On Zeroes of the Schwarzian Derivative

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1. The Schwarzian derivative

The main character of the present note is the Schwarzian derivative, and we start with a brief reminder of its definition and main properties.

Let $f : \mathbf{RP}^1 \to \mathbf{RP}^1$ be a projective line diffeomorphism. For every point $x \in \mathbf{RP}^1$ there exists a unique projective transformation $g_x : \mathbf{RP}^1 \to \mathbf{RP}^1$ whose 2-jet at x coincides with that of f. The Schwarzian derivative $S(f)$ measures the deviation of the 3-jet $j^3 f$ from $j^3 g_x$.

More specifically, let $x \in \mathbf{RP}^1$ and v be a tangent vector to $\mathbf{RP}^1$ at x. Extend v to a vector field in a vicinity of x and denote by ϕ_t the corresponding local one-parameter group of diffeomorphisms. Consider 4 points:

$$x, x_1 = \phi_\epsilon(x), x_2 = \phi_{2\epsilon}(x), x_3 = \phi_{3\epsilon}(x)$$

(ϵ is small) and compare their cross-ratio with that of their images under f. It turns out that the cross-ratio does not change in the first order in ϵ:

$$[f(x), f(x_1), f(x_2), f(x_3)] = [x, x_1, x_2, x_3] + \epsilon^2 S(f)(x) + O(\epsilon^3).$$

The ϵ^2–coefficient depends on the diffeomorphism f, the point x and the tangent vector v (but not on its extension to a vector field); it is a quadratic function in v. That is to say, $S(f)$, the Schwarzian derivative of a diffeomorphism f, is a quadratic differential on $\mathbf{RP}^1$.

By the very construction $S(g) = 0$ if g is a projective transformation, and $S(g \circ f) = S(f)$ if g is a projective transformation and f is an arbitrary diffeomorphism.

Choose a projective coordinate $x \in \mathbf{R}^1 \cup \{\infty\} = \mathbf{RP}^1$. Then a diffeomorphism can be considered as a function of x, and the projective transformations are identified with fraction-linear functions. The Schwarzian derivative is given by the formula

$$S(f) = \frac{f'''}{f'} - \frac{3}{2}\left(\frac{f''}{f'}\right)^2.$$

The Schwarzian derivative enjoys the following cocycle property:

$$S(g \circ f) = S(g)(f')^2 + S(f),$$

which means that S is a 1-cocycle of the group of the projective line diffeomorphisms with the values in quadratic differentials.

Let α be another parameter on $\mathbf{RP}^1$, that is, $x = g(\alpha)$ for some function g, and let $f : \mathbf{RP}^1 \to \mathbf{RP}^1$ be a diffeomorphism. The Schwarzian derivative of f is given, in terms of α, by the formula $S(g \circ f) - S(g)$ where, as before,

$$S(h) = \frac{h'''}{h'} - \frac{3}{2}\left(\frac{h''}{h'}\right)^2,$$

and where prime means now $d/d\alpha$. By the cocycle property,

$$S(g \circ f) - S(g) = ((f')^2 - 1)S(g) + S(f).$$

A particular case of interest is $x = \tan(\alpha/2)$ where $\alpha \in \mathbf{T}^1 = \mathbf{R}/2\pi\mathbf{Z}$ is an angle parameter. Then $S(g) = 1/2$ and the formula for the Schwarzian derivative reads:

$$S(f) = \frac{f'''}{f'} - \frac{3}{2}\left(\frac{f''}{f'}\right)^2 + \frac{1}{2}\left((f')^2 - 1\right).$$

2. Theorem on Zeroes of the Schwarzian Derivative

Recently E. Ghys discovered a beautiful theorem ([Gh]; see also [O-T]):

Theorem 2.1. *The Schwarzian derivative of every projective line diffeomorphism has at least 4 distinct zeroes.*

The proof outlined by Ghys is based on the similarity with the classical 4 vertex theorem:

Theorem 2.2. *A smooth closed convex plane curve has at least 4 distinct vertices (curvature extrema).*

The latter theorem has the following refinement (see [Bl]):

Theorem 2.3. *If a smooth closed convex plane curve transversally intersects a circle in N points then it has at least N distinct vertices (N is even).*

The analogy between zeroes of the Schwarzian derivative and curvature extrema suggest a refinement of the Ghys theorem. Call a fixed points x of a projective line diffeomorphism f *simple* if the graph of f transversally intersects the diagonal at point (x, x) and $S(f)(x) \neq 0$. A generic diffeomorhism has only simple fixed points. The next theorem is the main result of the present note.

