ON SCHUBERT'S METHOD OF DEGNERATION FOR COMPUTING DEGENERATION NUMBERS

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ON SCHUBERT'S METHOD OF DEGENERATION FOR COMPUTING GEOMETRIC NUMBERS

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1. Introduction

1.1. The goal of Enumerative Geometry is to study those concepts, principles and methods that allow to answer questions that ask how many figures of some kind satisfy a given list of conditions, the explicit contruction of the figures not being required (see Schubert [1879]). It is important to remark that these tasks include the formulation of criteria that guarantee a finite number of solutions.

1.2. In these notes we will be considering Schubert's method of degeneration to solve enumerative problems and also how to get about certain difficulties arising from its application in its original form, especially those related to the use of coincidence formulas as a means for deriving the key degeneration relations. Although Schubert's coincidence formulas are not hard to establish, the way he uses them leads to the computation of multiplicities which do not seem easy to handle. For example, formula 13) on page 45 in Schubert [1879] can be proved working on the blow up of $\mathbf{P}^3 \times \mathbf{P}^{3^*}$ along the diagonal, but its application to the three dimensional system Σ of bisecants to a 1-dimensional system Γ of twisted cubics (as suggested on p. 167) leads to the computation, among other terms, of $\epsilon g_p \Sigma$; set-theoretically this cycle represents the lines through p that are tangent to a cubic in Γ, but its multiplicities appear to be rather unclear.

1.3. In our discussion we will focus on the enumerative geometry of rational cubics. In particular we will consider the following two examples:

(a) How many twisted cubics are tangent to 12 quadrics? (see Schubert [1879], p. 184, and Kleiman-Strømme-Xambó [1987])

(b) How many cuspidal cubics in a plane go through 2 points, are tangent to 2 lines and are such that the cusp and the inflexion move each on a line and being at the same time colinear with a given point? (see Miret-Xambó [1988])

1.4. Remark. Problem (a) involves only one kind of condition, tangency to a quadric surface, but the figures are non-complete intersections in $\mathbf{P}^3$ and thus not easily amenable to analytic treatment. On the other hand, the figures in (b) are easier to parametrize, but in this case 5 kinds of conditions are involved.

1.5. The number in (a), whose value is 5819539783680, was computed by Schubert, as a consequence of his work on the theory of characteristics for twisted cubics. This work was awarded the Gold Medal of the Royal Danish Academy in 1875. It was recently verified in Kleiman-Strømme-Xambó [1987]. Here we will give a new and simpler proof of the key degeneration relations for the first order characteristic conditions and will also show that our approach yields all characteristic numbers, that is, including those involving the second order condition P (going through a point). The number in (b), whose value is 55, has been computed for the first time, as far as the author knows, in Miret-Xambó [1987, 88]. These two works summarize the research done to verify and extend the work of Schubert and others on the enumerative geometry of cuspidal cubics. They are quickly surveyed in Section 5.

1.6. Our discussion will also lead us (see Section 3) to consider the enumerative theory of quadratic varieties (or, more generally, quadratic chains) in projective space, to which we will add a few observations. In particular we give explicit expressions for the first order conditions in terms of the first order degenerations, and comment on a new way of deriving the expressions of the degenerations in terms of the first order conditions, starting with the case of quadratic varieties on a projective line and then using an inductive procedure, via hyperplane sections, for the general case. We also take care of how the characteristic conditions restrict to the corresponding first order degenerations. Thus the paper may as well serve as a rather straightforward introduction to a fascinating subject that has been studied following several approaches by many authors: Schubert [1870, 1879, 1894], Van der Waerden [1938], Severi [1940], Semple-Roth [1949], Tyrrell [1956], Kleiman [1980, 87], Vainsencher [1982], Finat [1983], Laksov [1982, 86, 87], De Concini-Procesi [1983, 85], De Concini-Gianni-Traverso [1985], Casas-Xambó [1986], De Concini-Goreski-MacPherson-Procesi [1987], Kleiman-Thorup [1987], Bifet [1988], Bifet-De Concini-Procesi [1988].

1.7. As it is well known, Schubert and other 19-th century geometers computed thousands of geometric numbers, but the concepts, principles and methods they used have generally been considered to lack a solid foundation and thus Hilbert included as Problem 15 in his famous list the quest for a "rigorous foundation of Schubert's enumerative calculus", explaining that this also includes "to establish rigorously and with an exact determination of the limits of their validity those geometric numbers which Schubert especially has determined ... ". For an excellent introduction to this problem see Kleiman [1987] and the references given therein.

1.8. Base field. For simplicity we will take $\mathbf{C}$, the field of complex numbers, although most of the arguments and conclusions are valid over an algebraically closed field $\mathbf{k}$ of characteristic $p \geq 0$, $p \neq 2, 3$. Only occasionally we will make a remark on the positive charactersitic case.

The n-dimensional projective space over the base field will be denoted $\mathbf{P}^n$ and $\mathrm{PGL}(\mathbf{P}^n)$ will be its group of automorphisms.

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2. Foundations

2.1. The study of foundations in this century, through the work of many mathematicians, has afforded the tools that allow to handle constructs such as "figures of some kind", "conditions" and "number of figures satisfying ..." that appear in the demarcation of enumerative geometry (see **1.1**) in a way that matches current standards.

Thus the totality of "figures of some kind" (or classes of such under some equivalence relation) will be in one to one correspondence with the (closed) points of some variety X and a "system of figures" of dimension r is just an irreducible subvariety of X of dimension r. For instance, when dealing with linear spaces or flags of such, we are lead to the Grassmannian and flag varieties. Similarly, smooth quadratic varieties in $\mathbf{P}^n$ may be identified with the open set of $\mathbf{P}(S^2 E^*_{n+1})$, if $\mathbf{P}^n = \mathbf{P}(E_{n+1})$; cuspidal cubics form an orbit of $\mathrm{PGL}(\mathbf{P}^2)$ under the natural action on the space $\mathbf{P}(S^3 E^*_3)$ of all plane cubics; twisted cubics form an orbit of $\mathrm{PGL}(\mathbf{P}^3)$ under its natural action on the Hilbert (or Chow) scheme of curves of degree 3 and arithmetic genus 0 in $\mathbf{P}^3$.

2.2. Conditions. On the other hand, a simple way to deal with the notion of condition is to think of a condition α as an algebraic (often rational) family of cycles $\alpha = \{\alpha_t\}_{t \in T}$ on X. For a given t, the points in the support $|\alpha_t|$ of α_t are to be interpreted as the figures satisfying α with datum t. The order of α is the codimension of the cycles α_t.

2.3. Example: Characteristic conditions. Given a d-dimensional irreducible family $X = \{V\}$ of n-dimensional projective varieties $V \subseteq \mathbf{P}^N$ and an i-dimensional linear space $L \subseteq \mathbf{P}^N$, let $\mu_L \subseteq X$ be the subfamily of varieties that have a contact with L, that is, such that there exists $P \in L \cap V$ and a hyperplane H containing L such that (P, L) lies in the conormal variety $CV \subset \mathbf{P}^N \times \mathbf{P}^{N^*}$ of V (CV is the closure of the set of pairs $(P, H) \in \mathbf{P}^N \times \mathbf{P}^{N^*}$ such that $P \in V$ is simple and $T_P V \subseteq H$). So μ_L is a cycle on X and the rational family $\mu_i = \{\mu_L\}_L$ is the i-th characteristic condition of the family X.

Notice that for $i \leq N-n-1$ the cycle μ_L is the cycle of $V \in X$ that intersect L. The order of μ_i will be denoted $\mathrm{ord}_X(\mu_i)$, if we want to declare to which family μ_i refers.

