Non-commutative geometry and deformations of A_∞-algebras and A_∞-categories

Yan Soibelman

July 10, 2003

1 Introduction

My talk is based on several joint projects with Maxim Kontsevich. The largest one aims to revisit deformation theory (at least in the case of characteristic zero) and will be presented in [KoSo2]. Today I am going to review the main ideas in the framework of A_∞-algebras and A_∞-categories.

2 Ind-affine schemes in tensor categories

Let $\mathcal{C}$ be a k-linear symmetric tensor category (for simplicity we will always assume that k is a field of characteristic zero).

Definition 1 *a) The category $Aff_{\mathcal{C}}$ of affine schemes in $\mathcal{C}$ is the one opposite to the category $ComAlg_{\mathcal{C}}$ of commutative associative unital algebras.*

b) The category $NAff_{\mathcal{C}}$ of non-commutative affine schemes in $\mathcal{C}$ is the one opposite to the category $Alg_{\mathcal{C}}$ of associative unital algebras.

Let $\mathcal{S}$ be either of these two categories. Then one can form the category $Ind(\mathcal{S})$ of ind-objects called ind-affine (resp. non-commutative ind-affine) schemes in $\mathcal{C}$. Let us assume from now on that $\mathcal{C} = Vect_k^{\mathbf{Z}}$ is the tensor category of $\mathbf{Z}$-graded vector spaces (we allow infinite sums). The commutativity constraint satisfies the Quillen rule of signs: $c(a \otimes b) = (-1)^{deg\, a \cdot deg\, b} b \otimes a$.

We will be mainly interested in the subcategories of $Ind(\mathcal{S})$ corresponding to the coalgebras (which we will assume to be cocommutative in the case of $Aff_{\mathcal{C}}$). Such ind-affine schemes will be called *special*. To save space we will only discuss non-commutative special schemes. The corresponding definitions in the more familiar case of commutative schemes are left as exercises to the reader.

Example 1 *Let Q be a quiver in $Vect_k^{\mathbf{Z}}$ (i.e. arrows form a $\mathbf{Z}$-graded vector space). Then the coalgebra B_Q of this quiver gives rise to a non-commutative special scheme $X_{B_Q} := X_Q$. It contains the non-commutative ind-subscheme X_I of disjoint vertices of Q. There are natural morphisms $\pi : X_Q \to X_I$ (projection) and $j : X_I \to X_Q$ (closed embedding).*

Let $Q = Q_V$ be a quiver with one vertex, and the space of loops isomorphic to $V \in Vect_k^{\mathbf{Z}}$. Then X_{Q_V} is called the formal neighborhood of zero in the non-commutative space V.

Non-commutative special schemes form a symmetric tensor category (the tensor product corresponds to the tensor product of coalgebras).

Theorem 1 *([KoSo3]) For any two non-commutative special schemes X and Y there exists their inner Hom denoted by $Maps(X, Y)$, i.e. for any Z one has*

$$Hom(Z, Maps(X, Y)) \simeq Hom(Z \otimes X, Y).$$

One can do differential geometry in the category of non-commutative special schemes. In particular, there are notions of smoothness (in its formal variant of Cuntz and Quillen), vector fields (they correspond to coderivations of coalgebras), differential forms, tangent and cotangent bundles, etc.

3 Non-commutative differential-graded schemes and A_∞-categories

Let (X, X_I, π, j) be a quadruple consisting of special schemes X, X_I, a closed embedding $j : X_I \to X$, and a projection $\pi : X \to X_I$ such that $\pi \cdot j = id$. We will consider quadruples isomorphic to those obtained from graded quivers (see Example 1).

Definition 2 *A non-commutative differential-graded manifold is a quadruple as above equipped with an odd vector field $d = d_X$ of degree $+1$, which is compatible with the projection and closed embedding, and such that $[d, d] = 0$.*

If the quadruple corresponds to a quiver Q with set of vertices I, then the corresponding non-commutative differential-graded manifold is called (a small) A_∞*-category with the set of objects I*. In the conventional language it is given by a set of objects I, $\mathbf{Z}$-graded spaces of morphisms $Hom(i, j), i, j \in I$, and for any $i_0, i_1, ..., i_n \in I$ collections of "higher compositions" $m_n : \otimes_{0 \le m \le n-1} Hom(i_m, i_{m+1}) \to Hom(i_0, i_n)[2-n]$ (i.e. $deg\, m_n =$

$n-2$) satisfying a system of quadratic relations (see for ex.[K], [KoSo4], Section 4, [Lyu]). The notion of A_∞-functor corresponds to the notion of morphism of non-commutative dg-manifolds compatible with the embedding and the projection.