Theorem 2.4. *If a projective line diffeomorphism has N fixed points of which at least one is simple then its Schwarzian derivative has at least N distinct zeroes.*

Notice that the number of fixed points of a projective line diffeomorphism, counted with multiplicities, is even.

Theorem 2.4 implies the existence of 4 zeroes of the Schwarzian derivative. Given a diffeomorphism f pick 3 generic points $x_1, x_2, x_3 \in \mathbf{RP}^1$, and let g be the projective transformation that takes each x_i to $f(x_i); i = 1, 2, 3$. Then $g^{-1} \circ f$ has 3 fixed points. The number of fixed points being even, there exists a fourth one. By Theorem 2.4 $S(g^{-1} \circ f)$ has at least 4 zeroes, and so does $S(f) = S(g^{-1} \circ f)$.

3. Proof of Theorem 2.4

The proof is elementary in that it boils down to a (somewhat messy) successive application of Rolle's theorem.

We assume that f is orientation–preserving; otherwise f has exactly 2 fixed points and the assertion of Theorem 2.4 is weaker than that of Theorem 2.1.

Without loss of generality, assume that $\infty \in \mathbf{RP}^1$ is a simple fixed point. The graph G of $f(x)$ is a smooth curve with everywhere positive slope that intersects the diagonal at $(N-1)$ points.

The following identity is straightforward:

$$((f')^{-1/2})'' = -\frac{1}{2}(f')^{-1/2} S(f).$$

Thus one wants to show that $((f')^{-1/2})''$ has N zeroes.

Rolle's theorem provides quite a few. Between every two consecutive intersections of G with the diagonal there is a point at which $f' = 1$. Hence $(f')^{-1/2}$ assumes the value 1 at least $(N-2)$ times, and therefore $((f')^{-1/2})''$ vanishes at least $(N-4)$ times. To account for the missing 4 zeroes we analyse the behaviour of f at infinity.

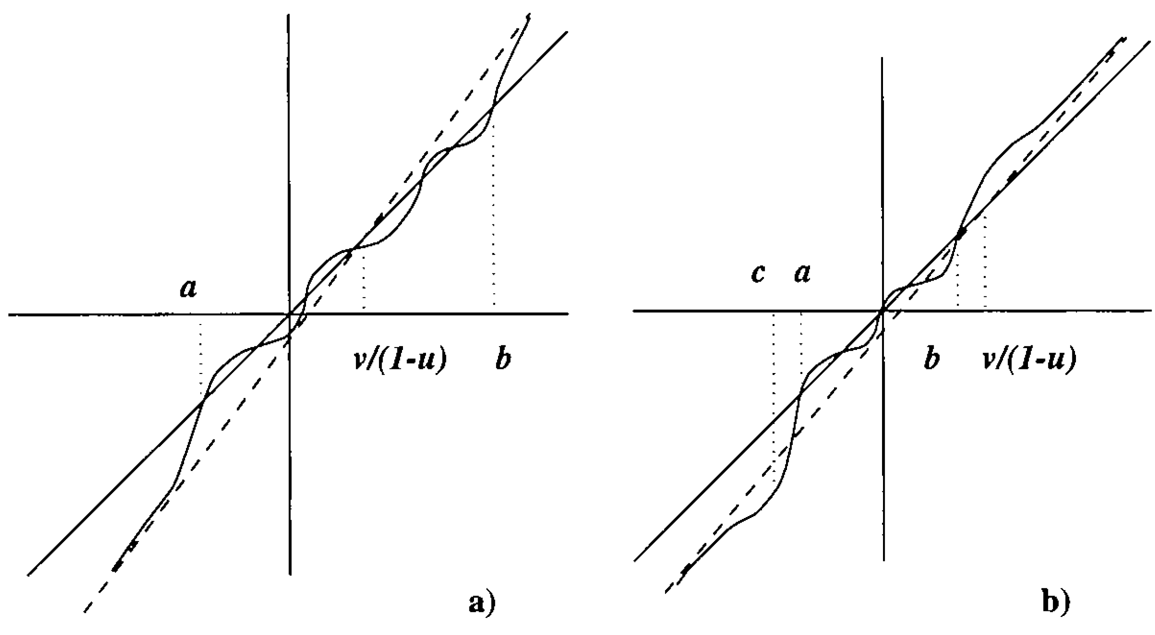


Figure 1

Let $t = 1/x$ be a local parameter at ∞; set $g(t) = 1/f(1/t)$. Then $g(0) = 0$; consider the Taylor expansion