The characteristic conditions for the twisted cubics are usually denoted P, ν, ρ (to go through a point, to intersect a line, to be tangent to a plane). Condition P has order 2 and μ, ν have order 1. In fact, quite generally, we have:

2.4. Proposition. *Let X be a family as above. Let V be a general member of X, P a general point on V and L a i-dimensional linear space chosen generically among those that have a contact with X. Let δ_i denote the dimension of the contact locus of L with V. Then we have:*

$$\mathrm{ord}_X(\mu_i) = \begin{cases} N-n-i & \text{if } i \leq N-n-1 \\ 1+\delta_i & \text{if } i \geq N-n \end{cases}$$

Proof: By "computo di constanti" one easily finds that

$$\mathrm{ord}_X(\mu_i) = \mathrm{cod}_X(\mu_L) = (i+1)(N-i) - n - d_i + \delta_i,$$

where d_i is the dimension of the variety Γ_i of i-dimensional linear spaces having a contact with a given n-dimensional variety V at a general point $P \in V$. If $i \leq N-n-1$, then Γ_i consists of the i-dimensional linear spaces L such that $P \in L$. It is therefore clear that

$$d_i = i(N-i).$$

Since in this case $\delta_i = 0$, we find that $\mathrm{ord}_X(\mu_i) = N-n-i$.

Assume now that $i \geq N-n$. In this case the condition that L has a contact with V at P is equivalent to the relations $P \in L$ and $\dim(L \cap T_P V) > i+n-N$. From this it is not hard to see that $d_i = (i+1)(N-i) - n - 1$ and hence that $\mathrm{ord}_X(\mu_i) = 1 + \delta_i$. □

2.5. Example: fundamental conditions for cuspidal cubics. For plane cuspidal cubics we have the characteristic conditions μ_0, μ_1 (going through a point and being tangent to a line; usually they are denoted μ and ν, respectively) and the conditions c, v, y, z, w and q which are defined as follows (see the picture page at the end): given a point P and a line L, c_L (resp. v_L, y_L) is the cycle of cuspidal cubics that have the cusp (resp. the inflexion, the intersection of the cuspidal and inflexional tangents) on L, and q_P (resp. w_P, z_P) is the cycle of cuspidal cubics whose cuspidal tangent (resp. inflexional tangent, line joining the cusp and the inflexion) goes through P. Such conditions will be called fundamental conditions for the cuspidal cubics.

2.6. A convenient way to understand the construct "number of figures satisfying n times α, m times β, ... ", $n \cdot \mathrm{ord}(\alpha) + m \cdot \mathrm{ord}(\beta) + \ldots = d$, is to take it to mean the degree N of the 0-cycle

$$(*) \qquad \alpha_{t_1} \cdots \alpha_{t_n} \cdot \beta_{t_1} \cdots \beta_{t_m} \cdots$$

(assuming that it is a 0-cycle; see **2.7**), where $(t_1, \ldots, t_n, s_1, \ldots, s_m, \ldots)$ is generic. This number is denoted

$$N = \alpha^n \beta^m \cdots$$

2.7. Transversality. In many cases it can be guaranteed a priori that **2.6**$(*)$ is a 0-cycle and that the multiplicities in the intersection are 1. In such a situation N is indeed the number of distinct figures satisfying the conditions, with data in general position.

An interesting example of this occurs under the following circumstances. Suppose the characteristic of the ground field is 0. Let G be a connected algebraic group and assume that G acts transitively on the parameter varieties $T = \{t\}$, $S = \{s\}$, ... and also on X. Assume moreover that the relations α, β, ... are G-invariant. Then the intersection is finite and the multiplicities are all equal to 1. Indeed, since for all $\sigma \in G$ and $t \in T$ we have $\sigma(\alpha_t) = \alpha_{\sigma^{-1}(t)}$, our intersection is of the form

$$\sigma_1(\alpha_t) \cdots \sigma_n(\alpha_t) \sigma'_1(\beta_s) \cdots \sigma'_m(\beta_s) \cdots,$$

where $\sigma_1, \ldots, \sigma_n$, $\sigma'_1, \ldots, \sigma'_m$, ... are general in G and so the claim follows from the principle of transversality of general translates (see Kleiman [1974]).

If the same conditions hold but $\mathrm{char}(\mathbf{k}) = p > 0$, then the multiplicities are all equal to a fixed power of p, say p^e, $e \geq 0$, so that $N = p^e N'$, N' the number of distinct solutions. Thus for this multiplicity to be > 1 it is necessary that $p|N$. This in practice allows to rule out multiplicities > 1 for all but a finite number of p.

These observations can be applied, for example, to the fundamental numbers of cuspidal cubics, taking $G = \mathrm{PGL}(\mathbf{P}^2)$. Indeed, in this case G acts transitively on $\mathbf{P}^2$, $\mathbf{P}^{2*}$ and on the family of cuspidal cubics. Moreover, the fundamental conditions are certainly projectively invariant. Similar remarks can be made for many other figures and conditions, like for the nodal and twisted cubics, taking as fundamental conditions those listed by Schubert [1879] (p. 144 and pp. 163-164, respectively; see also **5.1** below).

There are other more general transversalty statements which can be applied where the group does not act transitively, or even where there is no group acting at all: see Casas [1987], Speiser [1988].

2.8. Characteristic numbers. Given a d-dimensional family X of n-dimensional varieties V, the numbers

$$N_X(m_0, \ldots, m_{N-1}) = \mu_0^{m_0} \ldots \mu_{N-1}^{m_{N-1}},$$

where

$$(N-n)m_0 + \ldots + 2m_{N-n-2} + \sum_{N-n-1}^{N-1} (1+\delta_j)m_j = d,$$

will be called the characteristic numbers of the family X. Here μ_i are the characteristic conditions for the family X (see **2.3**).

For example, if X is a family of plane curves, such that the generic member is irreducible, then its characteristic numbers are $N_X(i,j) = \mu_0^i \mu_1^j$, $i+j=d$, for in this case μ_0 and μ_1 are simple conditions. The characteristic numbers of the family of twisted cubics are the numbers

$$N(i,j,k) = P^i \nu^j \rho^k, \quad i,j,k \geq 0, \quad 2i+j+k = 12.$$

2.9. Contacts. According to the contact theorem of Fulton-Kleiman-MacPherson [1982], the knowledge of the characteristic numbers of a family X suffices for the determination of the number, say $N_X(W_1,\ldots,W_d)$, of varieties in X that have a simultaneous contact with d given varieties $W_1,\ldots,W_d$, in general position in $\mathbf{P}^n$. In fact the contact theorem implies that

$$N_X(W_1,\ldots,W_d) = c_X\,(r_X(W_1)\ldots r_X(W_d)),$$

where, for a given projective variety W, $r_X(W)$ is the linear form in the variables $\mu_0,\ldots,\mu_{N-1}$ defined by the relation

$$r_X(W) = \sum_{\mathrm{ord}(\mu_j)=1} r_j(W)\mu_j$$

and c_X is the unique additive map from the group of homogeneous polinomials of degree d, with integer coefficients, to $\mathbf{Z}$ such that $c_X(\mu_0^{m_0}\cdots\mu_{N-1}^{m_{N-1}}) = N_X(m_0,\ldots,m_{N-1})$. The expression $r_j(W)$ in the formula denotes the j-th rank of W, that is, the number of $(n-j-1)$-dimensional linear spaces in a given general pencil that have a contact with W.

Thus the number N of twisted cubics tangent to 12 quadrics is given by

$$N = 2^{12}\cdot\sum_{j=0}^{12}\binom{12}{j}\nu^{12-j}\rho^j,$$

because if Q is a quadric then $r_j(Q) = 2$, $j = 0,1,2$. This reduces the computation of Schubert's number **1.2** (a) to the calculation of the characteristic numbers $n_j = \nu^{12-j}\rho^j$.

3. On the characteristic numbers of quadratic varieties.

3.1. Remark. If X is a complete variety, then our number $N = \alpha^n \beta^m \cdots$ is equal to the degree of the 0-dimensional rational class $[\alpha]^n [\beta]^m \cdots$, so that in this case we are lead to a computation in the intersection ring of X (such computations are often referred to as Schubert calculus). Important examples of this situation are the Grassmannian and flag varieties.

