, in order to prove Theorem 1, it suffices to show

$$(1+\varepsilon) c_0 c_1 c_3 c_4 v_n w_n / \sqrt{2\pi} \leq C_1(p,n) \quad , \tag{5.33}$$

where ε, c_0, c_1, c_3, c_4 are given in Proposition 3 and v_n, w_n are given in Lemma 5.1. We can easily prove (5.33) by a direct calculation, thereby complete the proof of Theorem 1.

Completion of the proof of Theorem 2

Theorem 2 is a direct consequence of Proposition 4, Lemma 5.1 and Lemma 2.7.

(1) $p = 2$.

If $2 \leq n \leq 17$, it suffices to show that

$$(1+\varepsilon) c_0 c_1 c_3^2 w_n^2 / \sqrt{2\pi} \leq C_2(2,n) \quad , \tag{5.34}$$

where c_0, c_1, c_3, ε are given by Proposition 4, (ii).

If $n \geq 18$, on noting that $w(n) \geq w(18)$, it suffices to show that

$$(1+\varepsilon)c_0c_1c_3^2(w(18))^2/\sqrt{2\pi} \leq C_2(2,n) \quad , \tag{5.35}$$

where c_0, c_1, c_3, ε are given by Proposition 4, (i), $w(18) \leq 2.1001457$ (see Lemma 5.1).

(2) $p = 3$. It suffices to show that

$$(1+\varepsilon)c_0c_1c_3^2w_n^2/\sqrt{2\pi} \leq C_2(3,n) \quad , \tag{5.36}$$

where c_0, c_1, c_3, ε are given by Proposition 4, (i).

(3) $p \geq 5$.

If $2 \leq n \leq 16$, it suffices to show that

$$(1+\varepsilon)c_0c_1c_3^2w_n^2/\sqrt{2\pi} \leq C_2(p,n) \quad , \tag{5.37}$$

where c_0, c_1, c_3, ε are given by Proposition 4, (ii).

If $n \geq 17$, on noting that $w(n) \geq w(17)$, it suffices to show that

$$(1+\varepsilon)c_0c_1c_3^2(w(17))^2/\sqrt{2\pi} \leq C_2(p,n) \quad , \tag{5.38}$$

where c_0, c_1, c_3, ε are given by Proposition 4, (i), and $w(17) \leq 2.1201893$ (see Lemma 5.1).

Now the inequalities (5.34)-(5.38) can be easily verified by a direct calculation. This completes the proof of Theorem 2.

Appendix. Hermite interpolation and a combinatorial identity

Let E be an algebraically closed field of characteristic 0. Suppose that $n \geq 2$, $\tau_1 > 0, \ldots, \tau_n > 0$ are integers,

$$T = \tau_1 + \ldots + \tau_n .$$

Let $\beta_1, \ldots, \beta_n$ ($\beta_i \neq \beta_j$ for $1 \leq i < j \leq n$) and $q_{i,t}$ ($1 \leq i \leq n$, $0 \leq t < \tau_i$) be given elements in E.

Theorem A. The unique polynomial $Q(z) \in E[z]$ of degree at most $T-1$ satisfying

$$Q^{(t-1)}(\beta_i) = q_{i,t-1} \quad (1 \leq i \leq n,\ 1 \leq t \leq \tau_i) \tag{1}$$

is given by the formula

$$\begin{aligned} Q(z) = \sum_{h=1}^{n} \sum_{t=1}^{\tau_i} q_{h,t-1} (-1)^{\tau_h - t} \frac{(z-\beta_h)^{t-1}}{(t-1)!} \left\{ \prod_{\substack{k=1 \\ k \neq h}}^{n} \left(\frac{z-\beta_k}{\beta_h - \beta_k} \right)^{\tau_k} \right\} . \\ \cdot \sum_{s=1}^{\tau_h - t} (-1)^{s-1} \sum_{\substack{\lambda_1 + \ldots + \lambda_{\tau_h - t} = \tau_h - t \\ \lambda_j = 0 (j<s), \lambda_j \geq 1 (j \geq s)}} \prod_{j=1}^{\tau_h - t} \frac{\left(\frac{\partial}{\partial \beta_h}\right)^{\lambda_j} \left\{ \left(z-\beta_h \right) \prod_{\substack{k=1 \\ k \neq h}}^{n} (\beta_h - \beta_k)^{\tau_k} \right\}}{\lambda_j ! \prod_{\substack{k=1 \\ k \neq h}}^{n} (\beta_h - \beta_k)^{\tau_k}} \end{aligned} \tag{2}$$

where the second line of (2) reads as 1 when $t = \tau_h$.

