Analytic Jacobi Eisenstein series and the Shimura method

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Abstract

In this paper it is proven that analytic Jacobi Eisenstein series always admit a meromorphic continuation on the whole complex plane and statements about the location of possible poles are given. Moreover a new interpretation of Shimura's approach to the standard L-function of a Siegel modular form is presented.

The purpose of this paper is twofold. First and mainly we study the analytic continuation and the location of poles of analytic Jacobi Eisenstein series. Secondly we present a new interpretation of Shimura's approach to automorphic L-functions and indicate several improvements and applications.

In [Sh75] Goro Shimura introduced a method to study in a precise way the poles of the Rankin L-function $D_f(s)$ attached to an elliptic cusp form f by integrating the cusp form against an Eisenstein series of half-integral weight multiplied with a theta series. Later on this method was generalized to standard L-functions of Siegel modular forms ([An79][A-K79],[Sh94]).

In this paper we show that Shimura's integral can be interpreted as the Petersson scalar product of an analytic Jacobi Eisenstein series (introduced by Arakawa [Ar90], [Ar92], [Ar94]) restricted to the Siegel variable against the modular form.

The reason for this is the following insight: an analytic Jacobi Eisenstein series can be interpreted as the product of a Siegel Eisenstein series (of integral or half integral weight) with a theta series, up to some trace. As a by-product we get the meromorphic continuation and a precise description of possible poles of analytic Jacobi Eisenstein series of every degree and index. To get the functional equation one may has to proceed in a different way. This has been done in the elliptic case by Tsuneo Arakawa in [Ar90] and [Ar92], using methods of Roelcke and Selberg. He worked with vector-valued half-integral Eisenstein series. But if one is satisfied with Jacobi-Eisenstein series attached to congruence subgroups of the Jacobi group (actually this is sufficient for the Shimura method), then the functional equation follows from the theory of Siegel Eisenstein series (of half-integral weight).

It should be mentioned that the Fourier-Jacobi expansion of an analytic Siegel Eisenstein series is not a Jacobi Eisenstein series in the sense of Arakawa. This is in contrast to the holomorphic situation.

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Notation: For an associative ring R with identity element, we denote by $R^\times$ the group of all invertible elements of R and by $R^{m,n}$ the module of all $m \times n$ matrices with entries in R. If M is a matrix, M^t, $\det(M)$, and $\mathrm{tr}(M)$ stand for its transpose, determinant, and trace. We put $M_n(R) = R^{n,n}$, $Gl_n(R) = M_n(R)^\times$, $Sl_n(R) = \{M \in Gl_n(R) | \det(M) = 1\}$ (when R is commutative). The identity and zero elements of $M_n(R)$ are denoted by 1_n and 0_n respectively (when n needs to be stressed). Let $J_n = \left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)$ then $Sp_n(R) = \{M \in Gl_{2n}(R) | M^t J_n M = J_n\}$.

$$\begin{aligned} P_{n,r}(R) &= \left\{ \begin{pmatrix} * & * \\ 0_{n+r,n-r} & * \end{pmatrix} \in Sp_n(R) \right\}, \\ P^J_{n,r}(R) &= \left\{ \begin{pmatrix} 1_{n-r} & * \\ 0_{n+r,n-r} & * \end{pmatrix} \in Sp_n(R) \right\}. \end{aligned}$$

For convenience let $\Gamma_n = Sp_n(\mathbb{Z})$, $P_{n,r} = P_{n,r}(\mathbb{Z})$, $\Gamma = \Gamma_1$ and $\Gamma_\infty = P_{1,0}$. For real symmetric matrices A and B, we write $A > B$ ($A \geq B$) to indicate that $A - B$ is positive definite (positive semi-definite) and $A[B] = B^t AB$ if A, B are suitable. If $A_1, A_2, \ldots, A_n$ are square matrices, $[A_1, A_2, \ldots, A_n]$ denotes the matrix with $A_1, A_2, \dots, A_n$ in the diagonal blocks and 0 in all other blocks. Let A_n^+ denote the set of all semi-integral positiv-definit matrices of size n. Let $Z \in \mathbb{C}^{n,n}$, then we put $e\{Z\} = e^{2\pi i\,(\mathrm{tr}\,Z)}$ and $\mathrm{Re}(Z), \mathrm{Im}(Z)$ for the real and imaginary part of Z and for $M \in \mathbb{C}^{m,n}$ we put occasionally $x_M = \mathrm{Re}(M)$ and $y_M = \mathrm{Im}(M)$. Further $\delta(Z) = \det(\mathrm{Im}\,(Z))$. By $\overline{a}$ we denote the complex conjugation of an element $a \in \mathbb{C}$.