3.2. Remark. Let us notice here, as did Schubert, that Schubert calculus often gives also insight into relations among conditions for figures other than linear spaces and flags. Usually this is accomplished by what we may call "transfer of relations" by a correspondence (a pattern that repeats itself in many other situations). Assume

$$X \xleftarrow{p} Z \xrightarrow{q} Y$$

is a correspondence between the smooth varieties X and Y, with q a local complete intersection morphism and p proper. Then for any relation $R = 0$ in the intersection ring of Y we get a relation $p_* q^* R = 0$ in the intersection ring of X.

For example, if X is a family of plane curves of degree m in $\mathbf{P}^3$ and if we let Y be the full flag variety of $\mathbf{P}^3$, then we have a correspondence between X and Y given by associating to each curve in the family the flags consisting of a point of the curve, the tangent there and the plane where the curve lies. So we can transfer relations on Y to relations on the family of curves X. In particular, it is not hard to see that from the relation

$$(*) \qquad p^3 - p^2 h + ph^2 - h^3 = 0$$

(p the condition that the point of the flag lies in a plane, h the condition that the plane of the flag goes through a point) gives the relation $P = h\nu - mh^2$. The formula $T = h^2\rho - mh^3$ for the triple condition T (tangency to a line) can be deduced similarly (see Schubert [1879], p. 40). The relation $(*)$ itself is equivalent to the fundamental relation satisfied by the hyperplane class on the projective bundle associated to the tautological rank three vector bundle on $\mathbf{P}^{3*}$.

3.3. When the variety X is not complete, then a natural strategy is to try to find a compactification X^* of X in such a way that our numbers N can be computed on X^*, that is, so that

$$N = \alpha^{*n} \beta^{*m} \cdots$$

where the star denotes closure in X^*. This strategy works remarkably well for the space of quadratic varieties of rank r $(1 \leq r \leq n+1)$. In this Section we will explain this example, stressing how the numbers may be computed by reduction to the boundary of the compactification (see **1.6** and the references given there).

3.4. Quadratics. We will write $S_{\mathbf{P}}$ to denote the space of non-degenerate quadratic varieties of $\mathbf{P}$. Notice that in a given projective reference,

$$S_{\mathbf{P}} \simeq \mathbf{P}^{\frac{n(n+3)}{2}} - V(\det(a_{ij})).$$

In particular $S_{\mathbf{P}}$ is smooth and

$$\operatorname{Pic}(S_n) = \mathbf{Z}/(n+1)$$

(see Fulton [1984]).

More generally, let $S_{\mathbf{P}}^{(r)}$ denote the variety of rank r quadratic hypersurfaces (quadratics for short) in $\mathbf{P}$, so that $S_{\mathbf{P}} = S_{\mathbf{P}}^{(n+1)}$. If Q is a quadratic, we shall write $W(Q)$ to denote its double locus and $d(Q) = \dim(W(Q))$, so that $r(Q) = n - d(Q)$ is the rank of Q. The variety $S_{\mathbf{P}}^{(r)}$ comes equipped with a natural map

$$\pi^{(r)} : S_{\mathbf{P}}^{(r)} \longrightarrow \operatorname{Gr}(n-r, n),$$

which assigns to a given rank r quadratic Q the linear space $W(Q)$. It follows that $S_{\mathbf{P}}^{(r)}$ is a smooth quasiprojective variety (see the proof of **3.5**). On $S_{\mathbf{P}}^{(r)}$ the characteristic conditions $\mu_0, \ldots, \mu_{r-2}$ have order 1. We shall let ν_{r-1} denote the Schubert condition that $W(Q)$ intersects a given $(r-1)$-dimensional linear space .

3.5. Proposition.

(a) $\operatorname{Pic}(S^{(r)})$ *is generated by* $[\mu_0]$ *and* $[\nu_{r-1}]$.

(b) *The following relations hold:*

$$r[\mu_0] = 2[\nu_{r-1}],$$
$$\mu_i = (i+1)\mu_0, \text{ for } i = 0, \ldots, r-2.$$

Proof: Let $\Gamma_r = \operatorname{Gr}_{n-r}(\mathbf{P})$ be the Grassmannian of $(n-r)$-dimensional linear spaces in $\mathbf{P}$, $E' \subset E|\Gamma_r$ the tautological bundle over Γ_r and F the quotient of $E|\Gamma_r$ by E', so that $\operatorname{rank}(E') = n - r + 1$ and $\operatorname{rank}(F) = r$. Then the composition

$$\mathbf{P}(S^2F^*) \hookrightarrow \mathbf{P}(S^2E^*) \times \Gamma_r \xrightarrow{\text{pr}} \mathbf{P}(S^2E^*)$$

gives a map

$$\mathbf{P}(S^2F^*) \longrightarrow \mathbf{P}(S^2E^*)$$

whose image is the subvariety V_r of quadratics of rank $\leq r$. The fiber over a point $Q \in V_r$ is naturally isomorphic to $\mathrm{Gr}_{n-r}(W(Q))$. In particular it follows that if $\Delta \subset \mathbf{P}(S^2F^*)$ is the irreducible subvariety of singular quadratics then we have an isomorphism

$$\mathbf{P} - \Delta \simeq V_r - V_{r-1} = S_{\mathbf{P}}^{(r)}.$$

Under this isomorphism the class $\mu = c_1(\mathcal{O}_{\mathbf{P}}(1))$ corresponds to the class of μ_0. Moreover, if ℓ is the class of a codimension one Schubert cycle on Γ_r (that is, the class in $A^1(\Gamma_r)$ of the variety ℓ_L of $(n-r)$-dimensional linear spaces that meet a given $(r-1)$-dimensional linear space L), then $\ell = c_1(F)$ and $\pi^*(\ell) = \nu_{r-1}$. Since $\mathrm{Pic}\,(\mathbf{P}(S^2F^*))$ is freely generated by μ and $\pi^*(\ell)$, (a) is now clear. To see (b) it will be enough to prove that on $P = \mathbf{P}(S^2F^*)$ we have

$$[\Delta] = r\mu - 2\pi^*\ell$$

To see this, consider the composition

$$\begin{aligned} \mathcal{O}_P(-1) \hookrightarrow S^2F^*|P &\hookrightarrow \mathrm{Hom}(F|P, F|P) \\ &\xrightarrow{\Lambda^r} \mathrm{Hom}(\Lambda^r F|P, \Lambda^r F^*|P) \\ &\simeq \pi^*(\Lambda^r F)^{-2}. \end{aligned}$$

Since it is r-linear, it gives rise to a linear map

$$\mathcal{O}_P(-r) \to \pi^*(\Lambda^r F)^{-2}$$

and hence a section of

$$\mathcal{O}_P(r) \otimes \pi^*(\Lambda^r F)^{-2}$$

whose zero locus is, by construction, the scheme Δ. This proves the first relation in (b).

To see the second relation, let $L = P(\widetilde{L})$, where L is a given general i-dimensional linear space . Then degenerate quadratics on L form an irreducible hypersurface Δ_i of degree $i+1$ in the space $P(S^2\widetilde{L})$ of all quadratics in L and this hypersurface pulls back to $(\mu_i)_L$ by the restriction map $P(S^2E^*) \to P(S^2\widetilde{L})$. $\square$

3.6. Quadratic chains. Let $\mathbf{P} = \mathbf{P}(E)$, E a vector space of dimension $n+1$. Given any strictly decreasing sequence $I = (i_1 > \ldots > i_k)$, $k \geq 0$, $i_1 \leq n-1$, $i_k \geq 0$, we will be interested in the space $S(I) = S_{\mathbf{P}}(I)$ whose points parametrize "quadratic chains of type I in $\mathbf{P}$". We define a quadratic chain of type I to be a sequence $Q = (Q_0, Q_1, \ldots, Q_k)$ with the following properties: (a) Q_0 is a quadratic of $\mathbf{P}$, (b) for $j = 1, \ldots, k$, Q_j is a quadratic in $W(Q_{j-1})$, and (c) Q_k is non-degenerate. For a given k, there are $\binom{n}{k}$ such types. The dual of a quadratic chain of type I is a quadratic chain of type I^*, where $i_j^* = n - i_j - 1$.