$$g(t) = pt + qt^2 + rt^3 + O(t^4).$$

Since ∞ is a simple fixed point, $p \neq 1$ and $S(g)(0) \neq 0$. It is straightforward that $S(g)(0) = 6(pr - q^2)/p^2$; thus $pr - q^2 \neq 0$.

One obtains the expansion of f at ∞:

$$f(x) = 1/g(1/x) = ux + v + w\frac{1}{x} + O\Big(\frac{1}{x^2}\Big)$$

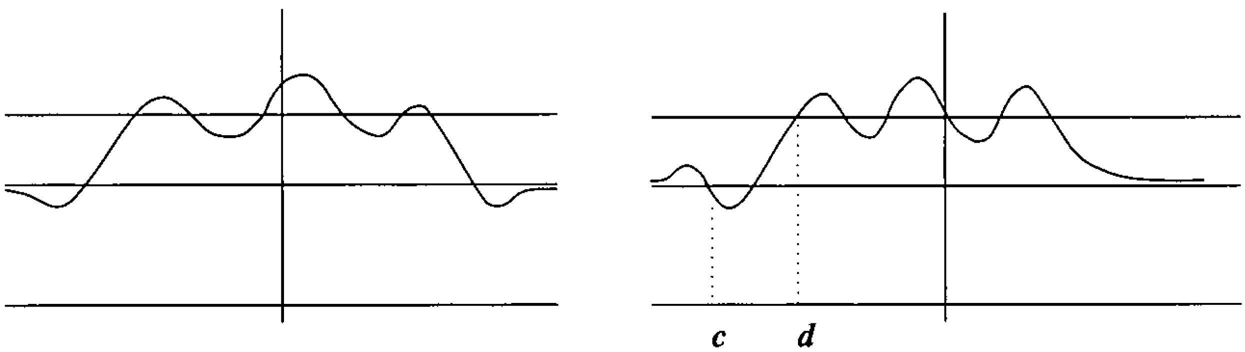
where $u = 1/p, v = -q/p^2$ and $w = (q^2 - pr)/p^3$. Thus $u \neq 1$, $w \neq 0$ and G has the asymptotic line $y = ux + v$ – see fig. 1.

Without loss of generality, assume that $u > 1$; otherwise consider f^{-1} instead of f. Let a and b be the x-coordinates of the left and rightmost intersection points of G with the diagonal.

Case 1: $w < 0$ (fig. 1 a).

By the Taylor expansion of f at infinity, $f'(x) > u$, and so $f'(x)^{-1/2} < u^{-1/2}$, for sufficiently great $|x|$. Consider the graph G' of $f'(x)^{-1/2}$.

By Rolle's theorem G' intersects the line $y = 1$ at least $(N-2)$ times, and this provides $(N-3)$ extrema of $(f')^{-1/2}$. Also G' lies below the line $y = u^{-1/2}$ for sufficiently great $|x|$, therefore $(f')^{-1/2}$ has a local minimum to the right of the rightmost intersection of G' with $y = 1$ and to the left of the leftmost such intersection – see fig. 2 a. The points $x = +\infty$ and $x = -\infty$ also qualify as extrema of $(f')^{-1/2}$. Altogether $(f')^{-1/2}$ has $(N+1)$ critical points, and by Rolle's theorem $((f')^{-1/2})''$ has N zeroes.



a)　　b)

Figure 2

Case 2: $w > 0$ (fig. 1 b).

In this case $f'(x) < u$, and so $f'(x)^{-1/2} > u^{-1/2}$, for sufficiently great $|x|$. Either $a < v/(1-u)$ or $b > v/(1-u)$ (or both). Consider the former case, the latter being completely analogous.

For x sufficiently close to $-\infty$ the graph G lies below the asymptotic line $y = ux + v$. Therefore G must intersect the asymptotic line to the left of the vertical line $x = a$. By Rolle's theorem there exists $c < a$ such that $f'(c) = u$, and so $f'(c)^{-1/2} = u^{-1/2}$ – see fig. 2 b.