If $Q = Q_{V[1]}$ (here $V[1]$ is V with the grading shifted by one: $(V[1])_n = V_{n+1}$) then the structure of a non-commutative differential-graded manifold on X_Q is the same as the structure of an A_∞-algebra on V. Equivalently, an A_∞-algebra is given by a vector field d of degree $+1$ in the formal neighborhood of zero in the non-commutative vector space $V[1]$, vanishing at zero and satisfying the condition $[d,d]=0$. Taking the Taylor decomposition $d = m_1 + m_2 + \ldots$ at zero, one can interpret $m_n : V^{\otimes n} \to V[2-n]$ as "higher products". Here $m_1^2 = 0$ and $m_2 : V \otimes V \to V$ defines a product, which is associative "up to higher homotopies" (hence another name: homotopy associative algebras). The graded vector space $V[1]$ is naturally interpreted as the tangent space $T_0V[1]$. Being equipped with the differential m_1, it is called *the tangent complex*. The same terminology applies to any non-commutative formal pointed differential-graded manifold.

The notion of quasi-isomorphism of non-commutative dg-manifolds corresponds to the notion of quasi-isomorphism of the corresponding differential coalgebras, which is compatible with the morphisms π and j (see above). It gives rise to the notions of *quasi-equivalence of A_∞-categories* and *quasi-isomorphisms of A_∞-algebras*. They are infinitesimal, due to the following result.

Proposition 1 *Two A_∞-algebras are quasi-isomorphic iff the corresponding tangent complexes are quasi-isomorphic.*

The usefulness of the geometric language is transparent, when one tries to speak about A_∞-functors. Let me illustrate it in an example.

Theorem 2 *If X and Y are non-commutative dg-manifolds then $Maps(X,Y)$ is also a non-commutative dg-manifold.*

Let $f \in Hom(Spec(k), Maps(X,Y)) = Hom(X,Y)$, where X and Y are non-commutative special schemes. If X and Y are non-commutative dg-manifolds and f is a morphism, then f defines a k-point of the non-commutative subscheme of zeros Z_{Maps} of the odd vector field d_{Maps} (which exists according to the previous theorem). The formal neighborhood of f in $Maps(X,Y)$ is a well-defined non-commutative formal pointed dg-manifold. Using this geometric construction one proves (see [KoSo3]) the following result.

Theorem 3 *If $\mathcal{A}$ and $\mathcal{B}$ are small A_∞-categories then the A_∞-functors $Funct(\mathcal{A},\mathcal{B})$ form an A_∞-category. The explicit A_∞-structure on $Funct(\mathcal{A},\mathcal{B})$ is described in terms of the sums over the set of planar trees.*

There is a notion of weak unit in an A_∞-algebra (and A_∞-category). It is given by the unit in the cohomology $H^0(V,m_1)$ of the tangent complex in the case of weakly unital A_∞-algebras and by the identity morphisms in the cohomology $H^0(Hom(X,Y),m_1)$ in the case of weakly unital A_∞-categories.

Theorem 4 *(Yoneda lemma) Let $\mathcal{C}$ be a weakly unital A_∞-category, $\mathbf{K}$ be the dg-category of complexes of k-vector spaces. Then the natural functor $Yo: \mathcal{C}^{op} \to Funct(\mathcal{C},\mathbf{K})$, such that $Yo(X)(Y) = Hom(X,Y)$ is a faithful A_∞-functor (i.e. it induces quasi-isomorphisms of Hom's).*

4 Triangulated and saturated A_∞-categories

The material of this section is borrowed from [KoSo3].

First we will define pairs of A_∞-categories (in fact dg-categories) $C_1^{(i)}, C_2^{(i)}, 1 \leq i \leq 5$.

1. The category $C_1^{(1)}$ is the empty category with no objects.

The category $C_2^{(1)}$ has one object F such that $Hom(F,F)=0$.

2. The category $C_1^{(2)}$ is an A_∞-category with two objects E_1, E_2 such that $Hom(E_i,E_j)=0$ for $i \neq j$, and $Hom(E_i,E_i)=k$ for all i,j.