Remark. Henceforth we write $\left(\frac{\partial}{\partial \beta_h}\right)^{\lambda_j} \left\{ (z-\beta_h) \prod_{\substack{k=1 \\ k \neq h}}^{n} (\beta_h - \beta_k)^{\tau_k} \right\}$ for the value $\left(\frac{\partial}{\partial y}\right)^{\lambda_j}\left\{(z-y)\prod_{\substack{k=1\\k\neq h}}^{n}(y-\beta_k)^{\tau_k}\right\}_{y=\beta_h}$.

The uniqueness of $Q(z)$ is proved, for example, in Davis [13], pp. 29-30. For the self-containess of our exposition, we reintroduce it here. Obviously it suffices to prove that if $Q(z) \in E[z]$ of degree at most $T-1$ satisfies

$$Q^{(t-1)}(\beta_i) = 0 \quad (1 \leq i \leq n,\ 1 \leqq t \leq \tau_i)\ , \tag{3}$$

then $Q(z) = 0$. The case $\tau_1 = \ldots = \tau_n = 1$ is trivial. So we may assume $\tau_n \geqq 2$. From (3) (except for $Q^{(\tau_n-1)}(\beta_n) = 0$) we see that there exists $A(z) \in E[z]$ such that

$$Q(z) = A(z)(z-\beta_1)^{\tau_1} \ldots (z-\beta_{n-1})^{\tau_{n-1}}(z-\beta_n)^{\tau_n-1} ,$$

by virtue of the hypothesis that the field E is algebraically closed. Since $Q(z)$ has degree at most $T-1 = \tau_1 + \ldots + \tau_{n-1} + \tau_n - 1$, it follows that $A(z) = A \in E$. Now by (3),

$$A(\tau_n-1)!(\beta_n-\beta_1)^{\tau_1} \ldots (\beta_n-\beta_{n-1})^{\tau_{n-1}} = Q^{(\tau_n-1)}(\beta_n) = 0 .$$

On noting that $\beta_i \neq \beta_j$ $(1 \leq i < j \leq n)$, we see that $A = 0$, whence $Q(z) = 0$. This proves the uniqueness of $Q(z)$. It remains to show that $Q(z)$ given by (2) satisfies (1).

Before doing so we introduce a result of van der Poorten [23] with slightly modified notations. For non-negative integers k and l we set $\binom{k}{l} = 0$ if $l > k$ and $\binom{0}{0} = 0$. Write

$$\Delta = \det\left((s-1)!\binom{\lambda-1}{s-1}\beta_k^{\lambda-s}\right),$$

which is a determinant of order T, whose rows are indexed by $\lambda = 1,\dots,T$ and columns indexed by T pairs (k,s) $(k=1,\dots,n$, $s=1,\dots,\tau_k)$ lexicographically ordered. Denote by $\Delta_{ks,\lambda}$ the cofactor of Δ of the element at λ-th row and (k,s)-th column. Further let β_{ks} $(k=1,\dots,n,\ s=1,\dots,\tau_k)$ be T independent indeterminates and

$$D = \det\left(\beta_{ks}^{\lambda-1}\right)$$

indexed by λ and (k,s) as in Δ. Denote by $D_{ks,\lambda}$ the cofactor of D. Now D is simply the Vandermonde determinant of $\beta_{ks}(k=1,\dots,n,\ s=1,\dots,\tau_k)$, so we can write

$$D = \prod_{(k,s)<(h,t)}(\beta_{ht}-\beta_{ks}) = \prod_{h=1}^{n}\prod_{t=1}^{\tau_h}\left(\left\{\prod_{l=1}^{t-1}(\beta_{ht}-\beta_{hl})\right\}\prod_{k=1}^{h-1}\prod_{s=1}^{\tau_k}(\beta_{ht}-\beta_{ks})\right), \tag{4}$$

where $\prod_{l=1}^{t-1}(\beta_{ht}-\beta_{hl})$ reads as 1 if $t=1$ and $\prod_{k=1}^{h-1}\prod_{s=1}^{\tau_k}(\beta_{ht}-\beta_{ks})$ reads as 1 if $h=1$. (In the sequel, the convention will be kept without mentioning). On noting that

$$\left\{\left(\frac{\partial}{\partial\beta_{ks}}\right)^{s-1}\beta_{ks}^{\lambda-1}\right\}_{\beta_{ks}=\beta_k} = (s-1)!\binom{\lambda-1}{s-1}\beta_k^{\lambda-s},$$

we get

$$\Delta = \lim\left(\prod_{k=1}^{n}\prod_{s=1}^{\tau_k}\left(\frac{\partial}{\partial\beta_{ks}}\right)^{s-1}\right)D, \tag{5}$$