1 Modular forms

1.1 Siegel modular forms

The group $Sp_n(\mathbb{R})$ acts on the Siegel upper half space $I\!H_n$ of degree n by

$$(M, Z) \mapsto M(Z) = (AZ + B)(CZ + D)^{-1},$$

where $M = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right)$ with $A, B, C, D \in \mathbb{R}^{n,n}$. We denote by M_n^k be the space of Siegel modular forms and S_n^k the subspace of cusp forms of degree n and weight k for the Siegel modular group Γ_n. Let $J(M, Z) = (CZ + D)$ and $j(M, Z) = \det(CZ + D)$. Then every $F \in M_n^k$ satisfies the transformation law

$$F(Z) = F(M(Z))\, j(M, Z)^{-k} \text{ for all } M \in \Gamma_n,\ Z \in I\!H_n. \tag{1}$$

The right side of the equation (1) is usually abbreviated by $(F|_k M)(Z)$. For arbitrary complex valued functions F and G on $I\!H_n$, which satisfy (1) for a congruence subgroup Γ' of Γ_n, we define the Petersson integral, convergence assumed, by

$$\langle F, G\rangle_{\Gamma'} = \int_{\Gamma' \backslash I\!H_n} F(Z)\, \overline{G(Z)}\, \delta(Z)^k d^*Z. \tag{2}$$

Here $Z = X + i\,Y$ and $d^*Z = \det(Y)^{-(n+1)} dX dY$ denotes the symplectic volume element. Every $F \in S_n^k$ has a Fourier expansion of the form $F(Z) = \sum_{T \in A_n^+} a^F(T) e\{TZ\}$. If σ is an automorphism of $\mathbb{C}$, then

$$F^\sigma(Z) = \sum_{T \in A_n^+} a^F(T)^\sigma e\{TZ\} \in S_n^k. \tag{3}$$

We denote by $\#$ the complex-conjugation in $\mathbb{C}$. If we assume that $F \in S_n^k$ is a Hecke eigenform with Satake-parameters $(\alpha_{0,p}; \alpha_{1,p}; \dots; \alpha_{n,p})$. Then the standard zeta function $D_F^n(s)$ of F is given by

$$D_F^n(s) = \prod_p \left\{ (1-p^{-s}) \prod_{j=1}^{n} (1-\alpha_{j,p}p^{-s})\,(1-\alpha_{j,p}^{-1}p^{-s}) \right\}^{-1}. \tag{4}$$

For more details the reader is referred to Andrianov [An87] or Klingen [Kl90].

1.2 Jacobi Eisenstein series

Every element $g \in P_{n+l,n}^J(\mathbb{R})$ can be parameterized by

$$((\lambda,\mu;\rho),M) \in \left(\mathbb{R}^{l,n} \times \mathbb{R}^{l,n} \times \mathbb{R}^{l,l}\right) \times Sp_n(\mathbb{R}) \tag{5}$$

determined by

$$g = \begin{pmatrix} 1_l & 0 & \rho & \mu \\ 0 & 1_n & \mu^t & 0 \\ 0 & 0 & 1_l & 0 \\ 0 & 0 & 0 & 1_n \end{pmatrix} \begin{pmatrix} 1_l & \lambda & 0 & 0 \\ 0 & 1_n & 0 & 0 \\ 0 & 0 & 1_l & 0 \\ 0 & 0 & -\lambda^t & 1_n \end{pmatrix} (1_{2l} \times M). \tag{6}$$

An important normal subgroup is given by the Heisenberg group $H_{n,l}(\mathbb{R})$, consisting of all elements h of the form $h = (\lambda,\mu;\rho) = ((\lambda,\mu;\rho),1_{2n})$. The group $P_{n+l,n}^J(\mathbb{R})$ acts on $D_{n,l} = \mathbb{H}_n \times \mathbb{C}^{l,n}$ by

$$P_{n+l,n}^J(\mathbb{R}) \times D_{n,l} \longrightarrow D_{n,l}$$

$$g(\tau,z) = (M(\tau), zJ(M,\tau)^{-1} + \lambda M(\tau) + \mu), \tag{7}$$

where $g = ((\lambda,\mu;\rho),M)$ and $(\tau,z) \in D_{n,l}$. It is sometimes convenient to regard $Sp_n(\mathbb{R})$ as a subgroup of $P_{n+l,n}^J(\mathbb{R})$.