The category $C_2^{(2)}$ has three objects F_1,F_2,F such that $Hom(F_i,F_j) \simeq Hom(E_i,E_j)$ and $Hom(F,F_1) = Hom(F,F_2) = Hom(F_1,F) = Hom(F_2,F) = k, Hom(F,F) = k \oplus k$.

3. The category $C_1^{(3)}$ has one object E_1 such that $Hom(E_1,E_1)=k$.

The category $C_2^{(3)}$ has three objects F_0,F_1,F_{-1} such that $Hom(F_i,F_j) = k[i-j]$ for all i,j.

The differentials on the morphisms are zero in all cases 1-3.

4. The category $C_1^{(4)}$ has two objects E_1,E_2 such that $Hom(E_1,E_1)=0$ and all other spaces of morphisms are isomorphic to the ground field k.

Differentials are trivial on the spaces of morphisms.

The category $C_2^{(4)}$ has three objects F_1,F_2,F such that $Hom(F_i,F_j) \simeq Hom(E_i,E_j)$ for all i,j. We set $Hom(F_1,F)=k$ with trivial differential, and $Hom(F_2,F) = k[1] \oplus k$. Furthermore $Hom(F,F_1) = k \oplus k[-1], Hom(F,F_2) = k[-1], Hom(F,F) = k \oplus k \oplus k[-1]$.

Differentials on the spaces of morphisms are defined in such a way that $F = F_1[1] \oplus F_2$ with differential corresponding to the cone of a morphism.

For example, $Hom(F_2, F) = k[1] \oplus k$ carries the differential d such that $d(1_{k[1]}) = 1_k$.

5. The category $C_1^{(5)}$ has one object E_1 such that $Hom(E_1, E_1) = k[p]/(p^2 - p)$, where $deg\, p = 0$.

The category $C_2^{(5)}$ has two objects F_1, F such that $Hom(F_1, F_1) = k[p]/(p^2 - p), deg\, p = 0$ and all other spaces of morphisms are equal to the ground field k.

The differentials are trivial for these categories.

Obviously there are A_∞-functors $\phi_i : C_1^{(i)} \to C_2^{(i)}, 1 \le i \le 5$, such that $\phi_i(E_j) = F_j$ for all j. The functors ϕ_i are faithful (i.e. induce embeddings on the zero cohomologies of the complexes of morphisms).

Definition 3 *We say that an A_∞-category $\mathcal{A}$* **satisfies the Axiom** *i, $1 \le i \le 5$, if the induced A_∞-functor $\phi_i^* : Funct(C_2^{(i)}, \mathcal{A}) \to Funct(C_1^{(i)}, \mathcal{A})$ is an equivalence of A_∞-categories of functors.*

We say that the category $\mathcal{A}$ is **triangulated** *if it satisfies Axioms 1-5.*

The Axioms 1-5 have special names:

Axiom 1: Existence of a zero object of the category.

Axiom 2: Existence of the direct sum of two objects.

Axiom 3: Existence of shifts of an object.

Axiom 4: Existence of a cone of a morphism.

Axiom 5: Splitting of the image of a projector.

Remark 1 *a) Using the Yoneda lemma (see above) one can give another definition of a triangulated A_∞-category $\mathcal{C}$ as a thick subcategory of $Funct(\mathcal{C}, \mathbf{K})$ (details can be found in [KoSo3]).*

b) For any A_∞-category $\mathcal{C}$ there is its triangulated envelope $\mathcal{C}^{tr}$, which can be explicitly described in terms of twisted complexes. Moreover, for a non-commutative dg-manifold X one can construct geometrically its triangulated envelope $D(X)$, which is also a non-commutative dg-manifold.

If W is a set of objects of a triangulated A_∞-category $\mathcal{C}$ then W^{tr} denotes its triangulated envelope, which is the smallest triangulated A_∞-subcategory containing W. It can be described explicitly in terms of extensions (see [KoSo3]).

Let $\mathcal{C}$ be a triangulated A_∞-category over k.