here and later we assume that the symbol lim means the substitution of β_k for β_{ks} $(k=1,\ldots,n,\ s=1,\ldots,\tau_k)$, and $\prod_{k=1}^{n}\prod_{s=1}^{\tau_k}\left(\frac{\partial}{\partial\beta_{ks}}\right)^{s-1}$ denotes the operation resulted from doing $\left(\frac{\partial}{\partial\beta_{ks}}\right)^{s-1}$ $(k=1,\ldots,n,\ s=1,\ldots,\tau_k)$ lexicographically. Since

$$\left(\frac{\partial}{\partial\beta_{ht}}\right)^{t-1}\prod_{l=1}^{t-1}(\beta_{ht}-\beta_{hl}) = (t-1)!\ ,$$

we obtain by (4), (5)

$$\Delta = \lim \prod_{h=1}^{n}\prod_{t=1}^{\tau_h}\left((t-1)!\prod_{k=1}^{h-1}\prod_{s=1}^{\tau_k}(\beta_{ht}-\beta_{ks})\right)$$

$$= \prod_{h=1}^{n}\prod_{t=1}^{\tau_h}\left((t-1)!\prod_{k=1}^{h-1}(\beta_h-\beta_k)^{\tau_k}\right)\ . \qquad (6)$$

(6) is due to van der Poorten (see [23] p. 282 and p. 283).

<u>Proof of Theorem A</u>. Let

$$H_{ht}(z) = \sum_{\lambda=1}^{T}\frac{\Delta_{ht,\lambda}\, z^{\lambda-1}}{\Delta}\ (h=1,\ldots,n,\ t=1,\ldots,\tau_h)\ , \qquad (7)$$

whose degree is at most $T-1$. Then by Cramer's rule for determinants, we see that

$$H_{ht}^{(s-1)}(\beta_k) = \delta_{hk}\delta_{ts}\ (1 \le h,k \le n, 1 \le t \le \tau_h, 1 \le s \le \tau_k)\ , \qquad (8)$$

where

$$\delta_{ij} = \begin{cases} 1\ , & \text{if } i = j \\ 0\ , & \text{if } i \neq j \end{cases}$$

is the Kronecker's symbol. Since the uniqueness of Q(z) has already been verified above, to prove the theorem it suffices, by (7), to show

$$H_{ht}(z) = (-1)^{\tau_h - t} \frac{(z-\beta_h)^{t-1}}{(t-1)!} \left\{ \prod_{\substack{k=1\\k\neq h}}^{n} \left(\frac{z-\beta_k}{\beta_h-\beta_k}\right)^{\tau_k} \right\} \cdot$$

$$\cdot \sum_{s=1}^{\tau_h - t} (-1)^{s-1} \sum_{\substack{\lambda_1+\dots+\lambda_{\tau_h-t}=\tau_h-t\\ \lambda_j=0\,(j<s),\,\lambda_j\geq 1\,(j\geq s)}} \prod_{j=1}^{\tau_h-t} \frac{\left(\frac{\partial}{\partial\beta_h}\right)^{\lambda_j} \left\{(z-\beta_h)\prod_{\substack{k=1\\k\neq h}}^{n}(\beta_h-\beta_k)^{\tau_k}\right\}}{\lambda_j!\prod_{\substack{k=1\\k\neq h}}^{n}(\beta_h-\beta_k)^{\tau_k}} \quad (9)$$

$$(h=1,\dots,n,\ t=1,\dots,\tau_h) \ .$$

Evidently we need only to verify the case $h=n$. (9) with $h=n$, $t=\tau_n$ is obvious (note that in this case the second line of (9) reads as 1). Further, we assert that (9) with $h=n, t=1$ implies (9) with $h=n$, $1 < t < \tau_n$. For it is easy to verify, by the uniqueness, that

$$H_{nt}(z) = \frac{(z-\beta_n)^{t-1}}{(t-1)!} \tilde{H}_{n1}(z) \ , \quad (10)$$

where $\tilde{H}_{n1}(z) \in E[z]$ is the unique polynomial of degree at most $T-t$ satisfying

$$\tilde{H}_{n1}^{(s-1)}(\beta_k) = \delta_{nk}\delta_{1s} \quad (1 \leq k \leq n-1,\ 1 \leq s \leq \tau_k;\ k = n,\ 1 \leq s \leq \tau_n-t+1) \ ,$$

and on applying (9) with $h=n, t=1$ for the points $\beta_1,\dots,\beta_n$ and the multiplicities $\tau_1,\dots,\tau_{n-1}$, τ_n-t+1 , and substituting the result for $\tilde{H}_{n1}(z)$ in (10), the above assertion follows at once. Thus the proof of the theorem is reduced to verifying (9) with $h=n$, $t=1$. This is obvious if $\tau_n=1$. Henceforth we assume $\tau_n > 1$. By (7), (4) and the Cramer's rule for determinants, we see that

$$H_{n1}(z) = \sum_{\lambda=1}^{T} \frac{\Delta_{n1,\lambda}}{\Delta} z^{\lambda-1}$$

$$= \lim \frac{\left(\prod_{(k,s)\neq(n,1)} \left(\frac{\partial}{\partial\beta_{ks}}\right)^{s-1}\right) \sum_{\lambda=1}^{T} D_{n1,\lambda}\, z^{\lambda-1}}{\Delta}$$

$$= \lim \frac{\left(\prod_{(k,s)\neq(n,1)} \left(\frac{\partial}{\partial\beta_{ks}}\right)^{s-1}\right)\left\{D \prod_{(k,s)\neq(n,1)} \frac{z-\beta_{ks}}{\beta_{n1}-\beta_{ks}}\right\}}{\Delta}. \quad (11)$$