3.7. Characteristic conditions for quadratic chains. We introduce contact conditions $\mu_0^I, \ldots, \mu_{n-1}^I$ on the variety $S(I)$ of quadratic chains $Q = (Q_0, \ldots, Q_k) = (Q_0, Q')$ of type I as follows. Let L be an i-dimensional linear space. If $i < n - i_1 - 1$, then $(\mu^I)_L$ is defined to be the condition that Q_0 has a contact with L. If $i = n - i_1 - 1$, then $(\mu^I)_L$ is defined to be $2\nu_{n-i_1-1}$, where ν_{n-i_1-1} is the Schubert condition that the linear space $W(Q_0)$ of double points of Q_0 intersects L. And if $i > n - i_1 - 1$, then $(\mu^I)_L$ is defined recursively to mean the condition that the quadratic chain Q' has a contact with $L \cap W(Q_0)$ (a condition μ_{i+n-i_1} for Q'). Then μ_i^I is the family $\{(\mu^I)_L\}_L$. They are all simple conditions.

3.8. Complete quadratics. In order to find the characteristic numbers of quadratic chains, we need to interprete them in terms of complete quadratics. Let

$$\lambda_i \colon S_{\mathbf{P}} \longrightarrow \mathbf{P}(S^2(\Lambda^i E))$$

be defined, in coordinates, as $A = (a_{ij}) \mapsto \Lambda^i A$. The reason for taking this map is that $\mathbf{P}(S^2(\Lambda^i E))$ parametrizes quadratics in $\mathbf{P}(\Lambda^i E)$, and the quadratic $\Lambda^i A$ intersects the Plücker embedding

$$\mathrm{Gr}_i(\mathbf{P}) \hookrightarrow \mathbf{P}(\Lambda^i E)$$

precisely along the cycle of i-linear spaces that are tangent to the quadratic A. Now the closure of the graph of $\lambda_2 \times \cdots \times \lambda_{n-1}$ is the space of complete quadratic varieties, $S_{\mathbf{P}}^*$. The group $G = \mathrm{PGL}(\mathbf{P}^n)$ acts on $S_{\mathbf{P}}^*$ in a natural way. It turns out that S_n^* has the following properties.

3.8.1. $S_{\mathbf{P}}^*$ is smooth and the codimension k orbits are in one to one correspondence with the strictly decreasing sequences $I = (i_1 > \ldots > i_k)$, where $i_1 \leq n-1$, $i_k \geq 0$, $0 \leq k \leq n$. Moreover, if $O(I)$ is the orbit corresponding to I, then there exists a natural isomorphism $O(I) \simeq S(I)$.

3.8.2. Let $D(I)$ be the closure of $S(I)$. Then the subvarieties $D(I)$ are smooth and an orbit $S(J)$ is contained in $D(I)$ if and only if $I \subseteq J$. Moreover, $D(I) \cap D(I') = D(I \cup I')$ and the intersection is transversal. In particular $D(I) = D(i_1) \cap \ldots \cap D(i_k)$.

3.8.3. The restriction of the (closures of the) conditions $\mu_0, \ldots, \mu_{n-1}$ to $D(I)$ are the (closures of the) conditions $\mu_0^I, \ldots, \mu_{n-1}^I$. In particular the characteristic conditions for $D(I)$ restrict to the characteristic conditions of $D(J)$ if $I \subseteq J$. This implies that in order to compute the characteristic numbers for the quadratics we can work on $S_{\mathbf{P}}^*$. And since this is complete, we can work directly with the rational classes of the characteristic conditions (see **3.1** and **3.3**).

3.9. Degeneration relations for quadratic chains. The closed orbits that have codimension 1 in $D(I)$ are the varieties $D(I)_j =: D(I \cup \{j\}) = D(I) \cap D(j)$, where $0 \leq j \leq n-1$, $j \notin I$. These $n-k$ varieties will be referred to as first order degenerations of $D(I)$.

Next statement gives a formula that yields the rational classes of such degenerations in terms of the characteristic conditions. A proof of it will be sketched at the end of this Section.

3.9.1. Theorem. *Let I be as before. For $0 \le j \le n-1$, let $\delta(I)_j$ be the rational class $D(I) \cdot D(j)$, so that for $j \notin I$ we have $\delta(I)_j = [D(I)_j]$. Then, with the convention that an expression vanishes if a subindex is out of the range $[0, n-1]$, we have:*

$$\delta(I)_{n-j-1} = -\mu^I_{j-1} + 2\mu^I_j - \mu^I_{j+1}.$$

3.10. Characteristic numbers of quadratic chains. Statement **3.9.1** gives a procedure for computing the characteristic numbers of all quadratic chains. In fact here we will show how the previous formula allows to transform the computation of a characteristic number on $D(I)$ into computations of characteristic numbers on the first order degenerations of $D(I)$, so that by iterating this step we may reduce the computation of any characteristic number for quadratic chains to the computation of characteristic numbers on $D(n-1, n-2, \ldots, 1, 0)$, which is the variety of complete flags (cf. De Concini-Procesi [1983, 85], De Concini-Gianni-Traverso [1985]).

In order to explain this we have to introduce some notations. Given a positive integer r, we shall denote by $A(r) = (a_{ij})$ the matrix defined as follows, where $0 \le i, j \le r-1$:

$$a_{ij} \doteq \begin{cases} 2 \text{ if } j = i \\ -1 \text{ if } j = i \pm 1 \\ 0 \text{ otherwise} \end{cases}$$

Then it is easy to check that $\det(A(r)) = r+1$, for all r. Let $B(r) = (b_{ij})$, $0 \le i, j \le r-1$, be the $r \times r$ matrix defined by the relation

$$B(r)A(r) = (r+1)I_r,$$

where I_r is the identity $r \times r$ matrix. Then it is easy to check that we have:

3.10.1. Lemma.

$$b_{ij} = \begin{cases} (i+1)(r-j) \text{ if } i \le j \\ (j+1)(r-i) \text{ if } j \le i \end{cases}$$

Let now I be a sequence as before. For each ℓ in the range $[1, k]$, let

$$I_\ell =: [i_\ell + 1, i_{\ell-1} - 1], \quad I^*_\ell =: [n - i_{\ell-1}, n - i_\ell - 2],$$

so that when j runs (increasing) on the range I^*_ℓ then $n - j - 1$ runs (decreasing) the range I_ℓ. We set $r_\ell = i_{\ell-1} - i_\ell - 1$, the number of elements in the interval I_ℓ (or I^*_ℓ). Let $\mu^I_{I^*_\ell}$ be the piece of μ^I with subindices in I^*_ℓ and $\delta(I)_{I_\ell}$ the piece of $\delta(I)$ involving only the integers in I_ℓ, arranged in decreasing order. Finally let Z_ℓ be the row $(\mu_{n-i_{\ell-1}-1}, 0, \ldots, 0) + (0, \ldots, 0, \mu_{n-i_\ell-1})$ (r_ℓ components in each summand). Then the expressions in **3.9.1** corresponding to j in the range I_ℓ can be written, in matrix form, as follows:

3.10.2. Lemma.
$$\delta(I)_{I_\ell} = \mu^I_{I^\bullet_\ell} A(r_\ell) - Z_\ell.$$

From this relation we obtain the following relation:

3.10.3. Proposition.

$$(r_\ell + 1)\mu^I_{I^\bullet_\ell} = \delta(I)_{I^\bullet_\ell} B(r_\ell) + Z_\ell B(r_\ell),$$

which written in explicit form is equivalent to the following relations:

$$\begin{aligned}(r_\ell + 1)\mu^I_{i_\ell+1+j} = & \sum_{0\le p\le j} (p+1)(r_\ell - j)\delta(I)_{n-i_\ell-2-p} \\ & + \sum_{j<p\le r_\ell - 1} (j+1)(r_\ell - p)\delta(I)_{n-i_\ell-2-p} \\ & + (r_\ell - j - 1)\mu_{n-i_{\ell-1}-1} + (j+1)\mu_{n-i_\ell-1}.\end{aligned}$$