Definition 4 *We say that $\mathcal{C}$ is* **saturated** *if the following properties are satisfied:*

1) For any $E, F \in Ob(\mathcal{C})$ one has: $\sum_{n \in \mathbf{Z}} dim(Hom(E, F[n])) < \infty$;

2) The identity functor $Id_{\mathcal{C}}$ belongs to the triangulated envelope of the set of images of the natural bifunctor $\Psi \in Bifunct((\mathcal{C}^{op}, \mathcal{C}), Funct(\mathcal{C}, \mathcal{C}))$ such that $\Psi(X, Y)(Z) = Hom_{\mathcal{C}}(X, Y) \otimes Z$ for any $X, Y, Z \in Ob(\mathcal{C})$.

If $W = \{F\}$ consists of one object F, and $W^{tr} = \mathcal{C}$, we say that F is a *generator* of the A_∞-category $\mathcal{C}$.

Proposition 2 *If $\mathcal{C}$ is saturated then it has a generator.*

5 Hochschild homology and degeneration *Hodge → de Rham*

Hochschild homology and cohomology complexes can be defined geometrically for any non-commutative dg-manifold X in terms of cyclic differential forms on X (see [KoSo3]). An algebraic translation is also easy, since the non-commutative dg-manifold $X_{\mathcal{A}}$ corresponding to a small A_∞-category $\mathcal{A}$ is an inductive limit $X = \varinjlim_{E_0, E_1, \dots, E_n \in Ob(\mathcal{A})} X_{E_0, \dots, E_n}$, where the non-commutative dg-manifold $X_{E_0, \dots, E_n}$ corresponds to the A_∞-algebra

$$\oplus_{0 \le i, j \le n} Hom_{\mathcal{A}}(E_i, E_j).$$

For example the *Hochschild chain complex* $C_\bullet(\mathcal{A}, \mathcal{A}) = C_\bullet(X, X)$ is given by

$$C_\bullet(\mathcal{A}, \mathcal{A}) = \oplus_{n \ge 0} \oplus_{E_0, E_1, \dots, E_n \in Ob(\mathcal{A})} (\otimes_{0 \le i \le n} Hom_{\mathcal{A}}(E_i, E_{i+1})[1])[-1].$$

In fact, it is a bicomplex with two commuting differentials: b (of degree $+1$) and B (of degree -1).

The Hochschild cochain complex $C^\bullet(\mathcal{A}, \mathcal{A}) = C^\bullet(X, X)$ is the same as $Vect(X)[-1]$ where $Vect(X)$ is the differential-graded Lie algebra of (graded) vector fields on X. We skip here the explicit formulas, referring to [KoSo3]. The homology $H_\bullet(\mathcal{A}, \mathcal{A})$ (resp. $H^\bullet(\mathcal{A}, \mathcal{A})$) of the above complexes are called *Hochschild homology (resp. Hochschild cohomology)* of the A_∞-category $\mathcal{A}$ (or the corresponding non-commutative dg-manifold).

Let $\mathcal{A}$ be an A_∞-category. Its triangulated envelope $\mathcal{A}^{tr}$ is a tringulated A_∞-category obtained from $\mathcal{A}$ by adding a zero object, shifts of all objects, cones of all objects, direct sums and images of all projectors. There is natural fully faithful A_∞-functor $\mathcal{A} \to \mathcal{A}^{tr}$ (embedding functor).

Proposition 3 *The embedding functor induces a quasi-isomorphism of Hochschild chain complexes* $C_\bullet(\mathcal{A},\mathcal{A}) \simeq C_\bullet(\mathcal{A}^{tr},\mathcal{A}^{tr})$.

Corollary 1 *Hochschild homology of* $\mathcal{A}$ *and its triangulated envelope are isomorphic.*

For any $n \geq 0$ we define $C_\bullet^{(n)}(\mathcal{A},\mathcal{A}) = C_\bullet(\mathcal{A},\mathcal{A}) \otimes \mathbf{Z}[u]/(u^n)$, where $deg\, u = 2$. Clearly it is a complex with the differential $b + uB$. Its cohomology will be denoted by $H^\bullet(C_\bullet^{(n)}(\mathcal{A},\mathcal{A}))$.

Proposition 4 *The previous proposition holds for all truncated complexes* $C_\bullet^{(n)}(\mathcal{A},\mathcal{A})$.

