Theta Functions and Weil Representations of Algebraic Loop Symplectic Groups

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Abstract: *We introduce Weil representations for loop symplectic groups and prove the convergence and modularity of the related theta functions.*

1 Introduction

The fundamental theta function on the n-th Siegel upper half space $\mathcal{H}_n$ is given by the series

$$\theta(\Omega) = \sum_{m \in \mathbf{Z}^n} e^{\pi i m \Omega m^T}. \tag{1.1}$$

$\theta(\Omega)$ is a Siegel modular form of half integral weight for a congruence subgroup $\Gamma_{1,2} \subset Sp_{2n}(\mathbf{Z})$ defined by the condition that $g \in Sp_{2n}(\mathbf{Z})$ is in $\Gamma_{1,2}$ iff $g \bmod 2$ is an isometry of the quadratic form Q on $(\mathbf{Z}/2\mathbf{Z})^{2n}$ given by $Q(x_1, x_2) = x_1 x_2^T$. In this work, we give a generalization of the theta function and other related theta functions to algebraic loop symplectic groups and prove that it is invariant under certain congruence subgroup Γ of $Sp_{2n}(\mathbf{Z}((t)))$ analogous to $\Gamma_{1,2}$. To describe our generalization, let $\widetilde{\mathcal{H}}$ denote the space of complex valued symmetric bilinear forms Ω on $(\mathbf{R}[t^{-1}]t^{-1})^{2n}$ such that the imaginary part of Ω is positively definite, one can show that $Sp_{2n}(\mathbf{R}((t))$ acts on $\widetilde{\mathcal{H}}$. For a $Sp_{2n}(\mathbf{R}((t)))$-stable subspace $\mathcal{H}$ of $\widetilde{\mathcal{H}}$, the theta series analogous to (1.1)

$$\theta(\Omega) = \sum_{m \in (\mathbf{Z}[t^{-1}]t^{-1})^{2n}} e^{\pi i (m,m)_\Omega}. \tag{1.2}$$

converges for every $\Omega \in \mathcal{H}$. Similar to the classical case, $\mathcal{H}$ is isomorphic to $Q \times Sp_{2n}(\mathbf{R}((t)))/K$, where K is a loop group analog of a maximal compact subgroup of $Sp_{2n}(\mathbf{R}((t)))$ and $G = \{qt \,|\, 0 < q < 1\}$ is a semi-subgroup of the reparametrization group for the variable t. The extra factor Q plays an important role for the convergence of (1.2) and the theta functions with characteristics. Note that it also plays a key role in the proof of the convergence of the Eisenstein series [4] [5] of loop groups. The modular invariance property may be termed that $\theta(\Omega)$ in (1.2) is a Siegel modular form for Γ. We also construct Weil representations of $Sp_{2n}(\mathbf{F}((t)))$ for a local field F. The modular invariance property means that the theta series gives an automorphic functional of the Weil representational for $Sp_{2n}(\mathbf{R}((t)))$.

It is well-known that $\mathcal{H}_n/Sp_{2n}(\mathbf{Z})$ parametrizes the principly polarized abelian varieties of dimension n. An obvious question is to obtain a loop group generalization, i.e., to interpret our infinite dimensional upper half space $\mathcal{H}$ as a moduli space. This question is related to Proposition 4.11, where we obtain an embedding of the infinite tori $\mathbf{C}[t]^{2n}/L(\Omega)$ into an infinite dimensional projective space, which suggests that the tori might have certain algebro-geometric structure compatible with its group structure.

2 Weil Representation of Loop Symplectic Groups

We fix a local field F of characteristic different from two and a non-trivial additive character ψ of F. Let dy denote the Haar measure on F such that the 4th power of the Fourier transform $f \mapsto (\mathcal{F}f)(x) = \int_F f(y)\psi(xy)dy$ is the identity map. Let V be a vector space over F and V^* be its dual vector space. We denote the pairing of $v \in V$ and $v^* \in V^*$ by (v, v^*) or (v^*, v). $X = V \oplus V^*$ has a symplectic form given by $\langle v_1 + v_1^*, v_2 + v_2^* \rangle = (v_1, v_2^*) - (v_2, v_1^*)$.

Later we shall specialize to the case that $X = F((t))^{2n}$ with $V = (F[t^{-1}]t^{-1})^{2n}$ and $V^* = F[[t]]^{2n}$. The symplectic form is given by

$$\langle w, v \rangle = \sum_{i=1}^{n} \mathrm{Res}(w_i v_{i+n} - w_{i+n} v_i).$$

where for $a(t) \in F((t))$, $\mathrm{Res}\, a(t)$ denotes the coefficient of t^{-1}. Then $X = F((t))^{2n}$ and the symplectic form on X is given by

$$\langle w, v \rangle = \sum_{i=1}^{n} \mathrm{Res}(w_i v_{i+n} - w_{i+n} v_i).$$

Let $Sp(X)$ denote the isometry group of the symplectic space X, we assume $Sp(X)$ acts on X from the right. Each $g \in Sp(X)$ has the matrix form

$$g = \begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix} \tag{2.1}$$

with respect to the decomposition $X = V \oplus V^*$. So it acts on $v + v^*$ as

$$(v + v^*)g = (v\alpha + v^*\gamma) + (v\beta + v^*\delta),$$

where $\alpha : V \longrightarrow V$, $\beta : V \longrightarrow V^*$, $\gamma : V^* \longrightarrow V$ and $\gamma : V^* \longrightarrow V^*$.

To characterize the elements in $Sp(X)$ in terms of its entries $\alpha, \beta, \gamma, \delta$, we first discuss the dual operators in the spaces that are possibly infinite dimensional. For each $\alpha \in \mathrm{Hom}_F(V, V)$, we have the dual operator $\alpha^* \in \mathrm{Hom}_F(V^*, V^*)$ characterized by the property $(v\alpha, v^*) = (v, v^*\alpha^*)$. For each $\beta \in \mathrm{Hom}_F(V, V^*)$, the dual operator is $\beta^* \in \mathrm{Hom}_F(V^{**}, V^*)$, we denote the restriction β^* on $V \subset V^{**}$ by the same symbol β^*, it satisfies the property that $(v_1\beta, v_2) = (v_1, v_2\beta^*)$ for all $v_1, v_2 \in V$. For $\gamma \in \mathrm{Hom}_F(V^*, V)$ (resp. $\delta \in \mathrm{Hom}_F(V^*, V^*)$), we look for $\gamma^* \in \mathrm{Hom}_F(V^*, V)$ (resp. $\delta^* \in \mathrm{Hom}_F(V, V)$) such that $(w^*, v^*\gamma^*) = (w^*\gamma, v^*)$ for all $w^*, v^* \in V^*$ (reps. $(v\delta^*, v^*) = (v, v^*\delta)$ for all $v \in V$ and $v^* \in V^*$). Such a γ^* (resp. δ^*) may not exist in general, when it exists it is unique. Let $\gamma' : V^* \longrightarrow V^{**}$ be the usual dual operator of γ, then γ^* exists iff $V^*\gamma' \subset V$. Let $\delta' : V^{**} \longrightarrow V^{**}$ be the usual dual operator of δ, then δ^* exists iff $V\delta' \subset V$, and δ^* is the restriction of δ' on V. And if $\delta = \alpha^*$ for some $\alpha \in \mathrm{Hom}_F(V, V)$, then δ^* exists and $\delta^* = \alpha$.

