Local class field theory: imperfect residue field case

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In this paper, which may be considered as a continuation of [F5], complete discrete valuation fields of rank n with a perfect residue field k of positive characteristic p are treated. We study Galois totally ramified p-extensions (with respect to the discrete valuation of rank n) and establish the reciprocity map

$$\Psi_F : VK_n^{top}(F) \to \mathrm{Hom}_{\mathbb{Z}_p}\left(\mathrm{Gal}\left(\tilde{F}/F\right), \mathrm{Gal}\left(F_p^{ab}/\tilde{F}\right)\right),$$

where $VK_n^{top}(F)$ is the subgroup in the topological Milnor K-group $K_n^{top}(F)$ generated by principal units with respect to the discrete valuation of rank n, $\tilde{F}/F$ is the maximal unramified with respect to the discrete valuation of rank n subextension in F_p^{ab}/F, F^{ab} is the maximal abelian p-extension of F. For an abelian totally ramified p-extension L/F the reciprocity map Ψ_F induces the isomorphism

$$VK_n^{top}(F)/N_{L/F}VK_n^{top}(L) \xrightarrow{\sim} \mathrm{Hom}_{\mathbb{Z}_p}\left(\mathrm{Gal}\left(\tilde{F}/F\right), \mathrm{Gal}(L/F)\right).$$

Our constructions work for the case of $\kappa = \dim_{\mathbb{F}_p} k/\wp(k) > 0$, where $\wp(x) = x^p - x$. In particular, they coincide with the known constructions in the case of $\kappa = 1$ ([F1–4]), and the constructions in the case of $n = 1$ ([F5]).

In the first section of this paper we establish a general construction of p-class field theory which develops the methods of J. Neukirch ([N1–3]) and the above-mentioned constructions. It should be stressed that this section (in such form of presentation) was written under the influence of the work [N4]. The second section deals with the specific features of the complete discrete valuation fields of rank n. The third section contains class field theory for such fields. We discuss ramification theory in the fourth section basing on the obtained results. In particular, the Hasse-Herbrand function is defined and its properties are studied. The material exposed may be considered as an explanation of some known phenomena described in the works of K. Kato, V.G. Lomadze and O. Hyodo ([K4–5], [L], {H]) in the case of the imperfect residue field. Finally, in the fifth section we clarify the properties of the reciprocity map exposing the description of norm subgroups in $VK_n^{top}(F)$.

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§ 1. p-class field theory.

1.1. Let K be a field and $G_p = \mathrm{Gal}(K_p/K)$ be the Galois group of the fixed maximal separable p-extension K_p of K over K. Let $Q(G_p)$ be the category, whose objects are finite subextensions F/K in K_p/K and the morphisms are the compositions of $\sigma : F \to \sigma F$ for $\sigma \in G_p$ and the inclusion $i_{\sigma F/L} : \sigma F \to L$.

Let A be a G_p-modulation (see § 4 of [N4], or s. 324 of [N3]), i.e. a double functor

$$A = (A^*, A_*) : Q(G_p) \to Ab,$$

where A^* is a covariant functor, A_* is a contravariant functor, such that $A^*(F) = A_*(F) = A_F$ for all $F \in Q(G_p)$, and such that: 1) $\sigma^*\sigma_* = \sigma_*\sigma^* = \mathrm{id}$ for $\sigma \in G_p$, $\sigma^* = A^*(\sigma)$, $\sigma_* = A_*(\sigma)$; 2) $N_{L/F} \circ i_{F/L} = |L : F|$ for $F, L \in Q(G_p)$, $F \subset L$

and $N_{L/F} = A_*(i_{F/L})$, $i_{F/L} = A^*(i_{F/L})$; 3) for any system R of representatives of $\mathrm{Gal}(K_p/L) \setminus G_p/\mathrm{Gal}(K_p/M)$, where $L, M \in Q(G_p)$, the formula

$$i_{K/L} \circ N_{M/K} = \sum_{\sigma \in R} N_{L\sigma M/L} \circ i_{\sigma M/L\sigma M} \circ \sigma^*$$

holds.

Further we will write σ instead of σ^* .

1.2. Assume that the field K possesses the following properties:

C1. There exists a Galois subsextension $\widetilde{K}/K$ in K_p/K, such that $\mathrm{Gal}\left(\widetilde{K}/K\right) \simeq \prod_{\kappa} \mathbf{Z}_p$, $\kappa > 0$. Put $\widetilde{F} = F\widetilde{K}$ for $F \in Q(G_p)$.

C2. There exists a valuation $v : A_K \to \mathbf{Z}$, such that $v(A_K) = \mathbf{Z}$, $v(N_{F/K}A_F) = |F \cap \widetilde{K} : K|\mathbf{Z}$, $N_{L/F}U_L = U_F$, where $F, L \in Q(G_p)$, $L \subset \widetilde{F}$ and

$$U_F = \{\alpha \in A_F : v(N_{F/K}\alpha) = 0\}.$$

Put $v_F = \frac{1}{|F \cap \widetilde{K} : K|} v \circ N_{F/K}$, then $v_F(A_F) = \mathbf{Z}$. An element $\pi_F \in A_F$ is called prime if $v_F(\pi_F) = 1$. The set U_F is called the group of units. Put

$$U_{1,F} = 1 + \pi_F \mathcal{O}_F, \ \mathcal{O}_F = \{\alpha \in A_F : v_F(\alpha) \geq 0\}.$$

Put $A_{\widetilde{F}} = \varinjlim A_{F_j}$, where F_j runs all finite extensions of F in $\widetilde{F}$ and the injective limit is taken with respect to $i_{F_j/F_{j'}}$. Then one can define the valuation $v_{\widetilde{F}} : A_{\widetilde{F}} \to \mathbf{Z}$ as the natural extension of the valuations $v_{F_j} : A_{F_j} \to \mathbf{Z}$.

A finite extension L/F, where $F, L \in Q(G_p)$, is called totally ramified if $L \cap \widetilde{F} = F$.

Now we assume that the following property holds also:

C3. Let L/F be a cyclic totally ramified extension, σ be a generator of $\mathrm{Gal}(L/F)$. Then the sequence

$$A_{\widetilde{L}} \xrightarrow{\sigma - 1} A_{\widetilde{L}} \xrightarrow{N_{\widetilde{L}/\widetilde{F}}} A_{\widetilde{F}} \to 0$$

is exact, where σ and $N_{\widetilde{L}/\widetilde{F}}$ are induced by $\sigma : A_{L_j} \to A_{L_j}$, $N_{L_j/F_j} : A_{L_j} \to A_{F_j}$, $L_j = LF_j$ and F_j runs all finite subextensions in $\widetilde{F}/F$; and if $\sigma(\alpha) = \alpha$ for $\alpha \in A_{\widetilde{L}}$, then $v_{\widetilde{L}}(\alpha) \in |L : F|\mathbf{Z}$.

1.3. Let L/F be a Galois totally ramified extension and $G = \mathrm{Gal}(L/F)$. Let $V(L|F)$ denote the subgroup in $U_{\widetilde{L}}$ generated by the elements $\sigma(\alpha) - \alpha$ with $\sigma \in G$, $\alpha \in U_{\widetilde{L}}$. For $\sigma \in G$ out

$$i(\sigma) = \sigma\left(\pi_{\widetilde{L}}\right) - \pi_{\widetilde{L}} \bmod V(L|F),$$

where $\pi_{\widetilde{L}}$ is a prime element in $\widetilde{L}$ (e.g., $\pi_{\widetilde{L}} = i_{L/\widetilde{L}}\pi_L$). In fact, i induces the homomorphism

$$i : \mathrm{Gal}(L/F)^{ab} \to U_{\widetilde{L}}/V(L|F),$$

since $\sigma\left(\pi_{\widetilde{L}}\right) - \pi_{\widetilde{L}} \in U_{\widetilde{L}}$.

Proposition. *The sequence*

$$1 \to \mathrm{Gal}(L/F)^{ab} \to U_{\widetilde{L}}/V(L/F) \overset{N_{\widetilde{L}/\widetilde{F}}}{\to} U_{\widetilde{F}} \to 0$$

is exact.

Proof: It can be carried out similarly to the proof of Theorem 2 in Chap. 2 of [I].