Conjecture 1 *(***Degeneration Hodge**$\to$ **de Rham***). Let* $\mathcal{A}$ *be a weakly unital saturated* A_∞*-category. Then for any* $n \geq 1$ *one has:* $H^\bullet(C_\bullet^{(n)}(\mathcal{A},\mathcal{A}))$ *is a flat* $k[u]/(u^n)$*-module.*

Remark 2 *If the main conjecture holds then* $\varprojlim_n H^\bullet(C_\bullet^{(n)}(\mathcal{A},\mathcal{A}))$ *is a flat* $k[[u]]$*-module.*

Definition 5 *Let us consider the complex* $\varprojlim_n H^\bullet(C_\bullet^{(n)}(\mathcal{A},\mathcal{A})) \otimes k((u)) = H^\bullet(\varinjlim \varprojlim (C_\bullet^{(n)}(\mathcal{A},\mathcal{A})))$. *We will call the space of elements of degree* 0 *the* **even periodic cyclic homology** *of the* A_∞*-category* $\mathcal{A}$ *and will denote it by* $HP_{ev}(\mathcal{A})$. *Similarly we will call the space of elements of degree* $+1$ *the* **odd periodic cyclic homology** *and will denote it by* $HP_{odd}(\mathcal{A})$. *The* $\mathbf{Z}/2$*-graded space* $HP_{ev}(\mathcal{A}) \oplus HP_{odd}(\mathcal{A})$ *is called the* **periodic cyclic homology** *of* $\mathcal{A}$ *and denoted by* $HP_\bullet(\mathcal{A})$.

We see that there is a G_m-equivariant bundle $\xi_{\mathcal{A}}$ over $Spf(k[[u]])$ with total space isomorphic to $\varprojlim_n C_\bullet^{(n)}(\mathcal{A},\mathcal{A}) := HC_\bullet(\mathcal{A})$. The latter space is called *cyclic homology* of the category $\mathcal{A}$. The fiber at $u = 0$ is isomorphic to the Hochschild homology $H_\bullet(\mathcal{A},\mathcal{A})$. The generic fiber is isomorphic to the periodic cyclic homology. It carries a filtration F (Hodge filtration). It is a filtration by even numbers on $HP_{ev}(\mathcal{A})$ and by odd numbers on $HP_{odd}(\mathcal{A})$. One has the following isomorphism

$$gr_F\, HP_\bullet(\mathcal{A}) \simeq H_\bullet(\mathcal{A},\mathcal{A}).$$

6 Serre functor and non-commutative smooth projective varieties

Let $\mathcal{A}$ be a saturated A_∞-category.

Definition 6 *A* **Serre functor** $S : \mathcal{A} \to \mathcal{A}$ *is an auto-equivalence of the triangulated category* $\mathcal{A}$ *such that for any two objects* $X, Y \in Ob(\mathcal{A})$ *one has a functorial isomorphism of complexes*

$$Hom_{\mathcal{A}}(X,Y)^* \simeq Hom_{\mathcal{A}}(Y, S(X)).$$

Example 2 *Let* M *be a projective variety over* k*,* $\mathcal{A} = D^b(Coh(M))$ *the bounded derived category of coherent sheaves on* M*. Then* $S(F^\bullet) = F^\bullet \otimes K_M[dim\, M]$ *where* K_M *is the canonical sheaf of* M*.*

Saturated A_∞-categories with Serre functor can be thought of as non-commutative analogs of smooth projective varieties.

If X is a Calabi-Yau manifold then the canonical sheaf K_X is trivial, and the Serre functor reduces to the shift by the dimension. This motivates the following definitions.

Definition 7 *a) A* **non-commutative Calabi-Yau manifold** *of dimension* d *is a saturated* A_∞*-category* $\mathcal{A}$ *which carries a Serre functor* S *such that* $S \simeq [d], d \in \mathbf{Z}_+$*.*

b) A **fractional non-commutative Calabi-Yau manifold** *of dimension* d *is defined as in a) with the last condition being replaced by* $S^m \simeq [dm]$ *for* $d \in \mathbf{Z}_+$ *and some integer* $m \geq 1$*.*

The notion of fractional Calabi-Yau manifold can be useful even in the commutative case. Fractional Calabi-Yau manifolds can appear as derived categories of coherent sheaves on orbifolds or derived categories of representations of oriented quivers.