Lemma 2.1 *Let $g \in Sp(X)$ with the matrix as in (2.1), then γ^* and δ^* exist.*

Proof. For all $v_1, v_2 \in V$, we have $\langle v_1 g, v_2 g \rangle = \langle v_1, v_2 \rangle = 0$. This implies $\langle v_1\alpha + v_1\beta, \langle v_2\alpha + v_2\beta \rangle = 0$, and then $(v_1\alpha, v_2\beta) - (v_2\alpha, v_1\beta) = 0$. So we have $\alpha\beta^* = \beta\alpha^*$. For all $v \in V, v^* \in V$, we have $\langle vg, v^*g\rangle = \langle v, v^*\rangle = (v, v^*)$. This implies that $\delta\alpha^* - \gamma\beta^* = 1_{V^*}$. These two relations are equivalent to that

$$\begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix} \begin{bmatrix} -\beta^* \\ \alpha^* \end{bmatrix} = \begin{bmatrix} 0 \\ 1_{V^*} \end{bmatrix}. \tag{2.2}$$

Let g^{-1} have the decomposition

$$g^{-1} = \begin{bmatrix} \alpha' & \beta' \\ \gamma' & \delta' \end{bmatrix},$$

then (2.2) implies that $\beta' = -\beta^*$ and $\delta' = \alpha^*$. In particular, δ'^* exists and $\delta'^* = \alpha$. Reverse the role of g and g^{-1}, we can prove δ^* exists. Next we prove γ^* exists. First using $\langle v_1^*g, v_2^*g\rangle = 0$ for all $v_1^*, v_2^* \in V^*$, we can prove $(\gamma\delta^*)^*$ exists, and $(\gamma\delta^*)^* = \gamma\delta^*$. Also we note that $(\gamma\beta^*)^*$ exists since $\gamma\beta^* = \delta\alpha^* - 1$. To prove γ^* exists, it suffices to prove that for every given $w^* \in V^*$, there is $w \in V$ such that $(v^*\gamma, w^*) = (v^*, w)$ for all $v^* \in V^*$ (then we define $w^*\gamma^* = w$). Since g is invertible, we can write $w^* = u\beta + u^*\delta$ for some $u \in V$ and $u^* \in V^*$, we have

$$\begin{aligned} (v^*\gamma, w^*) &= (v^*\gamma, u\beta + u^*\delta) \\ &= (v^*\gamma\beta^*, u) + (v^*\gamma\delta^*, u^*) \\ &= (v^*, u(\gamma\beta^*)^*) + (v^*, u^*(\gamma\delta^*)^*). \end{aligned}$$

So our desired w is $u(\gamma\beta^*)^* + u^*(\gamma\delta^*)^*$. This proves γ^* exists.

Lemma 2.2 *Let $g \in Sp(X)$, and it has the decomposition as in (2.1), then the following relations hold:*

$$\beta\alpha^* = \alpha\beta^*,\ \delta\gamma^* = \gamma\delta^*,\ \delta\alpha^* - \gamma\beta^* = 1_{V^*}.$$

and

$$\gamma^*\alpha = \alpha^*\gamma,\ \delta^*\beta = \beta^*\delta,\ \alpha^*\delta - \gamma^*\beta = 1_{V^*}.$$

Proof. The first three relations are already proved in the proof of Lemma 2.1. They imply

$$g^{-1} = \begin{bmatrix} \delta^* & -\beta^* \\ -\gamma^* & \delta^* \end{bmatrix}.$$

Apply the first three relations to g^{-1}, we get the last three relations.

$H = X \times F$ has a Heisenberg group structure given by

$$(x_1, k_1)(x_2, k_2) = (x_1 + x_2, \frac{1}{2}\langle x_1, x_2\rangle + k_1 + k_2).$$

$Sp(X)$ acts on H (from the right) by $(x, k) \cdot g = (x \cdot g, k)$.

We call a complex valued function on V a Schwartz function if its restriction to each finite dimensional subspace is a Schwartz function in the ordinary sense. Let $\mathcal{S}(V)$ denote the space of Schwartz functions on V. We view V, V^* and F as a subgroup of H by the embeddings $v \in V \mapsto (v, 0), v^* \in V^* \mapsto (v^*, 0), k \in F \mapsto (0, k)$. H acts on $\mathcal{S}(V)$ by the following:

$$
\begin{aligned}
(k\cdot f)(x) &= \psi(k)f(x)\\
(v\cdot f)(x) &= f(x+v)\\
(v^*\cdot f)(x) &= \psi(\langle x,v^*\rangle)f(x)
\end{aligned}
$$

Let $Sp(X,V^*)$ be a subset of $Sp(X)$ that contains the elements g satisfying the condition that $\dim Im(\gamma)<\infty$ (this condition is equivalent to that $V^*\sigma$ and V^* are commensurable, see e.g. [1]). $Sp(X,V^*)$ is a subgroup of $Sp(X)$. We prove that $\mathcal{S}(V)$ is a projective representation of $Sp(X,V^*)$ and the actions of $Sp(X,V^*)$ and H are compatible:

Proposition 2.3 *If* $g=\begin{bmatrix}\alpha & \beta\\ \gamma & \delta\end{bmatrix}\in Sp(X)$ *is an element in* $Sp(X,V^*)$*, we define an operator* T_g *on* $\mathcal{S}(V)$ *by*

$$(T_gf)(x)=\int_{V_g}S_g(x+x^*)f(x\alpha+x^*\gamma)d(x^*\gamma), \tag{2.3}$$

where

$$S_g(x+x^*)=\psi\left(\frac{1}{2}\langle x\alpha,x\beta\rangle+\frac{1}{2}\langle x^*\gamma,x^*\delta\rangle+\langle x^*\gamma,x\beta\rangle\right);$$

it is easy to see that $\langle x^*\gamma,x^*\delta\rangle$ *depends only on* $x^*\gamma$ *(not on the choice of* x^**), therefore* $S_g(x+x^*)$ *is a function of* x *and* $x^*\gamma$*. And the integration is for variable* $x^*\gamma$ *over* $V_g=\mathrm{Im}(\gamma)$ *with respect to a Haar measure. Then for any* $h\in H$*, we have* $hT_g=T_g(h\cdot g)$*.*

The operator T_g depends on the choice of Harr measure on V_g.

Since $\psi(\frac{1}{2}\langle x\alpha,x\beta\rangle+\frac{1}{2}\langle x^*\gamma,x^*\delta\rangle+\langle x^*\gamma,x\beta\rangle)$ is a quadratic character of variables x and $x^*\gamma$, so T_gf is a Schwartz function on each finite dimensional subspaces of V, therefore it is in $\mathcal{S}(V)$. This proposition is a straightforward generalization of Lemma 3.2 (3) in [10], It can be proved by a direct calculation or by a more conceptual method as follows. Let W be a Lagrangian subspace of X, then $W\times F$ is a maximal abelian subgroup of H, $(w,k)\mapsto\psi(k)$ is a character, the induced representation $H(W)$ consists of the functions $f:H\to\mathbf{C}$ such that $f((w,k)x)=\psi(k)f(x)$ for $(w,k)\in W\times F$ and with H-action given by $(\pi_W(h)f)(x)=f(x\cdot h)$, and we also require that the functions in $H(W)$, after identifying them with functions on the quotient space $W\setminus X$, are Schwartz functions. If W_1 and W_2 are commensurable Lagrangian subspaces, we define an intertwining operator $I(W_1,W_2):H(W_1)\to H(W_2)$ by

$$(I(W_1,W_2)f)(x)=\int_{W_1\cap W_2\setminus W_2}f(hx)dh,$$

here we use a Haar measure on $W_1\cap W_2\setminus W_2$. And for $g\in Sp(X,V^*)$. For $g\in Sp(X,V^*)$, V^* and V^*g^{-1} are commensurable, we define a map $g:H(V^*)\to H(V^*g^{-1})$ given by $(g\cdot f)=f(x\cdot g)$, then $\pi_{V^*g^{-1}}(h)\circ g=g\circ\pi_{V^*}(h\cdot g)$. Let $T(g)=I(g^{-1}V^*,V^*)\circ g:H(V^*)\to H(V^*)$, it is easy prove that $\pi(h)T(g)=T(g)\pi(h\cdot g)$. We identify the space $H(V^*)$ with $\mathcal{S}(V)$ in the obvious way and work out the formula for $T(g)$, then we get the expression as in the Proposition.

Lemma 2.4 *For $v = x + x^* \in X$, define $Q(v) = Q(x + x^*) = \psi(\frac{1}{2}\langle x, x^*\rangle)$, then $S_g(x + x^*) = Q((x + x^*)g)Q(x + x^*)^{-1}$, and*

$$S_{g_1g_2}(x + x^*) = S_{g_1}(x + x^*)S_{g_2}((x + x^*)g_1)$$

The statements equivalent to this lemma for finite dimensional case can be found in [10] and [12].

Proof.

$$Q((x + x^*)g)Q(x + x^*)^{-1} = \psi\left(\frac{1}{2}\langle x\alpha + x^*\gamma, x\beta + x^*\delta\rangle - \frac{1}{2}\langle x, x^*\rangle\right),$$

observe that $\langle x\alpha, x^*\delta\rangle - \langle x^*\gamma, x\beta\rangle - \langle x, x^*\rangle = 0$ (this follows from $\langle x, x^*\rangle = \langle xg, x^*g\rangle$), the first identity follows. The cocycle identity follows from the first one.