First assume that L/F is cyclic of degree p^n. Let σ be a generator of G. If $i(\sigma^m) \in V(L/F)$ for some m, then $\sigma\left(m\pi_{\widetilde{L}}\right) - m\pi_{\widetilde{L}} = \sigma(\varepsilon) - \varepsilon$ for a suitable $\varepsilon \in U_{\widetilde{L}}$. Then $\sigma\left(m\pi_{\widetilde{L}} - \varepsilon\right) = m\pi_{\widetilde{L}} - \varepsilon$. By the second condition of C3 we deduce that $p^n | m$ and i is injective. Further, let $N_{\widetilde{L}/\widetilde{F}}\alpha = 0$ for $\alpha \in U_{\widetilde{L}}$. Then there is an element $\beta \in A_{\widetilde{L}}$ such that $\alpha = (\sigma - 1)\beta$ because of the first condition of C3. Then we can write $\beta = a\pi_{\widetilde{L}} + \varepsilon$ with $\varepsilon \in U_{\widetilde{L}}$. Therefore, $\alpha \equiv i(\sigma^a) \bmod V(L/F)$.

Now let L/F be an arbitrary totally ramified Galois extension. Let M/F a subextension in L/F. As, by C3 $N_{\widetilde{L}/\widetilde{M}}U_{\widetilde{L}} = U_{\widetilde{M}}$, we get $N_{\widetilde{L}/\widetilde{M}}(V(L/F)) = V(M/F)$. Argue by induction on $|L : F|$, we can show the exactness of the sequence of C3 in the term $U_{\widetilde{L}}/V(L/F)$. The injectivity of i follows as well.

□

1.4. For a finite p-group G (of order of a power of p) let

$$G^* = \mathrm{Hom}_{\mathbf{Z}_p}\left(\mathrm{Gal}\left(\widetilde{F}/F\right), G\right)$$

denote the group of continuous homomorphisms of $\mathbf{Z}_p$-module $\mathrm{Gal}\left(\widetilde{F}/F\right)$ $(a \cdot \sigma = \sigma^a,\ a \in \mathbf{Z}_p)$ to the discrete $\mathbf{Z}_p$-module G. This group is isomorphic (non-canonically) with $\oplus_\kappa G$, where $x > 0$ was defined in C1.

Let L/F be a Galois totally ramified extension. Now we introduce the map $\Psi_{L/F} : U_F \to \left(\mathrm{Gal}(L/F)^{ab}\right)^*$. Let $\varepsilon \in U_F$. According to C3 there exists an element $\eta \in U_{\widetilde{L}}$, such that $N_{\widetilde{L}/\widetilde{F}}\eta = \varepsilon$. Let ψ be an extension of $\varphi \in \mathrm{Gal}\left(\widetilde{F}/F\right)$ on $\widetilde{L}$. As $N_{\widetilde{L}/\widetilde{F}}((\psi - 1)\eta) = 0$, we deduce from Proposition (1.3) that

$$(\psi - 1)\eta \equiv (1 - \sigma)\pi_{\widetilde{L}} \bmod V(L/F)$$

with a suitable $\sigma \in \mathrm{Gal}(L/F)^{ab}$, where $\pi_{\widetilde{L}}$ is a prime element in $A_{\widetilde{L}}$. It is easy to verify that σ doesn't depend on the choice of ψ and η. Put $\chi(\varphi) = \sigma$. One can immediately obtain that $\chi(\varphi_1\varphi_2) = \sigma_1\sigma_2$, i.e. $\chi \in \left(\mathrm{Gal}\,(L/F)^{ab}\right)^*$. Put $\Psi_{L/F}(\varepsilon) = \chi$.

Lemma. *The map* $\Psi_{L/F} : U_F/N_{L/F}U_L \to \left(\mathrm{Gal}(L/F)^{ab}\right)^*$ *is well defined and a homomorphism.*

Proof: If $\varepsilon = \varepsilon_1\varepsilon_2$, then one may assume $\eta = \eta_1\eta_2$. Therefore, $\sigma = \sigma_1\sigma_2$ and $\Psi_{L/F}(\varepsilon_1\varepsilon_2) = \Psi_{L/F}(\varepsilon_1)\Psi_{L/F}(\varepsilon_2)$.

□

1.5. Now we introduce the map $\Upsilon_{L/F}$, which is the inverse map to $\Psi_{L/F}$ as it will be shown later. Let L/F a Galois totally ramified extension. Let $\chi \in \mathrm{Gal}(L/F)^*$, and Σ_χ be the fixed field of all $\chi(\varphi)\varphi$, where φ runs $\mathrm{Gal}\left(\widetilde{L}/L\right)$. Put

$$\Upsilon_{L/F}(\chi) = N_{\Sigma_\chi/F}\pi_\chi - N_{L/F}\pi_L \bmod N_{L/F}U_L,$$

where π_χ and π_L are prime elements in A_{Σ_χ} and A_L.

Lemma. The map

$$\Upsilon_{L/F} : \mathrm{Gal}(L/F)^* \to U_F/N_{L/F}U_L$$

is well defined.

Proof: C.f. the proof of Lemma (1.2) in [F5].

□

1.6. Proposition. *Let L/F be a Galois totally ramified extension. Then $\Psi_{L/F} \circ Y_{L/F}$: $\left(\mathrm{Gal}(L/F)^{ab}\right)^* \to \left(\mathrm{Gal}(L/F)^{ab}\right)^*$ is the identity map.*

Proof: Let $\chi \in \mathrm{Gal}(L/F)^*$, and let π_x, π_L be prime elements in A_{Σ_χ}, A_L. Then

$$i_{\Sigma_\chi/\widetilde{L}}\pi_\chi = i_{L/\widetilde{L}}\pi_L + \eta \text{ with } \eta \in U_{\widetilde{L}}.$$

Let $\varphi \in \mathrm{Gal}\left(\widetilde{F}/F\right)$ and $\sigma = \chi(\varphi) \in \mathrm{Gal}(L/F)$. Then

$$(1-\sigma)i_{L/\widetilde{L}}\pi_L \equiv (\varphi - 1)\eta \bmod V(L/F)$$

and $N_{\widetilde{L}/\widetilde{F}}\eta = i_{F/\widetilde{F}}(N_{\Sigma_\chi/F}\pi_\chi - N_{L/F}\pi_L)$. Therefore, $\chi = \Psi_{L/F}(Y_{L/F}\chi)$.

□

Corollary. The homomorphism $\Psi_{L/F} : U_F/N_{L/F}U_L \to \left(\mathrm{Gal}(L/F)^{ab}\right)^*$ is surjective; the map $Y_{L/F}$: $\left(\mathrm{Gal}(L/F)^{ab}\right)^* \to U_F/N_{L/F}U_L$ is injective.

1.7. We are now in a position to formulate the last property to be satisfied.

C4. Let L/F be a cyclic totally ramified extension of degree p. Then the homomorphism $\Psi_{L/F}$ is injective, i.e. if $\varepsilon \in U_F$ and $N_{\widetilde{L}/\widetilde{F}}\eta = i_{F/\widetilde{F}}\varepsilon$ for $\eta \in U_{\widetilde{L}}$ with $(\varphi - 1)\eta \in V(L/F)$ for any $\varphi \in \mathrm{Gal}\left(\widetilde{L}/F\right)$, then $\varepsilon \in N_{L/F}U_L$.

Theorem. *Let L/F be a Galois totally ramified extension. Then the map*

$$\Upsilon_{L/F} : \left(\mathrm{Gal}(L/F)^{ab}\right)^* \to U_F/N_{L/F}U_L$$

is an isomorphism, and the map

$$\Psi_{L/F} : U_F/N_{L/F}U_L \to \left(\mathrm{Gal}(L/F)^{ab}\right)^*$$

is the inverse one.

Proof: First let L/F be a cyclic of degree p. Then it follows from C4 and Proposition (1.6) that $\Psi_{L/F}$ is an isomorphism and $\Upsilon_{L/F}$ is an isomorphism as well.

Now let M/F be a Galois subextension in L/F. The following diagram is commutative:

$$\begin{array}{ccccccccc} 1 & \to & \mathrm{Gal}(L/M)^* & \to & \mathrm{Gal}(L/F)^* & \to & \mathrm{Gal}(M/F)^* & \to & 1 \\ & & \downarrow \Upsilon_{L/M} & & \downarrow \Upsilon_{L/F} & & \downarrow \Upsilon_{M/F} & & \\ & & U_M/N_{L/M}U_L & \to & U_F/N_{L/F}U_L & \to & U_F/N_{M/F}U_M & & \end{array}$$

(the proof can be carried out similarly to the proof of Theorem 5.1 in [F1]). Then $\Upsilon_{L/F}$ is surjective. Proposition (1.6) implies now that $\Psi_{L/F}$ is injective.