In particular we have that

$$(n+1)\mu_j = \sum_{0\le p\le j} (p+1)(n-j)\delta_{n-p-1} + \sum_{j< \ \ p\le n} (j+1)(n-p)\delta_{n-p-1}.$$

These relations tell us, together with **3.8**, that we may substitute conditions μ_{n-j-1}, $j \notin I$, in a given characteristic number of $D(I)$, in terms of of first order degenerations of $D(I)$ and conditions μ_{n-j-1} with $j \in I$, which has the effect of transforming, if $|I| < n$, the given characteristic number into a linear combination of characteristic numbers on the first order degenerations and characteristic numbers for which the number of occurrences of conditions of the form μ_{n-i-1}, $i \notin I$, is one less that in the number we started with. Iterating, at the end we will be reduced to compute only characteristic numbers on the variety of complete flags. Notice that if $|I| < n$ then any characteristic number involving only conditions μ_{n-j-1} for $j \in I$ is 0, by reasons of dimension. In any case it is important to remember that $\mu_{n-i-1} = 2\nu_{n-i-1}$ for $i \in I$, where ν_{n-i-1} is the one dimensional Schubert condition on i-dimensional linear spaces.

3.11. Proof of Theorem 3.9. Since the left members of the relations are obtained intersecting $D(n-j-1)$ with $D(I)$ and the characteristic conditions restrict to the characteristic conditions it is enough to show that on S^* we have

$$D(n-j-1) = -\mu_{j-1} + 2\mu_j - \mu_{j+1}.$$

These are well know relations. In the rest of this section we sketch a simple proof by induction.

On $\mathbf{P}^1$ the quadratics are already complete and $D(0)$ is a conic, so that $\delta(0) = 2\mu_0$. The relations are therefore true for $n = 1$. Assume now $n > 1$. Take a hyperplane $\mathbf{P}'$ and consider the linear projection $S_{\mathbf{P}} \to S_{\mathbf{P}'}$ given by restriction. The center of this projection is the space of pairs of hyperplanes that contain $\mathbf{P}'$. For any linear space W we have also a linear projection $S_W \to S_{W \cap \mathbf{P}'}$ given by restriction from W to $W \cap \mathbf{P}'$. If Q is a quadratic on W that does not contain $W \cap \mathbf{P}'$ (that is, such that $\mathbf{P}'$ intersects Q properly) we will let $Q \cap \mathbf{P}'$ denote the image of Q by this map. Let now I be a sequence as before. Define I^- as follows:

$$I^- = \begin{cases} (i_1 - 1, \ldots, i_k - 1) \text{ if } i_k > 0 \\ (i_1 - 1, \ldots, i_{k-1} - 1) \text{ if } i_k = 0 \end{cases}$$

Then we have a rational map $S_{\mathbf{P}}(I) \to S_{\mathbf{P}'}(I^-)$ defined by

$$(Q_0, Q_1, \ldots, Q_k) \mapsto (Q_0 \cap \mathbf{P}', \ldots, Q_k \cap \mathbf{P}')$$

which is regular at all quadratic chains that are intersected properly by $\mathbf{P}'$. In particular it is regular for quadratic chains that do not satisfy $V = (\mu^I_{n-1})_{\mathbf{P}'}$. Away from (the closure of) this variety the map is smooth and pulls $\mu_j^{I^-}$ $(0 \leq j \leq n-2)$ back to μ_j^I. Moreover, these maps are compatible in the sense that they fit together into a smooth map of $S^*_{\mathbf{P}} - V$ to $S^*_{\mathbf{P}'}$ and $D(j)$ is, for $j > 0$, the pull-back of $D'(j-1)$, which gives the relation for $D(j)$, $j > 0$, up to terms in μ_{n-1}. Since by duality the matrix of the $(\delta_{n-1}, \ldots, \delta_0)$ is simmetric with respect to both diagonals, this leaves undetermined only the coefficient of μ_{n-1} in the expression of δ_0. But it is easy to see that this coefficient is -1 for $n = 2$ and 0 for $n > 2$ (for instance using **3.5**). $\square$

4. Characteristic numbers of twisted cubics

In this Section we derive in a new and more simple way the basic degeneration relations used in Kleiman-Strømme-Xambó [1987] to compute the characteristic numbers $\nu^j \rho^{12-j}$. Our approach also allows to compute all characteristic numbers $P^i \nu^j \rho^{12-2i-j}$.

4.1. Proposition. *Let $P_0, \ldots, P_5$ be 6 points in $\mathbf{P}^3$, no four of them in a plane. Then there is a unique twisted cubic C containing $P_0, \ldots, P_5$. In particular $P^6 = 1$.*

Proof: The quadrics containing a given twisted cubic form a 2-dimensional system which cuts out the cubic. Since six points in general linear position impose independent conditions on quadrics, we see that there is at most one twisted cubic containing the points $P_0, \ldots, P_5$, and that if it exists it must be the scheme cut out by the quadrics containing the points.

Now take $\Sigma = (P_0, \ldots, P_3; P_4)$ as projective reference of $\mathbf{P}^3$ and let $P_5 \equiv (a) = (a_0, a_1, a_2, a_3)$. By our hypothesis, $a_i \neq 0$ and if $i \neq j$ then $a_i \neq a_j$. Let

$$Y_i = \frac{a_0 X_i - a_i X_0}{a_0 - a_i}, \quad i = 1, 2, 3,$$

where $X_0, \ldots, X_3$ are the homogeneous coordinates of $\mathbf{P}^3$ with respect to Σ. Let $Q_{ij} = Q_{ij}^{(a)}$ be the quadric defined by the equation $X_i Y_j - X_j Y_i = 0$. Then Q_{12}, Q_{13}, Q_{23} are independent quadrics containing $P_0, \ldots, P_5$, so that if the cubic exists it must be cut out by them. Since the equation of $Q_{ij}^{(a)}$ may also be written in the form

$$a_0(a_i - a_j)X_i X_j + a_i(a_j - a_0)X_j X_0 + a_j(a_0 - a_i)X_0 X_j = 0,$$

we see that Q_{ij} is a cone with vertex at the point P_k such that $\{1,2,3\} - \{i,j,k\}$. Since the directrix of this cone is the conic through the projections of $\{P_j\}_{j \neq k}$ from P_k, we see that this cone has rank 3. On the other hand the lines L_i and L_j, where $L_i := \{X_0 = X_i = 0\}$, are rulings of Q_{ij}. From this it follows that $Q_{12} \cap Q_{13}$ is the union of the line L_1 and a cubic C that contains $P_0, \ldots, P_5$. Since Q_{23} does not contain L_1, it follows that C is $Q_{12} \cap Q_{13} \cap Q_{23}$. □

4.2. Remark. Assume now that P_5 moves on a general line L. Thus

$$a_i = \alpha_i \lambda + \beta_i \mu, \quad 0 \leq i \leq 3,$$

where $(\alpha), (\beta) \in L$, $(\alpha) \neq (\beta)$. Then it is easy to see that

$$(*) \qquad Q_{ij}^{(a)} = \lambda^2 Q_{ij}^{(\alpha)} + \lambda\mu Q_{ij}^{(\alpha,\beta)} + \mu^2 Q_{ij}^{(\beta)},$$

where

$$Q_{ij}^{(\alpha,\beta)} = (\alpha_0(\beta_i - \beta_j) + (\beta_0(\alpha_i - \alpha_j))X_i X_j + (\alpha_i(\beta_j - \beta_0) + (\beta_i(\alpha_j - \alpha_0))X_j X_0 + \\ (\alpha_j(\beta_0 - \beta_i) + (\beta_j(\alpha_0 - \alpha_i))X_0 X_i,$$

which also is a cone with vertex P_k (same notations as in **4.1**).

