Noncommutative local algebra and representations of certain rings of mathematical physics

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A family of Kähler-Einstein manifolds

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NONCOMMUTATIVE LOCAL ALGEBRA AND REPRESENTATIONS OF CERTAIN RINGS OF MATHEMATICAL PHYSICS.

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INTRODUCTION.

The Weyl algebra is one of the most important objects in mathematical physics and representation theory of Lie groups and Kac-Moody algebras.

Recently, T. Hayashi introduced a quantized version of the Weyl algebra [H]. This algebra can be thought as a deformation of the Weyl algebra which is compatible, in a certain sense, with the defined by Jimbo [J] and Drinfeld [D] 'quantization' of the universal enveloping algebras.

One of the purposes of this work is to find a way to describe the left spectrum and irreducible representations of the quantized (and non-quantized) Weyl and Heisenberg algebras. We show that this can be achieved for a much larger class of *hyperbolic* rings the simplified version of which appeared first in [R3]. The class of hyperbolic rings contains the (quantized) Weyl algebras, the introduced in [KS] quantized Heisenberg algebras, the (quantized) enveloping algebra of the Lie algebra sl(2), the coordinate ring of the quantum group $SL_q(2)$ and a lot more (cf. [R4]).

Since the language of rings and ideals is not convenient for the study of the spectrum, we switch from a hyperbolic ring H over a ring R to a *hyperbolic category* $\mathcal{H}$ (≃*H-mod*) over a category $\mathcal{A}$ (=*R-mod*). Here 'over' means that there is a natural (forgetting) functor $\mathcal{F}$ from $\mathcal{H}$ to $\mathcal{A}$. As a reward for changing the setting and means, we get a complete description of the spectrum of the functor $\mathcal{F}$. Reflected back to rings, these results provide, in particular, a complete description of the left spectrum of any hyperbolic ring over a commutative noetherian ring, and much more. I was not able to achieve this in [R4] using the ring-theoretical approach.

A brief outline of the contents:

Section 0 may be viewed as a continuation of Introduction: we recall the definitions of the quantized Weyl and Heisenberg algebras, and define the mentioned above hyperbolic rings.

Section 1 contains the necessary for what follows facts of the introduced in [R5] noncommutative local algebra: the spectrum of an abelian category, its connection with localizations etc..

In Sections 2, 3 and 4, we study the *skew polynomial* and the *skew Laurent* categories. The prototypes of these categories are the category of modules over the skew polynomial and the skew Laurent polynomial rings respectively. Although here these categories are investigated for the sake of hyperbolic categories, they are of independent interest.

In Section 5, we define a *hyperbolic category* over an abelian category and study some of its properties. Principal examples of hyperbolic categories are the categories of left modules over hyperbolic rings.

Section 6 fulfills the main goal of the work. It contains the description of the part of the spectrum of a hyperbolic category which is naturally related with the spectrum of the underlying abelian category.

In 'Complementary facts', we discuss iterated hyperbolic categories and some of the applications of the obtained in Section 5 general results to hyperbolic rings *of Weyl* and *Heisenberg type* which are natural generalizations of the corresponding quantized algebras.

A curious observation is that all hyperbolic rings of Weyl type are naturally related with the quantized enveloping algebras, exactly in the same sense as Hayashi's algebras are.

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0. Quantized Weyl and Heisenberg algebras and hyperbolic rings.

0.1. The quantized Heisenberg and Weyl algebras. Recall that the quantized Heisenberg algebra, $H_q(J)$, over a field k is generated by the set of elements $\{x_i, y_i, z_i \mid i \in J\}$ subject to the relations:

$$x_iz_i = qz_ix_i, \quad z_iy_i = qy_iz_i,$$

$$x_iy_i - q^{-1}y_ix_i = z_i$$

for all $i \in J$;

$$x_iy_j = y_jx_i, \quad x_ix_j = x_jx_i, \quad y_iy_j = y_jy_i, \quad z_jz_i = z_iz_j$$

for all $i, j \in J$, $i \neq j$.

The introduced by T.Hayashi [H] quantized Weyl algebra $W_q(J)$ is the quotient algebra of the algebra $H_q(J)$ obtained by adding the relations

$$(x_iy_i - qy_ix_i)z_i = 1 = z_i(x_iy_i - qy_ix_i)$$

for all $i \in J$.

Clearly $W_1(J)$ is the conventional Weyl algebra.

0.2. From the quantized Heisenberg algebra to hyperbolic rings. The spectral and representation theory of these algebras is pretty well understood in case when the set J consists of only one element, because in that case both, Heisenberg and Weyl quantized algebras happen to be *hyperbolic rings* over commutative noetherian rings. And the spectrum and irreducible representations of such rings are essentially (but not completely) described in [R3].

Recall that the introduced in [R3] (actually in [R2]), hyperbolic ring $R[\vartheta,\xi]$ is the associative ring generated by a commutative ring R and the indeterminables x, y which satisfy the relations:

$$xr = \vartheta(r)x, \quad ry = y\vartheta(r) \quad \text{for all} \quad r \in R,$$

$$xy = \xi, \quad yx = \vartheta^{-1}(\xi),$$

where ϑ is an automorphism of R, and ξ is an element of R.

The hyperbolic rings appeared as a result of an attempt to single out the biggest natural commutative subring in $U_q(sl(2))$, $U(sl(2))$ and some other algebras. One can try to do the same with the quantized Heisenberg algebra $H_q(J)$.

Notice that $(x_iy_i)z_j = z_j(x_iy_i)$ for all $i, j \in J$, and, moreover, the morphism of the ring $k[(z_i),(t_i)]$ of polynomials in variables z_i, t_i, $i \in J$, to the quantized Heisenberg algebra $H_q(J)$ which sends z_i into z_i and t_i into the product x_iy_i is injective. This implies that the ring $H_q(J)$ is generated by the commutative (polynomial) ring $R := k[(z_i),(t_i)]$ and the elements x_i, z_i, $i \in J$, subject to the relations:

$$x_ir = \vartheta_i(r)x_i, \quad ry_i = y_i\vartheta_i(r)$$

$$x_iy_i = t_i, \quad y_ix_i = q(t_i - z_i)$$

$$x_iy_j = y_jx_i$$

for all for all $r \in R$ and $i, j \in J$, where $j \neq i$.

Here ϑ_i is an automorphism of the ring R defined by

$$\vartheta_i(z_i) = qz_i, \quad \vartheta_i(t_i) = q^{-1}t_i + qz_i,$$

and

$$\vartheta_i(z_j) = z_j, \quad \vartheta_i(t_j) = t_j$$

if $j \neq i$.

Note that $\vartheta_i^{-1}(t_i) = q(t_i - z_i)$; i.e. $y_ix_i = \vartheta_i^{-1}(t_i)$.

These relations suggest the definition of (multi-dimensional) hyperbolic rings.

0.3. Hyperbolic rings. Let R be an associative ring with unity, $\{\vartheta_i | \ i \in J\}$ a family of pairwise commuting automorphisms of R. $\{\xi_i | \ i \in J\}$ a family of central elements of R such that $\vartheta_i(\xi_j) = \xi_j$ for any $i,j \in J$ such that $i \neq j$. Denote the data $\{\vartheta_i, \xi_i | \ i \in J\}$ by Θ, and let $R\{\Theta\}$ be the ring generated by R and by indeterminables x_i, y_i, $i \in J$, which satisfy the relations:

$$x_i r = \vartheta_i(r) x_i, \quad r y_i = y_i \vartheta_i(r) \tag{1}$$

for every $r \in R$;

$$x_i y_i = \xi_i, \quad y_i x_i = \vartheta_i^{-1}(\xi_i) \tag{2}$$

for every $i \in J$;

$$x_i y_j = y_j x_i, \quad x_i x_j = x_j x_i, \quad y_i y_j = y_j y_i \tag{3}$$

for any $i, j \in J$ such that $i \neq j$.

Due to the relations (2), we call the ring $R\{\Theta\}$ *hyperbolic.*

0.4. Examples. It was shown already that the quantized Heisenberg algebra is a hyperbolic ring. Now take as R the quotient ring of the polynomial ring $k[(z_i),(t_i)]$ by the relations

$$z_i(t_i(1 - q^2) + q^2 z_i) = 1, \quad i \in J.$$

Define the automorphisms ϑ_i, $i \in J$, by the same formulas, as for the Heisenberg quantized algebra $H_q(J)$; i.e.

$$\vartheta_i(z_i) = q z_i, \quad \vartheta_i(t_i) = q^{-1} t_i + q z_i$$

for all $i \in J$, and

$$\vartheta_i(z_j) = z_j, \quad \vartheta_i(t_j) = t_j$$

if $j \neq i$.

One can check that the morphism of the hyperbolic ring $R\{\Theta\}$ defined by the data $\Theta = \{\vartheta_i, t_i | \ i \in J\}$ to the Hayashi's Weyl algebra which sends z_i into z_i and t_i into $x_i y_i$, $i \in J$, is an isomorphism. ■

0.5. Iteration. For any subset $I \subseteq J$, denote by $\Theta | I$ the data $\{\vartheta_i, \xi_i | \ i \in I\}$. The ring $R\{\Theta | I\}$ shall be identified with the corresponding subring of $R\{\Theta\}$. Note that, for any $i \in J\text{-}I$, the automorphism ϑ_i can be extended to an automorphism, θ_i, of the ring $R\{\Theta | I\}$ by setting

$$\theta_i(x_j) = x_j \quad \text{and} \quad \theta_i(y_j) = y_j \quad \text{for any} \quad j \in I.$$

(The only thing to check is that the maps θ_i respect the relations. They do thanks to the commutativity of $\{\vartheta_i \mid i \in J\}$:

$$x_j\vartheta_i(r) = \theta_i(x_jr) = \theta_i(\vartheta_j(r)x_j) =$$
$$\vartheta_i \circ \vartheta_j(r)x_j = \vartheta_j \circ \vartheta_i(r)x_j = x_j\vartheta_i(r);$$

and similarly for y_j.)

Clearly the elements ξ_i, $i \in J\text{-}I$, belong the the center of the ring $R\{\Theta|I\}$, thanks to the equalities $\vartheta_j(\xi_i) = \xi_i$ for every $j \in I$, since

$$x_j\xi_i = \vartheta_j(\xi_i)x_j = \xi_ix_j \quad \text{for all} \quad j \in I.$$

This shows that the ring $R\{\Theta\}$ is naturally isomorphic to the ring $R'\{\Theta'\}$, where $R' = R\{\Theta|I\}$, $\Theta' = \{\theta_j, \xi_i \mid i \in J\text{-}I\}$.

Thus, if the set J is finite, the problem of description of the spectrum and representations of the ring $R\{\Theta\}$ can be reduced to the corresponding problems in case when $Card(J) = 1$.

But, using this reduction, we are facing the investigation of hyperbolic rings over *noncommutative* coefficient rings which does not look a priori very promising. It occurs, however, that the problem becomes much easier to handle if the language of rings and left ideals is replaced by the language of categories.

This transition leads to the notion of a *hyperbolic category*. A remarkable fact is that the switching to hyperbolic categories allows to obtain a complete description of the left spectrum of a hyperbolic ring over a commutative noetherian ring which I was not able to get in Chapter II using the language of rings and modules.

1. NONCOMMUTATIVE LOCAL ALGEBRA.

A detailed exposition (including proofs) of the presented in this section facts can be found in [R5].

1.1. A preorder in abelian categories. Fix an abelian category $\mathcal{A}$. For any two objects, X and Y, of the category $\mathcal{A}$ we shall write $X \succ Y$ if Y is a subquotient of a coproduct of a finite number of copies of X, i.e. if, for some finite k, there exists a diagram $(k)X \longleftarrow U \longrightarrow Y$, where the left arrow is a non-zero monomorphism, and the right one is an epimorphism; $(k)X$ is a direct sum of k copies of X. One can show that *the relation* $\succ$ *is a preorder on* $Ob\mathcal{A}$.

1.2. The spectrum of an abelian category. Let M be a nonzero object of the category $\mathcal{A}$. We write $M \in Spec\mathcal{A}$ if, for any nonzero subobject N of M, we have: $N \succ M$. Since $M \succ N$, we can say that $M \in Spec\mathcal{A}$ if and only if it is equivalent with respect to the preorder $\succ$ to any of its nonzero subobjects.

Denote by **Spec**$\mathcal{A}$ the ordered set of equivalence classes (with respect to $\succ$) of elements of $Spec\mathcal{A}$. The set **Spec**$\mathcal{A}$ shall be called the *spectrum of the category* $\mathcal{A}$.

1.3. Spectrum and simple objects. Clearly every simple object of the category $\mathcal{A}$ belongs to $Spec\mathcal{A}$. Moreover, we shall see in a moment that two simple objects are equivalent if and only if they are isomorphic.

1.3.1. Proposition. *Let M be a simple object of the category $\mathcal{A}$, and let N be an object of $\mathcal{A}$. Then the following conditions are equivalent:*

(a) N is isomorphic to $(k)M$ for some (finite) k;

(b) $M \succ N$.

In particular, if N and M are simple objects, then $N \succ M$ if and only if the objects M and N are isomorphic.

1.4. The spectrum and exact localizations. Recall that a full subcategory $\mathbb{S}$ of the category $\mathcal{A}$ is called *thick* if the following condition holds:

the object M in the exact sequence

$$0 \longrightarrow M' \longrightarrow M \longrightarrow M'' \longrightarrow 0$$

belongs to $\mathbb{S}$ if and only if M' and M'' are objects of $\mathbb{S}$.

It follows from the universal property of localizations that the map

$$Q \longmapsto KerQ$$

gives a bijection of the equivalence class of exact localizations of the category $\mathcal{A}$ onto the set of thick subcategories of $\mathcal{A}$.

Here (as everywhere) $KerQ$ is the full subcategory of $\mathcal{A}$ generated by all objects which are annihilated by Q.

1.4.2. Proposition. *Let $Q: \mathcal{A} \longrightarrow \mathcal{B}$ be an exact localization of an abelian category $\mathcal{A}$. For any $P \in Spec\mathcal{A}$, either $Q(P)$ equals to zero, or $Q(P)$ belongs to $Spec\mathcal{B}$.*

For any $M \in Ob\mathcal{A}$, consider the full subcategory $\langle M\rangle$ of $\mathcal{A}$ defined as follows: $Ob\langle M\rangle$ consists of all objects N such that the relation $N \succ M$ does not hold.

1.4.3.1. Lemma. *For any two objects, M and M', of the category $\mathcal{A}$, the fol-*

lowing conditions are equivalent:

(a) $M \succ M'$;

(b) $<M'> \subseteq <M>$.

Thus, the map $M \longmapsto <M>$ identifies the ordered set of equivalence classes of objects of $\mathcal{A}$ (the order is induced by $\succ$) with $(\{<M> \mid M \in Ob\mathcal{A}\}, \supseteq)$.

For any subcategory $\mathbb{T}$ of the category $\mathcal{A}$, let $\mathbb{T}^-$ denote the full subcategory of $\mathcal{A}$ generated by all objects M such that any nonzero subquotient of M has a nonzero subobject from $\mathbb{T}$.

1.4.3.2. Lemma. *For any subcategory* $\mathbb{T}$ *of* $\mathcal{A}$,

(a) the subcategory $\mathbb{T}^-$ *is thick;*

(b) $(\mathbb{T}^-)^- = \mathbb{T}^-$.

Call a subcategory $\mathbb{T}$ of $\mathcal{A}$ a *Serre subcategory* if $\mathbb{T} = \mathbb{T}^-$.

1.4.3.3. Proposition. *If an object* M *of the category* $\mathcal{A}$ *belongs to* $Spec\mathcal{A}$, *then* $<M>$ *is a Serre subcategory of* $\mathcal{A}$.

Thus, according to Proposition 2.3.2, to any point $<M>$ of **Spec**$\mathcal{A}$ an exact localization, $Q_{<M>}: \mathcal{A} \longrightarrow \mathcal{A}/<M>$, corresponds.

1.4.4. Local abelian categories and localizations at points of the spectrum. A nonzero object M of an abelian category $\mathcal{A}$ will be called *quasifinal* if $N \succ M$ for any nonzero object N of the category $\mathcal{A}$.