□

Corollary. Let M/F be the maximal abelian subextension in L/F. Then $N_{M/F}U_M = N_{L/F}U_L$.

1.8. In the same way as in the proof of Proposition (1.8) and its corollary of [F5] one can verify

Proposition. *1) Let L/F, $L'F'$ be Galois totally ramified, and F'/F, L'/L be totally ramified. Then the diagram*

$$\begin{array}{ccc} \mathrm{Gal}(L'/F')^* & \to & U_{F'}/N_{L'/F'}U_{L'} \\ \downarrow & & \downarrow N_{F'/F} \\ \mathrm{Gal}(L/F)^* & \to & U_F/N_{L/F}U_L \end{array}$$

is commutative, where the left vertical homomorphism is induced by the restriction $\mathrm{Gal}(L'/F') \to \mathrm{Gal}(L/F)$ *and the canonical isomorphism* $\mathrm{Gal}\left(\widetilde{L}'/L'\right) \simeq \mathrm{Gal}\left(\widetilde{L}/L\right)$.

2) Let L/F be a Galois totally ramified extension, and $\sigma \in G_p$. Then the diagram

$$\begin{array}{ccc} \mathrm{Gal}(L/F)^* & \to & U_F/N_{L/F}U_L \\ \sigma^* \downarrow & & \downarrow \\ \mathrm{Gal}(\sigma L/\sigma F)^* & \to & U_{\sigma F}/N_{\sigma L/\sigma F}U_{\sigma L} \end{array}$$

is commutative, where $(\sigma^*\chi)(\sigma\varphi\sigma^{-1}) = \sigma\chi(\varphi)\sigma^{-1}$.

3) Let L/F be a Galois totally ramified extension, and M/F be its subextension. Then the diagram

$$\begin{array}{ccc} \left(\mathrm{Gal}(L/F)^{ab}\right)^* & \to & U_F/N_{L/F}U_L \\ \mathrm{Ver}^* \downarrow & & \downarrow \\ \left(\mathrm{Gal}(L/M)^{ab}\right)^* & \to & U_M/N_{L/M}U_L \end{array}$$

is commutative.

Passing to the projective limit we obtain the *reciprocity map*

$$\Psi_F : U_F \to \mathrm{Hom}_{\mathbb{Z}_p}\left(\mathrm{Gal}\left(\widetilde{F}/F\right), \mathrm{Gal}\left(F_p^{ab}/\widetilde{F}\right)\right),$$

where F_p^{ab}/F is the maximal abelian subextension in F_p/F. The kernel of Ψ_F coincides with the intersection of all norm groups $N_{L/F}U_L$ for abelian totally ramified extensions L/F.

2. Complete discrete valuation fields of rank n.

In this section we treat the class of fields for which theory of section 1 can be applied later.

2.1. Let F be a field and

$$\mathfrak{v}_F : F^* \to (\mathbb{Z})^n = \underbrace{\mathbb{Z} \oplus \cdots \oplus \mathbb{Z}}_{n \text{ times}}$$

be a surjective valuation, where the additive group $(\mathbb{Z})^n$ is considered to be lexicographically ordered, i.e. $(m_1, \cdots, m_n) < \left(m_1', \cdots, m_n'\right)$ in $(\mathbb{Z})^n$ if $m_i < m_i'$ for the maximal i such that $m_i \neq m_i'$. The ring of integers of $\mathfrak{v}_F$, its maximal ideal and the group of units will be denoted as $\mathcal{O}_F$, $\mathcal{M}_F$, $\mathcal{U}_F$. Put $V_F = 1 + \mathcal{M}_F$, $U_{m_1,\cdots,m_n} = 1 + t_n^{m_n} \cdots t_1^{m_1} \mathcal{O}_F$. We will assume that the residue field $\mathcal{O}_F/\mathcal{M}_F = k$ is a perfect field of characteristic $p > 0$. The field F is said to be *complete* if it is complete with respect to the first component $v^{(n)}$ of $\mathfrak{v}_F = \left(v^{(1)}, \cdots, v^{(n)}\right)$ and the residue field $\overline{F}_{v^{(n)}}$ of F with respect to the discrete valuation $v^{(n)}$ of rank 1 is complete. The elements $t_n, \cdots, t_1$ of F such that

$$v_F(t_i) = (0, \cdots, 1, \cdots 0) ,$$

with 1 at the $(n + 1 - i)th$ position, are called *local parameters* of F.

If $\mathrm{char}(F) = p$, then it follows from the general theory that F is isomorphic and homeomorphic (with respect to the discrete valuation of rank n) to the field of formal power series $k((X_1)) \cdots ((X_n))$ (c.f. section 5 Chap. II of [FV]).

We are now going to introduce some special topology on F that takes into consideration the corresponding topology on $\overline{F}_{v^{(n)}}$. First assume that $\mathrm{char}(F) = p$. Let U_m, $m \in \mathbf{Z}$, be subgroups in $\overline{F}_{v^{(n)}}$, which are neighborhoods of zero in this topology (the topology coincides with the induced topology from the discrete valuation of rank 1 if $n = 1$). Assume that $U_m = \overline{F}_{v^{(n)}}$ for sufficiently large m. Put $U = \sum_m U_m t_n^m \cap F$, where t_n is a prime element of F with respect to $v^{(n)}$. All such subgroups U in F form a fundamental system of enighborhoods of zero in the topology of F. The so defined topology was introduced by A.N. Parshin, see [P4].

Now let $\mathrm{char}(F) = 0$ and $\mathrm{char}\left(\overline{F}_{v^{(n)}}\right) = p > 0$. According to the general theory there is a subfield F_0 in F which is a complete discrete valuation field of rank n under the induced valuation, and p is a prime element in F_0 with respect to the first component of the discrete valuation of rank n on F_0 (see section 5 Chap. II of [FV]). In this case F is a finite extension of F_0. One may assume that the field of fractions F_{00} of the Witt ring $W(k)$ is contained in F_0. Let U_m, $m \in \mathbf{Z}$, be subgroups in $\overline{F}_{v^{(n)}}$ as above. Let $\widetilde{U}_m$ be subgroups in F_0, such that the coefficients from k of elements of U_m are replaced by the ring of integers of F_{∞}. Then one can take $U = \sum_m \widetilde{U}_m t_n^m \cap F$ as a fundamental system of neighborhoods of zero in the topology of F_0. Define the topology on F as on the finite-dimensional vector space over F_0.

The multiplicative group F^* is isomorphic to the product of the cyclic subgroups $\langle t_i \rangle$ generated by t_i, where $t_n, \cdots, t_1$ are local parameters in F, the group of multiplicative representatives R^* of k^* in F, and the group V_F. If $\mathrm{char}\left(\overline{F}_{v^{(n)}}\right) = p$, then introduce the topology on F^* as the product of the topology on V_F induced from F and the discrete topology on $\langle t_n \rangle \times \cdots \times \langle t_1 \rangle \times R^*$. If $\mathrm{char}(F) = \cdots = \mathrm{char}\left(k^{(m+1)}\right) = 0$ and $\mathrm{char}\left(k^{(m)}\right) = p$, $m < n - 1$, where $k^{(n)} = F$ and $k^{(i)}$ is the residue field of $k^{(i+1)}$ with respect to the valuation of rank 1, then put $W_F = 1 + t_{m+1}\mathcal{O}_F$. The field F is isomorphic to the field $k^{(m+1)}((t_{m+1})) \cdots ((t_n))$, and $k^{(m+1)}$ is a complete discrete valuation field of rank $m + 1$ of the type considered above. We get the isomorphism

$$F^* \simeq k^{(m+1)*} \times W_F \times \langle t_{m+1} \rangle \times \cdots \times \langle t_n \rangle.$$

Introduce the topology on F^* as the product of the trivial topology on W_F, the discrete topology on $\langle t_{m+1}\rangle \times \cdots \times \langle t_n\rangle$, and the above-defined topology on $k^{(m+1)*}$. Note that the group W_F is uniquely divisible.

The so-defined topology on F^* doesn't depend on the choice of local parameters and an imbedding of the residue field into the field. The multiplication is sequentially continuous with respect to this topology.