Definition 1.1 *Let $k,l,n \in \mathbb{N}$ and $S \in A_l^+$. The factor of automorphy $j_{k,S}$ of index S and weight k for $P_{n+l,n}^J(\mathbb{R})$ is the map*

$$j_{k,S} : \begin{cases} P_{n+l,n}^J(\mathbb{R}) \times D_{n,l} & \longrightarrow & D_{n,l} \\ (g,(\tau,z)) & \mapsto & j_{k,S}(g,(\tau,z)) \end{cases}, \tag{8}$$

where

$$\begin{aligned} j_{k,S}(g,(\tau,z)) &= j(M,\tau)^k e\{-S\rho\} \\ & e\{-S[\lambda]\,M(\tau) - 2\lambda^t SzJ(M,\tau)^{-1} + S[z]J(M,\tau)^{-1}c\} \end{aligned} \tag{9}$$

for $g = ((\lambda, \mu; \rho), M)$ and $M = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$. Let $F : D_{n,l} \longrightarrow \mathbb{C}$, then the generalized Petersson slash operator $|_{J,k}$ is given by

$$(F|_{J,k}g)(\tau, z) := j_{S,k}(g, (\tau, z))^{-1} F(g(\tau, z)). \tag{10}$$

Let $\mathcal{J}^n_{k,S}$ and $\mathcal{J}^{n,\text{cusp}}_{k,S}$ denote the space of Jacobi forms and Jacobi cusp forms, respectively, of weight k, index S and degree (n, l) for $P^J_{n+l,n}(\mathbb{Z})$. We abbreviate $P^J_{n+l,n}(\mathbb{Z})$ by $\Gamma^J_{n,l}$. For more details see Arakawa [Ar94] (section 1.1 and 1.2).
Similar as Eisenstein series on the Siegel upper half plane, one defines Jacobi Eisenstein series. For each r with $0 \leq r \leq n$, let

$$\Gamma^J_{n,l,r} = \{((0\lambda_2, \mu; \rho), M) \mid \lambda_2 \in \mathbb{Z}^{l,r}, \mu \in \mathbb{Z}^{l,n}, \rho \in \mathbb{Z}^{l,l} \text{ and } M \in P_{n,r}\} \tag{11}$$

and

$$*_J : \begin{cases} D_{n,l} & \longrightarrow \quad D_{r,l} \\ (\tau, z) & \mapsto \quad (\tau \left[\left(\begin{smallmatrix} 0 \\ 1_r \end{smallmatrix}\right)\right], z\left(\begin{smallmatrix} 0 \\ 1_r \end{smallmatrix}\right)) \end{cases} . \tag{12}$$

Definition 1.2 *Let $k, l, n \in \mathbb{N}$ with k even, $0 \leq r \leq n$ and $S \in A^+_l$. To $\phi \in \mathcal{J}^{r,\text{cusp}}_{k,S}$ we attach an analytic Jacobi Eisenstein series of Klingen type on $D_{n,l}$ defined by:*

$$E^{S,k}_{n,r}((\tau, z), \phi); s) = \sum_{\gamma \in \Gamma^J_{n,l,r} \backslash \Gamma^J_{n,l}} \phi(\gamma(\tau, z)_{*_J})\ j_{k,S}(\gamma, (\tau, z))^{-1} \left(\frac{\delta(M(\tau))}{\delta(M(\tau)_*)}\right)^s, \tag{13}$$

here $\gamma = (h, M)$, $s \in \mathbb{C}$ with Re(s) sufficiently large and $\begin{pmatrix} \tilde{\tau}_1 & \tilde{\tau}_2 \\ * & \tilde{\tau}_4 \end{pmatrix}_* = \tilde{\tau}_4 \in \mathbb{H}_r$. If $r = 0$, we put $E^{S,k}_n((\tau, z); s) = E^{S,k}_{n,0}((\tau, z), 1; s)$.

Let $\phi \in \mathcal{J}^n_{k,S}$, then $(\mathcal{D}_0\phi)(\tau) := \phi(\tau, 0) \in M^k_n$.

Remark 1.3
a) The Eisenstein series is absolutely convergent for $k + 2$ Re$(s) > n + r + l + 1$. If $k > n + r + l + 1$, then $E^{S,k}_{n,r}(\phi, (\tau, z); 0) \in \mathcal{J}^n_{k,S}$.
b) Let us put

$$\chi^{k,s;J}_\mu(g, (\tau, z)) = j_{k,S}(g, (\tau, z))^{-1} |j(M, \tau)|^{-2s}, \tag{14}$$

then we have the cocycle relation

$$\chi^{k,s;J}_\mu(gh, (\tau, z)) = \chi^{k,s;J}_\mu(g, h(\tau, z))\ \chi^{k,s;J}_\mu(h, (\tau, z)) \tag{15}$$

for $g, h \in P^J_{n+l,n}(\mathbb{R})$ *and*

$$E^{S,k}_n((\tau,z),s) = \delta(\tau)^s \sum_{\gamma \in \Gamma^J_{(n,l),0} \backslash \Gamma^J_{n,l}} \chi^{k,s;J}_\mu(\gamma,(\tau,z)).$$

2 The MAIN FORMULA

The Jacobi Eisenstein series introduced in the last section, can be used to study analytic and arithmetic properties of the standard L-function attached to Siegel modular forms. In this section we present a formula, which can be used as a starting point in this direction.