Lemma 2.5 *Assume $g = \begin{bmatrix} \alpha_1 & \beta_1 \\ \gamma_1 & \delta_1 \end{bmatrix}$ and $p = \begin{bmatrix} \alpha_2 & \beta_2 \\ \gamma_2 & \delta_2 \end{bmatrix}$ are in $Sp(X, V^*)$ and $\gamma_2 = 0$.*

(1). We have $V_g = V_{pg}$. And if we choose the same Haar measure in V_g and V_{pg}, and the counting measure for $V_p = \{0\}$, then $T_pT_g = T_{pg}$.

(2). We have $V_g\alpha_2 = V_{gp}$. If we choose the Haar measures in V_g and V_{gp} such that $\alpha_2 : V_g \longrightarrow V_{gp}$ preserves the Haar measure, and choose the Haar measure in $V_p = \{0\}$ as the counting measure, then $T_gT_p = T_{gp}$.

Proof. (1). write $pg = \begin{bmatrix} \alpha_3 & \beta_3 \\ \gamma_3 & \delta_3 \end{bmatrix}$, where $\alpha_3 = \alpha_2\alpha_1 + \beta_2\gamma_1$, $\gamma_3 = \delta_2\gamma_1$,

$$\begin{aligned}&(T_pT_gf)(x) = S_p(x)(T_gf)(x\alpha_2) \\ &= \int_{V_g} S_p(x)S_g(x\alpha_2 + x^*)f(x\alpha_2\alpha_1 + x^*\gamma_1)d(x^*\gamma_1),\end{aligned}$$

change variable $x^*\gamma_1$ by $x^*\gamma_1 + x\beta_2\gamma_1$, it is equal to

$$\int_{V_g} S_p(x)S_g(x\alpha_2 + x\beta_2 + x^*)f(x\alpha_2\alpha_1 + x\beta_2\gamma_1 + x^*\gamma_1)d(x^*\gamma_1).$$

By Lemma 2.4, $S_p(x)S_g(x\alpha_2 + x\beta_2 + x^*) = S_{pg}(x + x^*\delta_2^{-1})$, the last integral is

$$\int_{V_{pg}} S_{pg}(x + x^*\delta_2^{-1})f(x\alpha_3 + x^*\delta_2^{-1}\gamma_3)d(x^*\gamma_1),$$

since our choice of Haar measure, it is $(T_{pg}f)(x)$. The second conclusion can be proved similarly.

For $g \in Sp(X)$, we denote by X^g the space of all fixed points of g in X

Let

$$Sp_{fin}(X) = \{g \in Sp(X) \,|\, X^g \,\text{is finite codimensional in}\, X\}.$$

$Sp_{fin}(X)$ is a subgroup of $Sp(X, V^*)$. We denote

$$P = \{g \in Sp(X, V^*) \,|\, V^*g = V^*\},$$

it is clear that $g \in P$ iff $\gamma = 0$.

Lemma 2.6 *Every $g \in Sp(X, V^*)$ can be factorized as $g = pg'$ with $p \in P$ and $g' \in Sp_{fin}(X)$.*

Proof. Let $Gr(X, V^*)$ denote the set of all the Lagrangian subspaces of X that is is commensurable with V^*. We first prove that $Sp_{fin}(X)$ acts on $Gr(X, V^*)$ transitively. Let $\pi_V : X \to V$ and $\pi_{V^*} : X \to V^*$ be the projection operators with respect to the decomposition $X = V + V^*$. For $W \in Gr(X, V^*)$, take a finite dimensional subspace K such that $W = W \cap V^* + K$ is a direct sum and

$$\pi_{V^*} K \cap (W \cap V^*) = \{0\}$$

Then we may choose M with $\pi_{V^*} K \subset M \subset V^*$ such that $V^* = W \cap V^* + M$ direct sum. Therefore $V = (W \cap V^*)^\perp + M^\perp$, where $M^\perp = \{v \in V \,|\, \langle v, m\rangle = 0 \, \text{for all} \; m \in M\}$, and $(W \cap V^*)^\perp$ has the similar meaning. Using the fact that W is a Lagrangian subspace, we can prove that $(W \cap V^*)^\perp = \pi_V(W)$. So we have a direct sum decomposition of symplectic spaces

$$X = (W \cap V^* + M^\perp) + (M + \pi_V W)$$

and $K \subset M + \pi_V W$ is a Lagrangian subspace. There is $g'' \in Sp(M + \pi_V W)$ such that $Mg'' = K$ and let g' be the extension of g'' to X that is identity map on $W \cap V^* + M^\perp$, then $V^* g' = W$. So the action of $Sp_{fin}(X)$ on $Gr(X, V^*)$ is transitive. Now for $g \in Sp(X, V^*)$, $V^* g \in Gr(X, V^*)$, so we find $g' \in Sp_{fin}(X, V)$ such that $V^* g = V^* g'$, so $gg'^{-1} \in P$. This proves the lemma.

We can also prove the following lemma which means that every two elements in $Sp_{fin}(X)$ lie in the same finite dimensional symplectic subgroup of $Sp_{fin}(X)$.

Lemma 2.7 *If $g_1, g_2 \in Sp_{fin}(X)$, then there are direct sum decompositions $V = V_0 + V_1$, $V^* = V_0' + V_1'$ with V_0 and V_0' finite dimensional, and $V_1' = V_0^\perp$, $V_1 = V_0'^\perp$ such that g_1 and g_2 fix $V_1 + V_1'$ pointwisely and preserve $V_0 + V_0'$.*

Proposition 2.8 *The action $g \mapsto T_g$ is a projective representation of $Sp(X, V^*)$.*

Proof. We need to prove that for $g_1, g_2 \in Sp(X, V^*)$, $T_{g_1} T_{g_2} = cT_{g_1 g_2}$ for some scalar c. By Lemma 2.6, we can write $g_1 = p_1 g_1'$ and $g_2 = g_2' p_2$ with $p_1, p_2 \in P$ and $g_1', g_2' \in Sp_{fin}(X, V^*)$. Using Lemma 2.5, we have $T_{g_1} T_{g_2} \sim T_{p_1} T_{g_1'} T_{g_2'} T_{p_2}$, where $\sim$ means the operators in the two sides are equal up to a scalar. Using Lemma 2.7 and using the results for finite dimensional Symplectic groups, we see that $T_{g_1'} T_{g_2'} \sim T_{g_1' g_2'}$. So $T_{g_1} T_{g_2} \sim T_{p_1} T_{g_1' g_2'} T_{p_2} \sim T_{g_1 g_2}$. This proves the lemma.

We specialize to the case that $V = (F[t^{-1}]t^{-1})^{2n}$ and $V^* = F[[t]]^{2n}$, and then $X = F((t))^{2n}$. It is clear that $Sp_{2n}(F((t))) \subset Sp(X, V^*)$, so we have

Corollary 2.9 *$Sp_{2n}(F((t)))$ acts on $\mathcal{S}(V)$ projectively.*

We call this representation the Weil representation of $Sp_{2n}(F((t)))$.

The reprametrization group of $F((t))$ is

$$\mathrm{Aut} F((t)) = \{\sum_{i=1}^{\infty} a_i t^i \,|\, a_1 \neq 0\},$$

with the group operation $(\rho_1 * \rho_2)(t) = \rho_1(\rho_2(t))$. It acts on $F((t))$ from the right by

$$a(t) \cdot \rho(t) = a(\rho(t)).$$

And it acts on the space $F((t))dt$ of formal 1-forms (from the right) by

$$a(t)dt \cdot \rho(t) = a(\rho(t))\rho'(t)dt.$$

We view the first n components in $X = F((t))^{2n}$ as elements in $F((t))$ and the last n components as elements in $F((t))dt$, then $\mathrm{Aut}F((t))$ acts on $F((t))^{2n}$ by

$$\begin{aligned} & (a_1(t), \ldots, a_n(t), a_{n+1}(t), \ldots, a_{2n}(t)) \cdot \rho(t) \\ = \ & (a_1(\rho(t)), \ldots, a_n(\rho(t)), a_{n+1}(\rho(t))\rho'(t), \ldots, a_{2n}(\rho(t))\rho'(t)). \end{aligned}$$

Since the residue of an one-form is independent of the local parameter, the action preserves the symplectic form, so we have an embedding $\mathrm{Aut}F((t)) \subset Sp(X)$. Because $V^*\rho(t) \subset V^*$, so $\mathrm{Aut}F((t)) \subset P \subset Sp(X, V^*)$. Apply Proposition 2.8, we have an operator T_ρ for each $\rho \in \mathrm{Aut}F((t))$. Since $V^*\rho(t) \subset V^*$, $V_\rho = \{0\}$, we take the Haar measure of V_ρ as the counting measure, then

$$T_{\rho_1} T_{\rho_2} = T_{\rho_1 * \rho_2}.$$

F^* is a subgroup of $\mathrm{Aut}F((t))$ by the embedding $c \mapsto ct$, the action of F^* on $\mathcal{S}(\mathcal{V})$ is given by

$$(c \cdot f)(x_1(t), \ldots, x_n(t), x_{n+1}(t), \ldots, x_{2n}(t)) \tag{2.4}$$
$$= f(x_1(ct), \ldots, x_n(ct), cx_{n+1}(ct), \ldots, cx_{2n}(ct)). \tag{2.5}$$

If we view the first n components in $X = F((t))^{2n}$ as elements in $F((t))(dt)^\Delta$ and the last n components as elements in $F((t))(dt)^{1-\Delta}$, then we have a different embedding of $\mathrm{Aut}F((t))$ in $Sp(X, V^*)$.