By (4) we get

$$D \prod_{(k,s)\neq(n,1)} \frac{z-\beta_{ks}}{\beta_{n1}-\beta_{ks}}$$

$$=\left\{\prod_{k=1}^{n-1}\prod_{s=1}^{\tau_k}\left(\left(\prod_{l=1}^{s-1}(\beta_{ks}-\beta_{kl})\right)\cdot\prod_{h=1}^{k-1}\prod_{t=1}^{\tau_h}(\beta_{ks}-\beta_{ht})\right)\right\}\cdot\left\{\prod_{k=1}^{n-1}\prod_{s=1}^{\tau_k}\frac{z-\beta_{ks}}{\beta_{n1}-\beta_{ks}}\right\}\cdot$$
$$\cdot\left\{\prod_{s=1}^{\tau_n}\left(\left(\prod_{l=1}^{s-1}(\beta_{ns}-\beta_{nl})\right)\cdot\prod_{h=1}^{n-1}\prod_{t=1}^{\tau_h}(\beta_{ns}-\beta_{ht})\right)\right\}\cdot\prod_{s=2}^{\tau_n}\frac{z-\beta_{ns}}{\beta_{n1}-\beta_{ns}}. \quad (12)$$

Note that the second line in the right-hand side of (12) equals to

$$(-1)^{\tau_n-1}\left\{\prod_{h=1}^{n-1}\prod_{t=1}^{\tau_h}(\beta_{n1}-\beta_{ht})\right\}\left\{\prod_{s=2}^{\tau_n}\prod_{l=2}^{s-1}(\beta_{ns}-\beta_{nl})\right\}\left\{\prod_{s=2}^{\tau_n}\left((z-\beta_{ns})\prod_{h=1}^{n-1}\prod_{t=1}^{\tau_h}(\beta_{ns}-\beta_{ht})\right)\right\}. \quad (13)$$

By (11), (12), (13), (6) and on operating first $\prod_{k=1}^{n-1}\prod_{s=1}^{\tau_k}\left(\frac{\partial}{\partial\beta_{ks}}\right)^{s-1}$ then $\prod_{s=2}^{\tau_n}\left(\frac{\partial}{\partial\beta_{ns}}\right)^{s-1}$, we obtain

$$H_{n1}(z) = (-1)^{\tau_n-1}\left\{\prod_{k=1}^{n-1}\left(\frac{z-\beta_k}{\beta_n-\beta_k}\right)^{\tau_k}\right\} \cdot \tag{14}$$

$$\cdot \lim \frac{\left(\prod_{s=2}^{\tau_n}\left(\frac{\partial}{\partial\beta_{ns}}\right)^{s-1}\right)\left(\left\{\prod_{s=2}^{\tau_n}\prod_{l=2}^{s-1}(\beta_{ns}-\beta_{nl})\right\}\left\{\prod_{s=2}^{\tau_n}\left((z-\beta_{ns})\prod_{h=1}^{n-1}\prod_{t=1}^{\tau_h}(\beta_{ns}-\beta_{ht})\right)\right\}\right)}{\prod_{s=2}^{\tau_n}\left((s-1)!\prod_{h=1}^{n-1}(\beta_n-\beta_h)^{\tau_h}\right)} .$$

If $\tau_n=2$, then (9) with $h=n,\ t=1$ follows from (14) immediately. So we may assume further that $\tau_n>2$. Now to simplify the notations, we write

$$m = \tau_n-1 \geq 2 ,$$

$$y_s = \beta_{n,s+1}(s=1,\ldots,m) ,$$

$$f(y) = f(y;z,\beta_{11},\ldots,\beta_{n-1,\tau_{n-1}}) = (z-y)\prod_{h=1}^{n-1}\prod_{t=1}^{\tau_h}(y-\beta_{ht}) ,$$