Let $S \in A^+_l$. Then we attach to S the theta series

$$\Theta^o_S(\tau) = \sum_{\lambda \in \mathbb{Z}^{l,n}} e\{S[\lambda]\tau\}, \qquad (\tau \in I\!H_n). \tag{16}$$

It is obvious, that one can choose

$$R_{l,n} = \left\{((\lambda,0;0),M) \mid \lambda \in \mathbb{Z}^{l,n} \text{ and } M \in P_{n,0}\backslash\Gamma\right\} \tag{17}$$

for a representative system of $\Gamma^J_{n,l,0} \setminus \Gamma^J_n$. Thus a simple pullback formula of the Jacobi Eisenstein series via the natural embedding

$$I\!H_n \longrightarrow I\!H_n \times \mathbb{C}^{l,n}, \qquad \tau \mapsto (\tau,0)$$

is given by

$$E^{S,k}_n\left((\tau,0),s\right) = \delta(\tau)^s \sum_{M \in P_{n,0}\backslash\Gamma_n} \chi^{k,s}_\mu(M,\tau)\ \Theta^o_S(M(\tau)). \tag{18}$$

Because the generalized automorphic factor $\chi^{k,s;J}_\mu$ splits into

$$\chi^{k,s}_\mu(M,\tau)\ e\{S[\lambda]M(\tau)\}. \tag{19}$$

For further simplifications we put: $\Gamma_n(s) = \prod_{\nu=1}^n \Gamma(s - \frac{\nu-1}{2})$ and

$$\mathcal{D}_0 E^{S,k}_n(s) := E^{S,k}_n((\tau,0),s).$$

Theorem 2.1 **(Main formula)** *Let $k, l, n \in \mathbb{N}$ and k even. Let $S \in A_l^+$ and $F \in S_n^k$. Let $s \in \mathbb{C}$ with $2\,Re(s) + k > n + l + 1$. Then*

$$\left\langle \mathcal{D}_0 E_n^{S,k}(s)\,;\,F \right\rangle_{\Gamma_n} = (4\pi)^{-n(k+s-\frac{n+1}{2})} \pi^{n(n-1)/4}\,\Gamma_n(s+k-\tfrac{n+1}{2}) \times D_S^{n,l}(F^{\#}, k+s-\tfrac{n+1}{2}). \tag{20}$$

Here

$$D_S^{n,l}(G,s) := \sum_{\lambda \in \mathbb{Z}^{l,n}/Gl_n(\mathbb{Z})} a^G(S[\lambda]) \ \ (\det(S[\lambda]))^{-s}$$

for $G \in S_n^k$.

Proof:

After all our previous observations, this can be done in a straightforward manner. Let $P_n = \{Y \in \mathbb{R}^{n,n} \mid iY \in I\!H_n\}$ and R_n a fundamental domain of the action of $Gl_n(\mathbb{Z})$ on P_n, via $Y \mapsto Y[U^t]$.

$$\left\langle \mathcal{D}_0 E_n^{S,k}(s)\,;\,F \right\rangle_{\Gamma_n} = \boxed{\boxed{\int_{\Gamma_{n,0}\backslash I\!H_n} \Theta_S^o(\tau)\overline{F(\tau)}\delta(\tau)^{k+s}\,d^*\tau}} \tag{21}$$

$$= \int_{X\in(\mathbb{R}/\mathbb{Z})^{n,n}} \int_{R_n} \Theta^o(X+iY)\,\overline{F(X+iY)}\delta(Y)^{k+s-n-1}\,dX\,dY \tag{22}$$

$$= \sum_{\lambda\in\mathbb{Z}^{l,n}/Gl_n(\mathbb{Z})} \overline{a(S[\lambda])} \int_{y>0} \det(y)^{k+s-n-1}\,e^{-4\pi \mathrm{tr}(S[\lambda]y)}\,dy. \tag{23}$$

Obviously, we have only to care for $S[\lambda] > 0$. But if this is the case, then $\int_y$ can be calculated by applying Lemma 2 given in [Kl90](Chapter 6), which proves the theorem.

□

Thus $\left\langle \mathcal{D}_0 E_n^{S,k}(s)\,;\,F \right\rangle_{\Gamma_n} \equiv 0$, if $l < n$. Let now F be a Hecke eigenform. It had been shown by Andrianov [An79], Böcherer [Boe86] and Shimura [Sh94], that $D_S^{n,n}(F,s)$ is essentially an Euler product obtained from the eigenvalues, i.e the standard L-function of F.

Moreover, Andrianov and Shimura used a similar expression as given in the box, given in (21), to study analytic properties of the L-function. Actually, this approach had been introduced by Shimura [Sh75] to give a precise description of the poles of the Rankin L-function related to $D_1^{1,1}(F,s)$. Hence the **Shimura method** can be interpreted as the Petersson scalar product of a cusp form with an on the Siegel variable restricted analytic Jacobi Eisenstein series. This view point, avoids somehow, the theory of modular forms of half-integral weight. But nevertheless, in the next section we will go somehow in the opposite direction to study analytic properties of the Jacobi Eisenstein series.