3 Steinberg Symbol for the Weil Representation

In this section we compute the Steinberg Symbol for the Weil Representation of $Sp(F((t))^{2n})$ constructed in the last section. Recall the Steinberg group $\widehat{Sp}_{2n}(K)$ for Sp_{2n} with $n \geq 2$ over a field K is generated by symbols $x_\alpha(a)$ where α is a root of Sp_{2n} and $a \in K$, the relations are
(A). For each α, $x_\alpha(a)$ is additive in a.
(B). If α and β are roots and $\alpha + \beta \neq 0$, then

$$x_\alpha(a)x_\beta(b)x_\alpha(a)^{-1}x_\beta(b)^{-1} = \Pi x_{i\alpha+j\beta}(c_{ij}a^i b^j)$$

see [11] for the meaning of the right hand side. We set for $a \in K^*$, $w_\alpha(a) = x_\alpha(a)x_{-\alpha}(-a^{-1})x_\alpha(a)$ and $h_\alpha(a) = w_\alpha(a)w_\alpha(1)^{-1}$. In the case $n = 1$, there are two roots α and $-\alpha$, (B) above is replaced by

$$w_\alpha(a)x_\alpha(b)w_\alpha(-a) = x_{-\alpha}(-a^{-2}b).$$

Let $\bar{x}_\alpha(a) \in Sp_{2n}(K)$ be the root vectors, the map $x_\alpha \mapsto \bar{x}_\alpha$ from $\widehat{Sp}_{2n}(K)$ to $Sp_{2n}(K)$ is a surjective homomorphism, and the kernel is in the center of $\widehat{Sp}_{2n}(K)$ and is generated by $h_\alpha(a)h_\alpha(b)h_\alpha(ab)^{-1}$. We denote this central element by $(a,b)_\alpha$). $(a,b)_{\alpha_1} = (a,b)_{\alpha_2}$ if α_1 and α_2 have equal length, we denote (a,b) the symbol for the long root. For K large (in particular K is infinite), $\widehat{Sp}(K)$ is a universal central extension of $Sp(K)$. The Weil representation in the last section is a representation of the Steinberg group $\widehat{Sp}(F((t))^{2n})$. We will prove the Steinberg symbol (a,b) acts as scalar (the action of the symbol for the short root is determined by (a,b), see). We first compute the action of (a,b) for the SL_2-case. And the computation in the following shows that the symbols (a,b) are the same for all $Sp_{2n}(F((t))$.

As in Section 2, the Weil representation of $SL_2(F((t)))$ acts on $\mathcal{S}(V)$ for $V = \{(x_1,x_2)\,|\,x_i \in F[t^{-1}]t^{-1}\}$. Suppose $\sigma \in F[[t]]$, then $g = \begin{pmatrix} 1 & \sigma \\ 0 & 1 \end{pmatrix}$ and $g' = \begin{pmatrix} 1 & 0 \\ \sigma & 1 \end{pmatrix}$ are in P, i.e. $V_g = V_{g'} = \{0\}$, we take the counting measure on V_g and $V_{g'}$, and definte $L_+(\sigma) = T_g$ and define $L_-(\sigma) = T_{g'}$. We also need operators T_g for $g = \begin{pmatrix} t^m & 0 \\ 0 & t^{-m} \end{pmatrix}$, we denote it by T_m. T_0 is the identity operator. The Haar measure used in (2.3) is as follows. For $m > 0$, $V_g = \{(0, a_1t^{-1} + \ldots + a_mt^{-m})\}$, which is isomorphic to $F^m = \{(a_1, a_2, \ldots, a_m)\}$, we use the product measure $da_1 \ldots da_m$ on V_g, where da_i is the standard Haar measure on F given in the beginning of Section 2. For $-m < 0$, V_g is is $\{(a_1t^{-1} + \ldots, a_mt^{-m}, 0)\}$, we use the Haar measure $da_1 \ldots da_m$.

To give a more explicit formula for T_m, we write $x_1 = \sum_{i=1}^\infty a_{-i}t^{-i}$ and $x_2 = \sum_{i=1}^\infty a_{i-1}t^{-i}$ (for a given (x_1,x_2) only finitely many a_i's are non-zero), we then write $f \in \mathcal{S}(V)$ as a function of variables a_i ($i \in \mathbf{Z}$): $f(x_1,x_2) = f(\ldots, a_{-2}, a_{-1}\,|\,a_0, a_1, \ldots)$. We introduce the operators τ_k ($k \in \mathbf{Z}$), for $m > 0$,

$$(\tau_m f)(\ldots, a_{-2}, a_{-1}\,|\,a_0, a_1, \ldots) = f(\ldots, a_{-2-m}, a_{-1-m}\,|\,a_{-m}, \ldots, a_{-1}, a_0, \ldots)$$

$$(\tau_{-m} f)(\ldots, a_{-2}, a_{-1}\,|\,a_0, a_1, \ldots) = f(\ldots, a_0, \ldots, a_{m-1}\,|\,a_m, \ldots),$$

and define $\tau_0 = id$. In other words, τ_m shifts the arguments to the right by m positions, and τ_{-m} shifts the arguments to the left by m positions, so we have $\tau_m\tau_n = \tau_{m+n}$. And we also introduce the operators $\mathcal{F}_i$ for $i \in \mathbf{Z}$, it is the Fourier transform on the i-th variable a_i, for example,

$$(\mathcal{F}_2 f)(\ldots, a_{-2}, a_{-1}\,|\,a_0, a_1, \ldots) = \int f(\ldots, a_{-1}\,|\,a_0, a_1, b, a_3, \ldots)\psi(a_2 b)db$$

It is easy to check that $\tau_m\mathcal{F}_i\tau_{-m} = \mathcal{F}_{i-m}$. And we can also check that for $m > 0$,

$$T_m = \mathcal{F}_{-1}^{-1}\mathcal{F}_{-2}^{-1} \ldots \mathcal{F}_{-m}^{-1}\tau_m, \quad T_{-m} = \mathcal{F}_0\mathcal{F}_1 \ldots \mathcal{F}_{m-1}\tau_{-m}.$$

Use these expressions, we show directly that $T_mT_n = T_{m+n}$ for all $m, n \in \mathbf{Z}$.