We do a similar bookkeeping of critical points of $(f')^{-1/2}$. The graph G' intersects the line $y = 1$ at least $(N-2)$ times on the interval $a \le x \le b$; let d be the x-coordinate of the leftmost such intersection. By Rolle's theorem $(f')^{-1/2}$ has $(N-3)$ critical points on the interval $[d, b]$. In addition $(f')^{-1/2}$ has a local maximum on the interval $(-\infty, c)$ and a local minimum on the interval $[c, d)$. The points $\pm\infty$ are critical points of $(f')^{-1/2}$ as well. Altogether $(f')^{-1/2}$ has $(N+1)$ extrema, thus $((f')^{-1/2})''$ has N zeroes.

4. Infinitesimal Version of Theorem 2.4

Let α be the angle parameter on $\mathbf{RP}^1$ introduced at the end of Section 1. Consider a diffeomorphism close to identity: $f(\alpha) = \alpha + \epsilon g(\alpha)$ where g is a 2π-periodic function and ϵ is small. Then, according to the last formula of Section 1,

$$S(f) = \epsilon\,(g''' + g') + O(\epsilon^2).$$

Thus the differential operator $(d/d\alpha)^3 + (d/d\alpha)$ is a linearization of the Schwarzian derivative. Fixed points of f are zeroes of g, and zeroes of $S(f)$ are, in the first approximation, zeroes of $g''' + g'$.

Let $g(\alpha) = 0$. Call α a *nondegenerate* zero if $g'(\alpha) \neq 0$ and $g'''(\alpha) + g'(\alpha) \neq 0$. The next statement is an infinitesimal version of Theorem 2.4.

Theorem 4.1. *If a smooth function g on the circle $\mathbf{T}^1$ has N zeroes of which at least one is nondegenerate then $g''' + g'$ has at least N distinct zeroes.*

We only outline the proof. Consider g as a function of $x = \tan(\alpha/2)$. Make use of the following identity between differential operators:

$$\left(\frac{d}{d\alpha}\right)^3 + \left(\frac{d}{d\alpha}\right) = \frac{1}{8}(1+x^2)^2\left(\frac{d}{dx}\right)^3(1+x^2).$$

Thus one wants to show that $((1+x^2)g(x))'''$ has N zeroes. Assume that π is a simple zero of $g(\alpha)$. One shows that an expansion at infinity holds:

$$(x^2+1)g(x) = ux + v + w\frac{1}{x} + O\left(\frac{1}{x^2}\right).$$

where $u \neq 0$ and $w \neq 0$. Therefore the graph of $(1+x^2)g(x)$ has an asymptotic line and intersects the x-axis $(N-1)$ times. The rest of the argument repeats the proof of Theorem 1 and we do not dwell on it.

Remark. $(d/d\alpha)^3 + (d/d\alpha)$ is a *disconjugate* differential operator on the circle. For an operator of odd degree k this means that every function in its kernel is 2π-periodic and has at most $(k-1)$ zeroes on $\mathbf{T}^1$, multiplicities counted. Could Theorem 4.1 be generalized to other disconjugate operators? For a disconjugate differential operator L of degree k on an interval I the following theorem holds: *if f has N zeroes on I then $L(f)$ has at least $N-k$ zeroes on I* (see [Po]). The proof consists in factorizing L into a product

$$v_0(x)\frac{d}{dx}v_1(x)\frac{d}{dx}v_2(x)...v_{k-1}(x)\frac{d}{dx}v_k(x),$$

where each function $v_i(x)$ is positive on I, and successively applying Rolle's theorem. Such a factorization is not available on the circle.

5. Application to Vertices of Plane Curves

The relation between Theorems 2.4 and 4.1 is another manifestation of the relation between zeroes of the Schwarzian derivative and curvature extrema of plane curves. We show here that Theorem 4.1 implies Theorem 2.3.

Let C be the circle that transversely intersects a smooth convex closed plane curve γ at N points. Denote by O the center and by r the radius of C. Let $f(\alpha)$ be the *support function* of γ, that is, the (signed) distance from O to an oriented tangent line to γ as a function of the direction of this line.