In other words, a nonzero object M is quasifinal if and only if

$$<M> = \{0\} = \bigcap_{N \in Ob\mathcal{A}-\{0\}} <N>.$$

Clearly a quasifinal object of the category $\mathcal{A}$ (if any) belongs to $Spec\mathcal{A}$, and every two quasifinal objects of $\mathcal{A}$ are equivalent.

1.4.4.1. Definition. An abelian category $\mathcal{A}$ is called *local* if it has a quasifinal object. ∎

1.4.4.2. Lemma. *The following properties of an abelian category* $\mathcal{A}$ *are equivalent:*

(a) $\mathcal{A}$ *is local and has simple objects;*

(b) any nonzero object of $\mathcal{A}$ *has a simple subquotient, and all simple objects of* $\mathcal{A}$ *are isomorphic one to another.*

1.4.4.3. Example. The category of left modules over a commutative ring k is local if and only if the ring k is local. ∎

1.4.4.4. Proposition. *Let* $\mathcal{A}$ *be an abelian category. For any object* M *of the category* $\mathcal{A}$ *such that* $<M>$ *is a thick subcategory of* $\mathcal{A}$*, the quotient category* $\mathcal{A}/<M>$ *is local.*

In particular, for any abelian category $\mathcal{A}$ *and any object* P *from* $Spec\mathcal{A}$*, the quotient category* $\mathcal{A}/<P>$ *is local.*

1.4.4.5. Corollary. *If* M *is a simple object of an abelian category* $\mathcal{A}$ *then* $\mathcal{A}/<M>$ *is a local category with a unique up to isomorphism simple object.*

The last assertion follows from the fact that if $Q: \mathcal{A} \longrightarrow \mathcal{B}$ is an exact localization and M a simple object of the category $\mathcal{A}$, then either $Q(M) = 0$, or $Q(M)$ is a simple object.

1.5. The topology τ and Zariski topology. The least requirement on the topology on **Spec**$\mathcal{A}$ is that it should be compatible with the preorder $\succ$. This means that the closure of any point $<P> \in$ **Spec**$\mathcal{A}$ should contain the set

$$\mathfrak{s}(<P>) := \{<P'> \mid \ <P'> \subseteq <P>\}$$

of specializations of that point. The topology τ as the strongest among the topologies which have this property.

Call a full subcategory $\mathcal{B}$ of the category $\mathcal{A}$ *topologizing* if it contains a taken in $\mathcal{A}$ coproduct of any two of its objects and the following condition holds:

if in the exact sequence $0 \longrightarrow M' \longrightarrow M \longrightarrow M'' \longrightarrow 0$ the object M belongs to $\mathcal{B}$, then M' and M'' belong to $\mathcal{B}$.

Call a full subcategory $\mathcal{B}$ of the category $\mathcal{A}$ *left closed* if it is topologizing, and the inclusion functor $\mathcal{B} \longrightarrow \mathcal{A}$ has a left adjoint functor. One can show that the subsets

$$\mathbf{Spec}\mathcal{B} = \{<P> \mid \ P \in Spec\mathcal{A} \cap Ob\mathcal{B}\},$$

where $\mathcal{B}$ runs through the family of all left closed subcategories of $\mathcal{A}$, is the set of closed subsets of a topology which is called (in [R5]) the *Zariski* topology and is denoted by $\mathfrak{Z}\mathfrak{t}$.

1.6. Supports. The *support* of an object M of an abelian category $\mathcal{A}$ is the set, $Supp(M)$, of all $\langle P\rangle \in \mathbf{Spec}\mathcal{A}$ such that $M \succ P$.

1.6.1. Proposition. *(a) For any short exact sequence*

$$0 \longrightarrow L \longrightarrow M \longrightarrow N \longrightarrow 0,$$

$$Supp(M) = Supp(L) \cup Supp(N).$$

(b) For any set Ξ *of objects such that there is a coproduct* $\bigoplus_{X \in \Xi} X$,

$$Supp\Big(\bigoplus_{X \in \Xi} X\Big) = \bigcup_{X \in \Xi} Supp(X).$$

1.6.2. Proposition. *For any subset* W *of* $\mathbf{Spec}\mathcal{A}$, *the full subcategory* $\mathcal{A}(W)$ *of* $\mathcal{A}$ *generated by all objects* M *such that* $Supp(M) \subseteq W$ *is a Serre subcategory.*

1.7. The left spectrum of a ring. Let $\mathcal{A}$ be the category $R\text{-}mod$ of left modules over an associative ring R with unity. Since each module from $Spec(R\text{-}mod)$ is equivalent to any of its cyclic submodules, we can take into consideration only the modules R/m, where m runs over the set $I_l R$ of left ideals of the ring R.

The set of all left ideals p of the ring R such that R/p belongs to $SpecR\text{-}mod$ is denoted by $Spec_l R$ and is called the *left spectrum* of R.

1.7.1. Lemma. *For any two left ideals* m *and* n *of the ring* R, *the relation* $R/m \succ R/n$ *is equivalent to the following condition:*

(#) there exists a finite set y *of elements of the ring* R *such that the ideal* $(m:y) := \{z \in R \mid zy \subset m\}$ *is contained in the ideal* n.

1.7.2. Corollary. *A left ideal* p *belongs to the left spectrum if and only if, for any* $x \in R\text{-}p$, *there exists a finite subset* y *of* R *such that*

$$((p:x):y) = (p:yx) \subseteq p.$$

1.7.3. Remark. If m is a two-sided ideal of the ring R, then, evidently, $R/m \succ R/m'$ if and only if m is conntained in m'. In particular, if the ring R is commutative, then the left spectrum $Spec_l R$ coincides with the set $SpecR$ of prime ideals of R. ■

1.8. Associated points. For any object M of an abelian category $\mathcal{A}$, denote by $Ass(M)$ the set of $\langle P\rangle \in \mathbf{Spec}\mathcal{A}$ such that P is a subobject of M. The points of $Ass(M)$ are called *associated to* M *elements of the spectrum.*

Here we need only the very first simple facts about this notion:

1.8.1. Lemma. *For any short exact sequence,* $0 \longrightarrow M' \longrightarrow M \longrightarrow M'' \longrightarrow 0$,

$$Ass(M') \subseteq Ass(M) \subseteq Ass(M') \cup Ass(M'').$$

1.8.2. Corollary. *For any finite set* Ω *of objects of an abelian category* $\mathcal{A}$, *we have:*

$$Ass\Big(\bigoplus_{X \in \Omega} X\Big) = \bigcup_{X \in \Omega} Ass(X).$$

1.9. The relative spectrum. *The spectrum of a functor* $\mathfrak{F}$ from an abelian category $\mathcal{B}$ to an abelian category $\mathcal{A}$ is the ordered set $\mathbf{Spec}(\mathfrak{F})$ of all pairs $(\langle M\rangle,\langle P\rangle)$ such that there is an object M' of $\mathcal{B}$ such that $\langle M\rangle = \langle M'\rangle$ and $\langle P\rangle \in Ass(\mathfrak{F}(M'))$. The order in $\mathbf{Spec}(\mathfrak{F})$ is induced from $\mathbf{Spec}\mathcal{B} \times \mathbf{Spec}\mathcal{A}$.

Note that, given a functor $\mathfrak{F}$, the description of $\mathbf{Spec}(\mathfrak{F})$ is reduced to the description, for any $\langle P\rangle \in \mathbf{Spec}\mathcal{A}$, of the fiber of $\mathbf{Spec}(\mathfrak{F})$ over $\langle P\rangle$ which is the set of all $\langle M\rangle \in \mathbf{Spec}\mathcal{B}$ such that $\langle P\rangle \in Ass(M)$.

Explicitly, the object of this work is the description of the spectrum of certain 'forgetting' functors.

2. SKEW POLYNOMIAL AND SKEW LAURENT CATEGORIES.

Fix an auto-equivalence ϑ of a category $\mathcal{A}$. Define the category $\mathcal{A}[\vartheta]$ as follows.

Objects of $\mathcal{A}[\vartheta]$ are all the pairs (M,v), where $M \in Ob\mathcal{A}$, v is an arrow $\vartheta M \longrightarrow M$. Morphisms from (M,v) to (M',v') are all the morphisms $f \in \mathcal{A}(M,M')$ such that the diagram

$$\begin{array}{ccc}
\vartheta M & \xrightarrow{\ \vartheta f\ } & \vartheta M' \\
{\scriptstyle v}\downarrow & & \downarrow{\scriptstyle v'} \\
M & \xrightarrow{\ \ f\ \ } & M'
\end{array}$$

is commutative.

2.1. Example. Let Φ be an automorphism of an associative ring with unity R. Let $R[x,\Phi]$ be the ring of polynomials in x with coefficients from R with the multiplication determined by the property

$$rx = x\Phi(r) \quad \textit{for any} \quad r \in R.$$

Let ϑ be the auto-equivalence of the category R-*mod* of left R-modules determined by the automorphism Φ; i.e. ϑ sends

a module $(M, \lambda: R \otimes M \longrightarrow M)$ into the module $(M, \lambda\circ(\Phi \otimes id_M))$.

It is easy to see that the category $R\text{-}mod[\vartheta]$ is equivalent to the category $R[x,\Phi]\text{-}mod$ of left $R[x,\Phi]$-modules. ∎

2.2. Example: filtrations and gradings. Fix an additive category $\mathcal{A}$. Denote by $\mathfrak{F}\mathcal{A}$ the category of $\mathbb{Z}$-filtered and by $\mathfrak{G}\mathcal{A}$ the category of $\mathbb{Z}$-graded objects over $\mathcal{A}$. There are two standart functors from $\mathfrak{F}\mathcal{A}$ to $\mathfrak{G}\mathcal{A}$:

the functor $\mathbb{G}: M \longmapsto \bigoplus_{n \in \mathbb{Z}} M_n/M_{n-1}$

and the functor $\mathbb{H}$ which assigns to a filtered object $M=(M_n \longrightarrow M_{n+1} \mid n\in\mathbb{Z})$ the graded object $\mathbb{H}(M):= \bigoplus_{n \in \mathbb{Z}} M_n$.

Denote by $\mathbb{T}$ the auto-equivalence (*translation*) $\mathfrak{G}\mathcal{A} \longrightarrow \mathfrak{G}\mathcal{A}$, $\mathbb{T}M_n = M_{n-1}$; and consider the category $\mathfrak{G}\mathcal{A}[\mathbb{T}]$.

Let $\mathfrak{I}\mathfrak{G}\mathcal{A}[\mathbb{T}]$ denote the full subcategory of the category $\mathfrak{G}\mathcal{A}[\mathbb{T}]$ generated by all the pairs (M,u), where $u: \mathbb{T}M \longrightarrow M$ is a monoarrow. There is a functor $\mathbb{H}': \mathfrak{F}\mathcal{A} \longrightarrow \mathfrak{I}\mathfrak{G}\mathcal{A}[\mathbb{T}]$ which assigns to any filtered object $M=(M_n \longrightarrow M_{n+1} \mid n\in\mathbb{Z})$ the graded object $\mathbb{H}(M):= \bigoplus_{n\in\mathbb{Z}} M_n$ and the canonical monomorphism $\mathbb{T}\mathbb{H}(M) \longrightarrow \mathbb{H}M$.

2.2.1. Lemma. *The functor* $\mathbb{H}'$ *is an equivalence of the category* $\mathfrak{F}\mathcal{A}$ *of filtered objects and the subcategory* $\mathfrak{I}\mathfrak{G}\mathcal{A}[\mathbb{T}]$ *of* $\mathfrak{G}\mathcal{A}[\mathbb{T}]$.

2.2.2. Note. If $\mathcal{A}$ is an abelian (or Grothendieck) category, then such is $\mathfrak{G}\mathcal{A}[\mathbb{T}]$. While the category $\mathfrak{F}\mathcal{A}$ is not abelian. ∎

2.3. The subcategory of 'skew double points' and the category of chain complexes. Given an auto-equivalence ϑ of a category $\mathcal{A}$, let $\mathfrak{D}\mathcal{A}[\vartheta]$ denote the full subcategory of the category $\mathcal{A}[\vartheta]$ generated by all objects (M,d) such that $d\circ\vartheta d = 0$. It is easy to check that $\mathfrak{D}\mathcal{A}[\vartheta]$ is a topologizing subcategory of $\mathcal{A}[\vartheta]$. We call $\mathfrak{D}\mathcal{A}[\vartheta]$ the subcategory of *skew double points*. We denote the cohomology functor $(M,d) \longmapsto Ker(d)/Im(\vartheta d)$ by H.

2.3.1. Example. The category $C\mathcal{A}$ of chain complexes is naturally identified with the subcategory $\mathfrak{D}\mathfrak{C}\mathcal{A}[\mathbb{T}]$ of double points of the category $\mathfrak{C}\mathcal{A}[\mathbb{T}]$ from Example 2.2. Clearly the defined above cohomology functor coincides, in this case, with the usual one. ∎

2.4. Functors F_ϑ and $F_\vartheta{}^\wedge$. Fix an additive category $\mathcal{A}$ and its auto- equivalence ϑ. Denote by F_ϑ the forgetting functor

$$\mathcal{A}[\vartheta] \longrightarrow \mathcal{A}, \quad (M,v) \longmapsto M, \quad Hom\mathcal{A}[\vartheta] \ni f \longmapsto f \in Hom\mathcal{A}.$$

Obviously, the functor F_ϑ is faithful and exact.

Let $\vartheta^\wedge$ be the left (and right) adjoint functor to ϑ; and let

$$\varphi : \vartheta^\wedge\vartheta \longrightarrow Id_{\mathcal{A}}, \quad \psi : Id_{\mathcal{A}(\vartheta)} \longrightarrow \vartheta\vartheta^\wedge$$

be adjunction isomorphisms.

Suppose that the category $\mathcal{A}$ has countable coproducts. Then this data provides us with the functor $F_\vartheta{}^\wedge$ from $\mathcal{A}$ to $\mathcal{A}[\vartheta]$ which assigns to any object M of the category $\mathcal{A}$ the pair $(\bigoplus_{n\geq 0} (\vartheta)^n M, v_\vartheta)$, where

$$v_\vartheta : \vartheta(\bigoplus_{n\geq 0} (\vartheta)^n M) \longrightarrow \bigoplus_{n\geq 0} (\vartheta)^n M = M \oplus \vartheta(\bigoplus_{n\geq 0} (\vartheta)^n M)$$

is the morphism $(0_M, id)$. The definition of $F_\vartheta{}^\wedge$ on morphisms is obvious.

2.4.1. Lemma. *The functor $F_\vartheta{}^\wedge$ is left adjoint to F_ϑ.*

Proof. For any object M of the category, $\mathcal{A}$ there is the canonical morphism

$$\gamma = \gamma(M) := (id_M, 0): M \longrightarrow F_\vartheta \circ F_\vartheta{}^\wedge(M) = M \oplus (\bigoplus_{n\geq 1} (\vartheta)^n M)$$

Let $(M,v) \in Ob\mathcal{A}[\vartheta]$. The morphisms

$$id_M, \quad \vartheta^{(n)}v := v \circ \vartheta v \circ \ldots \circ \vartheta^{n-1} v: \vartheta^n M \longrightarrow M \ , \quad n \geq 1,$$

determine the cannical morphism

$$\lambda = \lambda((M,v)): F_\vartheta{}^\wedge \circ F_\vartheta(M) = M \oplus (\bigoplus_{n\geq 1} (\vartheta)^n M) \longrightarrow (M,v)$$

It is left to the reader to check that thus defined functor morphisms,

$$\gamma : Id_{\mathcal{A}} \longrightarrow F_\vartheta \circ F_\vartheta{}^\wedge \quad \text{and} \quad \lambda : F_\vartheta{}^\wedge \circ F_\vartheta \longrightarrow Id_{\mathcal{A}(\vartheta)},$$

are the adjunction arrows; i.e.

$$F_\vartheta \lambda \circ \gamma F_\vartheta = Id_{F_\vartheta} \ , \quad \lambda F_\vartheta{}^\wedge \circ F_\vartheta{}^\wedge \gamma = Id_{F_\vartheta}.$$ ■

2.4.2. Corollary. *Suppose that the category $\mathcal{A}$ has countable products. Then the functor F_ϑ has a right adjoint functor.*

Proof. Note that the map which assigns to any object (M,u) of the category $\mathcal{A}[\vartheta]$ the pair $(M,u^\wedge)$, where $u^\wedge: M \longrightarrow \vartheta^\wedge(M)$ is the adjoint to $u^\wedge$ arrow, and acts identically on morphisms, induces an equivalence of categories,

$$\mathfrak{E}: \mathcal{A}[\vartheta]^{op} \longrightarrow \mathcal{A}^{op}[\vartheta^\wedge].$$

By Lemma 2.4.1, the forgetting functor $'F_{\vartheta^\wedge}: \mathcal{A}^{op}[\vartheta^{\wedge op}] \longrightarrow \mathcal{A}^{op}$ has a left adjoint. Therefore, since the diagram

$$\begin{array}{ccc} \mathcal{A}[\vartheta]^{op} & \xrightarrow{\mathfrak{E}} & \mathcal{A}^{op}[\vartheta^{\wedge op}] \\ & {\scriptstyle F_{\vartheta}^{op}} \searrow \quad \swarrow {\scriptstyle 'F_{\vartheta^{\wedge}}} & \\ & \mathcal{A}^{op} & \end{array}$$

is commutative, the functor $F_{\vartheta}{}^{op}$ has a left adjoint functor. which means that the functor F_{ϑ} has a right adjoint functor. ■

2.5. The functor J_{ϑ}. Consider the natural embedding

$$J_{\vartheta}: \mathcal{A} \longrightarrow \mathcal{A}[\vartheta]$$

which assigns to every $M \in Ob\mathcal{A}$ the pair $(M,0) \in Ob\mathcal{A}[\vartheta]$ and acts identically on morphisms.