We have already defined $L_+(\sigma)$ and $L_-(\sigma)$ for $\sigma \in F[[t]]$, to extend them to $\sigma \in F((t))$, we use the following lemma which is a direct corollary of Lemma 2.5,

Lemma 3.1 *For $\sigma \in F[[t]]$ and $m > 0$, we have $T_mL_+(\sigma) = L_+(t^{2m}\sigma)T_m, T_{-m}L_-(\sigma) = L_-(t^{2m}\sigma)T_{-m}$.*

For $\sigma \in F((t))$, we write it as $\sigma = t^{-2m}\sigma'$ for $\sigma' \in F[[t]]$ and $m \geq 0$, we define $L_+(\sigma) = T_{-m}L_+(\sigma')T_m$ and $L_-(\sigma) = T_mL_-(\sigma')T_{-m}$. Lemma 3.1 implies that they are independent of the way writing $\sigma = t^{-2m}\sigma'$. A direct calculation shows that

Proposition 3.2 *The map $L_\alpha(\sigma) \mapsto x_\alpha(\sigma)$ and $L_{-\alpha} \mapsto x_{-\alpha}(\sigma)$ gives a representation of the Steinberg group $\widehat{SL}_2(F((t)))$ on $\mathcal{S}(V)$.*

The action of Steinberg symbols (a, b) are described in terms of the Tame symbol and the Weil index. Recall that the Tame symbol for a pair of non-zero elements in $F((t))$: for σ_1, σ_2 in $F[[t]]^*$ with the constant terms c_1 and c_2 respectively, the Tame symbol $C(t^m\sigma_1, t^n\sigma_2) = (-1)^{mn}c_1^n/c_2^m$. Recall the Weil index: for $c \in F^*$, the quadratic character $\psi(\frac{1}{2}cx^2)$ gives a distribution, its Fourier transform $\mathcal{F}(\psi(\frac{1}{2}cx^2))$ is also a quadratic character, the Weil index $\gamma(c, \psi)$ is the norm 1 complex number defined by the equation

$$\mathcal{F}(\psi(\frac{1}{2}cx^2)) = \gamma(c, \psi)|c|^{-\frac{1}{2}}\psi(-\frac{1}{2}c^{-1}x^2),$$

It is proved in [12] that $\gamma(c, \psi)$ is an eighth root of 1. We have the following properties (see [12]) $\gamma(ca^2, \psi) = \gamma(c, \psi)$ and $\gamma(c, \psi)\gamma(-c^{-1}, \psi) = 1$, and it follows that $\gamma(c, \psi)\gamma(-c, \psi) = 1$.

Theorem 3.3 *Let σ_1, σ_2 be elements in $F[[t]]^*$ with the constant terms c_1 and c_2 respectively, then we have*

$$\begin{aligned}
(t^{2m}\sigma_1, t^{2n}\sigma_2) &= |C(t^{2m}\sigma_1, t^{2n}\sigma_2)|^{-\frac{1}{2}} && (3.1)\\
(t^{2m}\sigma_1, t^{2n+1}\sigma_2) &= \frac{\gamma(c_2, \psi)}{\gamma(c_1c_2, \psi)}|C(t^{2m}\sigma_1, t^{2n+1}\sigma_2)|^{-\frac{1}{2}} && (3.2)\\
(t^{2m+1}\sigma_1, t^{2n}\sigma_2) &= \frac{\gamma(c_1, \psi)}{\gamma(c_1c_2, \psi)}|C(t^{2m}\sigma_1, t^{2n+1}\sigma_2)|^{-\frac{1}{2}} && (3.3)\\
(t^{2m+1}\sigma_1, t^{2n+1}\sigma_2) &= \gamma(c_1, \psi)\gamma(c_2, \psi)|C(t^{2m+1}\sigma_1, t^{2n+1}\sigma_2)|^{-\frac{1}{2}} && (3.4)
\end{aligned}$$

The theorem is proved by a direct computation.

We like to make some comments about the topology aspects of our Weil representations. Motivated the work [6], in [2], the concept of smooth representations is introduced for certain category of groups including algebraic loop group $G(F((t)))$ (F is a non-Arichmedean local field) and its certain algebraic central extensions. A feature of the concept is that the representation space is no logner an ordinary vector space, but an object in the category of the provector spaces. When F is a non-Arichmedean local field, our Hisenberg loop group $F((t))^{2n} \times F$ is clearly an object studied in [2], and the representation $\mathcal{S}((F[t^{-1}]t^{-1})^{2n})$ is smooth representation in the sense of [2]. One may also show that the the central extension of $Sp_{2n}(F((t)))$ given by the symbol in Theorem 3.3 is also a group object in [2] and its weil representation is smooth. When F is real or complex, one can prove that the Weil representation is continuous in certain sense.

4 Upper Half Space and Theta Functions

In this section we assume $F = \mathbf{R}$ and take the additive character $\psi(x) = e^{2\pi ix}$, the corresponding Haar measure is the ordinary Lebegue measure on $\mathbf{R}$.

Recall the Siegel upper half space $\mathcal{H}_n$ for $Sp_{2n}(\mathbf{R})$ is the set of all $n \times n$ complex symmetric matrices Ω with $Im\Omega > 0$. $Sp_{2n}(\mathrm{R})$ acts on $\mathcal{H}_n$ transitively as $g \cdot \Omega = (a\Omega + b)(c\Omega + d)^{-1}$ for

$$g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

The stabilizer of $iI \in \mathcal{H}$ is $K = Sp_{2n}(\mathbf{R}) \cap SO_{2n}(\mathbf{R})$. The relation of $\mathcal{H}$ and the Weil representation of $Sp_{2n}(\mathbf{R})$ is as follows. Let $\mathbf{R^n}$ be the Lagrangian subspace $\mathbf{R}^n = \{x_1, \ldots, x_n, 0, \ldots, 0\}$, then $L^2(\mathbf{R^n})$ is a model of the Weil representation of $Sp_{2n}(\mathbf{R})$, the dense subspace $\mathcal{S}(\mathbf{R^n})$ is closed under the action. For $\Omega \in \mathcal{H}_n$, its corresponding Gaussian function $f_\Omega(x) = \exp(\pi i X \Omega X^T)$ is in $\mathcal{S}(\mathbf{R}^n)$ (here and later, $\exp(a) = e^a$), then $g \cdot f_\Omega = f_{g\cdot\Omega}$ up to a non-zero scalar. The Theta function $\theta : \mathcal{H} \to \mathbf{C}$ given by

$$\theta(\Omega) = \sum_{m \in \mathbf{Z}^n} f_\Omega(m) = \sum_{m \in \mathbf{Z}^n} e^{\pi i m \Omega m^T} \tag{4.1}$$

is automorphic for certain arithmetic subgroup of $Sp_{2n}(\mathbf{R})$. In this section we generalize the above to loop group $Sp_{2n}(\mathbf{R}((t)))$.

We first give a general discussion about $Sp(X)$. Let V be a vector space over $\mathbf{R}$ of countable dimension, and V^* be the dual space, and $X = V + V^*$ be the symplectic space as in Section 2. For $\alpha \in \mathrm{Hom}_{\mathbf{R}}(V, V^*)$, we have the corrresponding bilinear form on V given by $(v_1, v_2)_\alpha = (v_1, v_2\alpha)$. In this way we identify the space of bilinear forms on V with $\mathrm{Hom}_{\mathbf{R}}(V, V^*)$. An operator $\alpha \in \mathrm{Hom}_{\mathbf{R}}(V, V^*)$ corresponding to a symmetric form iff α^* exists (see Section 2 for the definition of α^*) and $\alpha = \alpha^*$, we call such an operator a self-dual operator. $X \in \mathrm{Hom}_{\mathbf{R}}(V, V^*)$ is called positively definite (we denote it by $X > 0$) if X is self-dual and $(v, vX) > 0$ for every $0 \neq v \in V$. Denote $V_{\mathbf{C}} = V + iV$ and $V^*_{\mathbf{C}} = V^* + iV^*$ the complexifications of V and V^*, then $\mathrm{Hom}_{\mathbf{C}}(V_{\mathbf{C}}, V^*_{\mathbf{C}}) = \mathrm{Hom}_{\mathbf{R}}(V, V^*) + i\mathrm{Hom}_{\mathbf{R}}(V, V^*)$. Let $\widetilde{\mathcal{H}}$ be the set of $\Omega = X + iY \in \mathrm{Hom}_{\mathbf{C}}(V_{\mathbf{C}}, V^*_{\mathbf{C}})$ such that X, Y are self-dual and $Y > 0$.