7 Deformation theory of A_∞-categories

We briefly recall here the main structures of deformation theory referring to [Ko1], [KoSo1], [KoSo2] for more details. The general deformation theory is described by the following data:

1) A formal pointed dg-manifold (X, x_0, d) in the category $Vect_k^{\mathbf{Z}}$. Now we consider *commutative* special schemes, which carry a vector field d of

degree $+1$ such that $d(x_0) = 0$ and $[d, d] = 0$. If X is modelled by a cofree cocommutative coalgebra generated by a graded vector space $V[1]$ then the Taylor coefficients $d = b_1 + b_2 + ...$ give rise to "higher Lie brackets"

$$b_n : \wedge^n V \to V[2 - n], n \geq 1$$

satisfying a system of quadratic equations. Then V is called an *L_∞-algebra (a.k.a. homotopy Lie algebra).*

2) A formal pointed dg-manifold (let us abbreviate it by X) gives rise to a deformation functor $Def_X : Artin_k \to Sets$, where $Artin_k$ is the category of local Artin algebras over k. Namely, let R be an Artin algebra with the maximal ideal m. Let us consider R-points $Z(R)$ of the ind-affine subscheme Z of zeros of the vector field d. There is an equivalence relation on $Z(R)$: two points p_1 and p_2 are equivalent if they can be joined by a polynomial path $f(t)$ tangent to the distribution $[d, \bullet]$. Then $Def_X(R)$ consists of the classes of equivalent points.

Remark 3 *It is well-known that any differential-graded Lie algebra $g = \oplus_{n\geq 0} g_n$ gives rise to a formal pointed dg-manifold modelled by the cofree coalgebra $C(g[1]) = \oplus_{n\geq 0} Sym(g[1]^{\otimes n})$. The set $Z(R)$ is isomorphic to the set of solutions $\gamma \in g_1 \otimes m$ of the Maurer-Cartan equation $d\gamma + \frac{1}{2}[\gamma, \gamma] = 0$. Equivalence of points corresponds to the gauge equivalence of solutions with respect to the natural action of the unipotent Lie group $exp(g_0 \otimes m)$.*

There is a notion of quasi-isomorphism of formal pointed dg-manifolds. It corresponds to the notion of quasi-isomorphism of differential-graded coalgebras.

Theorem 5 *(Main theorem of deformation theory) If X_1 is quasi-isomorphic to X_2 then the deformation functor Def_{X_1} is isomorphic to the deformation functor Def_{X_2}.*

In particular, quasi-isomorphic DGLAs give rise to isomorphic deformation functors. This fact, together with the formality theorem for the Hochschild cochain complex, explains why any Poisson bracket on a smooth manifold can be quantized (see [Ko1]).

The deformation theory of A_∞-algebras and A_∞-categories can be spelled out in terms of their Hochschild complexes (see [KoSo1]). The tangent complex to the moduli space of deformations of an A_∞-category $\mathcal{A}$ can be described more explicitly due to the following result (see [KoSo3]).

Theorem 6 *Let X be a non-commutative dg-manifold corresponding to a non-unital A_∞-algebra A, and $C^\bullet(A, A) := C^\bullet(X, X)$ be the Hochschild cochain complex. Then one has the following quasi-isomorphism of complexes*

$$C^\bullet(X, X)[1] \simeq T_{id_X}(Maps(X, X)),$$

where T_{id_X} denotes the tangent complex at the identity map.

Corollary 2 *Let $\mathcal{A}$ be an A_∞-category with finite-dimensional Hochschild cohomology. Then*

$$H^\bullet(\mathcal{A}, \mathcal{A}) \simeq Ext^\bullet(Id_{\mathcal{A}}, Id_{\mathcal{A}}),$$

where the RHS denotes the cohomology of the tangent complex at the identity functor $Id_{\mathcal{A}}$ to the non-commutative dg-manifold corresponding to the A_∞-category $Funct(\mathcal{A}, \mathcal{A})$.

The Hochschild cohomology is isomorphic to the tangent space to the moduli space of deformations of $\mathcal{A}$. Using the Yoneda product on the algebra $Ext^\bullet(Id, Id)$ in the case of the Fukaya category (see for ex. [KoSo4]) one derives from this corollary the quantum product on the cohomology of a Calabi-Yau manifold.

8 Remarks on Deligne's conjecture

Deformation theory of A_∞-algebras is a special case of the general deformation theory of algebras over operads (see [KoSo1]) and PROPs ([KoSo2], [KoSo5]). In [KoSo1], the following conjecture of Deligne was proved using the general theory developed there (since that time several other proofs have appeared in the literature, see [MS], [T]).