Clearly the functor J_{ϑ} is fully faithful.

2.5.1. Lemma. *The functor* J_{ϑ} *has left and right adjoint functors. In particular, the functor* J_{ϑ} *is exact.*

Proof. 1) Denote by $v^{\wedge}$ the morphism adjoint to v :

$$v^{\wedge} = \vartheta^{\wedge}v \circ \psi : M \longrightarrow \vartheta^{\wedge}M.$$

Let $^{\wedge}J_{\vartheta}$ be the map which sends any object (M,v) of the category $\mathcal{A}[\vartheta]$ into $Ker(v^{\wedge})$ and any morphism $f: (M,v) \longrightarrow (M',v')$ into the unique morphism $^{\wedge}J_{\vartheta}f : Ker(v^{\wedge}) \longrightarrow Ker(v'^{\wedge})$ such that the diagram

$$\begin{array}{ccc} Ker(v^{\wedge}) & \xrightarrow{^{\wedge}J_{\vartheta}f} & Ker(v'^{\wedge}) \\ {\scriptstyle \kappa(v^{\wedge})} \downarrow & & \downarrow {\scriptstyle \kappa(v'^{\wedge})} \\ M & \xrightarrow{f} & M' \end{array} \tag{1}$$

is commutative.

Evidently, $^{\wedge}J_{\vartheta} \circ J_{\vartheta} \simeq Id_{\mathcal{A}}$.

For any $(M,v) \in Ob\mathcal{A}[\vartheta]$, $\kappa(v)$ is a morphism from $(Ker(v),0)$ to (M,v); and the map $\kappa : (M,v) \longmapsto \kappa(v)$ is a functor morphism

$$J_{\vartheta} \circ {}^{\wedge}J_{\vartheta} \longrightarrow Id_{\mathcal{A}(\vartheta)}.$$

It is easy to verify that $(\kappa, id_{\mathcal{A}})$ are adjunction morphisms.

2) Dually, the functor $J_{\vartheta}{}^{\wedge}$ which sends an object (M,v) of the category $\mathcal{A}[\vartheta]$ into $Coker(v)$ and a morphism from (M,v) to (M',v') into the corresponding morphism from $Coker(v)$ to $Coker(v')$, is left adjoint to the functor J_{ϑ}. ■

2.6. Proposition. *1) If* $\mathcal{A}$ *is a Grothendieck category, then* $\mathcal{A}[\vartheta]$ *is a Grothendieck category for any auto-equivalence* ϑ.

2) If $\mathcal{A}$ *is a Grothendieck category of finite type, then the category* $\mathcal{A}[\vartheta]$ *is also of finite type.*

Proof. *1)* The implications

$$\mathcal{A} \text{ is abelian } \Rightarrow \mathcal{A}[\vartheta] \text{ is abelian}$$

$$\mathcal{A} \text{ satisfies the property AB5 } \Rightarrow \text{ so does } \mathcal{A}[\vartheta]$$

are straightforward.

If V is a generator of the category $\mathcal{A}$, then, since the functor $F_\vartheta{}^\wedge$ is left adjoint to the forgetting functor, $F_\vartheta{}^\wedge(V)$ is a generator of the category $\mathcal{A}[\vartheta]$.

2.1) If M' *is an object of finite type of the category* $\mathcal{A}$, *then* $F_\vartheta{}^\wedge M'$ *is of finite type.*

In fact, let Ω be a directed subset of subobjects of $F_\vartheta{}^\wedge M'$ such that $\sup \Omega = F_\vartheta{}^\wedge M'$. Since the functor F_ϑ is compatible with colimits (as any functor which has a right adjoint), the last equality means that

$$sup\{F_\vartheta N|\ N \in \Omega\} = F_\vartheta \circ F_\vartheta{}^\wedge M'.$$

Since the subobject

$$\gamma(M'):\ M' \longrightarrow F_\vartheta F_\vartheta{}^\wedge M'$$

is of finite type, it is 'contained' in

$$F_\vartheta i_N:\ F_\vartheta N \longrightarrow F_\vartheta F_\vartheta{}^\wedge M'$$

for a certain subobject $i_N:\ N \longrightarrow F_\vartheta{}^\wedge M'$ from Ω. Thanks to the universal property of the functor $F_\vartheta{}^\wedge$, the arrow i_N is an isomorphism.

2.2) Now let $(M,v) \in Ob\mathcal{A}[\vartheta]$. By hypothesis, M is the supremum of a directed subset $\Theta=\{i_V:\ V \longrightarrow M\}$ of its subobjects of finite type. Consider the directed diagram of morphisms $\lambda\Theta:=\{\lambda(M)\circ F_\vartheta{}^\wedge i_V : F_\vartheta{}^\wedge V \longrightarrow M : i_V \in \Theta\}$ (here $\lambda(M)$ is the adjunction morphism $F_\vartheta{}^\wedge \circ F_\vartheta M \longrightarrow M$). Since the functor $F_\vartheta{}^\wedge$ is compatible with colimits, the canonical morphism

$$colim\{F_\vartheta{}^\wedge V : i_V \in \Theta\} \longrightarrow F_\vartheta{}^\wedge M$$

is an epimorphism. Since the adjunction arrow $\lambda(M)$ is also an epimorphism, the canonical morphism $colim\lambda\Theta \longrightarrow (M,v)$ is an epimorphism. This means that M is a supremum of the family of the images of morphisms

$$\lambda(M)\circ F_\vartheta{}^\wedge i_V:\ F_\vartheta{}^\wedge V \longrightarrow M.$$

Since, according to the heading *2)* of the proof, the objects $F_\vartheta{}^\wedge V$ are of finite type, their images are also of finite type. ∎

3. THE SKEW LAURENT CATEGORY.

Define the *skew Laurent category,* $\mathcal{A}[\vartheta]/\mathcal{A}$, as the full subcategory of the category $\mathcal{A}[\vartheta]$ generated by all objects (M,u) such that u is an isomorphism.

3.1. Example. Let θ be an automorphism of an associative ring R and ϑ the induced by θ auto-equivalence of the category *R-mod* (cf. Example 2.1). The corresponding to this data skew Laurent category is naturally equivalent to the category of left modules over the ring $R[x,x^{-1};\theta]$ of skew Laurent polynomials. Recall that $R[x,x^{-1};\theta] = \sum_{n \in \mathbb{Z}} x^n R$ as a right R-module, with the multiplication determined by

$$rx = x\theta(r) \quad \text{for all} \quad r \in R$$

(cf. Example 2.1). ∎

3.2. Lemma. *The skew Laurent category* $\mathcal{A}[\vartheta]/\mathcal{A}$ *is equivalent to the quotient category* $\mathcal{A}[\vartheta]/(J_\vartheta \mathcal{A})^-$.

Proof. It is clear that the right adjoint to the localization

$$Q: \mathcal{A}[\vartheta] \longrightarrow \mathcal{A}[\vartheta]/(J_\vartheta \mathcal{A})^-$$

functor takes values in the subcategory $\mathcal{A}[\vartheta]/\mathcal{A}$.

On the other hand, if s is an arrow from $\mathcal{A}[\vartheta]/\mathcal{A}$ such that Qs is invertible, then, since the subcategory $\mathcal{A}[\vartheta]/\mathcal{A}$ is thick, both $Ker(s)$ and $Cok(s)$, belong to the intersection of $(J_\vartheta \mathcal{A})^-$ and $\mathcal{A}[\vartheta]/\mathcal{A}$ which is zero. Therefore s is an isomorphism. ∎

3.3. Proposition. *The forgetful functor* $\mathcal{A}[\vartheta]/\mathcal{A} \longrightarrow \mathcal{A}$ *is right adjoint to the functor* $\vartheta^\bullet = (\bigoplus_{n \in \mathbb{Z}} \vartheta^n, \iota)$, *where* $\vartheta^{-n} := \vartheta^{\wedge n}$ *for every positive integer* n, *and*

$$\iota: \vartheta(\bigoplus_{n \in \mathbb{Z}} \vartheta^n) \longrightarrow \bigoplus_{n \in \mathbb{Z}} \vartheta^n$$

is defined by the arrows

$$\iota_n: \vartheta \circ \vartheta^{n-1} \longrightarrow \vartheta^n, \quad n \in \mathbb{Z},$$

such that ι_n *is the identical morphism if* $n \geq 1$, *and* $\iota_n = \sigma\vartheta^n$, *where* σ *is the adjunction isomorphism* $\vartheta \circ \vartheta^{\wedge} \longrightarrow Id$, *if* $n \leq 0$.

Proof is left to the reader. ∎

4. A PART OF THE SPECTRUM OF A SKEW POLYNOMIAL CATEGORY.

What we really are going to study is a part of the spectrum of the forgetting functor $\mathfrak{F} : \mathcal{A}[\vartheta] \longrightarrow \mathcal{A}$, $(M,u) \longmapsto M$, $f \longmapsto f$ (cf. 1.9).

4.1. The case of local 'base' category. Suppose that the category $\mathcal{A}$ is local with a quasi-final object P.

4.1.1. Lemma. *Let (M,u) be an object of $Spec\mathcal{A}[\vartheta]$ such that $<P> \in Ass(M)$. Then $Supp(M) = \{<P>\}$.*

Proof. We can assume that there is a monoarrow $i: P \longrightarrow M$. Take the adjoint to i morphism $\hat{}i: \hat{}F_{\vartheta}(P) \longrightarrow (M,u)$. Since (M,u) is an object of the spectrum, the image of $\hat{}i$ is equivalent to (M,u). This implies that

$$Supp(M) \subseteq Supp\big(\bigoplus_{n\geq 0}\vartheta^n(P)\big) = \bigcup_{n\geq 0} Supp(\vartheta^n(P)) = \{<P>\}. \qquad \blacksquare$$

Consider the full subcategory $\mathcal{A}(<P>)$ of $\mathcal{A}$ generated by all objects M' such that $Supp(M') = \{<P>\}$. Since $\{<P>\}$ is a closed subset, $\mathcal{A}(<P>)$ is a Serre subcategory of $\mathcal{A}$. Thanks to the ϑ-stability of $<P>$, the subcategory $\mathcal{A}(<P>)$ is ϑ-stable. Therefore we can (and will) replace the local category $\mathcal{A}$ by its Serre subcategory $\mathcal{A}(<P>)$; i.e. we assume that **Spec**$\mathcal{A} = \{<P>\}$.

Suppose now that the category $\mathcal{A}$ has simple objects (or, equivalently, objects of finite type). Then we can assume that the quasi-final object P is simple. Clearly the object M, being equivalent to the image of

$$\hat{}i: \bigoplus_{n\geq 0}\vartheta^n(P) \longrightarrow M,$$

is semisimple (since the image of the semisimple object $\bigoplus_{n\geq 0}\vartheta^n(P)$ is semisimp le). Therefore we can replace the category $\mathcal{A}$ by its full (obviously, ϑ-stable) subcategory of semisimple objects.

4.2. Proposition. *Let $\mathcal{A}$ be a semisimple prime category; and let ϑ be an auto-equivalence of $\mathcal{A}$. Then there exist a skew field D and an automorphism ϑ'' of D such that the category $\mathcal{A}[\vartheta]$ is equivalent to the category $D[x,\vartheta'']$-mod of left $D[x,\vartheta'']$-modules.*

Proof. Fix a simple object W of the category $\mathcal{A}$. According to Lemma 1.3.1 the functor $\mathcal{A}(W, \ ): X \longmapsto \mathcal{A}(W,X)$ is an equivalence of the category $\mathcal{A}$ and the category Ω-*mod* of left modules over the skew field $\Omega = \mathcal{A}(W,W)$. The functor ϑ determines an auto-equivalence ϑ' of the category Ω-*mod*. Since

$\mathfrak{Q}$ is a skew field, any auto-equivalence of the category $\mathfrak{Q}$-*mod* is determined by an automorphism of the skew field $\mathfrak{Q}$. ∎

Call an abelian category $\mathcal{B}$ *prime* if $Spec\mathcal{B}$ contains a generator of $\mathcal{B}$.

4.3. Proposition. *Let $\mathcal{A}$ be a semisimple prime category; and let ϑ be an auto-equivalence of $\mathcal{A}$. Then the category $\mathcal{A}[\vartheta]$ is prime; and every object from $Spec\mathcal{A}[\vartheta]$ which is not a generator of the category $\mathcal{A}[\vartheta]$ is equivalent to a simple object.*

Proof. 1) According to Proposition 4.2, the category $\mathcal{A}[\vartheta]$ is equivalent to the category $\mathfrak{Q}[x,\vartheta'']$-*mod* for some automorphism ϑ'' of the skew field $\mathfrak{Q}$. The ring $\mathfrak{Q}[x,\vartheta'']$ is euclidean. In particular, it is a principal (left and right) ideal domain. Clearly the zero ideal of the ring $\mathfrak{Q}[x,\vartheta'']$ is completely prime; in particular, it belongs to the left (and right) spectrum of $\mathfrak{Q}[x,\vartheta'']$. Therefore the category $\mathfrak{Q}[x,\vartheta'']$-*mod* is prime (cf. Example 1.3.0).

2) By [R3, Proposition 10.1.1], if R is a left and right principal ideal domain, then every nonzero ideal from $Spec_l R$ is equivalent to a maximal left ideal; and every maximal left ideal is of the form Rg, where g is an irreducible element of the ring R. ∎

4.4. Points of $\mathbf{Spec}\mathcal{A}[\vartheta]$ over ϑ-stable elements of $\mathbf{Spec}\mathcal{A}$. Let (M,u) be an object of $Spec\mathcal{A}[\vartheta]$ such that there is a monoarrow $P \longrightarrow M$ for some $P \in Spec\mathcal{A}$ such that $P \approx \vartheta(P)$.

The Serre subcategory $<P>$ is invariant with respect to ϑ. This implies that the functor ϑ induces an auto-equivalence ϑ' of the quotient category $\mathcal{A}/<P>$ such that the diagram

$$\begin{array}{ccc} \mathcal{A} & \xrightarrow{\ \vartheta\ } & \mathcal{A} \\ {\scriptstyle Q}\downarrow & & \downarrow{\scriptstyle Q} \\ \mathcal{A}/<P> & \xrightarrow{\ \vartheta'\ } & \mathcal{A}/<P> \end{array}$$

(where $Q = Q_{<P>}$ is the locatization at $<P>$) is commutative. Clearly the functor ϑ' is determined uniqely up to isomorphism.

Evidently, the localization $Q : \mathcal{A} \longrightarrow \mathcal{A}/<P>$ determines a functor

$$Q' : \mathcal{A}[\vartheta] \longrightarrow \mathcal{A}/<P>[\vartheta']$$

which also is a localization. This implies that $(M',u') := Q'(M,u)$, being nonzero, belongs to $Spec\mathcal{A}/<P>[\vartheta']$.

Since the object (M,u), being from **Spec**$\mathcal{A}[\vartheta]$, is defined uniquely up to equivalence by its localization, (M',u'), and the category $\mathcal{A}/\langle P\rangle$ is local, the problem is reduced to the case of studied in 4.1.