Any element $\alpha \in V_F$ has precisely one expansion in the convergent product

$$\varepsilon = \varepsilon_1 \prod_{i_m \geq 0} \prod_{i_{m-1} \geq I_{m-1}(i_m)} \cdots \prod_{i_1 \geq I_1(i_m,\cdots,i_2)} \left(1 + \theta_{i_m,\cdots,i_1} t_m^{i_m} \cdots t_1^{i_1}\right),$$

where ε_1 is a divisible element in V_F, $\theta_{i_m,\cdots,i_1} \in R$, R is the set of multiplicative representatives of k in F, $m = n$ if $\operatorname{char}\left(k^{(n-1)}\right) = p$, and $I_{m-1}(0) \geq 0, \cdots, I_1(0,\cdots,0) > 0$.

2.2 Let $K_s(F)$ be the s th Milnor group of F. Introduce the topology on $K_m(F)$ as the strongest one such that the map $F^* \times \cdots \times F^* \to K_s(F)$ and the addition in $K_s(F)$ are sequentially continuous. Then the intersection of all neighborhoods of zero in $K_s(F)$ is a subgroup in $K_s(F)$. We will denote this subgroup as $\Lambda_s(F)$. In the same way as in section 2 of [F1] and section 5 of [F3] one can show that $\Lambda_s(F) = \bigcap_{l \geq 1} lVK_s(F)$, where $VK_s(F) = \{V_F\}K_{s-1}(F)$, in the case of $\operatorname{char}(F) = 0$. If $\operatorname{char}(F) = p$, then $\Lambda_s(F) \supset \bigcap_{l\geq 1} lVK_s(F)$. Put

$$K_s^{\text{top}}(F) = K_s(F)/\Lambda_s(F).$$

Let $UK_s^{\text{top}}(F)$, $VK_s^{\text{top}}(F)$, $U_IK_s^{\text{top}}(F)$ denote the subgroups in $K_s^{\text{top}}(F)$ generated by U_F, V_F, U_I respectively, where $I = (i_1,\cdots,i_n) \in (\mathbf{Z})^n$.

For the description of $K_s^{\text{top}}(F)$ one can apply generalizations of the pairings of section 2 [F2] and section 3 [F3].

2.3 Let $\widetilde{F}$ be the maximal abelian unramified p-extension of F with respect to the discrete valuation of rank n, i.e. $\widetilde{F} = F \otimes_{W(k)} W(k_p^{ab})$. Put

$$\kappa = \dim_{\mathbf{F}_p} k/wp(k),$$

where $p(x) = x^p - x$. We will assume further that $\kappa > 0$. The case $\kappa = 0$ requires special considerations taking into account the pro-quasi-algebraic structure of $VK_n^{\text{top}}(F)$ as a generalization of Serre's theory in the case of $n = 1$ (see [S]). The Witt theory implies that there is a non-canonical isomorphism $\operatorname{Gal}\left(\widetilde{F}/F\right) \simeq \Pi_x \mathbf{Z}_p$.

2.4. The first pairing we will employ now in the case of $\operatorname{char}(F) = p$ is the Artin-Schreier-Witt pairing. Let $\alpha_1,\cdots,\alpha_n \in F^*$, and let $(\beta_0,\cdots,\beta_s) \in W_s(F)$ be the Witt vector. Let $wp : W_s(F) \to W_s(F)$ be the operator defined as $wp(\beta_0,\cdots,\beta_s) = (\beta_0^p,\cdots,\beta_s^p) - (\beta_0,\cdots,\beta_s)$. For $\varphi \in \operatorname{Gal}\left(\widetilde{F}/F\right)$ put

$$(\alpha_1,\cdots,\alpha_n,(\beta_0,\cdots,\beta_s)]_s(\varphi) = \varphi(\gamma_0,\cdots,\gamma_s) - (\gamma_0,\cdots,\gamma_s),$$

where $wp(\gamma_0,\cdots,\gamma_s) = (\lambda_0,\cdots,\lambda_s)$ and the i th ghost component $\lambda^{(i)}$ of $(\lambda_0,\cdots,\lambda_s) \in W_s(F)$ is defined as $\operatorname{res}_k\left(\beta^{(i)}\alpha_1^{-1}d\alpha_1 \wedge \cdots \wedge \alpha_n^{-1}d\alpha_n\right)$. Then one can show similarly to section 2 of [F2] and (1.11) of [F5] that $(\cdot,\cdot]_s$ defines the non degenerate pairing

$$(\cdot,\cdot]_s : VK_n^{\mathrm{top}}(F)/p^s \times W_s(F)/wpW_s(F) + W_s(\overline{F}) \\ \to \mathrm{Hom}_{\mathbf{Z}_p}\Big(\mathrm{Gal}\Big(\widetilde{F}/F\Big), W_s(\mathbf{F}_p)\Big).$$

Applying this pairing in the same way as in [P4] or section 2 of [F2] one can prove

Proposition. *Let F be a complete discrete valuation field of rank n, $\mathrm{char}(F) = p$. Then any element $\alpha \in VK_n^{\mathrm{top}}(F)$ is uniquely expanded in the convergent series $\Sigma c_I x_I$ with $c_I \in \mathbf{Z}_p$,*

$$x_I = \Big\{1 + \theta t_n^{i_n} \cdots t_1^{i_1},\ t_{j_1}, \cdots, t_{j_{n-1}}\Big\},$$

where θ belongs to the fixed basis of k over $\mathbf{F}_p$, $p \nmid I = (i_1, \cdots, i_n) > 0$, the set $\{j_1, \cdots, j_{n-1}, j\}$ coincides with the set $\{1, \cdots, n\}$, where j is the minimal integer such that $p \nmid i_j$.

2.5 The second pairing is a generalization of the pairing introduced by S.V. Vostokov in the case of a finite k (see [V] and Appendix B of [FV]). Assume that $\mathrm{char}(F) = 0$, $\mathrm{char}\Big(k^{(n-1)}\Big) = p$, and a primitve p^r th root of unity ζ is contained in F. Let $\alpha = t_n^{a_n} \cdots t_1^{a_1}\theta\Big(1 + \Sigma\theta_{i_n,\cdots,i_1} t_n^{i_n} \cdots t_1^{i_1}\Big)$ be an element of F^*, where $\theta \in R^*$, $\theta_{i_n,\cdots,i_1}$ belongs to the ring of integers o of the field F_{00} (see (2.1)). Put $\alpha(X) = X_n^{a_n} \cdots X_1^{a_1}\theta\Big(1 + \Sigma\theta_{i_n,\cdots,i_1} X_n^{i_n} \cdots X_1^{i_1}\Big)$. Let $z(X) = \zeta(X)$, $s(X) = z(X)^{p^r} - 1$. Let the operator Δ act on elements of o as the Frobenius automorphism Fr and on X_i as raising to the pth power. For $\alpha \in F^*$ put

$$l(\alpha) = \frac{1}{p}\log \alpha(X)^{p-\Delta},\ \delta_i(\alpha) = \alpha^{-1}\frac{\partial \alpha}{\partial X_i},\ \eta_i(\alpha) = \delta_i(\alpha) - \frac{\partial l(\alpha)}{\partial X_i}.$$

For $\alpha_1, \cdots, \alpha_{n+1} \in F^*$ put

$$\Phi(\alpha_1, \cdots, \alpha_{n+1}) = l(\alpha_{n+1})D_{n+1} - l(\alpha_n)D_n + \cdots + (-1)^n l(\alpha_1)D_1,$$

where D_i is the discriminant of the matrix

$$\begin{pmatrix} \delta_1(\alpha_1) & \ldots & \delta_n(\alpha_1) \\ \delta_1(\alpha_{i-1}) & \ldots & \delta_n(\alpha_{i-1}) \\ \eta_1(\alpha_{i+1}) & \ldots & \eta_n(\alpha_{i+1}) \\ \eta_1(\alpha_{n+1}) & \ldots & \eta_n(\alpha_{n+1}) \end{pmatrix}$$

Let μ denote the cyclic group generated by ζ. Define the map $\Gamma_r : (F^*)^{n+1} \to \mathrm{Hom}_{\mathbf{Z}_p}\Big(\mathrm{Gal}\Big(\widetilde{F}/F\Big), \mu\Big)$ as

$$\Gamma_r(\alpha_1, \cdots, \alpha_{n+1})(\varphi) = \zeta^{\gamma},$$

where $\gamma = (\varphi - 1)\delta$ and $Fr(\delta) - \delta = \mathrm{res}_X \Phi(\alpha_1, \cdots, \alpha_{n+1})/s(X)$.