Let $C_{(\lambda,\mu)}$ be the twisted cubic going through the points $P_0, \ldots, P_5$, for general (λ, μ). When we let P_5 move on L then $C_{(\lambda,\mu)}$ sweeps out a surface S. By the expression of $Q_{ij}^{(a)}$ above we see that S is contained in the octic surfaces obtained solving any two of the three equations $Q_{ij}^{(a)} = 0$ for $(\lambda^2, \lambda\mu, \mu^2)$ and imposing the unique relation $(\lambda\mu)^2 = \lambda^2\mu^2$. From this it follows that $\deg(S) \leq 8$.

Another way of seeing this is the following. Take a second general line L'. Parametrize L' using points (α'), (β') and parameters (λ', μ'). Then the cubics through the five points

$P_0, \ldots, P_4$ that meet L and L' are in one to one correspondence with the intersections of the three curves C_{ij} on the quadric $L \times L'$ obtained replacing $(X_0, \ldots, X_3)$ in the equations $Q_{ij}^{(a)} = 0$ by $(\alpha'\lambda' + \beta'\mu')$. These curves are efective cycles of type $(2,2)$ on $L \times L'$. From this and the fact that the intersection number of any pair of such cycles is 8 it is not hard to see that there are at most 8 points in their intersection.

4.3. Schubert's space X^*. Let X^* be the "partial compactification" of the space of twisted cubics X obtained in Kleiman-Strømme-Xambó [1987] and let P^*, ν^*, ρ^* be the closures in X^* of the characteristic conditions P, ν, ρ. Given non-negative integers i, j such that $2i + j \leq 11$, we shall let $\Gamma_{ij} = P^i \nu^j \rho^{11-2i-j}$ be the 1-dimensional system of twisted cubics obtained intersecting the corresponding conditions with data in general position. By transversality of general translates, in characteristic 0 it is a reduced curve. We shall write $\Gamma_{ij}^* = P^{*i} \nu^{*j} \rho^{*11-2i-j}$ to denote the intersection of the corresponding closed conditions. We will refer to the systems Γ_{ij}, or Γ_{ij}^*, as characteristic systems of twisted cubics. Now we will use the following facts:

4.3.1. X^* is smooth.

4.3.2. $X^* - X$ is the union of two (disjoint) smooth hypersurfaces D_0 and D_1.

4.3.3. D_0 is isomorphic to the variety of planar nodal cubics, which is an orbit under the action of $\mathrm{PGL}(\mathbf{P}^3)$. We shall let P_0, ν_0 and ρ_0 denote the characteristic conditions on D_0.

4.3.4. $P^* \cdot D_0 = P_0, \quad \nu^* \cdot D_0 = \nu_0, \quad \rho^* \cdot D_0 = \rho_0.$

4.3.5. D_1 is isomorphic to the variety parametrizing cubics consisting of a conic C and a line L meeting C at a unique point Q. It is also an orbit under the action of $\mathrm{PGL}(\mathbf{P}^3)$. We shall write P_C, ν_C and ρ_C to denote the conditions on D_1 corresponding to the characteristic conditions of C; P_L and ν_L the conditions corresponding to L going through a point and meeting a line, respectively; and Q the condition that the distinguished point lies on a plane.

4.3.6. $P^* \cdot D_1 = P_C + P_L, \quad \nu^* \cdot D_1 = \nu_C + \nu_L, \quad \rho^* \cdot D_1 = \rho_C + 2Q.$

4.3.7. From **4.3.4** and **4.3.6** it follows that if Γ is a characteristic system then Γ^* is a closed curve in X^*, so that it coincides with the closure of Γ. It is a basic result in Kleiman-Strømme-Xambó [1987] that *these closed characteristic systems are complete.* As we shall see, this is one of the crucial facts that allow to determine the characteristic numbers of the twisted cubics without needing any further knowledge of the boundary of X in some compactification. In other words, it plays the role that the compactification does when the "compactify strategy" works.

4.4. Theorem (degeneration relations). *On X^* the following relations hold:*

$$2\nu \equiv 3D_0 + D_1$$
$$\rho \equiv D_0 + D_1$$

Proof: It will follow from paragraphs **4.5** to **4.10**. $\square$

4.5. We shall also use the fact, attributed to Ellingsrud in Piene [1985], that $\mathrm{Pic}(X) = \mathbf{Z}/(2)$. Then one easily sees that there exist integers a, b, c, d such that

$$2\nu^* \equiv aD_0 + bD_1, \quad 2\rho^* \equiv cD_0 + dD_1,$$

where $\equiv$ denotes the rational equivalence relation. As we shall see in next paragraph, the knowlewdge of the coefficients a, b, c, d, plus enumerative information on D_0 and D_1, allow us to compute all characteristic numbers.

4.6. Degeneration reduction. Let Γ be a characteristic system. Let α be either ν or ρ and $N = \alpha \cdot \Gamma$. By **4.3.4** and **4.3.6** we have that $N = \alpha^* \cdot \Gamma^*$. Now this intersection is the degree of the restriction of α^* to Γ^*. *Since Γ^* is complete, this degree coincides with the degree of the corresponding linear class*, that is, the degree of the restriction of the rational class of α. If $2\alpha \equiv pD_0 + qD_1$, p, q integers, then $2N$ will coincide with $p\deg([D_0]\cdot\Gamma^*) + q\deg([D_1]\cdot\Gamma^*)$. But since Γ^* intersects D_0 and D_1 properly the last expression is equal to $p(D_0\cdot\Gamma^*) + q(D_1\cdot\Gamma^*)$. Let $\Gamma_{D_k} = (D_k\cdot\Gamma^*)$. We will say that Γ_{D_0} and Γ_{D_1} are the degeneration numbers of Γ. It is clear then that

$$\Gamma_{D_k} = (P^*\cdot D_k)^i(\nu^*\cdot D_k)^j(\rho^*\cdot D_k)^{11-2i-j}$$

and hence

$$(*) \qquad \Gamma_{D_0} = P_0^i \nu_0^j \rho_0^{11-2i-j},$$

$$(**) \qquad \Gamma_{D_1} = (P_C + P_L)^i(\nu_C + \nu_L)^j(\rho_C + 2Q)^{11-2i-j},$$

so that the computation of the degeneration numbers is reduced to the solution of other suitable enumerative problems on the degeneration varieties D_0 and D_1, which are well understood.

4.7. Examples of degeneration numbers.

$$(P^5\nu)_{D_1} = (P_C + P_L)^5(\nu_C + \nu_L) = \binom{5}{3}P_C^3P_L^2\nu_C = 10.$$

The second step follows because all other terms in the expansion are easily seen to vanish; for instance, a conic does not go through 4 or more points in general position, and a line does not go through more than 2, so that the only possible choice from the first factor is $P_C^3 P_L^2$. But then from the other factor we have to choose ν_C, for the line determined by P_L^2 cannot meet another line in general position. Finally we are easily reduced to finding the number of conics through 5 points in a plane. The same argument gives also $(P^5\rho)_{D_1} = 20$.

Here are a few other degeneration numbers, that will be used below, and which can be obtained in a similar way:

$$\begin{array}{llll}
(P^3\nu^5)_{D_0} & = 12 & (P^3\nu^5)_{D_1} & = 344 \\
(P^3\nu^4\rho)_{D_0} & = 36 & (P^3\nu^4\rho)_{D_1} & = 604 \\
(P^3\nu^3\rho^2)_{D_0} & =100 & (P^3\nu^3\rho^2)_{D_1} & = 980
\end{array}$$

4.8. $P^5\nu^2$. Applying the principles explained in **4.5** we find that

$$2P^5\nu^2 = b(P^5\nu)_{D_1} = 10b.$$

Since, by **4.2**, $P^5\nu^2$ is at most 8, it follows that $P^5\nu^2 = 5$ and $b = 1$.