4.5. From Spec$\mathcal{A}[\vartheta]$ **to Spec**$\mathcal{A}[\vartheta]/\mathcal{A}$. The functor $J_\vartheta : \mathcal{A} \longrightarrow \mathcal{A}[\vartheta]$ identifies $Spec\mathcal{A}$ with the part of $Spec\mathcal{A}[\vartheta]$ which consists of all $(M,v) \in Spec\mathcal{A}[\vartheta]$ such that $v = 0$.

4.5.1. Lemma. *Let* $(M,v) \in Spec\mathcal{A}[\vartheta]$, *and* $v \neq 0$. *Then* v *is a monomorphism.*

Proof. Note that

v *is a monoarrow if and only if* $v^\wedge$ *is a monoarrow.*

Suppose that $Ker(v^\wedge) \neq \{0\}$. Since the object (M,v) belongs to $Spec\mathcal{A}[\vartheta]$, $(Ker(v),0)$ is equivalent to (M,v); i.e. (M,v) is a subquotient of the direct sum $(n)(Ker(v^\wedge),0)$ of n copies of $(Ker(v),0)$. But this means that v equals to zero. ■

4.5.2. Example. Consider the subcategory $\mathfrak{D}\mathcal{A}[\vartheta]$ of skew double points of the category $\mathcal{A}[\vartheta]$ (cf. 2.3). Clearly $\mathfrak{D}\mathcal{A}[\vartheta]$ contains the image of the embedding

$$J_\vartheta : \mathcal{A} \longrightarrow \mathcal{A}[\vartheta], \quad M \longmapsto (M,0).$$

On the other hand, since the subcategory $\mathfrak{D}\mathcal{A}[\vartheta]$ is topologizing,

$$Spec\mathfrak{D}\mathcal{A}[\vartheta] = Spec\mathcal{A}[\vartheta] \cap Ob\mathfrak{D}\mathcal{A}[\vartheta].$$

Evidently, for any $(M,u) \in Ob\mathfrak{D}\mathcal{A}[\vartheta]$, the arrow u cannot be monomorphic. Hence, by Lemma 4.5.1, $Spec\mathcal{A}[\vartheta] \cap Ob\mathfrak{D}\mathcal{A}[\vartheta]$ consists of all $(P,0)$, where $P \in Spec\mathcal{A}$; i.e. $Spec\mathfrak{D}\mathcal{A}[\vartheta]$ coincides with the image of $Spec\mathcal{A}$ in $Spec\mathcal{A}[\vartheta]$. ■

Thus, $\mathbf{Spec}\mathcal{A}[\vartheta] = \mathbf{V}(\mathcal{A}) \cup \mathbf{U}(\mathcal{A})$, where

$$\mathbf{V}(\mathcal{A}) = \{\langle P,0\rangle \mid P \in Spec\mathcal{A}\} \simeq \mathbf{Spec}\mathcal{A},$$

is the defined by the embedding J_ϑ Zariski closed subset, and

$$\mathbf{U}(\mathcal{A}) = \{\langle (M,u)\rangle \mid (M,u) \in Spec\mathcal{A}[\vartheta],\ u \text{ is a monoarrow}\}$$

is the complement to $\mathbf{V}(\mathcal{A})$ Zariski open subset.

The localization functor $Q: \mathcal{A}[\vartheta] \longrightarrow \mathcal{A}[\vartheta]/\mathcal{A}$ (cf. Lemma 3.2) defines an embedding of the open set $\mathbf{U}(\mathcal{A})$ into the spectrum of the skew Laurent category $\mathcal{A}[\vartheta]/\mathcal{A}$. Therefore a way to study $\mathbf{U}(\mathcal{A})$ is to investigate first the spectrum of the category $\mathcal{A}[\vartheta]/\mathcal{A}$, which is of independent interest, and then try to single out the image of $\mathbf{U}(\mathcal{A})$ in the spectrum of $\mathcal{A}[\vartheta]/\mathcal{A}$.

4.6. Skew Laurent category over a local category. Assume that the category $\mathcal{A}$

is local. Let $(M,u) \in Spec\mathcal{A}[\vartheta]/\mathcal{A}$ be such that a quasi-final object P of $\mathcal{A}$ is a subobject of M. This implies, as in 4.1, that there is a nonzero morphism

$$\hat{i}: \bigoplus_{n \in \mathbb{Z}} \vartheta^n(P) \longrightarrow M$$

(- the adjoint arrow to the monomorphism $i: P \longrightarrow M$) such that the image of $\hat{i}$ is equivalent to M. In particular, since $\vartheta(P)$ is equivalent to P, $Supp(M) = \{<P>\}$.

If the category $\mathcal{A}$ has simple objects, then the object M is semisimple, and it follows from Proposition 4.2 that the full subcategory of semisimple objects of the category $\mathcal{A}[\vartheta]/\mathcal{A}$ is equivalent to the category of left modules over the ring $D[x,x^{-1};\theta]$ of skew Laurent polynomials with coefficients in a skew field D.

4.7. Points of $Spec\mathcal{A}[\vartheta]/\mathcal{A}$ over ϑ-stable elements of $Spec\mathcal{A}$. Let $<P> \in$ **Spec**$\mathcal{A}$ be a ϑ-stable element; and let (M,u) be an object of $Spec\mathcal{A}[\vartheta]/\mathcal{A}$ such that $<P> \in Ass(M)$. The defined in 4.4 localization

$$Q': \mathcal{A}[\vartheta] \longrightarrow \mathcal{A}/<P>[\vartheta'],$$

where ϑ' is the induced by ϑ auto-equivalence of the category $\mathcal{A}<P>$ (cf. 4.4), induces the localization $Q'': \mathcal{A}[\vartheta]/\mathcal{A} \longrightarrow \mathcal{A}<P>[\vartheta']/\mathcal{A}<P>$.

Since the category $\mathcal{A}<P>$ is local, and the object (M,u) is determined (up to equivalence) by its image, $Q''(M,u)$, in $\mathcal{A}<P>[\vartheta']/\mathcal{A}<P>$, one can use the the result of 4.6 to obtain the description of the equivalence class of (M,u).

One of the consequences of these facts is the following

4.7.1. Proposition. *The canonical embedding* U$(\mathcal{A}) \longrightarrow$ **Spec**$\mathcal{A}[\vartheta]/\mathcal{A}$ *induces the bijection of the subset* $\{<(M,u)> \mid Ass(M)$ *contains a ϑ-stable element$\}$ onto the similar subset of* **Spec**$\mathcal{A}[\vartheta]/\mathcal{A}$.

Now, instead of going on with the study of the remaining part of **Spec**$\mathcal{A}[\vartheta]/\mathcal{A}$, we shall switch to the *hyperbolic* categories. The reason for this sudden abandon is that any skew Laurent category is equivalent to a hyperbolic category of a special type. Hence we shall safe some effort by investigating first the spectral picture of hyperbolic categories (which is, anyway, our first priority) and then applying it to the skew Laurent categories.

5. THE HYPERBOLIC CATEGORIES.

Let θ be an auto-equivalence of an additive category $\mathcal{A}$; and let ξ be an endomorphism of the identical functor of $\mathcal{A}$.

Denote by $\mathcal{A}\{\theta,\xi\}$ the category objects of which are triples (γ,M,η), where $M \in Ob\mathcal{A}$ and $\gamma\colon M \longrightarrow \theta(M)$, $\eta\colon \theta(M) \longrightarrow M$ are arrows such that

$$\eta\circ\gamma = \xi(M) \quad \text{and} \quad \gamma\circ\eta = \xi\theta(M).$$

Morphisms from (γ,M,η) to (γ',M',η') are those morphisms f from M to M' for which the diagram

$$\begin{array}{ccccc} M & \xrightarrow{\ \gamma\ } & \theta(M) & \xrightarrow{\ \eta\ } & M \\ {\scriptstyle f}\big\downarrow & & \big\downarrow{\scriptstyle \theta(f)} & & \big\downarrow{\scriptstyle f} \\ M' & \xrightarrow{\ \gamma'\ } & \theta(M') & \xrightarrow{\ \eta'\ } & M' \end{array}$$

is commutative.

The category $\mathcal{A}\{\theta,\xi\}$ will be called *hyperbolic.*

5.1. Example. Let R be an associative ring, ϑ an automorphism of R, $\xi' \in R$ a central element, and $R\{\vartheta,\xi'\}$ the related to this data hyperbolic ring (cf. 1.5); i.e. $R\{\vartheta,\xi'\}$ is generated by R and by two elements, x and y, subject to the relations:

$$xr = \vartheta(r)x, \quad ry = y\vartheta(r) \quad \text{for all} \quad r \in R;$$

$$xy = \xi', \quad yx = \vartheta^{-1}(\xi').$$

The category $R\{\vartheta,\xi'\}$*-mod is hyperbolic.*

Namely, $R\{\vartheta,\xi'\}$*-mod* is equivalent to the category $\mathcal{A}\{\theta,\xi\}$, where $\mathcal{A}$ = *R-mod*, θ is an auto-equivalence of the category $\mathcal{A}$ induced by the automorphism ϑ (cf. 2.1), ξ is the endomorphism of the identical functor, $Id_{\mathcal{A}}$, which assigns to every R-module M the action of the element ξ' on M; i.e. $\xi(w) := \xi'\cdot w$ for each $w \in M$. ∎

5.2. Lemma. *Suppose that the endomorphism* ξ *in the definition of the category* $\mathcal{A}\{\theta,\xi\}$ *is an isomorphism. Then the category* $\mathcal{A}\{\theta,\xi\}$ *is equivalent to the category* $\mathcal{A}[\theta]/\mathcal{A}$ *of skew Laurent polynomials.*

Proof. By definition, for every object (γ,M,η) in $\mathcal{A}\{\theta,\xi\}$, we have:

$$\eta\circ\gamma = \xi(M), \quad \text{and} \quad \gamma\circ\eta = \xi\theta(M) \tag{1}$$

If ξ is an isomorphism, then, in particular, $\xi(M)$ and $\xi\theta(M)$ are isomorphisms which implies that both γ and η are isomorphisms. So, the claimed equivalence $\mathcal{A}\{\theta,\xi\} \longrightarrow \mathcal{A}[\theta]/\mathcal{A}$ is given by

$$(\gamma,M,\eta)\longmapsto(\gamma,M). \qquad (2)$$

Clearly the quasi-inverse to (2) functor assigns to an object (γ,M) the object $(\gamma,M,\xi(M)\circ\gamma^{-1})$. ■

5.3. The monad associated with (θ,ξ). Assume that the category $\mathcal{A}$ has countable coproducts. Fix a quasi-inverse to θ functor, $\theta^\wedge$; and, for any negative integer n, denote by θ^n the functor $\theta^{\wedge -n}$. Let $\theta^\bullet$ denote the functor

$$\bigoplus_{n\in\mathbb{Z}}\theta^n:\ M\longmapsto\bigoplus_{n\in\mathbb{Z}}\theta^n(M)$$

(as usual, $\theta^0:=Id$).

And let ${}^{\lq}\xi$ denote the morphism $\theta^\bullet\circ\theta^\bullet\longrightarrow\theta^\bullet$ given by the arrows

$${}^{\lq}\xi_{n,m}:\ \theta^n\theta^m\longrightarrow\theta^{n+m},\quad n,m\in\mathbb{Z},$$

where

$${}^{\lq}\xi_{n,m}=id:\ \theta^n\theta^m\longrightarrow\theta^{n+m}$$

if n and m are both nonnegative or both nonpositive;

for $n\geq m\geq 1$, the arrows

$${}^{\lq}\xi_{n,-m}:\ \theta^n\theta^{\wedge m}\longrightarrow\theta^{n-m},\quad {}^{\lq}\xi_{m,-n}:\ \theta^m\theta^{\wedge n}\longrightarrow\theta^{\wedge n-m},$$

$${}^{\lq}\xi_{-n,m}:\ \theta^{\wedge n}\theta^m\longrightarrow\theta^{\wedge n-m},\quad {}^{\lq}\xi_{-m,n}:\ \theta^{\wedge m}\theta^n\longrightarrow\theta^{n-m}$$

are defined by the following recurrent formulas:

$${}^{\lq}\xi_{n,-m}={}^{\lq}\xi_{n-1,-m+1}\circ\theta^{n-1}(\lambda\circ\theta^\wedge\xi\theta)\theta^{\wedge m-1},$$

$${}^{\lq}\xi_{-n,m}={}^{\lq}\xi_{-n+1,m-1}\circ\theta^{\wedge n-1}(\xi\circ\lambda^{\wedge -1})\theta^{m-1},$$

where

$$\lambda:\ \theta\circ\theta^\wedge\longrightarrow Id\quad\text{and}\quad\lambda^\wedge:\ Id\longrightarrow\theta^\wedge\circ\theta$$

are adjunction isomorphisms.

One can check that $(\theta^\bullet,{}^{\lq}\xi)$ is a monad with the unity $e:\ Id\longrightarrow\theta^\bullet$ which identifies Id with the summand θ^0.

5.4. Proposition. *The hyperbolic category $\mathcal{A}\{\theta,\xi\}$ is equivalent to the category $(\theta^\bullet,{}^{\lq}\xi)$-mod of $(\theta^\bullet,{}^{\lq}\xi)$-modules.*

Proof. 1) To every object (γ,M,η) of the category $\mathcal{A}\{\theta,\xi\}$ one can assign a morphism $\nu:\ \theta^\bullet M\longrightarrow M$ given by the arrows

$$\eta_0:=id:\ \theta^0(M)=M\longrightarrow M,$$

$$\eta_n:=\eta\circ\ldots\circ\theta^{n-1}\eta:\ \theta^n(M)\longrightarrow M,$$

and

$${}^\wedge\gamma_n:={}^\wedge\gamma\circ\ldots\circ\theta^{n-1}{}^\wedge\gamma:\ \theta^n(M)\longrightarrow M$$

for $n \geq 1$. Here $^\wedge\gamma = {^\wedge\gamma_1}\colon \theta^\wedge(M) \longrightarrow M$ is the adjoint to γ morphism. Clearly the composition of the action ν and the unity of the monad $(\theta^\bullet, {}`\xi)$ is the identical arrow: $\nu \circ e(M) = id_M$.

The condition $\eta \circ \gamma = \xi(M)$ implies that (M, ν) is an $(\theta^\bullet, {}`\xi)$-module; i.e. that $\nu \circ {}`\xi(M) = \nu \circ \theta^\bullet \nu$.

In fact, the diagram

$$\begin{array}{ccc} \theta^n\theta^m(M) & \xrightarrow{\ `\xi_{n,m}\ } & \theta^{n+m}(M) \\ {\scriptstyle \theta^n\nu_m}\downarrow & & \downarrow{\scriptstyle \nu_{n+m}} \\ \theta^n(M) & \xrightarrow{\ \nu_n\ } & M \end{array}$$

is commutative without any conditions on γ and η if n and m are both nonnegative or both nonpositive.