$$V(y_1,\ldots,y_m) = \prod_{s=1}^{m}\prod_{l=1}^{s-1}(y_s-y_l) .$$

Then (14) implies that

$$H_{n1}(z)=(-1)^m\left\{\prod_{k=1}^{n-1}\left(\frac{z-\beta_k}{\beta_n-\beta_k}\right)^{\tau_k}\right\}\lim\frac{\left(\prod_{s=1}^{m}\left(\frac{\partial}{\partial y_s}\right)^s\right)\left\{V(y_1,\ldots,y_m)f(y_1)\ldots f(y_m)\right\}}{\prod_{s=1}^{m}\left(s!\prod_{h=1}^{n-1}(\beta_n-\beta_h)^{\tau_h}\right)} , \tag{15}$$

where the symbol lim denotes the substitution

$$y_s = \beta_n \quad (1 \leq s \leq m) ,$$

$$\beta_{ht} = \beta_h \quad (1 \leq h \leq n-1,\ 1 \leq t \leq \tau_h) .$$

Note that $V(y_1,\dots,y_m)$ is the Vandermonde determinant of $y_1,\dots,y_m$, so

$$\lim\left(\prod_{j=1}^{m}\left(\frac{\partial}{\partial y_j}\right)^{j}\right)\left\{V(y_1,\dots,y_m)f(y_1)\dots f(y_m)\right\}$$

$$=\lim\sum_{(\mu_1,\dots,\mu_m)\in M}\left(\prod_{j=1}^{m}\binom{j}{\mu_j}\right)\left(\frac{\partial}{\partial y_1}\right)^{\mu_1}\dots\left(\frac{\partial}{\partial y_m}\right)^{\mu_m}V(y_1,\dots,y_m)\cdot\prod_{j=1}^{m}\left(\frac{\partial}{\partial y_j}\right)^{j-\mu_j}f(y_j), \quad (16)$$

where M is the set of m-tuples $\boldsymbol{\mu} = (\mu_1,\dots,\mu_m)$ such that

$$0 \leq \mu_j \leq j \quad (1 \leq j \leq m)$$

and the set

$$\{\mu_1,\mu_2,\dots,\mu_m\} = \{0,1,\dots,m-1\} .$$

By an induction on m it is easy to see that the cardinal of M

$$\# M = 2^{m-1} . \quad (17)$$

Let

$$\mu^{(0)} = (\mu_1^{(0)},\mu_2^{(0)},\dots,\mu_m^{(0)}) = (0,1,\dots,m-1) .$$

Write

$$\sigma_1 = (12),\ \sigma_2 = (23),\dots,\sigma_{m-1} = (m-1\ m)$$

for transpositions of the set $\{1,2,\ldots,m\}$. Define for every permutation σ of $\{1,2,\ldots,m\}$

$$\sigma\mu = (\mu_{\sigma(1)},\ldots,\mu_{\sigma(m)}) .$$

Then for each product

$$\sigma_{i_k}\ldots\sigma_{i_1} \quad \text{with} \quad 0 \leq k \leq m-1 \ , \ 1 \leq i_1 < \ldots < i_k \leq m-1 \ ,$$

where we assume for $k=0$ the product reads as the identical permutation of $\{1,2,\ldots,m\}$, we have

$$\sigma_{i_k}\ldots\sigma_{i_1}\mu^{(0)} \in M .$$

Since all the 2^{m-1} products $\sigma_{i_k}\ldots\sigma_{i_1}$ $(0 \leq k \leq m-1$, $1 \leq i_1 < \ldots < i_k \leq m-1)$ are distinct permutations of $\{1,2,\ldots,m\}$, we obtain

$$M = \{\sigma_{i_k}\ldots\sigma_{i_1}\mu^{(0)} \,|\, 0 \leq k \leq m-1,\ 1 \leq i_1 < \ldots < i_k \leq m-1\} . \tag{18}$$

Suppose $\mu = (\mu_1,\ldots,\mu_m) = \sigma_{i_k}\ldots\sigma_{i_1}\mu^{(0)}$ is an element of M . Then we have

$$\left(\prod_{j=1}^{m}\left(\frac{\partial}{\partial y_j}\right)^{\mu_j}\right)V(y_1,\ldots,y_m) = (-1)^k\prod_{j=1}^{m}(j-1)! = (-1)^k\prod_{j=1}^{m}\mu_j \ ! \tag{19}$$

and

$$\mu_j = j \quad \text{if} \quad j \in \{i_1,\ldots,i_k\} \ ,$$

$$\mu_j < j \quad \text{if} \quad j \in \{1,\ldots,m\}\setminus\{i_1,\ldots,i_k\} .$$

Further we rearrange the components of

$$(1-\mu_1, \ldots, j-\mu_j, \ldots, m-\mu_m)$$

by putting the i_1-th,...,i_k-th components (that is, all the zero-components) in front of the non-zero ones and keeping the ordering among the non-zero ones, and then denote the result by $(\lambda_1, \ldots, \lambda_m)$. Obviously

$$\lambda_1 + \ldots + \lambda_m = \sum_{j=1}^{m} (j-\mu_j) = \sum_{j=1}^{m} j - \sum_{j=0}^{m-1} j = m$$

and

$$\lambda_j = 0 \quad \text{for} \quad 1 \leq j \leq k, \; \lambda_j \geq 1 \quad \text{for} \quad k+1 \leq j \leq m \; .$$