4.9. Now we get a relation between b and d expressing $P^5\nu\rho$ in two different ways:

$$2P^5\nu\rho = \begin{cases} (P^5\rho)_{D_1} = 20 \\ d(P^5\nu)_{D_1} = 10d \end{cases}$$

and so we get that $d = 2$.

4.10. To determine a and c we cannot use numbers containing P^4. Thus we look at $P^3\nu^5\rho$ and $P^3\nu^4\rho^2$. Proceeding, as in **4.7**, to express each of these two numbers in two ways we will get two relations that allow us to determine a and c. In fact the relations we obtain are the following:

$$3a = c + 7, \quad 25a = 9c + 57,$$

so that $a = 3$, $c = 2$.

4.11. With the degeneration relations **4.4** and the degeneration reduction **4.6** we can proceed to determine all characteristic numbers $P^i\nu^j\rho^{12-2i-j}$ of the twisted cubics, much as it is done in Kleiman-Strømme-Xambó [1987] for the 13 numbers $\nu^{12-i}\rho^i$, $0 \leq i \leq 12$. The actual computation is reduced to enumerative problems on D_0 and D_1 which are, as said above, well understood and verified. So altoghether we have:

4.12. Theorem. *The values of the characteristic numbers $P^i \nu^j \rho^{12-2i-j}$ for the family of twisted cubics agree with the values listed in* Schubert [1879], pp. 171-180.*

5. Fundamental numbers of plane cuspidal cubics

Here we give a quick survey of the results in Miret-Xambó [1987, 88] in order to form an idea of how one may deal with the second example in paragraph **1.2**.

5.1. Motivation. An important goal in the framework of Hilbert's 15th problem is the verification of all the fundamental numbers for the twisted cubics. These are the numbers involving, aside from P, ν and ρ, the conditions:

β to have a 2-secant in a pencil,

σ to have an osculating line in a pencil,

T to be tangent to a line,

and the dual conditions P', ν', ρ', β' (σ and T are self-dual). Recall that a line L through a point Q of a cubic C is an osculating line of C at Q if L is contained in the osculating plane to C at Q. Notice that β and σ are simple conditions and that T is a third order condition.

If we want to apply the method of degeneration, as explained in Section 4 for the characteristic conditions, it is necessary to know how to deal effectively with the enumerative geometry on the boundary components of some suitalbe partial compactification. Since among such components there is one whose enumerative geometry is equivalent to the enumerative geometry of cuspidal cubics, we see that a first step is to understand and verify the fundamental numbers for the cuspidal cubics. In what follows we will survey the results in this direction for plane cuspidal cubics and then return to example **1.2 b**). The enumerative geometry of space cuspidal cubics can be worked out similarly, but involving more computations.

5.2. Complete cuspidal cubics and degenerations. If we think of a cuspidal cubic as consisting not only of points, but also of the dual cubic and the complete triangle whose vertices are c, v, y (singular triangle) then we can define the space of compete cuspidal cubics S^* as the compactification of the space S of cuspidal cubics with respect to those structures, that is, S^* is the graph of the map that assigns to a cuspidal cubic the dual cubic and the singular triangle.

Schubert showed that $S^* - S$ contains at least 13 hypersurfaces, say $D_0, \ldots, D_{12}$ (see the picture page at the end). Now in Miret-Xambó [1987] it is proved that

*The first entry in the second table on p. 177 is misprinted as $P^4\nu^3 = 30$, instead of $P^4\nu^4 = 30$.

S^* is smooth in codimension 1,

by exhibiting S^* as the result of blowing up successively a smooth variety along centers that are regularly immersed, and that

$S^* - S$ does not contain components other than the D_i.

Then in Miret-Xambó [1988] we give a detailed geometric description of all the D_i and work out the enumerative geometry for the corresponding fundamental conditions.

5.3. Fundamental systems and degeneration numbers. We define fundamental systems of cuspidal cubics as 1-dimensional systems defined by fundamental conditions. By the nature of S^* it follows that the closed fundamental systems are complete curves and so we can try to follow a procedure as the one we have explained in Section 4. This has been done in Miret-Xambó [1988]. The key points are the computation of degeneration numbers of fundamental systems (Section 9) and the derivation of the degeneration relations for the first order fundamental conditions (Section 10). In this case the Picard group of S is and extension of $\mathbf{Z}/(5)$ by an infinite cyclic group generated by c (Theorem 1.3), so that in particular for any first order condition on S, 5α is rationally equivalent to a multiple of c. This facts manifiests itself, on S^*, with the presence in the degeneration relations of a term in c, in addition to a linear combination of the degenerations with integer coefficients. For example, the degeneration relation for 5μ is the following:

$$5\mu \equiv 3c + 2D_0 + 3D + 6D_4 + 2D_5 + 3D_6 + 4D_7 + 3D_8 + 9D_9 + 9D'$$

where $D = D_1 + D_2 + D_3$ and $D' = D_{10} + D_{11} + D_{12}$.

5.5. $\mu^2\nu^2cvz = 55$. Indeed, let $N = \mu^2\nu^2cvz$. Then

$$5N = (5\mu)(\mu\nu^2cvz) = \sum a_i(\mu\nu^2cvz)_{D_i} + a(\mu\nu^2c^2vz),$$

where a_i, a are the coefficients of the degeneration relation given in the previous paragraph.

The number $N' = \mu\nu^2c^2vz$ can be obtained similarly. Notice that the term in c^3 that comes from the c term in the degeneration relation is 0, so that N' is already a linear combination of degeneration numbers only. Thus N itself can be expressed as a linear combination of degeneration numbers. We find $N' = 9$. Now the non-zero degeneration numbers of $\Gamma = \mu\nu^2cvz$ turn out to be $\Gamma_{D_0} = 2$, $\Gamma_{D_5} = 27$, $\Gamma_{D_8} = 1$, $\Gamma_{D_4} = 13$, $\Gamma_{D_{10}} = 6$ and $\Gamma_{D_{12}} = 9$, which finally gives

$$5N = 2\cdot 2 + 2\cdot 27 + 3\cdot 1 + 4\cdot 13 + 9\cdot 6 + 9\cdot 9 + 3N' = 275,$$

so $N = 55$.

5.6. Final remark. The list of all fundamental numbers for plane cuspidal cubics is tabulated in the last Section in Miret-Xambó [1988]. This not only has verified Schubert's results,* we believe for the first time, but has also gone deeper into the geometry of cuspidal cubics and has completed the tables that Schubert gave more than a century ago.

References

Ballico, E., Ciliberto, C. [1988], *Algebraic curves and projective geometry, Trento 1988*, to appear in LNiM.

Bifet, E. [1988], *On complete symmetric varieties*, Preprint Univ. Roma II, 1988.

Bifet, E., De Concini, C., Procesi, C. [1988], *Cohomology of complete symetric varieties*, Preprint 1988.

Bloch, S. (ed.) [1987], *Algebraic Geometry, Bowdoin 1985*, Proc. of Symp. in Pure Math. 46 (2 vols.), AMS, 1987.

Casas, E. [1987], *A transversality theorem and an enumerative calculus for proper solutions*, Preprint Univ. Barcelona, 1987.

Casas, E., Xambó, S. [1986], *The enumerative theory of conics after Halphen*, Lect. Notes in Math. 1196, Springer-Verlag, 1986.

De Concini, C., Gianni, P., Traverso, C. [1985], *Computation of new Schubert tables for quadrics and projectivities*, in: Hotta [1985], 515-523.

De Concini, C., Goreski, M., MacPherson, R., Procesi, C. [1986], *On the geometry of quadrics and their degenerations*, Preprint IHES, 1986.

De Concini, C., Procesi, C. [1983], *Complete symmetric varieties*, in: Gherardelli [1983].

De Concini, C., Procesi, C. [1985], *Complete symmetric varieties, II – Intersection theory*, in: Hotta [1985], 481-513.

Finat, J. [1983], Thesis, Univ. Valladolid (Spain).