We have:

(a) $\nu_0 \circ {}`\xi_{-1,1}(M) := \lambda(M) \circ \theta^\wedge\xi\theta(M)\colon \theta^\wedge\theta(M) \longrightarrow M$,

$$\nu_{-1} \circ \theta^\wedge\nu_1 := \gamma^\wedge \circ \theta^\wedge\eta = \lambda \circ \theta^\wedge\gamma \circ \theta^\wedge\eta = \lambda \circ \theta^\wedge\xi\theta(M);$$

(b) if $m \geq 2$,

$$\nu_{m-1} \circ {}`\xi_{-1,m}(M) := \eta \circ \ldots \circ \theta^{m-2}\eta \circ (\lambda \circ \theta^\wedge\xi\theta)\theta^{m-1}(M)$$

$$\nu_{-1} \circ \theta^\wedge\nu_m := \gamma^\wedge \circ \theta^\wedge(\eta \circ \ldots \circ \theta^{m-1}\eta) = \lambda \circ \theta^\wedge\xi\theta(M) \circ \theta^\wedge\theta(\eta \circ \ldots \circ \theta^{m-2}\eta).$$

(c) Similarly,

$$\nu_0 \circ {}`\xi_{1,-1}(M) = {}`\xi_{1,-1}(M) := \xi(M) \circ \lambda^{\wedge-1}(M),$$

$$\nu_1 \circ \theta\nu_{-1} := \eta \circ \theta\gamma^\wedge = \eta \circ \theta(\lambda \circ \theta^\wedge\gamma) = \eta \circ \lambda^{\wedge-1}\theta \circ \theta\theta^\wedge\gamma = \lambda^{\wedge-1} \circ \theta\theta^\wedge(\eta \circ \gamma) = \xi(M) \circ \lambda^{\wedge-1}(M)$$

(we have used here the equality $\theta\lambda = \lambda^{\wedge-1}\theta$);

$$\nu_{1-m} \circ {}`\xi_{1,-m}(M) = \gamma^\wedge \circ \ldots \circ \theta^{\wedge m-2}\gamma^\wedge \circ (\xi \circ \lambda^{\wedge-1})\theta^{\wedge m-1}(M),$$

and

$$\nu_1 \circ \theta\nu_{-m} := \eta \circ \theta(\gamma^\wedge \circ \ldots \circ \theta^{\wedge m-1}\gamma^\wedge) = \xi \circ \lambda^{\wedge-1}(M) \circ \theta\theta^\wedge(\gamma^\wedge \circ \ldots \circ \theta^{\wedge m-2}\gamma^\wedge)$$

$$= \gamma^\wedge \circ \ldots \circ \theta^{\wedge m-2}\gamma^\wedge \circ (\xi \circ \lambda^{\wedge-1})\theta^{\wedge m-1}(M) = \nu_{1-m} \circ {}`\xi_{1,-m}(M).$$

This implies that $\nu_{n+m} \circ {}`\xi_{n,m} = \nu_n \circ \theta^n\nu_m$ for any $n, m \in \mathbb{Z}$.

2) On the other hand, a $(\theta^\bullet, {}`\xi)$-module, (M, ν), is uniquely defined by the arrows $\nu_{-1}\colon \theta^\wedge(M) \longrightarrow M$ and $\nu_1\colon \theta(M) \longrightarrow M$.

Denote by γ the adjoint to v_{-1} morphism $M \longrightarrow \theta^\wedge(M)$ which is uniquely defined by the equality $v_{-1} = \lambda(M)\circ\theta^\wedge\gamma$. We have:

$$v_{-1}\circ\theta^\wedge v_1 = {}'\xi_{-1,1}(M) := \lambda(M)\circ\theta^\wedge\xi\theta(M),$$

$$v_{-1}\circ\theta^\wedge v_1 = \lambda(M)\circ\theta^\wedge\gamma\circ\theta^\wedge v_1 = \lambda(M)\circ\theta^\wedge(\gamma\circ v_1).$$

Therefore, since $\lambda(M)$ is an isomorphism and $\theta^\wedge$ is an equivalence (actually, we need only monomorphness of λ and faithfulness of $\theta^\wedge$), $\gamma\circ v_1 = \xi\theta(M)$.

Similarly, it follows from the equalities

$$v_1\circ\theta v_{-1} = {}'\xi_{1,-1}(M) := \xi(M)\circ\lambda^{\wedge -1}(M)$$

and

$$v_1\circ\theta v_{-1} = v_1\circ\theta(\lambda\circ\theta^\wedge\gamma) = v_1\circ\lambda^{\wedge -1}\theta\circ\theta\theta^\wedge\gamma = (v_1\circ\gamma)\circ\lambda^{\wedge -1}(M).$$

Therefore, since $\lambda^{\wedge -1}$ is an isomorphism, the equality

$$(v_1\circ\gamma)\circ\lambda^{\wedge -1}(M) = \xi(M)\circ\lambda^{\wedge -1}(M)$$

implies that $v_1\circ\gamma = \xi(M)$. ∎

5.5. Corollary. *The forgetting functor* $\mathfrak{F} : \mathcal{A}(\theta,\xi) \longrightarrow \mathcal{A}$ *has a left adjoint functor.*

Proof. The forgetting functor $(\theta^\bullet, {}'\xi)$*-mod* $\longrightarrow \mathcal{A}$ is right adjoint to the functor which assigns to an object V in $\mathcal{A}$ the $(\theta^\bullet, {}'\xi)$-module $(\theta^\bullet(V), {}'\xi(V))$. ∎

5.6. The category $\mathcal{A}(\theta,\xi)^{op}$**.** The category $\mathcal{A}(\theta,\xi)^{op}$ is naturally isomorphic to the category $\mathcal{A}^{op}(\theta^\circledcirc,\xi^\circ)^{op}$, where ξ° is the image of ξ in $\mathcal{A}^{op}$, $\theta^\circledcirc := \theta^{\wedge\circ}$ the dual to $\theta^\wedge$ auto-equivalence. The isomorphism in question assigns to an object $(\gamma,M,\eta)^\circ$ of the category $\mathcal{A}(\theta,\xi)^{op}$ the object $(\gamma^{\wedge\circ},M^\circ,\eta^{\wedge\circ})$ of the category $\mathcal{A}^{op}(\theta^\circledcirc,\xi^\circ)$ and acts in an obvious way on morphisms.

5.7. The adjoint hyperbolic category. We name this way the category $\mathcal{A}(\theta^\wedge,\xi^\wedge)$, where $\xi^\wedge = \sigma\circ\theta^\wedge\xi\theta\circ\sigma^{-1}$, σ is the adjunction isomorphism $\theta^\wedge\theta \longrightarrow Id$. It is easy to check that the map that assigns to the triple $(\gamma,M,\eta) \in Ob\mathcal{A}(\theta,\xi)$ the triple $(\eta^\wedge,M,\gamma^\wedge)$, where $\eta^\wedge := \theta^\wedge\eta\circ\sigma^{-1}(M)$ and $\gamma^\wedge := \sigma(M)\circ\theta^\wedge\gamma$, is an equivalence of categories, $\Psi: \mathcal{A}(\theta,\xi) \longrightarrow \mathcal{A}(\theta^\wedge,\xi^\wedge)$.

5.8. An analog of Verma modules. Fix an autoequivalence ϑ of an abelian category $\mathcal{A}$ and an element ξ of $\mathfrak{z}(\mathcal{A}) := End(Id_{\mathcal{A}})$.

Let $\mathcal{A}_\xi$ denote the full subcategory of $\mathcal{A}$ generated by all objects V of $\mathcal{A}$ such that $\xi(V) = 0$. It is clear that $\mathcal{A}_\xi$ is a topologizing subcategory.

Set

$$\theta_+ := \bigoplus_{i\geq 0} \theta^i : \mathcal{A} \longrightarrow \mathcal{A};$$

and let g_+ be the functor morphism $\theta_+ \longrightarrow \theta\circ\theta_+$ which is defined by morphisms

$$g_{+i} = \xi\theta^i\colon \theta^i \longrightarrow \theta\circ\theta^{i-1} \quad \text{for} \quad i \geq 1, \quad \text{and} \quad g_{+0} = 0\colon Id \longrightarrow \theta.$$

And let h_+ denote the functor morphism $\theta\circ\theta_+ \longrightarrow \theta_+$ defined by

$$h_{+i} = id\colon \theta\circ\theta^{i-1} \longrightarrow \theta^i \quad \text{for} \quad i \geq 1.$$

5.8.1. Lemma. *The function that assigns to each* $V \in Ob\mathcal{A}_\xi$ *the triple* $(g_+(V),\theta_+(V),h_+(V))$ *and to each arrow* $f\colon V \longrightarrow V'$ *from* $Hom\mathcal{A}_\xi$ *the arrow*

$$\theta_+f\colon (g_+(V),\theta_+(V),h_+(V)) \longrightarrow (g_+(V'),\theta_+(V'),h_+(V'))$$

is a functor from $\mathcal{A}_\xi$ *to* $\mathcal{A}/\theta,\xi/$.

Proof. In fact,

$$h_{+i}\circ g_{+i} = \xi\theta^i \quad \text{if} \quad i \geq 1, \quad \text{and} \quad h_{+0}\circ g_{+0} = 0;$$

$$g_{+i}\circ h_{+i} = \xi\theta^i = \xi\theta(\theta^{i-1}) \quad \text{if} \quad i \geq 1, \quad \text{and} \quad g_{+1}\circ h_{+0} = \xi\theta.$$

Thus, $g_+\circ h_+ = \xi\theta\theta_+$, and $h_+\circ g_+ = \xi\theta_+(V)$ if $\xi(V) = 0$ which shows that $(g_+(V),\theta_+(V),h_+(V)) \in Ob\mathcal{A}/\theta,\xi/$ for every $V \in \mathcal{A}_\xi$. ■

We denote the functor $\mathcal{A}_\xi \longrightarrow \mathcal{A}/\theta,\xi/$ by $\mathfrak{B}$ and call it the *Verma functor.*

If $P \in Ob\mathcal{A}_\xi \cap Spec\mathcal{A}$, then the object $\mathfrak{B}(P)$ will be called the *Verma object with the highest weight* P.

5.9. About the subcategory $\mathcal{A}_\xi$**.** Let $J\xi$ denote the inclusion functor

$$\mathcal{A}_\xi \longrightarrow \mathcal{A}.$$

5.9.1. Proposition. *The embedding* $J\xi$ *has right and left adjoint functors.*

Proof. For each object M of the category $\mathcal{A}$, denote by $Ker\xi(M)$ and by $Cok\xi(M)$ respectively the kernel and the cokernel of the morphism $\xi(M)$. The maps $Ker\xi$ and $Cok\xi$ are extended uniquely to functors $\mathcal{A} \longrightarrow \mathcal{A}$, and both take values in the subcategory $\mathcal{A}_\xi$. Denote the corestrictions of $Ker\xi$ and

$Cok\xi$ on $\mathcal{A}_\xi$ by $K\xi$ and $C\xi$ respectively. It is easy to see that $K\xi$ is the right and $C\xi$ is the left adjoint functor to the embedding $J\xi: \mathcal{A}_\xi \longrightarrow \mathcal{A}$. ■

5.9.2. Corollary. *The full embedding* $J\xi: \mathcal{A}_\xi \longrightarrow \mathcal{A}$ *is an exact functor. In particular, it sends* $Spec\mathcal{A}_\xi$ *into* $Spec\mathcal{A}$.

The last assertion means that

$$\mathbf{Spec}\mathcal{A} \cap \mathcal{A}_\xi = \mathbf{Spec}\mathcal{A}_\xi = \mathbf{V}(\mathcal{A}_\xi) := \{\langle P\rangle \mid P \in Ob\mathcal{A}_\xi\}.$$

5.9.3. Lemma. *If* $V \in Spec\mathcal{A}$, *then either* $V \in \mathcal{A}_\xi$, *or* $\xi(V)$ *is a monomorphism.*

Proof. If $Ker\xi(M)$ is nonzero, then, since M is from the spectrum, $Ker\xi(M)$ is equivalent to M. This implies immediately that M is also in the subcategory $\mathcal{A}_\xi$. ■

5.10. The subcategories $\mathcal{A}_{\xi,n}$. Fix a nonzero integer n, and denote by $\mathcal{A}_{\xi,n}$ the full subcategory of the category $\mathcal{A}$ such that $Ob\mathcal{A}_{\xi,n}$ consists of all $V \in Ob\mathcal{A}$ for which $\xi(V) = 0$ and $\xi\theta^n(V) = 0$. Denote by $J_{\xi,n}$ the natural embedding $\mathcal{A}_{\xi,n} \longrightarrow \mathcal{A}_\xi$.

5.10.1. Lemma. *The functor* $J_{\xi,n}$ *has both right and left adjoint functors.*

Proof. For every $V \in Ob\mathcal{A}_\xi$, set $K_{\xi,n}(V) := \theta^{-n}Ker\xi\theta^n(V)$.

Clearly $K_{\xi,n}(V) \in Ob\mathcal{A}_{\xi,n}$, and the map $V \longmapsto K_{\xi,n}(V)$ defines a functor

$$K_{\xi,n}: \mathcal{A}_\xi \longrightarrow \mathcal{A}_{\xi,n}$$

which is right adjoint to the embedding $J_{\xi,n}$.

Similarly, the map $V \longmapsto C_{\xi,n}(V) = \theta^{-n}Cok\xi\theta^n(V)$ defines a functor,

$$C_{\xi,n}: \mathcal{A}_\xi \longrightarrow \mathcal{A}_{\xi,n},$$

which is left adjoint to the embedding $J_{\xi,n}$. ■

5.10.2. Corollary. *a) The embedding* $J_{\xi,n}: \mathcal{A}_{\xi,n} \longrightarrow \mathcal{A}_\xi$ *is an exact functor.*

b) $Spec\mathcal{A}_{\xi,n} = Spec\mathcal{A}_{\xi,n} \cap \mathcal{A}_\xi = Spec\mathcal{A}_{\xi,n} \cap Spec\mathcal{A}$.

5.10.3. The functors $\Psi_{\xi,n}$. Let $\theta_{1,n} := \bigoplus_{0\le i<n} \theta^i$; and let g_n be the functor

morphism $\theta_{1,n} \longrightarrow \theta\circ\theta_{1,n}$ which is defined by the arrows

$$g_{n,i} = \xi\theta^i\colon \theta^i \longrightarrow \theta\circ\theta^{i-1} \quad \text{for} \quad n-1 \geq i \geq 1,$$

$$g_{n,0} = 0\colon Id \longrightarrow \theta^n.$$

And let h_n be the functor morphism $\theta\circ\theta_n \longrightarrow \theta_n$ defined by

$$h_{n,i} = id\colon \theta\circ\theta^{i-1} \longrightarrow \theta^i \quad \text{for} \quad n-1 \geq i \geq 1,$$

$$h_{n,n} = 0\colon \theta\circ\theta^{n-1} \longrightarrow Id.$$

5.10.4. Lemma. *The function which assigns to each* $V \in Ob\mathcal{A}_{\xi,n}$ *the triple* $(g_n(V),\theta_n(V),h_n(V))$ *and to each arrow* $f\colon V \longrightarrow V'$ *from* $Hom\mathcal{A}_{\xi,n}$ *the arrow*

$$\theta_n f\colon (g_n(V),\theta_n(V),h_n(V)) \longrightarrow (g_n(V'),\theta_n(V'),h_n(V'))$$

is a functor, $\Psi_n = \Psi_{\xi,n}$, *from* $\mathcal{A}_{\xi,n}$ *to* $\mathcal{A}(\theta,\xi)$.

Proof. We have:

$$h_{n,i}\circ g_{n,i} = \xi\theta^i \quad \text{for} \quad 1 \leq i \leq n-1,$$

$$h_{n,n-1}\circ g_{n,0} = 0;$$

$$g_{n,i}\circ h_{n,i} = \xi\theta^i = \xi\theta\theta^{i-1} \quad \text{for} \quad 1 \leq i \leq n-1,$$

$$g_{n,0}\circ h_{n,n} = 0.$$

This implies that, if $\xi(V) = 0$ and $\xi\theta^n(V) = 0$, then

$$h_n(V)\circ g_n(V) = \xi\theta_n(V), \quad \text{and} \quad g_n(V)\circ h_n(V) = \xi\theta\theta_n(V);$$

i.e. $(g_n(V),\theta_n(V),h_n(V)) \in Ob\mathcal{A}(\theta,\xi)$ for every $V \in Ob\mathcal{A}_{\xi,n}$. ∎

5.10.4. $\mathcal{A}$-finite objects in $\mathcal{A}(\theta,\xi)$ and the functors $\Psi_{\xi,n}$. We shall say that an object (γ,M,η) of the category $\mathcal{A}(\theta,\xi)$ is of *$\mathcal{A}$-finite type* if the object M is of finite type.

Clearly if V is an object of finite type in $\mathcal{A}$, then the object $\Psi_{\xi,n}(V)$ is of $\mathcal{A}$-finite type.

Thus, the objects $\Psi_{\xi,n}(V)$ with V being of finite type are straightforward analogs of finite dimensional representations.

5.11. The degenerate part of a hyperbolic category. Consider now the category

$\mathcal{A}_{\xi,1}$ and the functors

$$\Psi_{\xi,1}: \mathcal{A}_{\xi,1} \longrightarrow \mathcal{A}/\theta,\xi/.$$

By definition, $Ob\mathcal{A}_{\xi,1}$ consists of all objects V of the category $\underline{A}$ such that $\xi(V) = 0$ and $\xi\theta(V) = 0$. The functor $\Psi_{\xi,1}$ assigns to every object V of $\mathcal{A}_{\xi,1}$ the triple $(0,V,0)$ which happens to be an object of the hyperbolic category $\mathcal{A}/\theta,\xi/$, and to every morphism $f: V \longrightarrow W$ the morphism

$$f: (0,V,0) \longrightarrow (0,W,0).$$

6. The spectrum of a hyperbolic category.

A hyperbolic category $\mathcal{A}/\theta,\xi/$, as any category modules over a monad, is defined, uniquely up to equivalence, by the forgetting functor

$$\mathfrak{F}: \mathcal{A}/\theta,\xi/ \longrightarrow \mathcal{A}, \quad (\gamma,M,\eta) \longmapsto M.$$

The object of this section is to get a description of the spectrum of the functor $\mathfrak{F}$.