In this way, we have defined a map from M into the set Λ of $(\lambda_1, \ldots, \lambda_m) \in \mathbb{N}^m$ having the two properties:

i) $\lambda_1 + \ldots + \lambda_m = m$,

ii) there exists k with $0 \leq k \leq m-1$ such that

$$\lambda_j = 0 \; (1 \leq j \leq k), \; \lambda_j \geq 1 \; (k+1 \leq j \leq m) \; .$$

It is readily verified that this map is injective. Furthermore, the cardinal of Λ

$$\# \Lambda = \sum_{k=0}^{m-1} \binom{m-1}{k} = 2^{m-1} \; ,$$

since for each k with $0 \leq k \leq m-1$, the equation

$$x_{k+1} + \ldots + x_m = m$$

has $\binom{m-1}{k}$ solutions in positive integers $x_{k+1},\ldots,x_m$. By (17), we get $\# M = \# \Lambda$, whence this map is bijective. By (19) and the definition of this map, we see that for every
$\mu = \sigma_{i_k}\ldots\sigma_{i_1}\mu^{(0)} = (\mu_1,\ldots,\mu_m) \in M$,

$$\lim \frac{\prod_{j=1}^{m}\binom{j}{\mu_j}\cdot\left(\left(\frac{\partial}{\partial y_1}\right)^{\mu_1}\cdots\left(\frac{\partial}{\partial y_m}\right)^{\mu_m} V(y_1,\ldots,y_m)\right)\prod_{j=1}^{m}\left(\frac{\partial}{\partial y_j}\right)^{j-\mu_j} f(y_j)}{\prod_{j=1}^{m}\left(j!\prod_{h=1}^{n-1}(\beta_n-\beta_h)^{\tau_h}\right)}$$

$$= (-1)^k\prod_{j=1}^{m}\frac{\left(\frac{\partial}{\partial\beta_n}\right)^{j-\mu_j}\left\{(z-\beta_n)\prod_{h=1}^{n-1}(\beta_n-\beta_h)^{\tau_h}\right\}}{(j-\mu_j)!\prod_{h=1}^{n-1}(\beta_n-\beta_h)^{\tau_h}}$$

$$= (-1)^k\prod_{j=1}^{m}\frac{\left(\frac{\partial}{\partial\beta_n}\right)^{\lambda_j}\left\{(z-\beta_n)\prod_{h=1}^{n-1}(\beta_n-\beta_h)^{\tau_h}\right\}}{\lambda_j!\prod_{h=1}^{n-1}(\beta_n-\beta_h)^{\tau_h}} \quad . \tag{20}$$

Now on combining (15), (16), (18), (20), noting the fact that the map from M to Λ defined above is bijective, recalling $m = \tau_n-1$, we obtain (9) with $h=n$, $t=1$ immediately. This completes the proof of the theorem.

<u>Remark</u>. van der Poorten [24] gives a similar formula, but it is false; a simple counterexample can be obtained in the case $n=2$,

$\tau(1) = \tau(2) = 2$. Consequently, the interpolation formula in Lemma 1 of van der Poorten [25] is also false. The sum

$$\sum_{\lambda=1}^{T} \frac{\Delta_{ht,\lambda}}{\Delta} z^{\lambda-1}$$

and (11) have appeared in van der Poorten [24], the former of which derived from Mahler's ideas. But [24] gives an incorrect expression for (7).

There are other kinds of Hermite interpolation formulas. A well-known one is the following (see, for example, Berezin and Zhidkov [9], pp. 145-147). Take $E = \mathbb{C}$ and let

$$\Omega(z) = \prod_{j=1}^{n} (z-\beta_j)^{\tau_j} .$$

Denote again by $H_{ht}(z) \in \mathbb{C}[z]$ $(1 \leq h \leq n , 1 \leq t \leq \tau_h)$ the polynomials, of degree at most $T - 1 = \sum_{j=1}^{n} \tau_j - 1$, determined uniquely by the conditions (8). It is proved in [9] that

$$H_{ht}(z) = \frac{(z-\beta_h)^{t-1}}{(t-1)!} \cdot \frac{\Omega(z)}{(z-\beta_h)^{\tau_h}} \sum_{k=0}^{\tau_h - t} \frac{1}{k!} \frac{d^k}{dz^k} \left\{ \frac{(z-\beta_h)^{\tau_h}}{\Omega(z)} \right\}_{z=\beta_h} (z-\beta_h)^k \tag{21}$$

$$(1 \leq h \leq n , 1 \leq t \leq \tau_h) .$$

It may be interesting to deduce a combinatorial identity from comparing the two kinds of Hermite interpolation formula, namely (21) and our (9).