Proposition 4.1 *For $\Omega \in \widetilde{\mathcal{H}}$ and $g \in Sp(X)$ as in (2.1), then $\gamma\Omega + \delta$ is invertible. $g \cdot z = (\alpha\Omega + \beta)(\gamma\Omega + \delta)^{-1}$ gives an action of $Sp(X)$ on $\widetilde{\mathcal{H}}$, and the action is transitive.*

Proof. We will use Lemma 2.1 and Lemma 2.2. Using the relations in Lemma 2.2, we can prove that

$$(\gamma\bar{\Omega} + \delta)^*(\alpha\Omega + \beta) - (\alpha\bar{\Omega} + \beta)^*(\gamma\Omega + \delta) = 2iY.$$

Suppose $v \in V_{\mathbf{C}}$ is in the kernel of $(\gamma\bar{\Omega} + \delta)^*$, then $\bar{v}$ is in the kernel of $(\gamma\Omega + \delta)^*$, then

$$2i(vY, \bar{v}) = (v(\gamma\bar{\Omega} + \delta)^*(\alpha\Omega + \beta), \bar{v}) - (v(\alpha\bar{\Omega} + \beta)^*(\gamma\Omega + \delta), \bar{v}) = 0.$$

Since Y is positively definite, we have $v = 0$. This proves that $(\gamma\bar{\Omega} + \delta)^*$ is injective, therefore $(\gamma\Omega + \delta)^*$ is injective. Next we prove $\gamma\Omega + \delta$ is injective. Suppose $v^* \in V^*_{\mathbf{C}}$ is in the kernel. Then $(\bar{v^*}\gamma, v^*(\gamma\Omega + \delta)) = 0$. This implies that

$$(\bar{v^*}\gamma, v^*\gamma X) + i(\bar{v^*}\gamma, v^*\gamma Y) + (\bar{v^*}\gamma, v^*\delta) = 0. \tag{4.2}$$

The first term is in $\mathbf{R}$ since X is symmetric. The third is also in $\mathbf{R}$ since $\delta\gamma^*$ is self-dual. And $(\bar{v^*}\gamma, v^*\gamma Y) \geq 0$ since Y is positively definite. So (4.2) implies that $\bar{v^*}\gamma = 0$. Since

$v^*(\gamma\Omega+\delta)=0$, we have $v^*\delta=0$. Using $1_{V^*}=\delta\alpha^*-\gamma\beta^*$, we further derive $v^*=0$. This proves that the kernel of $\gamma\Omega+\delta$ is $\{0\}$. Because $(\gamma\Omega+\delta)^*$ is injective, so there is $U\in \mathrm{Hom}_{\mathbf{C}}(V_{\mathbf{C}},V_{\mathbf{C}})$ such that $(\gamma\Omega+\delta)^*U=1_V$, taking dual, we have $U^*(\gamma\Omega+\delta)=1_{V^*}$. This means $\gamma\Omega+\delta$ is surjective, therefore $\gamma\Omega+\delta$ is invertible. The rest of the proof is similar to the finite dimensional case.

Next we study the stabilizer of $iY\in H$ for $Y\in \mathrm{Hom}_{\mathbf{R}}(V,V^*)$ positively definite. (v,vY) defines an inner product on V. Clearly Y is injective, let $V'\subset V^*$ be the image of Y, we define an inner on V' such that $Y:V\to V'$ is an isomorphism of inner product spaces. Let $X'=V\oplus V'$ be the direct sum of inner product spaces (X' is a subspace of X). Let $O(X')$ denote the group of the isometries of X'.

Proposition 4.2 *$g\in Sp(X)$ as in (2.1) is in the stabilizer of iY iff $X'g=X'$ and $g\in O(X')$.*

The proof is similar to the finite dimensional case.

For $\Omega\in\widetilde{\mathcal{H}}$, then $f_\Omega(v)=\exp(\pi i(v,v\Omega))\in\mathcal{S}(V)$. And for $g\in Sp(X,V^*)$, it is easy to see that $T_g f_\Omega=cf_{\Omega'}$ for some scalar c and $\Omega'\in\widetilde{\mathcal{H}}$. This gives an action of $Sp(X,V^*)$ on $\widetilde{\mathcal{H}}$: $g\circ\Omega=\Omega'$.

Lemma 4.3 *The action $g\circ\Omega$ is the same as the action in the Proposition 4.1.*

Proof. We need to prove

$$g\circ\Omega=g\cdot\Omega \tag{4.3}$$

for all $g\in Sp(X,V^*)$. Since $Sp(X,V^*)$ is generated by P and $Sp_{fin}(X)$, we may assume $g\in P$ or $g\in Sp_{fin}(X)$. If $g\in P$, then $V_g=\{0\}$, using the formula for T_g in Lemma 2.8, we prove (4.3) by a direct calculation. For $g\in Sp_{fin}(X)$, we apply the result for the finite dimensional case.

Now we return to the case $V=(\mathbf{R}[t^{-1}]t^{-1})^{2n}$, $V^*=\mathbf{R}[[t]]^{2n}$, and $X=\mathbf{R}((t))^{2n}$. As a subgroup of $Sp(X)$, $Sp_{2n}(\mathbf{R}((t)))$ acts on the upper half space $\widetilde{\mathcal{H}}$. Let $(\,,\,)_0$ be the inner product on V such that $(0,\dots,0,t^{-n}(\mathrm{k-th\,position}),0,\dots,0)$ ($k=1,\dots,2n$, $n=1,2,\dots$) is an orthonormal basis. $I\in\mathrm{Hom}(V,V^*)$ be the corresponding operator, so $(t^{-k_1},\dots,t^{-k_{2n}})I=(t^{k_1-1},\dots,t^{k_{2n}-1})$. iI is in $\widetilde{\mathcal{H}}$, we find now its stabilizer in $Sp_{2n}(\mathbf{R}((t)))$.

Let $K=\{g\in Sp_{2n}(\mathbf{R}[t,t^{-1}])\mid gg^*=1\}$, where for $g=g(t)=(a_{ij}(t))\in Sp_{2n}(\mathbf{R}[t,t^{-1}])$, g^* is defined as $g(t^{-1})^T$. For example, the following elemept is in K:

$$U=\cos\theta E_{1,1}+t\sin\theta E_{1,n+1}-t^{-1}\sin\theta E_{1,1}+\cos\theta E_{n+1,n+1}+\sum_{i=2}^{n}E_{i,i}+\sum_{i=n+2}^{2n}E_{i,i} \tag{4.4}$$

where $E_{i,j}$ denote the matrix with (i,j)-entry 1 and all other entries 0, i.e., U is blockwise diagonal such that the $(1,n+1)\times(1,n+1)$-block is

$$\begin{pmatrix}\cos\theta & t\sin\theta\\ -t^{-1}\sin\theta & \cos\theta\end{pmatrix}$$

and $(2,\ldots,n)\times(n+2,\ldots 2n)$-block is the identity matrix. Let B_0 denote the Borel subgroup of $Sp_{2n}(\mathbf{R})$ given by

$$B_0=\{\begin{pmatrix} A & C \\ 0 & (A^{-1})^T \end{pmatrix} \in Sp_{2n}(\mathbf{R}); A\,\text{is upper triangular}\}. \tag{4.5}$$

And we define

$$B=\{g\in Sp_{2n}(\mathbf{R}[[t]])\mid g \bmod t\,\text{is in}\, B_0\}.$$

Proposition 4.4 *(1). $Sp_{2n}(\mathbf{R}((t)))=BK$. (2). K is generated by $Sp_{2n}(\mathbf{R})\cap SO_{2n}(\mathbf{R})$ and element U in (4.4).*

This proposition is a symplectic case of the Iwasawa decomposition for loop groups in [3] (the formulation here is different from [3]).

It is easy to see that $B\cap K$ consists of the diagonal matrices in $Sp_{2n}(\mathbf{R})$ with diagonal entries 1 or -1, so the decomposition $Sp_{2n}(\mathbf{R}((t)))=BK$ is almost a unique decomposition.

Proposition 4.5 *The stabilizer of iI in $Sp_{2n}(\mathbf{R}((t)))$ is K.*

Proof. We first observe that in our case iI, $X'=\mathbf{R}[t,t^{-1}]^{2n}$ and $K\subset O(X)$. By Proposition 4.2, we see that K fixes iI. Suppose $g\in Sp_{2n}(\mathbf{R}((t)))$ fixes iI, we write $g=bk$ with $b\in B$ and $k\in K$, since K fixes iI, so b fixes iI. Now $B\subset P$, an easy computation shows that b is diagonal with diagonal entries 1 or -1, so it is in K. This proves the proposition.