Then one can show similarly to section 3 of [F1] that Γ_r induces the non degenerate pairing (for $p > 2$)

$$\Gamma_r : K_n^{\mathrm{top}}(F)/p^r \times F^*/F^{*p^r} \to \mathrm{Hom}_{\mathbf{Z}_p}\Big(\mathrm{Gal}\Big(\widetilde{F}/F\Big), \mu\Big).$$

Applying this pairing in the same way as in section 3 of [F1] (for $r = 1$), one can prove

Proposition. *Let F be a complete discrete valuation field of rank n, $\mathrm{char}(F) = 0$, $\mathrm{char}\left(k^{(n-1)}\right) = p$. Let $p = \theta t_n^{en} \cdots t_1^{e_1} + \cdots$ with $\theta \in R^*$. Then any element $\alpha \in VK_n^{\mathrm{top}}(F)/p$ is uniquely expanded into the convergent series $\Sigma c_I x_I$ with $c_I \in \mathbf{Z}/p$, and*

$$x_I = \left\{1 + \theta t_n^{i_n} \cdots t_1^{i_1}, t_{j_1}, \cdots, t_{j_{n-1}}\right\},$$

if $0 < I < p(e_1, \cdots, e_n)/(p-1)$, $p \nmid I$, $\theta \in R$, $j_1 < \cdots < j_{n-1}$, set $\{j_1, \cdots, j_{n-1}, j\}$ coincides with the set $\{1, \cdots, n\}$, where j is the minimal index such that i_j is not divisible by p, and

$$x_I = \left\{w_*, t_{k_1}, \cdots, t_{k_{n-1}}\right\},$$

where $w_* = 1$, $c_I = 0$ when a primitive p th root of unity doesn't belong to F; $w_* \in V_F$ such that $F\left(\sqrt[p]{w_*}\right)/F$ are non-trivial subextensions in $\widetilde{F}/F$, and $k_1 < \cdots < k_{n-1}$, $\{k_1, \cdots, k_{n-1}\}$ is a subset in $\{1, \cdots, n\}$, if a primitive p th root of unity belongs to F.

In the case of $\mathrm{char}\left(k^{(m+1)}\right) = 0$, $\mathrm{char}\left(k^{(m)}\right) = p$ it is easy to deduce similar assertions in the way as in section 5 of [F3].

3. Multidimensional local p-class field theory

Let F be a complete discrete valuation field of rank n with the residue field k. Assume that k is a perfect field of characteristic p and $\kappa = \dim_{\mathbf{F}_p} k/\wp(k) > 0$. In this section we will show that F and $A_F = K_n^{\mathrm{top}}(F)$ satisfy C1–C4 of § 4 and therefore, obtain class field theory for F. This theory may be regarded as a generalization of the known results in the case of a finite k ([P1–5], [K1–3], [F1–4]).

3.1 It is well-known that $A_F = K_n^{\mathrm{top}}(F)$ is a G_p-modulation. C1 is satisfied with $\widetilde{K}$ defined in (2.3). For the valuation $v_F : A_F \to \mathbf{Z}$ one can take the composition

$$K_n^{\mathrm{top}}(F) \xrightarrow{\partial} K_{n-1}^{\mathrm{top}}(F) \to \cdots \to K_0(k) \xrightarrow{\sim} \mathbf{Z},$$

where ∂ is the well-known homomorphism in K-theory, c.f. section 2 Chap. IX of [FV]. Then U_F of section 1 coincides with $UK_n^{\mathrm{top}}(F)$ of section 2. A prime element π_F of A_F can be written as $\{t_1, \cdots, t_n\} + \varepsilon$ with a suitable $\varepsilon \in UK_n^{\mathrm{top}}(F)$, where $t_n, \cdots, t_1$ are local parameters of F. The norm map N_{L_F} maps $UK_n^{\mathrm{top}}(L)$ onto $UK_n^{\mathrm{top}}(F)$ as immediately follows.

3.2. In order to verify the third condition C3 we need the following description of the norm map (analogously to Proposition 4.1 of [F1] and Proposition 3.1 of [F3]):

Proposition. *Let L/F be a cyclic totally ramified extension of degree p, σ be a generator of $\mathrm{Gal}(L/F)$. Let $L = F\left(t_{s,L}\right)$ for some s, $1 \le s \le n$. Take local parameters $t_n, \cdots, t_{s,F} = N_{L/F} t_{s,L}, \cdots, t_1$ in F and $t_n, \cdots, t_{s,L}, \cdots, t_1$ in L, and assume that*

$$\frac{\sigma t_{s,L}}{t_{s,L}} \equiv 1 + \theta_0 t_n^{r_n} \cdots t_{s,L}^{r_s} \cdots t_1^{r_1} \bmod U_{r_1+1,\cdots,r_n}$$

with $\theta_0 \in R^*$. *Let* $\widehat{U}_{i_1,\cdots,i_n} = U_{i_1,\cdots,i_n}/U_{i_1+1,\cdots i_n}$.
This group is isomorphic to $k : 1 + \theta t_n^{i_n} \cdots t_1^{i_1} \to \overline{\theta} \in k$. *Then*

1) if $(i_1, \cdots, i_n) < (r_1, \cdots, r_n)$, *then the diagram*

$$\begin{array}{ccc} \widehat{U}_{i_1,\cdots,i_n,L} & \to & k \quad \overline{\theta} \\ N_{L/F} \downarrow & & \downarrow \quad \downarrow \\ \widehat{U}_{pi_1,\cdots,i_s,\cdots,pi_n,F} & \to & k \quad \overline{\theta}^p \end{array}$$

is commutative;

2) if $(i_1, \cdots, i_n) = (r_1, \cdots, r_n)$, *then the diagram*

$$\begin{array}{ccc} \widehat{U}_{r_1,\cdots,r_n,L} & \to & k \qquad \overline{\theta} \\ N_{L/F} \downarrow & & \downarrow \qquad \downarrow \\ \widehat{U}_{pr_1,\cdots,r_s,\cdots,pr_n,F} & \to & k \quad \overline{\theta}^p - \overline{\theta}\,\overline{\theta}_0^{p-1} \end{array}$$

is commutative;

3) if $(i_1, \cdots, i_n) > 0$, *then the diagram*

$$\begin{array}{ccc} \widehat{U}_{r_1+i_2,\cdots,r_s+pi_s,\cdots,r_n+i_n,L} & \to & k \qquad \overline{\theta} \\ N_{L/F} \downarrow & & \downarrow \qquad \downarrow \\ \widehat{U}_{pr_1+i_1,\cdots,r_s+i_s,\cdots,pr_n+i_n,F} & \to & k \quad -\overline{\theta}\,\overline{\theta}_0^{p-1} \end{array}$$

3.2. We need also the following assertion which is proved similarly to Theorem 4.2 of [F1] and Theorem 3.2 of [F3] (using the bijectivity of the norm residue symbol):

Proposition. *Let* L/F *be as just above. Then the sequence*

$$K_n^{\text{top}}(L) \overset{\sigma-1}{\to} K_n^{\text{top}}(L) \overset{N_{L/F}}{\to} K_n^{\text{top}}(F)$$

is exact.

Now the sequence of C3 is exact for a cyclic totally ramified extension L/F of degree p, since the surjectivity of the norm map $N_{\widetilde{L}/\widetilde{F}} : A_{\widetilde{L}} \to A_{\widetilde{F}}$ follows from Proposition (3.2). The exactness of the sequence of C3 in the general case of a cyclic totally ramified p-extension is proved now by induction.