<u>Theorem B</u>. Suppose $m, n, \tau_1, \ldots, \tau_n$ are positive integers. For every $(\rho_1, \ldots, \rho_n) \in \mathbb{N}^n$ satisfying

$$\rho_1 + \dots + \rho_n = m$$

we have

$$\sum_{s=1}^{m} (-1)^{s-1} \sum_{\substack{\lambda_s+\dots+\lambda_m=m \\ \lambda_j \geq 1 (s \leq j \leq m)}} \sum_{(k_{ij})} \prod_{i=1}^{n} \prod_{j=s}^{m} \binom{\tau_i}{k_{ij}} = \prod_{i=1}^{n} \binom{\tau_i+\rho_i-1}{\rho_i} ,$$

where $\sum_{(k_{ij})}$ ranges over all $n \times (m-s+1)$ matrices (k_{ij}) satisfying

$$\left. \begin{array}{l} k_{ij} \in \mathbb{N} \ (1 \leq i \leq n,\ s \leq j \leq m),\ \sum_{j=s}^{m} k_{ij} = \rho_i \ (1 \leq i \leq n) , \\ \sum_{i=1}^{n} k_{ij} = \lambda_j \ (s \leq j \leq m) . \end{array} \right\} \quad (22)$$

<u>Proof</u>. We apply (9) (with $E = \mathbb{C}$) and (21) to $n+1$ points

$$\beta_1 = -\delta_1, \dots, \beta_n = -\delta_n , \ \beta_{n+1} = 0$$

with multiplicities

$$\tau_1, \dots, \tau_n , \ m+1$$

respectively, where $\delta_1, \dots, \delta_n \in \mathbb{C}$ are algebraically independent over $\mathbb{Q}$: for instance, we may take, by Lindermann's theorem (see Baker [7], pp. 6-8), $\delta_1 = e^{\gamma_1}, \dots, \delta_n = e^{\gamma_n}$ with $\gamma_1, \dots, \gamma_n$ being algebraic numbers linearly independent over $\mathbb{Q}$. Take $h = n+1$, $t=1$ in (9) and (21). Then (9) and (21) imply

$$(-1)^m \prod_{i=1}^{n} \left(\frac{z+\delta_i}{\delta_i}\right)^{\tau_i} \cdot \sum_{s=1}^{m} (-1)^{s-1} \sum_{\substack{\lambda_1+\dots+\lambda_m=m \\ \lambda_j=0\,(j<s),\,\lambda_j\geq 1\,(j\geq s)}} \prod_{j=1}^{m} \frac{\left(\frac{\partial}{\partial y}\right)^{\lambda_j} \left\{ (z-y) \prod_{k=1}^{n} (y+\delta_k)^{\tau_k} \right\}_{y=0}}{\lambda_j! \prod_{k=1}^{n} \delta_k^{\tau_k}}$$

$$= \prod_{i=1}^{n} (z+\delta_i)^{\tau_i} \sum_{k=0}^{m} \frac{1}{k!} \frac{d^k}{dz^k} \left\{ \prod_{i=1}^{n} (z+\delta_i)^{-\tau_i} \right\}_{z=0} z^k . \tag{23}$$

On comparing the leading coefficients of both sides of (23), we obtain

$$\sum_{\substack{\rho_1+\dots+\rho_n=m \\ \rho_1\geq 0,\dots,\rho_n\geq 0}} \left(\sum_{s=1}^{m} (-1)^{s-1} \sum_{\substack{\lambda_s+\dots+\lambda_m=m \\ \lambda_s\geq 1,\dots,\lambda_m\geq 1}} \sum_{(k_{ij})} \prod_{i=1}^{n} \prod_{j=s}^{m} \binom{\tau_i}{k_{ij}} \right) \alpha_1^{-\rho_1} \dots \alpha_n^{-\rho_n}$$

$$= \sum_{\substack{\rho_1+\dots+\rho_n=m \\ \rho_1\geq 0,\dots,\rho_n\geq 0}} \left(\prod_{i=1}^{n} \binom{\tau_i+\rho_i-1}{\rho_i} \right) \alpha_1^{-\rho_1} \dots \alpha_n^{-\rho_n} , \tag{24}$$

where $\sum_{(k_{ij})}$ ranges over all $n \times (m-s+1)$ matrices (k_{ij}) satisfying (22). Recalling the fact that $\delta_1,\dots,\delta_n$ are algebraically independent, the theorem follows from (24) at once.

References

[1] W.W. Adams, Transcendental numbers in the p-adic domain, Amer. J. Math. 88 (1966), 279-308.