Next we construct the analog of the theta functions. The analog of lattice $\mathbf{Z^n}$ is $L=(\mathbf{Z}[t^{-1}]t^{-1})^{2n}$. For general $\Omega\in\widetilde{\mathcal{H}}$, the summation similar to (4.1) over $(\mathbf{Z}[t^{-1}]t^{-1})^{2n}$ is not convergent, we need to find a smaller subspace $\mathcal{H}$ in $\widetilde{\mathcal{H}}$ which satisfies the two conditions: (1) it is preserved by the $Sp_{2n}(\mathbf{R}((\mathbf{t})))$-action and (2). The summation (4.1) converges absolutely. Let $\mathrm{Aut}_0(\mathbf{R}((\mathbf{t})))$ denote the subgroup $\mathrm{Aut}(\mathbf{R}((\mathbf{t})))$ that consisits of the elements of the form $\rho(t)=t+\sum_{i=2}^{\infty}a_it^i$ and $Q=\{qt\,|\,0<q<1\}\subset\mathrm{Aut}(\mathbf{R}((\mathbf{t})))$. Q is a semigroup and is ismorphic to the multiplicative semigroup $\{q\in\mathbf{R}\,|\,0<q<1\}$. Since $\mathrm{Aut}(\mathbf{R}((t)))$ normalizes $Sp_{2n}(\mathbf{R}((t)))$, $\mathrm{Aut}_0(\mathbf{R}((t)))Sp_{2n}((\mathbf{R}((t)))$ is a subgroup of $Sp(X,V^*)$ and $Q\mathrm{Aut}_0(\mathbf{R}((t)))Sp_{2n}((\mathbf{R}((t)))$ closed under the multiplication. We put $\mathcal{H}=Q\mathrm{Aut}_0(\mathbf{R}((t)))Sp_{2n}((\mathbf{R}((t))\cdot(iI)$ and call it the upper half space for $Sp_{2n}((\mathbf{R}((t)))$. Clearly it satisfies the first condition above. We want to prove now for $\Omega\in\mathcal{H}$, the sum like (4.1) over L converges. We need a few lemmas first

Lemma 4.6 *For $c>0$, the one variable function $f(x)=\sum_{m\in\mathbf{Z}}\exp(-\pi c(x+m)^2)$ has the absolute minimum at $x=0$.*

Proof. By the Poisson summation formula, we have

$$f(x)=\frac{1}{\sqrt{c}}\sum_{m\in\mathbf{Z}}\exp(-\pi c^{-1}m^2)\exp(2\pi imx),$$

since $|\exp(2\pi imx)\leq 1|$, $f(x)\leq f(0)$.

Lemma 4.7 *Let C be an $k\times k$ positively definite matrix, then for any $x\in\mathbf{R}^k$,*

$$\sum_{m\in\mathbf{Z^k}}\exp(-\pi(m+x)C(m+x)^t)\leq\sum_{m\in\mathbf{Z^k}}\exp(-\pi mCm^t).$$

The proof of this lemma is similar to Lemma 4.6, we use the Poisson summation formula for $(\mathbf{R}^k, \mathbf{Z}^k)$.

Lemma 4.8 *Let D be a $k \times k$ diagonal matrix with positive diagonal entries, and N be a $k \times k$ upper triangular real unipotent matrix, then*

$$\sum_{m \in \mathbf{Z^k}} \exp(-\pi m N D N^t m^t) \leq \sum_{m \in \mathbf{Z^k}} \exp(-\pi m D m^t).$$

Proof. The left side is

$$\sum_{m_1, \ldots m_k \in \mathbf{Z}} \exp(-\pi(d_1 m_1^2)) \exp(-\pi(d_2(m_2 + N_{12} m_1)^2) \cdots$$
$$\exp(-\pi(d_k(m_k + N_{1k} m_1 + N_{2k} m_2 + \ldots + N_{k-1,k} m_{k-1})^2).$$

Apply Lemma 4.6, we have

$$\sum_{m_k \in \mathbf{Z}} \exp(-\pi(d_k(m_k + N_{1k} m_1 + N_{2k} m_2 + \ldots + N_{k-1,k} m_{k-1})^2) \leq \sum_{m_k \in \mathbf{Z}} \exp(-\pi d_k m_k)^2),$$

we then apply Lemma 4.6 to the sum over $m_{k-1}, \ldots, m_2$ successively, we get the inequality in the lemma.

Now for $\Omega \in \mathcal{H}$ and $a \in (\mathbf{R}[t^{-1}]t^{-1})^{2n}$ and $b \in \mathbf{R}[[t]]^{2n}$, and $z \in \mathbf{R}[[t]]^{2n} + (\mathbf{R}[t^{-1}]t^{-1})^{2n}\Omega \subset \mathbf{C}[[t]]^{2n}$, we define the theta series

$$\theta \begin{bmatrix} a \\ b \end{bmatrix} (z, \Omega) = \sum_{m \in L} \exp\left(\pi i(m + a, (m + a)\Omega) + 2\pi i(m + a, z + b)\right). \tag{4.6}$$

Theorem 4.9 *(4.6) converges absolutely, and let C be any compact subset of a finitely dimensional subspace of $(\mathbf{R}[t^{-1}]t^{-1})^{2n}$, the convergence is uniform for $z \in \mathbf{R}[[t]]^{2n} + C\Omega$.*

Proof. Let $\Omega = X + iY$ and $z = w + v\Omega$ with $w \in \mathbf{R}[[t]]^{2n}$ and $v \in (\mathbf{R}[t^{-1}]t^{-1})^{2n}$. The absolute value of $\exp\left(\pi i(m + a, (m + a)\Omega) + 2\pi i(m + a, z + b)\right)$ is

$$\exp\left(-\pi(m + a, (m + a)Y) - 2\pi(m + a, vY)\right).$$

We need to prove

$$\sum_{m \in L} \exp\left(-\pi(m + a, (m + a)Y) - 2\pi(m + a, vY)\right)$$

converges absolutely, it is equivalent to

$$\sum_{m \in L} \exp\left(-\pi(m + a + v, (m + a + v)Y)\right) \tag{4.7}$$

converges. Using Lemma 4.7, we see that (4.7) is bounded by

$$\sum_{m \in L} \exp\left(-\pi(m, mY)\right). \tag{4.8}$$

Let $\Omega = (qt) \cdot \rho(t) \cdot b \cdot (iI)$, for $0 < q < 1$, $\rho(t) \in \mathrm{Aut}_0(\mathbf{R}((t)))$ and $b \in B$. Let $b \bmod t$ be as (4.5), and let $d_1, d_2, \ldots, d_n$ be the diagonal entries of A, and D be the $n \times n$ diagonal matrix with diagonals $d_1, d_2, \ldots, d_n$, let b' be

$$\begin{pmatrix} D & 0 \\ 0 & D^{-1} \end{pmatrix}.$$

We consider the sum (4.8) over each finite rank sub-lattice $(\mathbf{Z}t^{-1} + \ldots + \mathbf{Z}t^{-k})^{2n}$ of L and apply Lemma 4.8, then we let $k \to \infty$, we see that (4.8) is bounded by

$$\sum_{m \in L} \exp\left(-\pi i(m, m\Omega')\right) \tag{4.9}$$

with Ω' equal to $(qt) \cdot b' \cdot (iI)$. (4.9) can be computed explicitly as

$$\Pi_{i=1}^{n} \Pi_{k=1}^{\infty} \left(\sum_{m \in \mathbf{Z}} \exp(-\pi d_i^2 m^2 q^{-2k}) \right) \left(\sum_{m \in \mathbf{Z}} \exp(-\pi d_i^2 m^2 q^{-2k+2}) \right),$$

which is convergent. The the statement about the uniform convergence also follows from the above proof.

The quasi-periodicity also holds for $\theta \begin{bmatrix} a \\ b \end{bmatrix}$:

Proposition 4.10 *For $k \in \mathbf{Z}[[t]]^{2n}$ and $m \in L$, we have*

$$\theta \begin{bmatrix} a \\ b \end{bmatrix} (z + k, \Omega) = \exp(2\pi i(a, k)) \theta \begin{bmatrix} a \\ b \end{bmatrix} (z, \Omega),$$

$$\theta \begin{bmatrix} a \\ b \end{bmatrix} (z + m\Omega, \Omega) = \exp(-2\pi i(m, b)) \exp(-\pi i(m, m\Omega) - 2\pi i(m, z)) \theta \begin{bmatrix} a \\ b \end{bmatrix} (z, \Omega).$$

It is easy to see that if a (resp. b) is increased by an element in L (resp. $L\Omega$), $\theta \begin{bmatrix} a \\ b \end{bmatrix}$ changes by a non-zero scalar.