Theorem 7 *Let A be an A_∞-algebra. Then its Hochschild complex $C^\bullet(A, A)$ carries a structure of an algebra over the operad $Chains(C_2)$, where C_2 is the operad of little discs in the plane.*

One can try to generalize Deligne's conjecture to higher dimensions, so that the operad of d-dimensional discs will emerge for $d > 2$ (see [Ko2] and an attempt to prove the conjecture from [Ko2] in [HKV]). I would like to mention here two possible generalizations suggested in [KoSo6] and [KoSo7]. In [KoSo7] we used the notion of d-algebra from [Ko2] as a local model for what we called d-manifolds. By definition, a d-algebra is an algebra over the operad $Chains(C_d)$, where C_d is the operad of d-dimensional discs (or cubes). For example, A_∞-algebras correspond to $d = 1$. One can speak about moduli spaces of d-manifolds (one can imagine that locally a d-manifold is something like $Spec(A)$, where A is a d-algebra).

Conjecture 2 *The formal moduli space of deformations of a d-manifold X is a $(d+1)$-manifold.*

A different generalization was suggested in [KoSo6]. It is based on the notions of homotopy category and homotopy monoidal category introduced there. Informally, they satisfy the usual axioms "up to higher homotopies". We can consider homotopy monoidal categories enriched over homotopy monoidal categories (i.e. we assume that morphisms are not sets but objects of a homotopy monoidal category). In particular, we can start with the (small) homotopy monoidal category $\mathcal{C}$ and consider all small homotopy categories enriched over $\mathcal{C}$. Let us call this symmetric monoidal homotopy category $\mathcal{C}^{(1)}$. Iterating we get a sequence of symmetric monoidal homotopy categories $\mathcal{C}^{(n)}$ such that morphisms in $\mathcal{C}^{(n)}$ are objects in $\mathcal{C}^{(n-1)}$. Objects of $\mathcal{C}^{(n)}$ are called (small) homotopy $(n+1)$-categories. Then the generalized Deligne's conjecture says that if X is an object of $\mathcal{C}^{(n)}$ then $End(id_X)$ is a monoid in $\mathcal{C}^{(n+1)}$. In particular, starting with the homotopy monoidal category of complexes, we conclude that the Hochschild complex responsible for the deformations of the identity morphism of an object in the corresponding homotopy n-category is an $(n+1)$-algebra. More generally, endomorphisms of a monoid in a homotopy n-category should form a monoid in a homotopy $(n+1)$-category. Applying this to the identity functor, considered as a monoid in the homotopy monoidal category of functors, we get the generalized Deligne conjecture.

References

[HKV] P. Hu, I. Kriz, A. Voronov, On Kontsevich's Hochschild cohomology conjecture, preprint, http://www.math.umn.edu/ voronov/publicat.html

[Ke] B. Keller, Introduction to A_∞-algebras and modules, Homology,Homotopy and Applications, 3:1(2001), 1-35.

[Ko1] M. Kontsevich, Deformation quantization of Poisson manifolds, math.QA/9709040.

[Ko2] M. Kontsevich, Operads and motives in deformation quantization, math.QA/9904055.

[KoSo1] M. Kontsevich, Y. Soibelman, Deformations of algebras over operads and Deligne conjecture, math.QA/0001151, published in Lett. Math. Phys. 21:1 (2000), 255-307.

[KoSo2] M. Kontsevich, Y. Soibelman, Deformation theory, book (in preparation).

[KoSo3] M. Kontsevich, Y. Soibelman, A_∞-categories and non-commutative geometry (in preparation).

[KoSo4] M. Kontsevich, Y. Soibelman, Mirror symmetry and torus fibrations, math.SG/0011041.

[KoSo5] M. Kontsevich, Y. Soibelman, Deformation theory of bialgebras, Hopf algebras and tensor categories (in progress).

[KoSo6] M. Kontsevich, Y. Soibelman, Homotopy n-categories and Deligne's conjecture (in progress).

[KoSo7] M. Kontsevich, Y. Soibelman, d-manifolds and Deligne's conjecture (in progress).

[Lyu] V.Lyubashenko, Category of A_∞-categories, math. CT/0210047.

[MS] J. McClure, J. Smith, A solution of Deligne's conjecture, math.QA/9910126.

[T] D. Tamarkin, Another proof of M. Kontsevich formality theorem, math.QA/9803025.