Let $\widetilde{L}/\widetilde{F}$ be a cyclic totally ramified extension of degree p. Let $\sigma\alpha = \alpha$ for $\alpha \in K_n^{\text{top}}\left(\widetilde{L}\right)$. Let $(r_1, \cdots, r_n)$ be as in Proposition (3.2). If $p \nmid r_s$, then it immediately follows from Proposition (2.4) and Proposition (2.5) that $p | v_{\widetilde{L}}(\alpha)$. If $p | r_s$, then $\text{char}(F) = 0$ and a primitive pth root of unity ζ_p belongs to F. Let ζ be a primitive root of the maximal index p^r, $r \geq 1$, which is contained in $\widetilde{L}$. In the case under consideration $\widetilde{L} = \widetilde{F}(\sqrt[p]{t_s})$ for a suitable local parameter t_s. One may take $t_n, \cdots, t_s, \cdots, t_1$ as local parameters in $\widetilde{F}$, such that $t_n, \cdots, \sqrt[p]{t_s}, \cdots, t_1$ are local parameters in $\widetilde{L}$ and $\sigma(\sqrt[p]{t_s})/\sqrt[p]{t_s} = \zeta_p$. We get $\alpha = a\{t_1, \cdots, \sqrt[p]{t_s}, \cdots t_n\} + \varepsilon$ with $a \in \mathbf{Z}$, $\varepsilon \in UK_n^{\text{top}}\left(\widetilde{L}\right)$. As $\zeta_p \in \widetilde{F}^*$ and $\zeta_p \notin \widetilde{L}^{*p^r}$, there exists an element $\beta \in \widetilde{L}$, such that

$$\Gamma_r(\{t_1, \cdots, \zeta_p, \cdots, t_n\}, \beta) \neq 1.$$

Let $\psi \in \text{Gal}\left(\widetilde{L}/L\right)$ be such that $\psi(\beta) = \beta \bmod \widetilde{L}^{*p^r}$. Put $(\sigma - 1)\{t_1, \cdots, \sqrt[p]{t_s}, \cdots, t_n\} = (\psi - 1)\gamma$ with $\gamma \in V(L/F)$. Then, if $p \nmid a$, we obtain that

$$\Gamma_r(\{t_1, \cdots, \zeta_p, \cdots, t_n\}, \beta) = \Gamma_r\left(\gamma, \beta^{\gamma^{-1}}\beta^{-1}\right) = 1,$$

a contradiction. Thus, $v_{\widetilde{L}}(\alpha)$ is divisible by p.

Now let $\widetilde{L}/\widetilde{F}$ be a cyclic totally ramified extension of degree p^m and $(\sigma-1)\alpha=0$ for a generator $\sigma\in\operatorname{Gal}\left(\widetilde{L}/\widetilde{F}\right)$, $\alpha\in K_n^{\mathrm{top}}\left(\widetilde{L}\right)$. Then, by the inductional assumption, $\alpha=ap^{m-1}\pi_{\widetilde{L}}+\varepsilon$, where $\pi_{\widetilde{L}}$ is a prime element in $K_n^{\mathrm{top}}\left(\widetilde{L}\right)$, $\varepsilon\in UK_n^{\mathrm{top}}\left(\widetilde{L}\right)$. Then

$$a\left(\tau\pi_{\widetilde{L}}-\pi_{\widetilde{L}}\right)\equiv 0 \bmod V(L|F),$$

where $\tau=\sigma^{p^{m-1}}$. Let M/F be the subextension in L/F of degree p^{m-1}. As $V(L/F)=(\psi-1)V(L/F)$ for any $\psi\in\operatorname{Gal}\left(\widetilde{L}/L\right)$, we obtain that $a\left(\tau\pi_{\widetilde{L}}-\pi_{\widetilde{L}}\right)$ is $(\psi-1)$-divisible in $VK_n^{\mathrm{top}}\left(\widetilde{L}\right)$. Now by the same reasons as above, we deduce that $p|a$, $p^m|v_{\widetilde{L}}(\alpha)$. Thus, C3 also holds.

3.4. Note that the quotient group $UK_n^{\mathrm{top}}(F)/VK_n^{\mathrm{top}}(F)$ is isomorphic to k, and hence, is p-divisible. Thus, we get the maps (according to section 1)

$$\Psi_{L/F}: VK_n^{\mathrm{top}}(F)/N_{L/F}VK_n^{\mathrm{top}}(L)\to\left(\operatorname{Gal}(L/F)^{\mathrm{ab}}\right)^*,$$
$$\Upsilon_{L/F}:\left(\operatorname{Gal}(L/F)^{ab}\right)^*\to VK_n^{\mathrm{top}}(F)/N_{L/F}VK_n^{\mathrm{top}}(L)$$

for a Galois totally ramified p-extension L/F.

3.5. It remains to varify C4. Assume that L/F is a cyclic totally ramified extension of degree p. Let $(r_1,\cdots,r_n)$ be as in (3.2). By employing the commutative diagrams of Proposition (3.2) it suffices to show that if

$$\varepsilon\equiv\left\{1+\theta t_n^{pr_n}\cdots t_{s,F}^{r_s}\cdots t_1^{pr_1},t_{j_1},\cdots,t_{j_{n-1}}\right\} \bmod N_{L/F}VK_n^{\mathrm{top}}(L)$$

with $\overline{\theta}\notin\overline{\theta}_0^p\wp(k)$, then $\Psi_{L/F}(\varepsilon)\neq 1$. In terms of C4 we obtain that

$$\eta=\left\{1+\theta' t_n^{r_n}\cdots t_{s,L}^{r_s}\cdots t_1^{r_1},t_{j_1},\cdots,t_{j_{n-1}}\right\}+\cdots,$$

where $\overline{\theta}'^{p}-\overline{\theta}_1^{p-1}\overline{\theta}'=\overline{\theta}$. Let L_1/L be a subextension in $\widetilde{L}/L$ of degree p, such that $\overline{\theta}'\in\overline{L}_1$. Let $\psi\in\operatorname{Gal}\left(\widetilde{L}/L\right)$ be such that $\psi|_{L_1}$ is a generator of $\operatorname{Gal}(L_1/L)$. Now, if $\Psi_{L/F}(\varepsilon)=1$, then $\left\{1+\theta_0 t_n^{r_n},\cdots,t_{s,L}^{r_s},\cdots,t_1^{r_1},\ t_{j_1},\cdots,t_{j_{n-1}}\right\}$ belongs to $(\varphi-1)VK_n^{\mathrm{top}}\left(\widetilde{L}\right)$ for any $\varphi\in\operatorname{Gal}\left(\widetilde{L}/L\right)$. Then, by the same arguments as in (3.3), one obtains a contradiction. Therefore, $\Psi_{L/F}$ is injective and C4 is true.

3.6. According to Theorem (1.7) we obtain

Theorem. *Let F be a complete discrete valution field of rank n with a perfect residue field k of characteristic $p>0$, $\kappa=\dim_{\mathbb{F}_p}k/\wp(k)>0$. Then for a Galois totally ramified extension L/F, $L\subset F_p$, the map*

$$\Upsilon_{L/F}:\left(\operatorname{Gal}(L/F)^{ab}\right)^*\to VK_n^{\mathrm{top}}(F)/N_{L/F}VK_n^{\mathrm{top}}(L)$$

is an isomorphism and the map

$$\Psi_{L/F}: VK_n^{\mathrm{top}}(F)/N_{L/F}K_n^{\mathrm{top}}(L)\to\left(\operatorname{Gal}(L/F)^{ab}\right)^*$$

is the inverse one. The map $\Psi_{L/F}$ *determines the reciprocity map*

$$\Psi_F : VK_n^{\text{top}}(F) \to \text{Gal}\left(F_p^{ab}/\widetilde{F}\right),$$

possessing natural functorial properties of section 1.

Remark. One can show that if $\text{char}(F) = p$, then for $\alpha \in VK_n^{\text{top}}(F)$, $\beta \in W_s(F)$

$$(\alpha, \beta]_s(\varphi) = \Psi_F(\alpha)(\varphi)(\gamma) - \gamma,$$

where $\varphi \in \text{Gal}\left(\widetilde{F}/F\right)$, and γ is the root of the polynomial $wp(X) - \beta$. If $\text{char}(F) = 0$, then for $\alpha \in VK_n^{\text{top}}(F)$, $\beta \in F^*$

$$\Gamma_r(\alpha, \beta)(\varphi) = \gamma^{\Psi_F(\alpha)(\varphi)-1},$$

where $\varphi \in \text{Gal}\left(\widetilde{F}/F\right)$ and $\gamma^{p^r} = \beta$.

4. Ramification theory.

Let F be a complete discrete valuation field of rank n as in section 3.

4.1. We get the filtration

$$U_{1,0,\cdots,0}K_n^{\text{top}}(F) \supset U_{2,0,\cdots,0}K_n^{\text{top}}(F) \supset \cdots$$

on $VK_n^{\text{top}}(F)$. Using the pairings of (2.4) and (2.5) in the context of Proposition (2.4) and Proposition (2.5), one can show that if $\text{char}(F) = p$, then for $I > 0$

$$U_I K_n^{\text{top}}(F)/U_{I+1}K_n^{\text{top}}(F) \simeq k,$$

where $1 = (1, 0, \cdots 0)$. If $\text{char}(F) = 0$, then for $I > 0$ and $E = (e_1, \cdots, e_n)$ as in (2.5) we get that

$$U_I K_n^{\text{top}}(F) + pK_n^{\text{top}}(F)/U_{I+1}K_n^{\text{top}}(F) + pK_n^{\text{top}}(F)$$

is 0 if $p|I$, $I < pE/(p-1)$ or $I > pE/(p-1)$ or $I = pE/(p-1) \in (\mathbb{Z})^n$ and a primitive p root of unity doesn't belong to F; is isomorphic to k if $p \nmid I$, $I < pE/(p-1)$; is isomorphic to $(k/wp(k))^n$ if $I = pE/(p-1)$ and a primitive p th root of unity belongs to F.