For most of the assertions of this section, it is required from the abelian category $\mathcal{A}$ to have a countable version of the property (Ab5):

(Ab5ω) The category $\mathcal{A}$ has countable coproducts and, for any countable family Ω of subobjects of the object X, and for any monoarrow $Y \longrightarrow X$, the canonical arrow $\sup_{M\in\Omega}(M \cap Y) \longrightarrow sup(\Omega) \cap Y$ is an isomorphism.

6.1. Theorem. *Let* $P \in Spec\mathcal{A}$.

1) If $\xi(P) = 0$, $\xi\theta^n(P) = 0$, *but* $\xi\theta^i(P) \neq 0$ *for* $1 \le i < n$, *then the object* $\Psi_{\xi,n}(P)$ *belongs to* $Spec\mathcal{A}/\theta,\xi/$.

In the following assertions, $\mathcal{A}$ *has the property (Ab5ω).*

2) If $\xi(P) = 0$, *but* $\xi\theta^i(P) \neq 0$ *for* $i \ge 1$, *then* $\mathfrak{B}(P) \in Spec\mathcal{A}/\theta,\xi/$.

3) If $\xi\theta(P) = 0$ *and* $\xi\theta^i(P) \neq 0$ *for* $i \le 0$, *then* $\mathfrak{B}^\wedge(P) \in Spec\mathcal{A}/\theta,\xi/$.

Proof. 1) Let, under the conditions of 1), $\iota: (\gamma,M,\eta) \longrightarrow \Psi_{\xi,n}(P)$ be an arbitrary nonzero monomorphism. To prove the assertion, we need to show the existence of a diagram $(I)(\gamma,M,\eta) \longleftarrow\!\!< \mathfrak{X} \longrightarrow\!\!\!\!\rightarrow \Psi_{\xi,n}(P)$ in the category $\mathcal{A}/\theta,\xi/$.

Suppose that $Im(\iota)$ is a subobject of $\Psi_{\xi,n}(P)_i := \bigoplus_{0\le j\le i} \theta^j(P)$, but is not a subobject of $\Psi_{\xi,n}(P)_{i-1}$, $i \le n$. This means that

$$\gamma^{\wedge i+1} := \gamma^\wedge \circ \theta^\wedge\gamma^\wedge \circ \ldots \circ \theta^i\gamma^\wedge: \theta^{\wedge i+1}(M) \longrightarrow M$$

is zero, and

$$\gamma^{\wedge i}: \theta^{\wedge i}(M) \longrightarrow M$$

is a nonzero morphism. (Here $\gamma^\wedge\colon \theta^\wedge(M) \longrightarrow M$ denotes the adjoint to γ morphism.) Therefore the canonical monoarrow $Im(\iota\circ\gamma^{\wedge i}) \longrightarrow \Psi_{\xi,n}(P)$ is factorized by $P \longrightarrow \Psi_{\xi,n}(P)$; i.e. ι induces a monoarrow

$$\iota_0\colon M_0 := Im(\iota\circ\gamma^{\wedge i}) \longrightarrow P.$$

Since ι_0 is a nonzero subobject of P, and $P \in Spec\mathcal{A}$, there exists a diagram

$$(l)M_0 \leftarrow\!\!< N \twoheadrightarrow P \tag{1}$$

for some finite l. Since $P \in Ob\mathcal{A}_{\xi,n}$, the object M_0, being a subobject of P, also belongs to the subcategory $\mathcal{A}_{\xi,n}$. This implies that (1) is the diagram in $\mathcal{A}_{\xi,n}$. So, we can apply to (1) the exact functor $\Psi_{\xi,n}$.

Note now that, since $\Psi_{\xi,n}(\iota_0)\colon \Psi_{\xi,n}(M_0) \longrightarrow \Psi_{\xi,n}(P)$ is a monoarrow, it follows from the commutative diagram

$$\begin{array}{ccc} \Psi_{\xi,n}(M_0) & \longrightarrow & (\gamma,M,\eta) \\ & {\scriptstyle \Psi_{\xi,n}(\iota_0)}\searrow \quad \swarrow{\scriptstyle \iota} & \\ & \Psi_{\xi,n}(P) & \end{array}$$

that the canonical arrow $\Psi_{\xi,n}(M_0) \longrightarrow M$ is a monomorphism.

Thus, we have come to the diagram

$$(l)(\gamma,M,\eta) \leftarrow\!\!< (l)\Psi_{\xi,n}(M_0) \leftarrow\!\!< \Psi_{\xi,n}(N) \twoheadrightarrow \Psi_{\xi,n}(P)$$

which proves the assertion.

2) Let, under the conditions of *2)*, $\iota\colon (\gamma,M,\eta) \longrightarrow \mathfrak{B}_\xi(P)$ be an arbitrary nonzero monomorphism.

There exist a nonzero monoarrow $\upsilon\colon L \longrightarrow M$ and $n \geq 1$ such that $Im(\iota\circ\upsilon)$ is a subobject of $\mathfrak{B}_\xi(P)_n := \bigoplus_{0\leq j\leq n} \theta^j(P)$.

In fact, let M_n denote the pullback of

$$M \xrightarrow{\ \iota\ } \bigoplus_{i\geq 0} \theta^i(P) \longleftarrow \bigoplus_{0\leq i\leq n} \theta^i(P) \tag{1}$$

and ι_n the canonical monoarrow

$$M_n \longrightarrow \bigoplus_{i\geq 0} \theta^i(P).$$

By assumption on the category $\mathcal{A}$, $M_n \neq 0$ for some $n \geq 0$. Clearly this n can be chosen in such a way that ι_n is not a subobject of $\mathfrak{B}(P)_{n-1}$. This means that $M_0 := Im(g_+{}^{\wedge n}\circ\theta^{\wedge n}(\iota\circ\upsilon))$ is a nonzero subobject of P (cf. the argument in 1)).

The rest of the proof of *2)* is the repetition of the corresponding part the argument in $1)$.

3) The assertion *3)* is equivalent to the assertion *2)* for the adjoint category $\mathcal{A}/\theta^\wedge,\xi^\wedge/$ (cf. 7). ∎

All the listed in Theorem 6.1 objects are not equivalent one to another (at least, under some mild conditions). Explicitly, there is the following 'uniqueness' theorem:

6.2. Theorem *(a) Let V and W be objects from $Spec\mathcal{A}$ such that*

$$\xi(V) = 0, \quad \xi\theta^i(V) \neq 0 \ \text{ for } \ 1 \leq i \leq n\text{-}1, \quad \text{and} \quad \xi\theta^n(V) = 0;$$

and

$$\xi(W) = 0, \quad \xi\theta^i(W) \neq 0 \ \text{ for } \ 1 \leq i \leq m\text{-}1, \quad \text{and} \quad \xi\theta^m(V) = 0.$$

Then

(i) the relation $\Psi_{\xi,n}(V) \succ \Psi_{\xi,m}(W)$ implies that

$$n \geq m \quad \text{and} \quad \theta^s(V) \succ W \ \text{ for some } \ 0 \leq s \leq n\text{-}1;$$

(ii) $\Psi_{\xi,n}(V) \approx \Psi_{\xi,m}(W)$ if and only if $V \approx W$.

(b) Let $V \in Spec\mathcal{A}$ and $W \in Ob\mathcal{A}$ be such that $\xi(V)=0$, $\xi\theta^n(V)=0$, $\xi(W)=0$, and $\xi\theta^i(W)\neq 0$ for all $i \geq 1$. Then it cannot be that $\Psi_{\xi,n}(V) \succ \mathfrak{B}(W)$.

(c) Let objects $V, W \in Spec\mathcal{A}$ have the property:

$\xi(V) = 0$, $\xi(W) = 0$; and $\xi\theta^i(V)\neq 0$, $\xi\theta^i(W)\neq 0$ for all positive i.

And suppose that the relation $V \succ \theta^n(V)$ implies that V is equivalent to $\theta^n(V)$. Then $\mathfrak{B}(V) \approx \mathfrak{B}(W)$ if and only if $V \approx W$.

(d) Let $V \in Ob\mathcal{A}$ and $W \in Spec\mathcal{A}$ be such that $\xi(V) = 0$, $\xi\theta(W) = 0$ and $\xi\theta^i(W) \neq 0$ for all $i \leq 0$. Then the relation $\mathfrak{B}(V) \succ \mathfrak{B}^\wedge(W)$ does not hold.

Proof. (a) Let objects V and W satisfy the assumptions of *(a)*.

(i) Suppose that $\Psi_{\xi,n}(V) \succ \Psi_{\xi,m}(W)$; i.e. there exists the diagram

$$(I)\Psi_{\xi,n}(V) \xleftarrow{\ i\ }\!\!< (\gamma,M,\eta) \xrightarrow{\ e\ }\!\!\!\!\rightarrow \Psi_{\xi,m}(W).$$

Note that, since $\langle W\rangle \in Supp(\mathfrak{F}\circ\Psi_{\xi,n}(V))$, where $\mathfrak{F}$ is the forgetting functor $\mathcal{A}/\theta,\xi/ \longrightarrow \mathcal{A}$, and

$$Supp(\mathfrak{F}\circ\Psi_{\xi,n}(V)) = Supp(\bigoplus_{0\leq\nu<n} \theta^\nu(V)) = \bigcup_{0\leq\nu<n} Supp(\theta^\nu(V))$$

$\langle W\rangle \in Supp(\theta^\nu(V))$ for some ν, $0 \leq \nu \leq n\text{-}1$. This implies that

$$\langle\theta^{n-\nu}(W)\rangle \in Supp(\theta^n(V)). \tag{2}$$

Since $\xi\theta^n(V)) = 0$, and the inclusion (2) means exactly that $\theta^n(V) \succ \theta^{n-\nu}(W)$, we obtain that $\xi\theta^{n-\nu}(W) = 0$ which, together with the condition $\xi\theta^i(W) \neq 0$ if $0 \leq i \leq m\text{-}1$, provides the inequality: $m \leq n\text{-}\nu$. In particu'

$m \leq n$.

(ii) Suppose now that $\Psi_{\xi,n}(V)$ is equivalent to $\Psi_{\xi,m}(W)$. Then, obviously, $m = n$ which implies, in the preceeding argument, that $\nu = 0$; i.e. $<W> \in Supp(V)$. By symmetry, $<V> \in Supp(W)$. Thus, we have: $V \succ W \succ V$; i.e. V and W are equivalent.

(b) Let objects V and W of $\mathcal{A}$ satisfy the conditions of the assertion *(b)*; i.e. $\xi(V) = 0$, $\xi\theta^n(V) = 0$, $\xi(W) = 0$, and $W \neq 0$.

Note that the action of θ^n on $\Psi_{\xi,n}(V)$, $\theta^n(\mathfrak{F}\circ\Psi_{\xi,n}(V)) \longrightarrow \mathfrak{F}\circ\Psi_{\xi,n}(V)$, is zero. Clearly this property is inherited by any object $\mathbb{M}$ such that $\Psi_{\xi,n}(V) \succ \mathbb{M}$. Since, the action of θ^n on $\mathfrak{F}\circ\mathfrak{B}(W)$ is not zero, this implies that it cannot be that $\Psi_{\xi,n}(V) \succ \mathfrak{B}(W)$.

(c) Consider now the case, when the objects $V, W \in Spec\mathcal{A}$ have the property: $\xi(V) = 0$, $\xi(W) = 0$; and both $\xi\theta^i(V)$ and $\xi\theta^i(W)$ are nonzero for any positive i.

The relations $\mathfrak{B}(V) \succ \mathfrak{B}(W) \succ \mathfrak{B}(V)$ imply that $\theta^n(V) \succ W$ and $\theta^m(W) \succ V$ for some $m \geq 0$ and $n \geq 0$. Thus, $V \succ \theta^{-\nu}(V)$, where $\nu = m + n$. By condition, V is equivalent to $\theta^{-\nu}(V)$, or, which is the same, $V \approx \theta^{\nu}(V)$. Since $\xi(V) = 0$, the relation $V \approx \theta^{\nu}(V)$ imply that $\xi\theta^{\nu}(V) = 0$. But, by hypothesis, $\xi\theta^i(V) \neq 0$ for every $i \geq 1$. Therefore $\nu = 0$ which means that $m = n = 0$; i.e. $V \approx W$.

(d) Let objects V and W satisfy the assumptions of the assertion *(d)*.

The relation $\mathfrak{B}(V) \succ \mathfrak{B}^\wedge(W)$ implies that $\theta^n(V) \succ W$ for some $n \geq 0$, or, equivalently, $V \succ \theta^{-n}(W)$. It follows from the last relation and the equality $\xi(V) = 0$ that $\xi\theta^{-n}(W) = 0$ which contradicts to the assumption that $\xi\theta^i(W) \neq 0$ for all $i \leq 0$. ∎

6.3. Theorem. *Let* $(\gamma, M, \eta) \in Spec\mathcal{A}(\theta, \xi)$; *and suppose that there exists* $<P> \in Ass(M)$ *such that* $\xi\theta^n(P) = 0$ *for some* $n \in \mathbb{Z}$ *and the relation* $P \succ \theta^\nu(P)$ *implies that* $\nu = 0$. *Then* (γ, M, η) *is equivalent to one of the objects of Theorem 6.1.*

Proof. Let $\mathfrak{v}: P \longrightarrow M$ be a monoarrow the existence of which is assumed, and let $\mathfrak{v}^\wedge: (g(P), \theta^\bullet(P), h(P)) \longrightarrow (\gamma, M, \eta)$ be the adjoint to $\mathfrak{v}$ morphism.

1) Consider the case when $n = 0$; i.e. $\xi(P) = 0$.

The equality $\xi(P)=0$ implies that $\theta_-(P) := \bigoplus_{i \leq -1} \theta^i(P)$ is a submodule of the $(\theta^\bullet, `\xi)$-module $(\theta^\bullet(P), `\xi(P))$ (cf. 5.3); or, equivalently, the natural embedding $\iota_-: \theta_-(P) \longrightarrow \theta^\bullet(P)$ is a subobject of $(g(P), \theta^\bullet(P), h(P))$.

(a) The composition, $\mathfrak{v}_-: (g_-, \theta_-(P), h_-) \longrightarrow (\gamma, M, \eta)$ *of* ι_- *and* $\mathfrak{v}^\wedge$ *is equal to zero.*

In fact, if $\mathfrak{v}_- \neq 0$, then $Im(\mathfrak{v}_-)$ is equivalent to M which implies that

$$Supp(M) = Supp(Im(\mathfrak{v})) \subseteq \bigcup_{i \leq -1} Supp(\theta^i(P)). \tag{1}$$

Since $<P> \in Supp(M)$, (1) implies that $\theta^i(P) \succ P$ for some $i \leq -1$ which contradicts to the assumption of the theorem we are proving.

Thus, the morphism $\mathfrak{v}^\wedge$ induces a morphism

$$\mathfrak{v}_+: (g_+(P), \theta_+(P), h_+(P)) \longrightarrow (\gamma, M, \eta). \tag{2}$$

(b) Suppose that $\xi\theta^m(P) = 0$ for some $m \geq 1$. Then

$$\iota_m: \bigoplus_{i \geq m} \theta^i(P) \longrightarrow \theta_+(P)$$

is a subobject of $(g_+(P), \theta_+(P), h_+(P))$.

The composition $\mathfrak{v}_m: \bigoplus_{i \geq m} \theta^i(P) \longrightarrow \theta_+(P)$ *of* ι_m *and* $\mathfrak{v}_+$ *is zero.*

If $\mathfrak{v}_m \neq 0$, then $Im(\mathfrak{v}_m) \approx M$ which implies (as in *(a)*) that $<P>$ belongs to $\bigcup_{i \geq m} Supp(\theta^i(P))$; i.e. $\theta^i(P) \succ P$ for some $i \geq 1$ which, again, contradicts to the assumption.