Fulton, W. [1984], *Intersection Theory*, Ergebnisse 2 (3. Folge), Springer-Verlag (1984).

Fulton, W., Kleiman, S., MacPherson, R. [1982], *About the enumeration of contacts*, in: *Proceedings Ravello 1982*, Springer-Verlag, Lect. Notes 997.

*The first number on the fourth row on p. 141 of Schubert [1879] should be 15, not 17; taking into account the last entry in the row $\mu^3 q_e w$ on p. 138, it looks like it is just a simple misprint.

Gherardelli, F. (ed.) [1983], *Invariant theory – Proceedings Montecatini*, Lect. Notes 996, Springer-Verlag, 1983.

Ghione, F., Peskine, C., Sernesei, E. (eds.) [1987], *Space curves* (Proceedings, Rocca di Papa 1985), Lect. Notes 1266, Springer-Verlag, 1987.

Holme, A., Speiser, B. (eds.) [1988], *Algebraic Geometry, Sundance 1986*, Lect. Notes in Math. 1311, Springer-Verlag 1988.

Hotta, R. (ed.) [1985], *Algebraic groups and related topics*, Advanced studies in Pure Math. 6, North-Holland, 1985.

Kleiman, S. [1974], *The transversality of a general translate*, Compositio 38 (1974), 287-297.

Kleiman, S. [1980], *Chasles enumerative theory of conics. A historical introduction,* in: *Studies in Algebraic Geometry*, Studies in Math. 20, M. A. A. (1980).

Kleiman, S., Strømme, A., Xambó, S. [1987], *Sketch of a verification of Schubert's number 5819539783680 of twisted cubics*, in: Ghione-Peskine-Sernesi [1987], 156-180.

Kleiman, S. [1987], *Intersection theory and enumerative geometry. A decade in review*, in: Bloch [1987], Part 2, 321-370 (with the collaboration of A. Thorup in Section 3).

Kleiman, S., Thorup, A. [1988], *Complete bilinear forms*, in: Holme-Speiser [1988], 253-320.

Laksov, D. [1982], *Notes on the evolution of complete correlations*, in: Enumerative Geometry and Classical Algebraic Geometry, Progress in Math. 24, Birkhäuser (1982), 107-132.

Laksov, D. [1986], *Complete linear maps, I: The geometry*, Trita-Mat-1986-5, Royal Institute of Technology, Stockholm, 1982.

Laksov, D. [1987], *Completed quadrics and linear maps*, in: Bloch [1987], Part 2, 371-387.

Miret, J. M., Xambó, S. [1987], *On Schubert's degenerations of cuspidal cubics*, Preprint, Univ. Barcelona 1987.

Miret, J. M., Xambó, S. [1988], *Geometry of complete cuspidal cubics*, in: Ballico-Ciliberto [1988].

Piene, R. [1985], *On the problem of enumerating twisted cubics,* in: *Algebraic Geometry – Sitges 1983* (Casas-Welters-Xambó eds.), Lect. Notes 1124, Springer-Verlag 1985.

Schubert, H. C. H. [1870], *Zur Theorie der Charakteristiken*, Crelles Journ. 71 (1870), 366-386.

Schubert, H. C. H. [1879], *Kalkül der abzählenden Geometrie*, Teubner 1879; Reprinted in 1979 by Springer-Verlag, with an introduction by S. Kleiman.

Schubert, H. C. H. [1894], *Allgemeine Anzahfunctionen für Kegelschnitte, Flächen und Räume zweites Grades in n Dimensionen,* Math. Ann. 45 (1894), 153-206.

Semple, J.; Roth, L. [1949], *Introduction to Algebraic Geometry*, Clarendon Press, 1949 (paperback corrected reprint, with new index, 1985).

Severi, F. [1940], *I fondamenti della geometria numerativa*, Ann. di Mat. (4)19 (1940), 151-242.

Speiser, B. [1988], *Transversality theorems for families of maps*, in: Holme-Speiser [1988], 235-252.

Tyrrell, J. A. [1956], *Complete quadrics and collineations in* S_n, Mathematika 3 (1956), 69-79.

Vainsencher, I. [1982], *Schubert calculus for complete quadrics*, in: *Enumerative Geometry and Classical Algebraic Geometry*, Proc. Nice 1980 (P. Le Barz, Y. Hervier, eds.), Birkhäuser (1982), 199-235.

Van der Waerden, B. L. [1938], *Zur algebraische Geometrie XV: Lösung des charakteristiken Problem für Kegelschnitte*, Math. Ann. 115 (1938), 645-655.

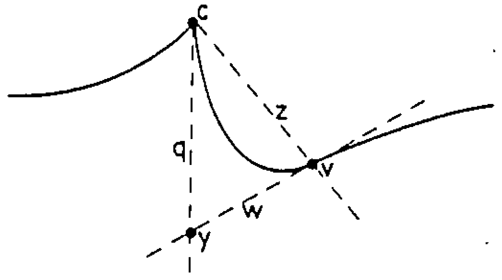
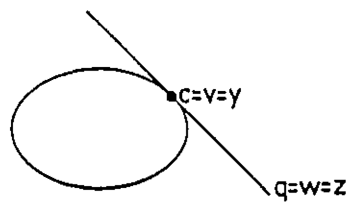


Fig.1

D_0



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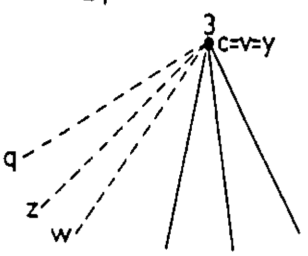

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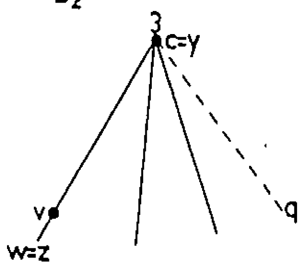

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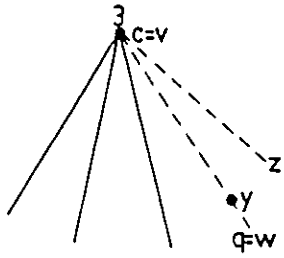

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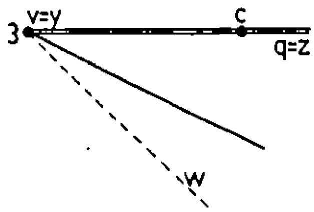

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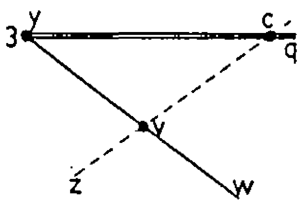

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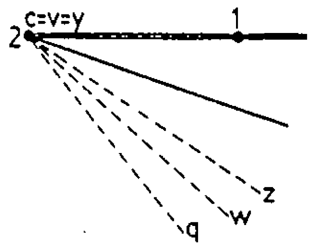

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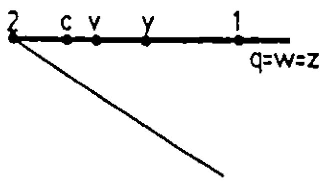

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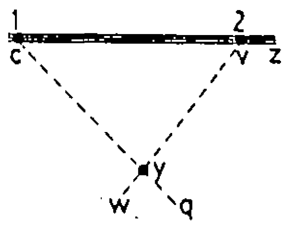

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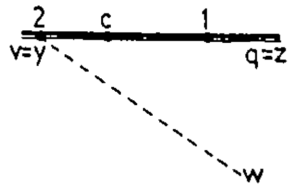
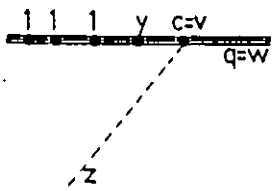


D_{10}



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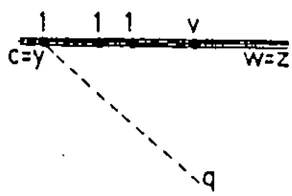

D_{12}