Now let L/F be a Galois totally ramified extension, $L \subset F_p$. The norm map $N_{L/F} : VK_n^{\text{top}}(L) \to VK_n^{\text{top}}(F)$ can be described on the base of Proposition (3.2). However, the behavior of $N_{L/F}$ is more complicated then in the case of $n = 1$. For instance, let L/F be of degree $p, n = 2$. Let $L = F(t_2, L)$. We take $t_{2,F} = N_{L/F}t_{2,L}, t_1$ as local parameters of F and $t_{s,L}, t_1$ as local parameters of L. Then $N_{L/F}\{1 + t_{2,L}t_1, t_1\} = \{1 + t_{2,F}t_1^p, t_1\}$ and $N_{L/F}\{1 + t_{2,F}t_1, t_{2,L}\} = \{1 + t_{2,F}t_1, t_{2,F}\}$, but $v_L(t_{2,L}t_1) < v_L(t_{2,F}t_1)$, $v_F(t_{2,F}t_1^p) > v_F(t_{2,F}t_1)$.

4.2. Applying the construction of the reciprocity map of sections 1 and 3,one can describe the image of $U_I K_n^{\text{top}}(F)$ in $\left(\text{Gal}(L/F)^{ab}\right)^*$. Then

$$\Psi_{L/F}(U_I K_n^{\text{top}}(F)) =$$

$$\left\{\chi \in \left(\text{Gal}(L/F)^{ab}\right)^* : \chi(\varphi)\pi_L - \pi_L \in (\varphi - 1)N^{-1}_{\widetilde{L}/\widetilde{F}} i_{F/\widetilde{F}} U_I K_n^{\text{top}}(F) \text{mod } V(L/F)\right\}$$

where φ runs over $\mathrm{Gal}\left(\widetilde{F}/F\right)$ and π_L is a prime element in $K_n^{\mathrm{top}}(L)$.

In the case of $n=1$ it is well-known that $(\varphi-1)N^{-1}_{\widetilde{L}/\widetilde{F}}U_{i,F} \bmod V(L/F)$ can be replaced by $U_{h(i),L} \bmod V(L/F)$ and $\Psi_{L/F}(U_{i,F})$ is equal to the ramification group $\mathrm{Gal}(L/F)^{ab}_{h(i)}$, where $h = h_{L/F}$ is the Hasse-Herbrand function defined as the maximal integer j such that (in the case of an infinite residue field)

$$N_{L/F}U_{j,L} \subset U_{i,F}, \not\subset U_{i+1,F} \quad N_{L/F}U_{j+1,L} \subset U_{i+1,F}$$

(cf. section 3 Chap. III of [FV]). It is not true in the general case of $n>1$.

4.3. For $\alpha \in VK_n^{\mathrm{top}}(L)$ put

$$w_L(\alpha) = \min\left\{I : x \in U_I K_n^{\mathrm{top}}(L)\right\}.$$

Let L/F be a Galois totally ramified extension, $L \subset F_p$. We will assume that the surjective discrete valuation of rank n on F is induced by the surjective discrete valuation on L. Let Z be the subset of indices $I \in (\mathbf{Z})^n$, $I > 0$, such that

$$U_I K_n^{\mathrm{top}}\left(\widetilde{L}\right) \subset V(L|F) + U_{I+1}K_n^{\mathrm{top}}\left(\widetilde{L}\right).$$

Define the Hasse-Herbrand function (which doesn't coincide with the classical one in the case of $n = 1$)

$$h = h_{L/F} : (\mathbf{Z})^n_+ \to (\mathbf{Z})^n_+,$$

where $(\mathbf{Z})^n_+$ is the subset of indices $I>0$ in $(\mathbf{Z})^n$, as

$$h(I) = \min\{w_L(\alpha) \notin Z : \alpha \in (\varphi-1)N^{-1}_{\widetilde{L}/\widetilde{F}} i_{F/\widetilde{F}} U_I K_n^{\mathrm{top}}(F)$$
$$\text{for some } \varphi \in \mathrm{Gal}\left(\widetilde{F}/F\right)$$

if its minimum exists, and $h(I) = +\infty$ otherwise. The so-defined function doesn't depend on the choice of the discrete valuations on L and F. The equality $h_{L/F} = h_{L/M} \circ h_{M/F}$ is not true in the general case. However,

$$h_{M/F} = n_{L/M} \circ h_{L/F}, \tag{*}$$

where $n_{L/M} : (\mathbf{Z})^n_+ \to (\mathbf{Z})^n_+$ is the function connected with the norm map $N_{L/M}$ and defined as

$$n_{L/M}(I) = \min\left\{w_M\left(N_{L/M}\alpha\right) : w_L(\alpha) = I\right\}.$$

This is the consequence of the relations $V(M/F) = N_{\widetilde{L}/\widetilde{M}}V(L/F)$ and

$$(\varphi-1)N^{-1}_{\widetilde{M}/\widetilde{F}} i_{F/\widetilde{F}} U_I K_n^{\mathrm{top}}(F) = N_{\widetilde{L}/\widetilde{M}}\left((\varphi-1)N^{-1}_{\widetilde{L}/\widetilde{F}} i_{F/\widetilde{F}} U_I K_n^{\mathrm{top}}(F)\right).$$

4.4. Put for $J \in (\mathbf{Z})^n_+$

$$\mathrm{Gal}(L/F)_J = \left\{\sigma \in \mathrm{Gal}(L/F) : \sigma\pi_L - \pi_L \in U_J K_n^{\mathrm{top}}(L)\right\}.$$

Then we deduce from (4.2) that $\Psi_{L/F}\left(U_I K_n^{\mathrm{top}}(F)\right) = \left(\mathrm{Gal}(L/F)^{\mathrm{ab}}_{n_{L/F}}(I)\right)^*$. As it follows from class field theory of section 3, if L/F abelian and $\mathrm{Gal}(L/F)_J \neq \mathrm{Gal}(L/F)_{J+1}$, then $J = h_{L/F}(I)$ for some $I \in (\mathbb{Z})^n_+$. This assertion may be treated as a direct generalization of the Hasse-Arf theorem.

4.5. Let L/F be a finite Galois totally ramified extension, $L \subset F_p$, and M/F be a Galois subextension in L/F. Put $G = \mathrm{Gal}(L/F)$, $H = \mathrm{Gal}(L/M)$. Then the formula (*) implies

$$(G/H)_{h_{M/F}(I)} = G_{h_{L/F}(I)}H/H.$$

This equality is an analog of the Herbrand theorem.

4.6. Finally we note that the Galois group of a totally ramified p-extension L of a complete disrete valuation field F of rank 1 with an arbitrary residue field of characteristic p is the Galois group of a totally ramified extension L' of a complete discrete valuation field F' of rank n with the perfect residue field for a suitable n, F'. Thus, the properties of $\mathrm{Gal}(L/F)$ from the standpoint of ramification theory can be reduced to the properties of $\mathrm{Gal}(L'/F')$ and studied by class field theory exposed above.

5. Existence theorem

Let L be as in section 3.

5.1. A polynomial $p(X)$ over k is called *k-decomposable* if it is additive, i.e. $p(a+b) = p(a)+p(b)$ for all $a, b \in k$, and all its roots belong to k (cf. section 2 of [F5]). A subgroup $\mathcal{N}$ in $VK_n^{\mathrm{top}}(F)$ is called *normic* if 1) $\mathcal{N}$ is open; 2) for any $I > 0$ there exists a polynomial $f_I(X) \in \mathcal{O}_F[X]$ such that the residue polynomial $\overline{f}_I \in k[X]$ is non zero k-decomposable and

$$\left\{1 + f_I(\mathcal{O}_F)t_1^{i_n}\right\} K_{n-1}^{\mathrm{top}}(F) \subset \mathcal{N},$$

where $t_n, \cdots, t_1$ are local parameters of F, $I = (i_1, \cdots, i_n)$; 3) for any $I > 0$ there exists a polynomial $g_I(X) \in \mathcal{O}_F[X]$ such that its residue $\overline{g}_I$ is non-zero k-decomposable,

$$\begin{aligned}&\mathcal{N} \cap U_I K_n^{\mathrm{top}}(F) + U_{I+1}K_n^{\mathrm{top}}(F)\\ &= \left\{1 + g_I(\mathcal{O}_F)t_n^{i_n}\cdots t_1^{i_1}\right\} K_{n-1}^{\mathrm{top}}(F) + U_{I+1}K_n^{\mathrm{top}}(F),\end{aligned}$$

and for almost all I the polynomial $g_I(X)$ is equal to X.