As a function of z, the domain of $\theta \begin{bmatrix} a \\ b \end{bmatrix} (z, \Omega)$ is $\mathbf{R}[[t]]^{2n} + (\mathbf{R}[t^{-1}]t^{-1})^{2n}\Omega$. For general Ω, it has no natural complex vector space structure. But if $\Omega \in (qt) \cdot Sp(\mathbf{R}[t, t^{-1}]) \cdot iI$, it is easy to verify that $(\mathbf{R}[t^{-1}]t^{-1})^{2n}\Omega \subset \mathbf{C}[t]^{2n}$ and $\mathbf{C}[t]^{2n} \subset \mathbf{R}[[t]]^{2n} + (\mathbf{R}[t^{-1}]t^{-1})^{2n}\Omega$. We consider $\theta \begin{bmatrix} a \\ b \end{bmatrix} (z, \Omega)$ as a function for $z \in \mathbf{C}[t]^{2n}$, then it is holomorphic in z (this means that it is holomorphic for z in any finite dimensional subspace of $\mathbf{C}[t]^{2n}$). Let $L(\Omega) = \mathbf{Z}[t]^{2n} + (Z[t^{-1}]t^{-1})^{2n}\Omega$. Fix a positive integer l, let S be the set of $a = (a_1(t), \ldots, a_{2n}(t)) \in (\mathbf{Z}[t^{-1}]t^{-1})^{2n}$ such that all the coefficients of a_i's are in $\{0, 1, \ldots, l-1\}$. Each $a \in S$ gives a theta function $f_a(z) = \theta \begin{bmatrix} a/l \\ 0 \end{bmatrix} (lz, l\Omega)$. It follows from Proposition 4.10 that for all $a \in S$,

$$f_a(z + k) = f_a(z)$$

$$f_a(z + m\Omega) = \exp(-\pi i l(m, m\Omega) - 2\pi i l(z, m)) f_a(z).$$

We call

$$B = \{z \in \mathbf{C}[t]^{2n} \mid f_a(z) = 0 \text{ for all } a \in S/L(\Omega)\}$$

the set of "base points" in the complex torus $\mathbf{C}[t]^{2n}/L(\Omega)$. Let P^S denote the infinite projective space $\mathbf{P}^S = (\mathbf{C}^S - \{0\})/\mathbf{C}^*$, we have a canonically defined holomorphic map

$$\phi : \mathbf{C}[t]^{2n}/L(\Omega) - B \to \mathbf{P}^S$$

given by $\phi(z) = \{f_a(z)\}$.

Proposition 4.11 *For $l \geq 2$, B is empty, and for $l \geq 3$, ψ is an embedding.*

The proof of this Proposition is similar to the finite dimensional case. In the finite dimensional case, Proposition 4.11 implies that the complex torus is an abelian variety, it would be interesting to generalize the notion "abelian variety" to infinite dimensional tori such as $\mathbf{C}[t]^{2n}/L(\Omega)$.

If $\Omega = (qt) \cdot iI$ with $q = \frac{1}{N}$ for a positive integer $N \geq 2$, then $\mathbf{C}[t]^{2n}/L(\Omega)$ is an infinite product of elliptic curves that has a complex multiplication by the ring $\mathbf{Z}[t^{-1}]$.

5 Modularity of Theta Functions

We want to consider now the dependence of the theta functions on Ω. For simplicity, we consider only $\theta(\Omega)$ given by

$$\theta(\Omega) = \theta \begin{bmatrix} 0 \\ 0 \end{bmatrix} (0, \Omega) = \sum_{m \in L} \exp\left(\pi i (m, m\Omega)\right). \tag{5.1}$$

As in the finite dimensional case, we prove that $\theta(\Omega)$ is modular invariant for certain subgroup of $Sp_{2n}(\mathbf{Z}((t)))$. Let

$$\Gamma_2 = \{g \in Sp_{2n}(\mathbf{Z}((t))) \mid g = 1 \bmod 2\},$$

let Q be the quadtratic form on $(\mathbf{Z}/2\mathbf{Z})((t))^{2n}$ given by $Q(x_1, x_2) = x_1 x_2^t$ ($x_1, x_2 \in (\mathbf{Z}/2\mathbf{Z})((t))^n$), define Γ by

$$\Gamma = \{g \in Sp_{2n}(\mathbf{Z}((t))) \mid g \bmod 2 \text{ preserves } Q\}.$$

It is clear that $\Gamma_2 \subset \Gamma \subset Sp_{2n}(\mathbf{Z}((t)))$. For each $g \in Sp_{2n}(\mathbf{Z}((t)))$, $V_g \cap L$ is a lattice in V_g. We choose the Haar measure on V_g such that the covolume of $V_g \cap L$ is 1, so we have an operator T_g given by (2.3) on $\mathcal{S}(V)$. For $\Omega \in \mathcal{H}$, by Lemma 4.3, $T_g f_\Omega = c(g, \Omega) f_{g \cdot \Omega}$ for some scalar $c(g, \Omega)$.

Theorem 5.1 *For $g \in \Gamma$, we have*

$$c(g, \Omega)\theta(g \cdot \Omega) = \theta(\Omega).$$

It is easy to see from the proof of Theorem 4.9 that $\theta(\Omega)$ is of moderate growth in certain sense, this with Theorem 5.1 means that $\theta(\Omega)$ is a "Siegel modular form" for Γ.

The proof of Theorem 5.1 is similar to the finite dimensional case, we first need to find certain generators of Γ.

Lemma 5.2 Γ *is generated by elements*

$$\begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}, \begin{pmatrix} A & 0 \\ 0 & (A^T)^{-1} \end{pmatrix}, \begin{pmatrix} I & B \\ 0 & I \end{pmatrix},$$

where $A \in GL_n(\mathbf{Z}((t)))$ *and* $B \in M_n(\mathbf{Z}((t)))$ *is symmetric and with diagonal entries in* $2\mathbf{Z}((t))$.

We sketch a proof of the lemma. First we define $N : \mathbf{Z}((t)) \to \mathbf{Z}_{\geq 0}$ as follows, for $a(t) \in \mathbf{Z}((t)) - \{0\}$, $N(a(t))$ is the absolute value of the leading coefficient of $a(t)$ and we put $N(0) = 0$. For $b(t) \in \mathbf{Z}((t))$, there is a unique $q(t) \in \mathbf{Z}((t))$ such that $N(b(t) - q(t)a(t)) < N(a(t))$. So $\mathbf{Z}((t))$ is an Euclidean domain, in particular, it is a principal ideal domain. Therefore $GL_n(\mathbf{Z}((t)))$ is generated by elements of type

$$I + a(z)E_{ij} \ \ (i \neq j), \ \ \mathrm{diag}(d_1(t), \ldots, d_n(t)),$$

where $a(t) \in \mathbf{Z}((t))$ and $d_i(t) \in \mathbf{Z}((t))^*$. $Sp_{2n}(\mathbf{Z}((t)))$ is generated by

$$\begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}, \begin{pmatrix} A & 0 \\ 0 & (A^T)^{-1} \end{pmatrix}, \begin{pmatrix} I & B \\ 0 & I \end{pmatrix},$$

with $A \in GL_n(\mathbf{Z}((t)))$ and $B \in M_n(\mathbf{Z}((t)))$ is symmetric. Γ_2 is generated by

$$\begin{pmatrix} A & 0 \\ 0 & (A^T)^{-1} \end{pmatrix}, \begin{pmatrix} I & 2B \\ 0 & I \end{pmatrix}, \begin{pmatrix} I & 0 \\ 2B & I \end{pmatrix} \tag{5.2}$$

with $A \in GL_n(\mathbf{Z}((t)))$ such that $A = 1 \bmod 2$, and $B \in M_n(\mathbf{Z}((t)))$ is symmetric. Note that the generators in (5.2) are contained in the generators in Lemma 5.2. Γ_2 is a normal subgroup of $Sp_{2n}(\mathbf{Z}((t)))$, the quotient group Γ/Γ_2 is isomorphic to the orthogonal group of the quadratic form Q, since the generators in Lemma 5.2 $\bmod 2$ contain all the standard root subgroup of the orthogonal group, therefore they generate Γ/Γ_2.

The generators in Lemma 5.2 can be further written as products of elements $\mathrm{diag}(t^{k_1}, \ldots, t^{k_n}, t^{-k_1}, \ldots, t^{-k_n})$ with elements in $P \cap \Gamma$, and it is easy to see that Theorem 5.1 is true for these elements, this proves Theorem 5.1.

Theorem 5.2 also implies that $Sp_{2n}(\mathbf{Z}((t)))$ acts on $\mathcal{S}(V)$ (with no central extension).

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