3 Jacobi Eisenstein series and theta series

Shimura established an isomorphism between $\mathcal{J}_{k,S}^n$ and a certain space of vector-valued Siegel modular forms [Zi89] (Theorem 3.3). Main ingredients of the isomorphism are theta series with characteristic (a,b) on $D_{n,l}$ of the following type:

$$\Theta_{S,a,b}(\tau,z) := \sum_{\lambda\in\mathbb{Z}^{l,n}} e\{S[\lambda+a]\tau + 2(\lambda+a)^t S(z+b)\} \tag{24}$$

with $(a,b)\in\mathbb{Q}^{l,n}\times\mathbb{Q}^{l,n}$. For $n=l=1$ see [E-Z85] and for simplification of the notation let $\Theta_S=\Theta_{S,0,0}$.
In a non-holomorphic setting (e.g. Jacobi Eisenstein series $E_{n,r}^{S,k}(\,*\,,s)$) the situation becomes much more complicated. A first approach to connect non-holomorphic Jacobi Eisenstein series with Siegel Eisenstein series has been established by Arakawa [Ar90] (Proposition 3.1) and [Ar92] (Theorem 6.2), at least in the case $n=1$. He related the Jacobi Eisenstein series to Eisenstein series with theta multiplier system on $I\!H_1$. By making use of a general theory for Eisenstein series of $\mathsf{SL}_2(\mathbb{Z})$ due to Selberg and Roelcke, he was able to establish the meromorphic continuation and the functional equation of the Jacobi Eisenstein series. Here we would like to state a slidly different approach, which reduces the problem of meromorphic continuation to that of certain Siegel Eisenstein series (of half-integral weight).

Remark 3.1 *Let $g=((\lambda,0;0),M)\in\Gamma_{n,l}^J$. Then we have*

$$\begin{aligned} J_{k,S}(g,(\tau,z)) = j(M,\tau)^k\, e\{S[z]J(M,\tau)^{-1}\,c\} \\ \times\, e\{-S[\lambda]M(\tau) - 2\lambda^t SzJ(M,\tau)^{-1}\}. \end{aligned} \tag{25}$$

This gives us the following description of analytic Jacobi Eisenstein series.

$$
\begin{aligned}
\boxed{E_n^{S,k}((\tau,z),s)} &= \delta(\tau)^s \sum_{M=\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)\in\Gamma_{n,0}\backslash\Gamma_n} \chi_\mu^{k,s}(M,\tau)\ e\{S[z]J(M,\tau)^{-1}\,c\} \\
&\quad \times \sum_{\lambda\in\mathbb{Z}^{l,n}} e\{S[\lambda]M(\tau) + 2\lambda^t SzJ(M,\tau)^{-1}\} \\
&= \boxed{\delta(\tau)^s \sum_{M=\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)\in\Gamma_{n,0}\backslash\Gamma_n} \chi_\mu^{k,s}(M,\tau)\ \ e\{-S[z]J(M,\tau)^{-1}\,c\}\ \ \Theta_S\left(M(\tau,z)\right).}
\end{aligned}
\tag{26}
$$

To employ the transformation law of the theta series Θ_S we decompose $\Gamma_{n,0}\backslash\Gamma_n$ into

$$
\Big(P_{n,0}\Big\backslash\Gamma_0^n(q)\Big)\times\Big(\Gamma_0^n(q)\Big\backslash\Gamma_n\Big).
$$

Here q denotes the level of the quadratic form $2S$, and

$$
\Gamma_0^n(q) = \left\{\gamma = \begin{pmatrix} A & B \\ C & D \end{pmatrix}\in\Gamma_n \,\Big|\, C\equiv 0\ (q)\right\}.
$$

By $\chi_S^{(n)}(M)$ we denote the multiplier, an eighth root of unity, attached to $S\in A_l^+$ and $\Gamma_0^n(q)$ ([A-M75]). For example in the case $l=n=1$ and $S=1$ we have

$$
\chi_1^{(1)}(M) = \left(\tfrac{c}{d}\right)\ \epsilon_d^{-1} \qquad \left(M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\in\Gamma_0(4)\right), \tag{27}
$$

here $\epsilon_d = 1$ if $d\equiv 1(4)$ and i if $d\equiv 3(4)$ and $\left(\frac{c}{d}\right)$ is a generalized quadratic residue symbol (Jacobi symbol, for more details [Sh73]). Let $\tau\in\mathbb{C}^\times$. For the square root $\tau^{1/2}$ we always choose the branch with $-\pi/2 < \arg(\tau^{1/2}) \le \pi/2$ and put $\tau^{r/2} = (\tau^{1/2})^r$ for all $r\in\mathbb{Z}$.

$$
\left(\left(\frac{c}{d}\right)\epsilon_d^{-1}(c\tau+d)^{1/2}\right)^2 = \left(\frac{-1}{d}\right)\ j(M,\tau). \tag{28}
$$

More general (e.g. [Boe83], Proposition 9) we have

Lemma 3.2 *Let $S\in A_l^+$ and let q be the level of S. Then for $M\in\Gamma_0^n(q)$ we have*

$$
\Theta_S(M(\tau,z)) = e\{S[z]\,J(M,\tau)^{-1}c\}\ h_S(M,\tau)\,\Theta_S(\tau,z), \tag{29}
$$

with $h_S(M,\tau) = j(M,\tau)^{l/2}\chi_S^{(n)}(M)$ and $(\tau,z)\in D_{n,l}$.