We will show that the class of normic subgroups coincides with the class of norm groups $N_{L/F}VK_n^{\mathrm{top}}(L)$ of Galois totally ramified extensions L/F, $L \subset F_p$.

5.2. It follows from the definition that the notion of a normic subgroup doesn't depend on the choice of local parameters in F.

Proposition. *Let L/F be a Galois totally ramified p extension, $F \subset L_p$. Then $N_{L/F}VK_n^{\mathrm{top}}(L)$ is a normic subgroup in $VK_n^{\mathrm{top}}(F)$.*

Proof: The first property for $N_{L/F}VK_n^{\mathrm{top}}(L)$ is evident. To verify the second and third properties, one can proceed by induction on degree of L/F. If L/F of degree p then all follows from Proposition (3.2). In the general case let M/F be a subextension in L/F of degree p. Let σ be a generator of $\mathrm{Gal}(M/F)$ and $M = F(t_{s,M})$. Let

$$\frac{\sigma t_{s,M}}{t_{s,M}} = 1 + \theta_0 t_n^{r_n}\cdots t_{s,M}^{r_s}\cdots t_1^{r_1} \bmod U_{r_1+1,\cdots,r_n}$$

with $\theta_0 \in R^*$. According to Proposition (3.2) the unique non trivial polynomial arises from the norm map $N_{M/F}$ is $f_2(X) = \theta_0^p wp(\theta_0^{-1}X)$. Now let π_L be a prime element in $K_n^{\mathrm{top}}(L)$, and $\sigma \in \mathrm{Gal}(L/F)$ be an extension of σ on L. Then

$$N_{L/M}(\sigma\pi_L - \pi_L) - \left\{1 + \theta_0 t_n^{r_n} \cdots t_{s,M}^{r_s} \cdots t_1^{r_1}, t_{j_1}, \cdots, t_{j_{n-1}}\right\}$$

belongs to $U_{r_1+1,\cdots,r_n}K_n^{\mathrm{top}}(M)$, where the set $\{j_1, \cdots, j_{n-1}, s\}$ coincides with the set $\{1, \cdots, n\}$ and $t_n, \cdots, N_{M/F}t_{s,M}, \cdots, t_1$ are local parameters of F. Therefore, by the inductional assumption,

$$\begin{aligned} &N_{L/M}VK_n^{\mathrm{top}}(L) \cap U_R K_n^{\mathrm{top}}(M) + U_{R+1}K_n^{\mathrm{top}}(M) \\ &= \left\{1 + f_1(\mathcal{O}_F) t_n^{r_n} \cdots t_{s,M}^{r_s} \cdots t_1^{r_1}\right\} K_{n-1}^{\mathrm{top}}(M) + U_{R+1}K_n^{\mathrm{top}}(M), \end{aligned}$$

where $R = (r_1, \cdots, r_n)$, and $\theta_0 \in f_1(\mathcal{O}_F)$, $\overline{f}_1$ is k-decomposable. Thus, $\overline{\theta}_0 \in \overline{f}_1(k)$ and the polynomial $\overline{f_2 \circ f_1}$ is non zero k-decomposable. The second property for $N_{L/F}VK_n^{\mathrm{top}}(L)$ can be verified now similarly to the proof of Proposition 15 of [W].

□

5.3. Proposition. *Let L/F be an abelian totally ramified extension, $L \subset F_p$. Let $\mathcal{N}$ be a normic subgroup in $VK_n^{\mathrm{top}}(F)$. Then $N_{L/F}^{-1}(\mathcal{N})$ is a normic subgroup in $VK_n^{\mathrm{top}}(L)$.*

Proof: It is carried out in the same way as the proof of Proposition (3.2) of [F5] using Proposition (3.2).

□

5.4. Let π be a prime element in $K_n^{\mathrm{top}}(F)$. Let $\mathcal{E}_\pi$ denote the set of abelian totally ramified extensions L/F, $L \subset F_p$, such that $\pi \in N_{L/F}K_n^{\mathrm{top}}(L)$. Then, if L_1/F, $L_2/F \in \mathcal{E}_\pi$, $L_1 \cap L_2/F \in \mathcal{E}_\pi$. Indeed, let $M = L_1 \cap L_2$ and $N_{L_1/F}\pi_1 = N_{L_2/F}\pi_2 = \pi$. Then $N_{M/F}\varepsilon = 0$ for $\varepsilon = N_{L_1/M}\pi_1 - N_{L_2/M}\pi_2$. Now it follows from the first commutative diagram of Proposition (1.8) that $\varepsilon \in N_{L/M}VK_n^{\mathrm{top}}(L)$. Therefore, there is a prime element π_M in $K_n^{\mathrm{top}}(M)$ such that

$$N_{M/F}\pi_M = \pi, \ \pi_M \in N_{L_1/M}K_n^{\mathrm{top}}(L_1) \cap N_{L_2/M}K_n^{\mathrm{top}}(L_2).$$

Thus, it suffices to consider the case when $L_1 \cap L_2 = F$ and L_1/F, L_2/F are cyclic extensions of degree p. Assume that $L_1/L_2/F$ is not totally ramified. Then there is an unramified cyclic extension E/F of degree p, $E \subset L_1L_2$. Let σ_1 and σ_2 be elements of $\mathrm{Gal}(L_1L_2/F)$ such that $\sigma_1|_{L_2}$ and $\sigma_2|_{L_1}$ are trivial and $\sigma_1|_{L_1}$ and $\sigma_2|_{L_2}$ are generators of $\mathrm{Gal}(L_1/F)$ and $\mathrm{Gal}(L_2/F)$ respectively. One may assume that E is the fixed field of $\sigma_3 = \sigma_1\sigma_2$. Let

$$\pi = N_{L_1/F}\pi_1 = N_{L_2/F}\pi_2$$

for some $\pi_1 \in K_n^{\mathrm{top}}(L_1)$, $\pi_2 \in K_n^{\mathrm{top}}(L_2)$. Then

$$N_{L_1L_2/E}\left(i_{L_1/L_1L_2}\pi_1 - i_{L_2/L_1L_2}\pi_2\right) = 0.$$

Then, by Proposition (3.3)

$$i_{L_1/L_1L_2}\pi_1 - i_{L_2/L_1L_2}\pi_2 = \sigma_3(\gamma) - \gamma$$

for some $\gamma \in VK_n^{\rm top}(L_1L_2)$. Put $\beta = \pi_1 + \gamma - \sigma_1(\gamma)$. Then $\sigma_3(\beta) = \beta$ and $i_{F/L_1L_2}\pi = \left(1 + \sigma_1 + \dots + \sigma_1^{p-1}\right)\beta$. Therefore, $v_{L_1L_2}(\beta) = 1$. It follows from (3.3) that this is impossible. Thus, L_1L_2/F is totally ramified. Now similarly with (3.3) of [F5] we get $L_1L_2/F \in \mathcal{E}_\pi$.

5.5. In the same way as in (3.4) of [F5] one can prove

Proposition. *Let* π *be a prime element in* $K_n^{\rm top}(F)$. *Let* $\mathcal{N}$ *be a normic subgroup in* $VK_n^{\rm top}(F)$. *Then there is precisely one abelian totally ramified* p*-extension* L/F *such that* $\mathcal{N} = N_{L/F}VK_n^{\rm top}(L)$ *and* $\pi \in N_{L/F}K_n^{\rm top}(L)$.

As a corollary, we obtain

Existence Theorem. *There is an order reversing bijection between the lattice of normic subgroups in* $VK_n^{\rm top}(F)$ *with respect to the intersection and sum and the lattice of extensions* $L/F \in \mathcal{E}_\pi$ *with respect to the intersection and composition:* $\mathcal{N} = N_{L/F}VK_n^{\rm top}(L) \leftrightarrow L$.

Finally, in the same way as in (3.4) of [F5] one can show that for the composition F_π of all fields L with $L/F \in \mathcal{E}_\pi$

$$F_\pi \cap \widetilde{F} = F \quad \text{and} \quad F_\pi \widetilde{F} = F_p^{\rm ab}.$$

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