[2] M. Anderson and D.W. Masser, Lower bounds for heights on elliptic curves, Math. Zeitschrift, 174 (1980), 23-34.

[3] A. Baker, Linear forms in the logarithms of algebraic numbers I, II, III, Mathematika, 13 (1966), 204-216; 14 (1967), 102-107, 220-228.

[4] A. Baker, Linear forms in the logarithms of algebraic numbers IV, Mathematika, 15 (1968), 204-216.

[5] A. Baker, A sharpening of the bounds for linear forms in logarithms II, Acta Arith. 24 (1973), 33-36.

[6] A. Baker, The theory of linear forms in logarithms, Transcendence theory: advances and applications (Academic Press, London, 1977) pp. 1-27.

[7] A. Baker, Transcendental number theory, Cambridge University Press, Cambridge, 1979.

[8] A. Baker and J. Coates, Fractional parts of powers of rationals, Math. Proc. Camb. Phil. Soc. 77 (1975), 269-279.

[9] I.S. Berezin and N.P. Zhidkov, Computing methods, Vol. 1, Pergamon Press, Oxford, 1965.

[10] D. Bertrand, Approximations diophantiennes p-adiques sur les courbes elliptiques admettant une multiplication complexe, Compositio Math. 37 (1978), 21-50.

[11] A. Brumer, On the units of algebraic number fields, Mathematika, 14 (1967), 121-124.

[12] J. Coates, An effective p-adic analogue of a theorem of Thue I; II: The greatest prime factor of a binary form, Acta Arith. 15 (1969), 279-305; 16 (1970), 399-412.

[13] P.J. Davis, Interpolation and approximation, Blaisdell Publishing Company, New York-Toronto-London, 1963.

[14] N.I. Feldman, An improvement of the estimate of a linear form in the logarithms of algebraic numbers, Mat. Sbornik, 77 (1968), 423-436; = Math. USSR Sbornik, 6 (1968), 393-406.

[15] A.O. Gelfond, Sur la divisibilité de la différence des puissances de deux nombres entières par une puissance d'un idéal premier, Mat. Sbornik, 7 (1940), 7-26.

[16] H. Hasse, Number theory, Springer-Verlag, Berlin Heidelberg New York 1980.

[17] R.M. Kaufman, Bounds for linear forms in the logarithms of algebraic numbers with p-adic metric, Vestnik Moskov. Univ. Ser. I, 26 (1971), 3-10.

[18] J.H. Loxton, M. Mignotte, A.J. van der Poorten and M. Waldschmidt, A lower bound for linear forms in the logarithms of algebraic numbers.

[19] K. Mahler, Ein Beweis der Transzendenz der P-adischen Exponentialfunktion, J. reine angew. Math. 169 (1932), 61-66.

[20] K. Mahler, Über transzendente P-adische Zahlen, Compositio Math. 2 (1935), 259-275.

[21] K. Mahler, On some inequalities for polynomials in several variables, J. London Math. Soc. 37 (1962), 341-344.

[22] M. Mignotte and M. Waldschmidt, Linear forms in two logarithms and Schneider's method, Math. Annalen, 231 (1978), 241-267.

[23] A.J. van der Poorten, Some determinants that should be better known, J. Austral.Math.Soc. 21 (1975), 278-288.

[24] A.J. van der Poorten, Hermite interpolation and p-adic exponential polynomials, J. Austral. Math. Soc. 22 (1976), 12-26.

[25] A.J. van der Poorten, Linear forms in logarithms in the p-adic case, Transcendence theory: advances and applications (Academic Press, London, 1977) pp. 29-57.

[26] A. Schinzel, On two theorems of Gelfond and some of their applications, Acta Arith. 13 (1967), 177-236.

[27] L.G. Schnirelman, On functions in normed algebraically closed fields, Izv. Akad. Nauk SSSR, Ser.Mat. 5/6, 23 (1938), 487-496.

[28] V.G. Sprindžuk, Concerning Baker's theorem on linear forms in logarithms, Dokl.Akad.Nauk BSSR, 11 (1967), 767-769.

[29] V.G. Sprindžuk, Estimates of linear forms with p-adic logarithms of algebraic numbers, Vescī Akad. Nauk BSSR, Ser. Fīz-Mat. (1968), no. 4, 5-14.

[30] R. Tijdeman, On the equation of Catalan, Acta Arith. 29 (1976), 197-209.

[31] M. Waldschmidt, A lower bound for linear forms in logarithms, Acta Arith. 37 (1980), 257-283.

[32] G. Wüstholz, A new approach to Baker's theorem on linear forms in logarithms I, II, to appear; III, to appear in the Proceedings of Durham Symposium on Transcendental Number Theory, July 1986.

[33] K.R. Yu, Linear forms in logarithms in the p-adic case, to appear in the Proceedings of Durham Symposium on Transcendental Number Theory, July 1986.