Definition 3.3 *Let $k, l, n \in \mathbb{N}$ and k even and let $S \in A_l^+$ with level q. Let $s \in \mathbb{C}$ with $2\,Re(s) + k > n + l + 1$. An (Siegel-)Eisenstein series of general type is defined by*

$$\begin{aligned} E_n^{S,k-l/2,q}(\tau, s) &= \delta(\tau)^s \sum_{\gamma \in P_{n,0} \backslash \Gamma_0^n(q)} \chi_\mu^{k,s}(\gamma, \tau)\, h_S(\gamma, \tau) \\ &= \delta(\tau)^s \sum_{\gamma \in P_{n,0} \backslash \Gamma_0^n(q)} \chi_S^{(n)}(\gamma)\ \chi_\mu^{k-l/2,s}(\gamma, \tau). \end{aligned} \tag{30}$$

Let us recall some well known results on multiplier systems coming from theta series ([An87], Chapter 1). Let $S \in A_l^+$ with level q.

- Let $l = 2g$ with $g \in \mathbb{N}$. Then $\chi_S^{(n)}(M) = \psi_S(\det(D)$, where ψ_S is a certain real Dirichlet character modulo q.
- Let $l = 2g + 1$ with $g \in \mathbb{N}$. Then $\chi_S^{(n)}(M) = \chi_{1_n}^{(n)}(M)\ \psi_S'(\det(D)$, where ψ_S' is again a certain real Dirichlet character modulo q. Further let

  $$(\chi_{1_n}^{(n)}(M))^2 = \mathrm{sgn}(\det D)\ \left(\frac{-4}{|\det(D)|}\right) \text{ and } (\chi_{1_n}^{(n)}(M))^4 = 1,$$

  here $\mathrm{sgn}(x) = x/|x|$ and $\left(-\right)$ denotes the Jacobi symbol.

Remark 3.4 *Let $l = 2g$ then the function defined in (30) is an ordinary Eisenstein series (of integer weight) with real Dirichlet character ψ_S. If $l = 2g + 1$, then it is a Eisenstein series of half-integral weight with Dirichlet character ψ_S'. Both Eisenstein series posses a meromorphic continuation on the whole $s-$plane (Langlands [La76], see also Shimura [Sh94], the remark given in the last paragraph on page 562).*

Thus the $E_n^{S,k-l/2,q}(\tau, s)$ is absolute convergent and behaves like an automorphic form with respect to $\Gamma_0^n(q)$. Further it admits a meromorphic continuation on the whole $s-$plane.

Definition 3.5 *Let F be a complex-valued function on $D_{n,l}$, which transforms like a Jacobi form (of weight k and index S) with respect to a congruence subgroup Γ' of Γ_n. Then the n^{th}-trace of F is defined by*

$$\left(\mathbf{Tr}_{\Gamma'}^{\Gamma_n} F\right)(\tau, z) = \sum_{M \in \Gamma' \backslash \Gamma_n} \left(F|_{J,k} M\right)(\tau, z). \tag{31}$$

This new obtained function transforms like a Jacobi form with respect to Γ_n. Of course, we can also apply this construction to functions which are restricted to the symplectic variable τ (i.e. $z = 0$).

Theorem 3.6 *Let $k, l, n \in \mathbb{N}$ and $S \in A_l^+$. Let k be even and q be the level of S and $s \in \mathbb{C}$ with Re$(s) + k > n + l + 1$. Then we have for $(\tau, z) \in D_{n,l}$:*

$$E_n^{S,k}\left((\tau, z), s\right) = \mathbf{Tr}_{\Gamma_0^n(q)}^{\Gamma_n}\left(E_n^{S,k-l/2,q}(\tau, s)\,\Theta_S(\tau, z)\right). \tag{32}$$

Theorem 3.6 gives us a tool to study analytic properties of the Jacobi Eisenstein series, by reducing everything to the theory of (Siegel-)Eisenstein series of integral or half-integral weight and properties of the trace map $\mathbf{Tr}$.
not known to everybody. Let Γ' be a congruence subgroup of Γ_n and let us assume that F behaves like automorphic forms with respect to Γ' and G with respect to Γ. Then we have, if the scalar products exist,

$$\langle \mathbf{Tr}_{\Gamma'}^{\Gamma_n} F, G\rangle_{\Gamma_n} = \langle F, G\rangle_{\Gamma'}. \tag{33}$$