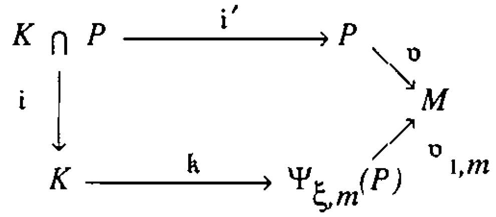
Thus, $\mathfrak{v}_m = 0$ which means that the morphism $\mathfrak{v}_+$ induces a morphism

$$\mathfrak{v}_{1,m}: \Psi_{\xi,m}(P) \longrightarrow (\gamma, M, \eta).$$

If m here is a minimal positive integer such that $\xi\theta^m = 0$, then $\Psi_{\xi,m}(P) \in Spec\mathcal{A}(\theta, \xi)$.

Note now that $\mathfrak{v}_{1,m}$ *is a monomorphism.*

Indeed, if $K := Ker(\mathfrak{v}_{1,m})$ is nonzero, then, according to the proof of Theorem 6.1, $K \cap P$ is nonzero. So, in the commutative diagram



$\upsilon \circ i'$, being a composition of nonzero monomorphisms, is nonzero. On the other hand, $\upsilon \circ i' = \upsilon_{1,m} \circ k \circ i = 0 \circ i = 0$.

The monomorphness of $\upsilon_{1,m}: \Psi_{\xi,m}(P) \longrightarrow (\gamma,M,\eta)$ implies that $\Psi_{\xi,m}(P) \approx (\gamma,M,\eta)$.

(c) Suppose now that $\xi\theta^m(P) \neq 0$ for all $m \geq 1$. Then, by Theorem 6.1, the Verma object $\mathfrak{B}(P) = (g_+(P),\theta_+(P),h_+(P))$ is in $Spec\mathcal{A}[\theta,\xi]$. Moreover, the canonical arrow (2) is a monomorphism, because, if the kernel, K, of υ_+ is nonzero, then $K \cap P \neq 0$ (cf. the proof of Theorem 6.1) which leads to a contradiction (cf. the part *(b)* of this proof).

2) Suppose now that $\xi\theta^n(P) = 0$ for some $n \geq 1$, and $\xi\theta^i$ is nonzero for $0 \leq i \leq n-1$. Then $\bigoplus_{m \geq n} \theta^m(P)$ is a subobject of $(g(P),\theta^\bullet(P),h(P))$, and the same argument as above shows that this subobject is annihilated by the canonical morphism $\upsilon^\wedge: (g(P),\theta^\bullet(P),h(P)) \longrightarrow (\gamma,M,\eta)$; i.e. $\upsilon^\wedge$ induces a morphism

$$\upsilon_{-,n}: \mathfrak{B}_-(\theta^n(P)) \longrightarrow (\gamma,M,\eta).$$

If $\xi\theta^i(P) \neq 0$ for all $i < n$, then $\upsilon_{-,n}$ is a monoarrow; hence (γ,M,η) is equivalent to $\mathfrak{B}_-(\theta^n(P))$.

If $\xi\theta^m(P) = 0$ for some (necessarily negative) m such that $\xi\theta^i(P) \neq 0$ for $m < i < n$, then $Ker(\upsilon_{-,n})$ coincides with the subobject

$$\bigoplus_{s<m} \theta^s(P) \longrightarrow \theta_-(\theta^n(P))$$

which implies that (γ,M,η) is equivalent $\Psi_{\xi,l}(\theta^m(P))$, where $l = n - m$.

The proof of these assertions follows the same pattern as the corresponding parts of the argument above. ∎

Consider now the case, when $\xi\theta^n(P) \neq 0$ for all $n \in \mathbb{Z}$.

6.4. Theorem. *Let the category $\mathcal{A}$ have the property (Ab5ω). And let an object $P \in Spec\mathcal{A}$ be such that*

(a) $\xi\theta^n(P) \neq 0$ *for all* $n \in \mathbb{Z}$;

(b) $\theta^n(P) \succ P$ *only if* $n = 0$.

Then

1) $\theta^\bullet(P) = (g, \bigoplus_{i\in\mathbb{Z}} \theta^i(P), h) \in Spec\mathcal{A}(\theta,\xi)$.

2) For any object (γ,M,η) *of the category* $\mathcal{A}(\theta,\xi)$ *such that* $Ass(M) \ni P$, *the canonical arrow* $\theta^\bullet(P) \longrightarrow (\gamma,M,\eta)$ *is a monomorphism.*

In particular, if $(\gamma,M,\eta) \in Spec\mathcal{A}(\theta,\xi)$, *and* $P \in Ass(M)$, *then* (γ,M,η) *is equivalent to* $\theta^\bullet(P)$.

If $(\gamma,M,\eta) \in Ob\mathcal{A}(\theta,\xi)$, $P \in Ass(M)$, *and the image of the canonical arrow* $\theta^\bullet(P) \longrightarrow (\gamma,M,\eta)$ *is equivalent to* (γ,M,η), *then* $\theta^\bullet(P)$ *is equivalent to* (γ,M,η).

Proof. *1)* Fix a nonzero monoarrow $\iota: (\gamma,M,\eta) \longrightarrow \theta^\bullet(P)$.

(i) There is a subobject $\mu: M_s \longrightarrow M$ of M such that the image of $\iota\circ\mu$ is contained in $\bigoplus_{0\le i\le s} \theta^i(P)$ for some $s \ge 0$, but is not contained neither in $\bigoplus_{1\le i\le s} \theta^i(P)$, nor in $\bigoplus_{0\le i<s} \theta^i(P)$.

In fact, let $M_{m,n}$ denote the pullback of

$$M \xrightarrow{\ \iota\ } \bigoplus_{i\in\mathbb{Z}} \theta^i(P) \longleftarrow \bigoplus_{-m\le i\le n} \theta^i(P) \qquad (1)$$

and $\iota_{m,n}$ the canonical monoarrow $M_{m,n} \longrightarrow \bigoplus_{i\in\mathbb{Z}} \theta^i(P)$.

Since the category $\mathcal{A}$ has the property $(Ab5\omega)$, $M_{m,n} \neq 0$ for some $m, n \ge 0$. And we take as m and n the minimal nonnegative numbers having this property.

Note that, since $\xi\theta^n(P) \neq 0$ for all n, and $\theta^n(P)$ is in $Spec\mathcal{A}$ for every $n \in \mathbb{Z}$, all arrows $\xi\theta^n(P)$ are monomorphisms. The monomorphness of $\xi\theta^n(P)$, $n \in \mathbb{Z}$, implies that the action

$$\mu_m: \theta^m(\bigoplus_{i\in\mathbb{Z}} \theta^i(P)) \longrightarrow \bigoplus_{i\in\mathbb{Z}} \theta^i(P)$$

is a monomorphism for any m (this follows from the description of the action in Section 3). Hence the composition of $\theta^m\iota_{m,n}$ and the action μ_m is a monomorphism the image of which is 'contained' in $\bigoplus_{0\le i\le s} \theta^i(P)$, but is not contained neither in $\bigoplus_{1\le i\le s} \theta^i(P)$, nor in $\bigoplus_{0\le i<s} \theta^i(P)$, where $s = n + m$.

On the other hand, this monomorphism is the composition of

$$\theta^m\iota': \theta^m M_{m,n} \longrightarrow \theta^m M,$$

where ι' is the canonical monoarrow, the action $\theta^m M \longrightarrow M$, and the monomorphism ι. This shows that the composition of $\theta^m\iota'$ and the action

$$\theta^m M \longrightarrow M$$

is a required monoarrow, μ, from $M_s := \theta^m(M_{m,n})$ to M.

(ii) Take the minimal s satisfying the conditions of the assertion *(i)*. The claim is that $s = 0$.

Suppose that, on the contrary, $s \geq 1$. Denote by $M(\nu)$ the kernel of the composition of the arrow

$$\mu\colon M_s \longrightarrow \bigoplus_{0 \leq i \leq s} \theta^i(P)$$

and the projection

$$\bigoplus_{0 \leq i \leq s} \theta^i(P) \longrightarrow \theta^\nu(P), \quad 0 \leq \nu \leq s.$$

Note that, thanks to the minimality of s, both $M(s)$ and $M(0)$ are zero objects; i.e. M_s is a subobject of $\theta^s(P)$ and of P. Since P and $\theta^s(P)$ belong to $Spec\mathcal{A}$, this means that M_s is equivalent to P and to $\theta^s(P)$. But this cannot happen, since, by hypothesis, P is not equivalent to $\theta^s(P)$.

Thus, the assumption $s \geq 1$ leads to the contradiction; i.e. $s = 0$, or, in other words,

$$\iota\circ\mu\colon M_s \longrightarrow \bigoplus_{i \in \mathbb{Z}} \theta^i(P)$$

is the composition of
a monoarrow $\nu\colon M_s \longrightarrow P$ and the natural morphism $P \longrightarrow \bigoplus_{i \in \mathbb{Z}} \theta^i(P)$.

(iii) So, we have the commutative diagram

$$\begin{array}{ccc} (\eta, M, \gamma) & \xrightarrow{\quad \iota \quad} & \theta^\bullet(P) \\ & \nwarrow \qquad \nearrow_{\theta^\bullet\nu} & \\ & \theta^\bullet(M_s) & \end{array}$$

in which ι is a monomorphism by assumption, $\theta^\bullet\nu$ is a monoarrow because ν is a monoarrow, and the functor $\theta^\bullet$ is left (and right) exact. Therefore the canonical morphism $\theta^\bullet(M_s) \longrightarrow (\gamma, M, \eta)$ is also a monoarrow.

Since $P \in Spec\mathcal{A}$, and $\nu\colon M_s \longrightarrow P$ is a nonzero monoarrow, M_s is equivalent to P (with respect to $\succ$). The functor $\theta^\bullet$, being exact, respects this equivalence: $\theta^\bullet(M_s) \approx \theta^\bullet(P)$. Since $\theta^\bullet(M_s)$ is a subobject of (γ, M, η), and (γ, M, η) is a subobject of $\theta^\bullet(P)$, we have:

$$\theta^\bullet(P) \succ (\gamma, M, \eta) \succ \theta^\bullet(M_s) \succ \theta^\bullet(P)$$

which shows that $\theta^\bullet(P) \in Spec\mathcal{A}/\theta, \xi/$.

2) Let (γ, M, η) be an object of the category $\mathcal{A}/\theta, \xi/$, and let

$$\upsilon\colon P \longrightarrow M$$

be a monoarrow, where $P \in Spec\mathcal{A}$.

Suppose that the kernel (g,K,h) of the morphism

$$\theta^{\bullet}(P) \longrightarrow (\gamma,M,\eta)$$

induced by the monoarrow $P \longrightarrow M$ is nonzero. Then, according to (the proof of) the heading $\imath$), there is a nonzero monomorphism $\sigma: W \longrightarrow K$ such that the composition of σ with the canonical monomorphism $\mathfrak{k}: K \longrightarrow \bigoplus_{i\in\mathbb{Z}} \theta^{i}(P)$ is a nonzero subobject of $\mathfrak{v}: P \longrightarrow \theta^{\bullet}(P)$. This means that the composition of $\mathfrak{k}\circ\sigma$ and the canonical morphism

$$\mathfrak{p}: \bigoplus_{i\in\mathbb{Z}} \theta^{i}(P) \longrightarrow (\gamma,M,\eta)$$

is nonzero. But, this cannot happen, since $\mathfrak{p}\circ\mathfrak{k} = 0$.

So, we have come to the contradiction with the assumption that $\mathfrak{p}$ has a nonzero kernel. ∎

6.5. The degenerate part of a hyperbolic category. Consider the full subcategory, $\mathcal{A}/\theta,\xi\,|\,0/$, of the category $\mathcal{A}/\theta,\xi/$ generated by all objects (γ,M,η) such that $\eta\circ\gamma = 0$, or, equivalently, $\xi(M) = 0$. Clearly the subcategory $\mathcal{A}/\theta,\xi\,|\,0/$, being a preimage of a thick (even closed) subcategory $\mathcal{A}_{\xi}$ (cf. Proposition 5.9.1 and Corollary 5.9.2) under an exact (- forgetting) functor, is thick. In particular, $Spec\mathcal{A}/\theta,\xi\,|\,0/ = Spec\mathcal{A}/\theta,\xi/ \cap Ob\mathcal{A}/\theta,\xi\,|\,0/$.

We have a cartesian square of exact fully faithful functors:

$$\begin{array}{ccc} \mathcal{A} & \longrightarrow & \mathcal{A}[\theta] \\ \downarrow & & \downarrow \\ \mathcal{A}[\theta^\wedge] & \longrightarrow & \mathcal{A}/\theta,\xi\,|\,0/ \end{array} \qquad (1)$$

where $\mathcal{A}[\theta^\wedge] \longleftarrow \mathcal{A} \longrightarrow \mathcal{A}[\theta]$ are natural embeddings:

$$V \longmapsto (V,0), \quad f \longmapsto f;$$

$\mathcal{A}[\theta] \longrightarrow \mathcal{A}/\theta,\xi\,|\,0/$ is the embedding

$$(M,\theta(M) \xrightarrow{u} M) \longmapsto (0,M,u), \quad f \longmapsto f;$$

$\mathcal{A}[\theta^\wedge] \longrightarrow \mathcal{A}/\theta,\xi\,|\,0/$ assigns to an object $(M,\theta^\wedge(M) \xrightarrow{v} M)$ the object $(v^\wedge,M,0)$ of the category $\mathcal{A}/\theta,\xi\,|\,0/$ and, again, acts identically on morphisms. To the cartesian square of functors (1), there corresponds a cartesian square of the embeddings:

$$
\begin{array}{ccc}
\mathbf{Spec}\mathcal{A} & \longrightarrow & \mathbf{Spec}\mathcal{A}[\theta] \\
\downarrow & & \downarrow \\
\mathbf{Spec}\mathcal{A}[\theta^\wedge] & \longrightarrow & \mathbf{Spec}\mathcal{A}/\theta,\xi\,|\,0/
\end{array}
\qquad (2)
$$

6.5.1. Lemma. *Let* $(\gamma,M,\eta) \in \mathit{Spec}\mathcal{A}/\theta,\xi/$. *The following properties are equivalent:*

(a) $\xi(M) = 0$;

(b) $\xi\theta(M) = 0$;

(c) either $\gamma = 0$, *or* $\eta = 0$, *or both.*

Proof. Clearly *(c)* implies *(a)* and *(b)*.

(a) $\Rightarrow$ *(c)*. Take the adjoint to (γ,M,η) object - $(\eta^\wedge,M,\gamma^\wedge)$. We have: $\eta^\wedge\circ\gamma^\wedge = \theta^\wedge\eta\circ\theta^\wedge\gamma = \theta^\wedge\xi(M) = 0$. Consider $Ker(\eta^\wedge)$. It is clear from the commutative diagram

$$
\begin{array}{ccccc}
\theta^\wedge(M) & \xrightarrow{\ \gamma^\wedge\ } & M & \xrightarrow{\ \eta^\wedge\ } & \theta^\wedge(M) \\
\theta^\wedge\iota \uparrow & \searrow & \uparrow \iota & & \uparrow \theta^\wedge\iota \\
\theta^\wedge(Ker\eta) & \xrightarrow{\ \sigma\ } & Ker\eta^\wedge & \xrightarrow{\ 0\ } & \theta^\wedge(Ker\eta^\wedge)
\end{array}
$$

where ι is the canonical monomorphism, and the diagonal arrow is due to the equality $\eta^\wedge\circ\gamma^\wedge = 0$, that $(0,Ker\eta^\wedge,\sigma)$ is a subobject of $(\eta^\wedge,M,\gamma^\wedge)$.

If $Ker\eta^\wedge \neq 0$, then, since $(\eta^\wedge,M,\gamma^\wedge)$ is in the spectrum, $(0,Ker\eta^\wedge,\sigma) \succ (\eta^\wedge,M,\gamma^\wedge)$. In particular, $\eta^\wedge = 0$; i.e. $\eta = 0$.

If $Ker\eta^\wedge = 0$, then the equality $\eta = \lambda\circ\theta\eta^\wedge$, where λ is the adjunction isomorphism $\theta\circ\theta^\wedge \longrightarrow Id_{\mathcal{A}}$, shows that η is also a monoarrow. The monomorphness of η and the equality $\eta\circ\gamma = \xi(M) = 0$ implies that $\gamma = 0$.