Support functions of circles are linear combinations of first harmonics: $a + b\cos\alpha + c\sin\alpha$. These functions generate the kernel of the differential operator $(d/d\alpha)^3 + (d/d\alpha)$. A vertex of a curve is its third order tangency with a circle. Therefore vertices of γ are zeroes of the function $f'''(\alpha) + f'(\alpha)$.

Lemma 5.1. *There exists at least N common support lines to γ and C.*

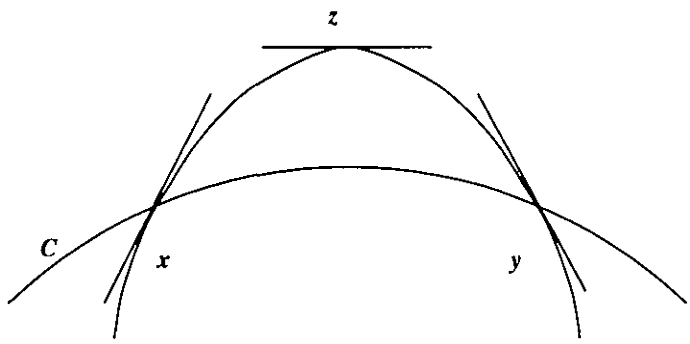


Figure 3

Proof. Consider an arc δ of the curve that lies outside C and is bounded by two consecutive intersection points $x, y \in \gamma \cap C$ – see fig.3. There are at least $N/2$ such arcs.

The distance function from O assumes maximum on δ at some point z. The support line to γ at z does not intersect C, while the support line at x does. It follows that there is a point on the arc xz such that the support line at this point is tangent to C. Likewise such a point exists on the arc yz. Altogether one gets $2 \cdot N/2 = N$ common support lines.

Lemma 5.1 implies that the function $f(\alpha)$ assumes the value r at least N times, so the function $g(\alpha) = f(\alpha) - r$ has N zeroes. Making a small perturbation of the circle C, if necessary, one may assume that at least one of these zeroes is nondegenerate. By Theorem 4.1 $g''' + g'$ nas N zeroes, and so does $f''' + f'$. Thus γ has at least N vertices.

6. Lorentz Caustics and Wave Fronts

According to E. Ghys ([Gh]), the theorem on 4 zeroes of the Schwarzian derivative *is* a 4-vertex theorem in *Lorentzian geometry*. In this last section we discuss this viewpoint.

As before, G denotes the graph of an orientation-preserving projective line diffeomorphism $f(x)$. The graphs of (orientation-preserving) fraction-linear functions are either straight lines or hyperbolas $(x-a)(y-b)=-s^2$; in the latter case $|s|$ is called the *radius* of the hyperbola. Abusing the language we will call these curves simply "hyperbolas".

For every point of G there exists a unique hyperbola second order tangent to G at this point; call it the *osculating hyperbola*. Zeroes of $S(f)$ are moments of third order tangency between G and its osculating hyperbola.

Following Ghys, introduce the Lorentz metric $g = dx\ dy$ in the (x, y) plane. Hyperbolas are circles in this metric. The graph G is a "space-like" curve. All geometrical terms such as "length", "orthogonality", etc, refer to the metric g. Let J be the linear operator $(x,y) \to (-x,y)$. Then $J(v)$ is orthogonal to v, and $g(J(u), J(v)) = -g(u,v)$ for every vectors u and v.

Let t be a (Lorentz) arc-length parameter along G. Denote d/dt by dot and d/dx by prime. One easily finds that

$$\ddot{G}(t) = \frac{1}{2}\ f''(x(t))\ f'(x(t))^{-3/2}\ J(\dot{G}(t)).$$

Call the coefficient $k = f''(f')^{-3/2}/2$ the (Lorentz) *curvature* of G. Then $\dot{k} = (f')^{-1/2}S(f)$ (the observation due to L. Guieu and V. Ovsienko). Thus curvature extrema are zeroes of the Schwarzian derivative.

Definition. The envelope Γ of g-orthogonal lines to G is called its *Lorentz caustic*.

Note that Γ is a time-like curve. Note also that at inflection points of G the caustic goes to infinity. If G is the graph of a projective line diffeomorphism with a fixed point at infinity then, according to the Moebius theorem, G has at least 3 inflections.

Definition. A *wave front* of G is obtained by moving every point $x \in G$ the same distance in the direction g-orthogonal to G at x.