This principle works of course in much more general situations. A first application is the following. If we make the same assumptions as in Theorem 2.1, then we have

$$I(F, s) := \left\langle \mathcal{D}_0 E_n^{S,k}(s)\,;\, F\right\rangle_{\Gamma_n} = \left\langle E_n^{S,k,q}(*, s)\left(\mathcal{D}_0(\Theta_S)\right)\,;\, F)\right\rangle_{\Gamma_0^n(q)}. \tag{34}$$

Thus the meromorphic continuation (and functional equation) of $\mathcal{D}_0 E_n^{S,k}(s)$ and $E_n^{S,k,q}(\tau, s)$ transfer directly to $I(F, s)$. Moreover one gets a sharp picture about the possible poles of $I(F, s)$. It is of course also interesting to compare directly the properties of the corresponding Eisenstein series of Jacobi and Siegel type.

4 Applications

4.1 Analytic properties of Jacobi Eisenstein series

Let $U \subseteq \mathbb{C}$ be an open subset and $F(\tau, z, s)$ a function $F : D_{n,l} \times U \longrightarrow \mathbb{C}$ holomorphic in s. Further, let Γ' a congruence subgroup of Γ and $F|_{J,k}\gamma = F$ for $S \in A_l^+$ and $\gamma \in \Gamma'$. Then $\mathbf{Tr}_{\Gamma'}^{\Gamma} F$ remains holomorphic. Hence we have the following

Theorem 4.1 *Let $l, n \in \mathbb{N}$ and $k \in 2\mathbb{Z}$ and $S \in A_l^+$. Then $E_n^{S,k}((\tau, z), s)$ has a meromorphic continuation to the whole $s-$plane with possible poles coming from Eisenstein series of integral or half-integral weight $E_n^{S,k-l/2,q}((\tau, z), s)$ (the latter can be studied by [Sh75],[Sh94] (Theorem 7.3)).*

Let us be more precise, at least in the simplest (non-trivial) case $n = l = 1$. Let $K \in \mathbb{N}$ odd, $N \in 4\mathbb{N}$ and ω be a Dirichlet character modulo N which satisfies $\omega(-1) = 1$. Further, let $L(s, \omega) = \sum \omega(n)\, n^{-s}$, $\tilde{j}(\gamma, z) = \Theta_1^o(\gamma(z)) \,/\, \Theta_1^o(z)$ for $\gamma \in \Gamma_0^1(4)$ and

$$E(z, s, K, \omega) = \delta(z)^{s/2} \sum_{\gamma \in P_{1,0} \backslash \Gamma_0^1(N)} \omega(d)\, \tilde{j}(\gamma, z)^{-K}\, |\tilde{j}(\gamma, z)|^{-2s}. \tag{35}$$

We put $\lambda_0 = 0$ or 1 according $\lambda = \frac{1-K}{2}$ is even or odd. Then it is well known [Sh75] (Proposition3), that

$$\begin{aligned} F(z, s, K, \omega) = \Gamma(\tfrac{s+K}{2})\Gamma(\tfrac{s-\lambda+\lambda_0}{2}) \\ L(2s - 2\lambda, \omega^2)E(z, s, K, \omega), \end{aligned} \tag{36}$$

multiplied with $(s - \lambda - 1)$ can be continued to a holomorphic function in the whole $s-$plane. The factor $(s - \lambda - 1)$ is unnecessary either if $\frac{|K|+1}{2}$ is even or if ω^2 is non-trivial. Hence we get

Corollary 4.2 *Let $k \in 2\mathbb{N}$ and $\chi_0(N)$ be the principal Dirichlet character modulo N, then*

$$\Gamma\left(s + k - 1/2\right) \Gamma\left(s + k/2\right)\ L(4s + 2k - 2, \chi_0(4))\ E_1^{1,k}((\tau, z), s) \tag{37}$$

can be continued to a holomorphic function in the whole $s-$plane. In particular $E_1^{1,k}((\tau, z), \frac{3-2k}{4}) \equiv 0$.

Next we would like to compare theses results with [Ar90]. For simplicity let $\kappa = \frac{k}{2} - \frac{1}{4}$ and $E^A(s) = E_1^{1,k}((\tau, z), s - \kappa)$. From the work of Arakawa one can deduce the following

Proposition 4.3 *The Eisenstein series $E^A(s)$ admits a meromorphic continuation on the whole $s-$plane with functional equation*

$$E^A(1 - s) = \Phi(1 - s)E^A(s), \tag{38}$$

here

$$\Phi(s) = 2^{3/2-2s} e^{-\pi i k/2} \frac{\Gamma(2s-1)}{\Gamma(s+\kappa)\Gamma(s-\kappa)} \frac{\zeta(4s-2)}{\zeta(4s-1)}. \tag{39}$$

Poles of the Eisenstein series can only occur in the stripe $\frac{1}{4} < Re(s) < \frac{1}{2}$, when $\zeta(2-4s)$ has (non-trivial) zeroes.

The result about possible poles is equivalent to the statement, that the function

$$\tilde{E}^A(s) = \Gamma(2s + \tfrac{1}{2})\,\zeta(4s-1)\,(s - \tfrac{1}{2})\,E^A(s)$$

is holomorphic. Which is again the same as to say that

$$\Gamma(2s+k)\,\zeta(4s+2k-2)\,(2s+k-3/2)\ E_1^{1,k}((\tau,z),s) \tag{40}$$

is holomorphic.

Remark 4.4 *Corollary 4.2 and Proposition 4.3 give the same candidates for possible poles of the analytic Jacobi Eisenstein series. But of course, Arakawa's approach has the advantage of giving a functional equation. Supplementary the Corollary determines a value of s where the Eisenstein series vanishes, which is not obvious from the Proposition 4.3. Hence (40) can be improved to the statement that*

$$\Gamma(2s+k)\,\zeta(4s+2k-2)\ E_1^{1,k}((\tau,z),s) \tag{41}$$

is holomorphic.

4.2 Properties of the standard L-function

As already stated, Theorem 2.1 can be used to study properties of the standard L-function $D_F^n(s)$ attached to a Hecke eigenform $F \in S_n^k$. For simplicity and because Arakawa's result is at the moment only available for Eisenstein series on $I\!H_1 \times \mathbb{C}^l$ we demonstrate this in the case of elliptic cusp forms.

Let us recall, that $D_1^{1,1}(F,s)/2 = \sum_{\lambda\in\mathbb{N}} a^F(\lambda^2)\ \lambda^{-2s}$, which is equal to

$$\zeta(4s-2k+2)^{-1} D_F^1(2s),$$

here $D_F(s)$ is the Euler product of degree 3 attached to a normalized Hecke eigenform $F \in S_1^k$. The *Main formula*, given in Theorem 2.1, specialized to this case leads to

$$\begin{aligned}\langle \mathcal{D}_0 E_1^{1,k}(s)\, ,\, F\rangle_\Gamma &= (4\pi)^{-(k+s-1)}\,\Gamma(s+k-1)\,D_1^{1,1}(F, k+s-1)\\ &= (4\pi)^{-(k+s-1)}\Gamma(s+k-1)\zeta(4s+2k-2)^{-1}\\ &\quad\times D_F^1(2s+2k-2).\end{aligned}$$

This integral representation of $D_F^1(s)$ and the holomorphy of the expression given in (41) leads to the holomorphy of

$$\Gamma(s-k+2)\ \Gamma(s/2-k)\ D_F^1(s). \tag{42}$$

Thus $D_F^1(s)$ is a holomorphic function. Proposition 4.3 allows the standard zeta function to have only one possible pole (at $s = k - 1/2$), but from Corollary 4.2, we know that 'there' the Eisenstein series has a zero, which gives the desired result. Hence the holomorphy of the L-function can be deduced directly from the analytic behavior of an (Jacobi) Eisenstein series, which can be seen as a slid refinement of the Shimura method (e.g.[Sh75]).

Remark 4.5 *These observations suggest, that generalizing Proposition 4.3 to Jacobi Eisenstein series of degree n should lead to a better understanding of the analytic behavior of the standard zeta function attached to a Siegel Hecke eigenform of arbitrary degree ([Mi91]).*

4.3 Analytic properties of Jacobi Eisenstein series of Klingen type

In this last subsection we would like to indicate briefly, how one can also study analytic properties of Jacobi Eisenstein series of Klingen type (attached to Jacobi cusp forms). The recipe, which worked already for Eisenstein series of Klingen type attached to the Siegel modular group, could be described as follows.

It starts with the fact, that a result in the style of the **Theorem of Garrett and Böcherer** [Boe85b] (Theorem 1.8), [He96] (Theorem 1.11), an important tool in the arithmetic theory of Siegel modular forms, is available. A proof and a precise description of this is given in [Ar94] (Theorem 2.9), see also the introduction of [Du97]. Thus let $\Phi \in \mathcal{J}_{k,S}^{r,\mathsf{cusp}}$ be a Hecke-Jacobi eigenform and

$Z_n(s,\Phi)$ the standard zeta function related to Φ (cf. [Ar94], section 2.2 and Theorem 2.9). Then the analytical properties of

$$Z_n(2s+k,\Phi)\, E^{S,k}_{m,n}(\star,\Phi,s)$$

are directly related with the ones of $E^{S,k}_{n+m}(\star,s)$: notice that $Z_n(s,\Phi)$ does not depend on m. Thus one can also study this function alone, because $E^{S,k}_{n,n}(z,1,s) = f(z)$.

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