If $G(t)$ is an arc-length parameterized curve then its s-front is the curve

$$G_s(t) = G(t) - s\ J(\dot{G}(t)).$$

Example. Let G be the upper branch of the hyperbola $xy = -s^2, s > 0$. The arc-length parameterization is

$$G(t) = (-e^{-t/s},\ \ s^2\ e^{t/s}).$$

The curvature is constant: $k = 1/s$ (and $k = -1/s$ on the lower branch of the same hyperbola). The acceleration vector $\ddot{G}(t)$ is collinear to $G(t)$ for all t, so all normals pass through the origin. The caustic degenerates to a point, the center of the hyperbola. The wave fronts are concentric hyperbolas, and in particular, G_s degenerates to the center of the original hyperbola G.

Lorentz caustics and fronts enjoy the familiar properties of their Euclidean counterparts. We summarize them in the next theorem.

Theorem 6.1. a). *The Lorentz caustic* Γ *of a curve* G *is the locus of its osculating hyperbolas' centers.* Γ *is also the locus of the singularities of the fronts* G_s.
b). *Singularities of* Γ *correspond to curvature extrema of* G. *If* G *is the graph of a projective line diffeomorphism* f *then singularities of* Γ *correspond to zeroes of* $S(f)$.
c). *(Huygens' Principle). The front* G_s *is the envelope of the hyperbolas of radii* s *with the centers at points of* G. *The caustic of each front* G_s *coincides with the caustic of* G. *The fronts enjoy the evolution property: the* s*-front of the* t*-front of* G *coincides with the* $(s+t)$*-front* G_{s+t}.
d). *Each front* G_s *is the locus of free ends of a stretched string of constant Lorentz length unwinding from* Γ.

Proof. To prove the assertions a) and b) approximate G by its osculating hyperbolas and use the properties of hyperbolas described in the Example.

To prove c) consider 3-dimensional contact space of space-like contact elements in the Lorentz plane with coordinates (x, y, u) and the contact 1-form $\lambda = (e^u\ dx - e^{-u}\ dy)/2$. A space-like curve in the (x, y) plane lifts to contact space as a Legendrian curve satisfying $2u = \log\ (dy/dx)$.

Let $\xi = e^{-u}\ d/dx - e^u\ d/dy$ be the Reeb vector field, that is, $\lambda(\xi) = 1$ and $i_\xi d\lambda = 0$. The flow of ξ parallel translates a contact element with a constant velocity in the direction of its Lorentz normal. Denote by $\{\phi_s\}$ the corresponding 1-parameter group of diffeomorphisms. Each ϕ_s preserves the contact structure (and even the contact form).

The s-front of a curve G is obtained as follows: lift G to contact space, apply ϕ_s to the lifted curve and project back to the plane. The evolution property follows. It follows also that all the fronts have the same family of orthogonal lines, therefore they share the same caustic.

To prove that the front G_s is the envelope of the hyperbolas of radii s centered at points of G, consider a point $x \in G$. Its lift to contact space is the curve γ of all space-like contact elements with the base-point x. The lift $\tilde{G}$ of G intersects γ, so $\phi_s(\tilde{G})$ intersects $\phi_s(\gamma)$. The projection of $\phi_s(\gamma)$ to the plane is the hyperbola of radius s with the center at x. Since $\phi_s(\tilde{G})$ intersects $\phi_s(\gamma)$ this hyperbola is tangent to G_s.

To prove d) let t be the arc-length parameter along Γ, i.e., $g(\dot{\Gamma}(t), \dot{\Gamma}(t)) = -1$. Then $\ddot{\Gamma}(t)$ is orthogonal to $\dot{\Gamma}(t)$. The free end of a string unwinding from Γ is the point

$$X(t) = \Gamma(t) + (c - t)\dot{\Gamma}(t)$$

where c is a constant, "length of the string". Then $\dot{X}(t) = (c - t)\ddot{\Gamma}(t)$, which is orthogonal to $\dot{\Gamma}(t)$. Therefore the normals to the curve X are tangent to Γ, that is, X is a front of G.

Remark. The above contact space is contactomorphic to the standard contact space of 1-jets $J^1\mathbf{R}^1$; indeed $\lambda = dq - pdu$ where $p = e^u x + e^{-u} y$ and $q = e^u x - e^{-u} y$.

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