The implication *(b)* $\Rightarrow$ *(c)* coincides with *(a)* $\Rightarrow$ *(c)* for the object $(\eta^\wedge,M,\gamma^\wedge)$ of the adjoint hyperbolic category. ∎

6.5.2. Corollary. *The square (2) is not only universal, but also couniversal; i.e.* $\mathbf{Spec}\mathcal{A}/\theta,\xi\,|\,0/ \simeq \mathbf{Spec}\mathcal{A}[\theta] \coprod_{\mathbf{Spec}\mathcal{A}} \mathbf{Spec}\mathcal{A}[\theta^\wedge]$.

6.6. The case of a local category $\mathcal{A}$**.** Suppose that the category $\mathcal{A}$ is local; and let P be the unique up to isomorphism quasi-final object of $\mathcal{A}$. Clearly $\theta(P) \approx P$.

6.6.1. Lemma. *Let* $\mathcal{A}$ *be a local category with a quasi-final object* P. *Then an* *endomorphims* ξ *of* $Id_{\mathcal{A}}$ *is an isomorphism if and only if* $\xi(P) \neq 0$.

Proof. Suppose that, for some object M, the morphism $\xi(M)$ has a nontrivial kernel. Clearly $\xi(Ker(\xi(M))) = 0$ which implies, since $Ker(\xi(M)) \succ P$, that $\xi(P) = 0$ which is not the case.

Similarly, the assumption that $\xi(M)$ is not epimorphic implies the same contradiction: $\xi(P) = 0$. ∎

6.6.2. Corollary. *The ring* $\mathfrak{z}(\mathcal{A})$ *of all endomorphisms of the identical functor* $Id_{\mathcal{A}}$ *- the center of the category* $\mathcal{A}$ *- is local if the category* $\mathcal{A}$ *is local.*

Proof. Consider the ideal $\mu(\mathcal{A})$ formed by all $\xi \in \mathfrak{z}(\mathcal{A})$ such that $\xi(P) = 0$, where P is a quasi-final object (clearly $\mu(\mathcal{A})$ does not depend on the choice of the object P). According to Lemma 6.6.1, $\mathfrak{z}(\mathcal{A})-\mu(\mathcal{A})$ consists of (all) invertible objects which means that $\mu(\mathcal{A})$ is the unique maximal ideal in $\mathfrak{z}(\mathcal{A})$. ∎

6.7. θ-invariant points. Suppose that $P \in Spec\mathcal{A}$ is θ-stable; i.e. $P \approx \theta(P)$.

6.7.1. Lemma. *Let* $<P> \in$ **Spec**$\mathcal{A}$ *be θ-stable. And let* $<(\gamma,M,\eta)>$ *be an element of* **Spec**$\mathcal{A}[\theta,\xi]$ *such that* $<P> \in Ass(M)$.

Then $Supp(M) = [<P>]^{-}$.

Proof. Indeed, $[<P>]^{-} \subseteq Supp(M)$, because $<P> \in Ass(M) \subseteq Supp(M)$.

The inclusion $<P> \in Ass(M)$ means that there is a monoarrow $\iota: P' \longrightarrow M$ for some $P' \approx P$. Being in the spectrum, the object $\mathbb{M} = (\gamma,M,\eta)$ is equivalent to the image of the adjoint morphism

$$\iota^{\wedge}: \mathbb{F}(P') = (g(P'),\theta^{\bullet}(P'),h(P')) \longrightarrow \mathbb{M}$$

which implies the inclusion $Supp(M) \subseteq Supp(\theta^{\bullet}(P'))$.

Now, since $P' \approx \theta(P')$,

$$Supp(\theta^{\bullet}(P')) = \bigcup_{n \in \mathbb{Z}} Supp(\theta^{n}(P')) = [<P>]^{-}$$

which implies the required inclusion $Supp(M) \subseteq [<P>]^{-}$. ∎

According to Lemma 5.9.3, there are only two possibilities:

either $\xi(P) = 0$, *or* $\xi(P)$ *is a monomorphism.*

The following Proposition takes care about the first one.

6.7.2. Proposition. *Let* $<P>$ *be a θ-stable point of* **Spec**$\mathcal{A}$ *such that* $\xi(P) =$

0. Then the set

$$\mathbf{Spec}_{<P>}\mathcal{A}[\theta,\xi] = \{<(\gamma,M,\eta)> \in \mathbf{Spec}\mathcal{A}[\theta,\xi] \mid <P> \in Ass(M)\}$$

has the following decomposition:

$$\mathbf{Spec}_{<P>}\mathcal{A}[\theta,\xi] \simeq \mathbf{Spec}_{<P>}\mathcal{A}[\theta] \coprod_{<P>} \mathbf{Spec}_{<P>}\mathcal{A}[\theta^{\wedge}].$$

Proof. The equality $\xi(P) = 0$ and θ-stability of $<P>$ imply that $\xi(\theta^{\bullet}(P)) = 0$. If $(\gamma,M,\eta) \in Spec_{<P>}\mathcal{A}[\theta,\xi]$, then there is a nonzero morphism $\theta^{\bullet}(P') \longrightarrow M$ for some $P' \approx P$, and M is equivalent to the image of this morphism (cf. the proof of Lemma 6.7.1) which implies that $\xi(M) = 0$; i.e. $(\gamma,M,\eta) \in Ob\mathcal{A}[\theta,\xi \mid 0]$.

Now the assertion follows from Corollary 6.5.2. ■

In order to study the nondegenerate case, $\xi(P) \neq 0$, as well as to finish the investigation of the degenerate one, we need to make some simplifications.

First, note that the equivalence $P \approx \theta(P)$ implies that the closure, $[<P>]^{-}$, of the point $<P>$ in the topology τ (i.e. the set of all specializations of $<P>$) is θ-stable. Therefore the thick subcategory $\mathcal{A}([<P>]^{-})$ of $\underline{A}$ is θ-stable which means that the preimage of $\mathcal{A}([<P>]^{-})$ under the forgetting functor $\mathcal{A}[\theta,\xi] \longrightarrow \mathcal{A}$ coincides with the hyperbolic category $\mathcal{A}'[\theta',\xi']$, where $\mathcal{A}' = \mathcal{A}([<P>]^{-})$, θ' is the induced by θ auto-equivalence of the category $\mathcal{A}'$, ξ' is the restriction of ξ on $\mathcal{A}'$.

It follows from Lemma 6.7.1 that $\mathbf{Spec}_{<P>}\mathcal{A}[\theta,\xi] = \mathbf{Spec}_{<P>}\mathcal{A}'[\theta',\xi']$.

Therefore, being interested in the subset $\mathbf{Spec}_{<P>}\mathcal{A}[\theta,\xi]$ of $\mathbf{Spec}\mathcal{A}[\theta,\xi]$, we replace the category $\mathcal{A}$ by its thick subcategory $\mathcal{A}' = \mathcal{A}([<P>]^{-})$ and the category $\mathcal{A}[\theta,\xi]$ by $\mathcal{A}'[\theta',\xi']$.

Futher, since the thick subcategory $<P>$ is θ-stable, the functor θ induces an auto-equivalence, θ', of the quotient category $\mathcal{A}' := \mathcal{A}/<P>$. Let ξ' denote the induced by ξ endomorphism of the identical functor from $\mathcal{A}'$ to $\mathcal{A}'$; and let P' be a (unique up to equivalence) quasi-final object of the local category $\mathcal{A}'$.

The equality $\mathcal{A} = \mathcal{A}([<P>]^{-})$ implies that the spectrum of the category $\mathcal{A}' = \mathcal{A}/<P>$ consists of only one point.

Clearly the localization at $<P>$ maps bijectively the set $\mathbf{Spec}_{<P>}\mathcal{A}[\theta,\xi]$ we are studying onto $\mathbf{Spec}'\mathcal{A}'[\theta',\xi'] := \{<(\gamma,M,\eta)> \mid Ass(M) \neq \emptyset\}$.

(a) Suppose now that $\xi(P) \neq 0$. This implies that $\xi'(P')$ is nonzero. According to Lemma 6.6.1, ξ' is an automorphism of the identical functor $Id_{\mathcal{A}'}$. This, in turn, means that the category $\mathcal{A}'[\theta',\xi']$ is equivalent to the category

$\mathcal{A}'[\theta']/\mathcal{A}'$ (cf. Lemma 5.2). Therefore the results of subsections 4.6 and 4.3 provide a description of $\mathbf{Spec}\mathcal{A}/\theta',\xi'/$; at least in the case when $\mathcal{A}'$ has simple objects.

(b) Suppose that $\xi(P) = 0$. This implies that $\xi'(P') = 0$. Hence, by Proposition 6.7.2, $\mathbf{Spec}\mathcal{A}'/\theta',0/ = \mathbf{Spec}\mathcal{A}'[\theta'] \coprod_{<P'>} \mathbf{Spec}\mathcal{A}'[\theta'^{\wedge}]$.

So, if the category $\mathcal{A}'$ has nonzero objects of finite type, we can use the obtained in Section 4 description of the spectrum of a skew polynomial category over a local semisimple category.

6.8. θ^N-invariant points. Let now $\theta^N(P) \approx P$ for some positive integer N, but $\theta^i(P)$ is not equivalent to P if $1 \le i < N$.

Denote by Θ the functor θ^N and by ζ the endomorphism of the identical functor, $Id_{\mathcal{A}}$ which is defined by the following recurrent relations:

$$\xi_1 = \xi, \quad \xi_{n+1} = \lambda\circ\theta^{\wedge}\xi_n\theta\circ\lambda^{-1}, \quad \zeta := \xi_N \tag{1}$$

and consider the category $\mathcal{A}/\Theta,\zeta/$.

Let R_N denote the map which assigns to any object (γ,M,η) of the category $\mathcal{A}/\theta,\xi/$ the triple (γ_N,M,η_N), where

$$\gamma_N := \theta^{N-1}\gamma\circ\ldots\circ\gamma, \quad \eta_N := \eta\circ\theta\eta\circ\ldots\circ\theta^{N-1}\eta,$$

and acts identically on morphisms: $f \longmapsto f$.

6.8.1. Lemma. *The map R_N is a functor from $\mathcal{A}/\theta,\xi/$ to $\mathcal{A}/\Theta,\zeta/$.*

Proof. 1) Clearly

$$\gamma_N\circ\eta_N = \theta^N\xi\circ\theta^{N-1}\xi\theta\circ\ldots\circ\xi\theta^N,$$

and one can show (by induction) that

$$\theta^N\xi\circ\theta^{N-1}\xi\theta\circ\ldots\circ\xi\theta^N = \zeta\theta^N := \zeta\Theta.$$

2) On the other hand,

$$\begin{aligned}\eta_N\circ\gamma_N &= \eta\circ\theta\eta\circ\ldots\circ\theta^{N-1}\eta\circ\theta^{N-1}\gamma\circ\ldots\circ\gamma = \\ &\eta\circ\theta\eta\circ\ldots\circ\theta^{N-2}\eta\circ\theta^{N-1}\xi\circ\theta^{N-2}\gamma\circ\ldots\circ\gamma = \\ &\eta\circ\theta\eta\circ\ldots\circ\theta^{N-2}(\eta\circ\theta\xi\circ\gamma)\circ\ldots\circ\gamma.\end{aligned} \tag{1}$$

Now note that

$$\theta^{\wedge}(\eta\circ\theta\xi\circ\gamma) = \eta^{\wedge}\circ\lambda^{\wedge-1}\circ\theta^{\wedge}\theta\xi\circ\lambda^{\wedge}\circ\gamma^{\wedge} = \eta^{\wedge}\circ\xi\circ\gamma^{\wedge} = \xi\theta^{\wedge}\circ\eta^{\wedge}\circ\gamma^{\wedge}$$

where $\eta^{\wedge}$ and $\gamma^{\wedge}$ are, as before, the adjoint to η and γ morphisms; i.e.

$$\eta^{\wedge} = \theta^{\wedge}\eta\circ\lambda, \quad \gamma^{\wedge} = \lambda^{-1}\circ\theta^{\wedge}\gamma.$$

This implies that

$$\eta\circ\theta\xi\circ\gamma = \lambda\circ\theta\xi\theta^{\wedge}\circ\lambda^{-1}\circ\eta\circ\gamma = \lambda\circ\theta\xi\theta^{\wedge}\circ\lambda^{-1}\circ\xi \; (= \xi\circ\lambda\circ\theta\xi\theta^{\wedge}\circ\lambda^{-1}). \qquad (2)$$

Therefore we can continue (1) as follows:

$$\eta\circ\theta\eta\circ\ldots\circ\theta^{n-3}\eta\circ\theta^{N-2}(\eta\circ\theta\xi\circ\gamma)\circ\theta^{n-3}\gamma\circ\ldots\circ\gamma =$$

$$\eta\circ\theta\eta\circ\ldots\circ\theta^{n-3}\eta\circ\theta^{N-2}(\lambda\circ\theta\xi\theta^{\wedge}\circ\lambda^{-1}\circ\xi)\circ\theta^{n-3}\gamma\circ\ldots\circ\gamma =$$

$$\eta\circ\theta\eta\circ\ldots\circ\theta^{n-3}(\eta\circ\theta(\lambda\circ\theta\xi\theta^{\wedge}\circ\lambda^{-1}\circ\xi)\circ\gamma)\circ\ldots\circ\gamma,$$

and, according to (2),

$$\eta\circ\theta(\lambda\circ\theta\xi\theta^{\wedge}\circ\lambda^{-1}\circ\xi)\circ\gamma = \lambda\circ\theta(\lambda\circ\theta\xi\theta^{\wedge}\circ\lambda^{-1}\circ\xi)\theta^{\wedge}\circ\lambda^{-1}\circ\eta\circ\gamma =$$

$$\lambda\circ\theta(\lambda\circ\theta\xi\theta^{\wedge}\circ\lambda^{-1}\circ\xi)\theta^{\wedge}\circ\lambda^{-1}\circ\xi.$$

Repeating this, we come to the required formula: $\eta_N\circ\gamma_N = \zeta(M)$.

Clearly the map R_N is functorial. ■

6.8.2. A general observation. Let $\mathbb{F} = (F,\mu)$ and $\mathbb{G} = (G,\nu)$ be monads in $\mathcal{A}$, and $h\colon \mathbb{G} \longrightarrow \mathbb{F}$ a morphism of monads. The morphism h induces the functor

$$h_*\colon \mathbb{F}\text{-}mod \longrightarrow \mathbb{G}\text{-}mod, \quad (M,\mathfrak{m}) \longmapsto (M,\mathfrak{m}\circ h(M)), \quad f \longmapsto f.$$

6.8.3. Lemma. *The functor* h_* *has a left adjoint functor.*

Proof. Given a $\mathbb{G}$-module $\mathbb{V} = (V,\mathfrak{v})$, denote by $\mathbb{F}\otimes_{\mathbb{G}}\mathbb{V}$ the $\mathbb{F}$-module $(M,\mathfrak{m})$, where M is the coequalizer of the pair

$$F\mathfrak{v},\ \mu\circ Fh(V)\colon FG(V) \longrightarrow F(V),$$

$\mathfrak{m}\colon F(M) \longrightarrow M$ the unique arrow which makes the diagram

$$\begin{array}{ccc} FFG(V) & \xrightarrow{\;Fc\;} & F(M) \\ \mu\big\downarrow & & \big\downarrow \mathfrak{m} \\ FG(V) & \xrightarrow{\;\;c\;\;} & M \end{array}$$

commute. ■

6.8.4. Corollary. *The functor* R_N *is exact and faithful and has a left adjoint functor,* $L_N\colon \mathcal{A}(\Theta,\zeta) \longrightarrow \mathcal{A}(\theta,\xi)$.

Proof. Take as $\mathbb{F}$ the monad $(\theta^{\bullet},{}^{\backprime}\xi)$ and as $\mathbb{G}$ the monad $(\Theta^{\bullet},\zeta')$ (cf. 3); and let $h\colon \mathbb{G} \longrightarrow \mathbb{F}$ be the morphism which identifies Θ^m with θ^{Nm}.

It is easy to see that the functor R_N can be defined by the commutativity of the diagram

$$\begin{array}{ccc} \mathcal{A}(\theta,\xi) & \longrightarrow & \mathbb{F}\text{-}mod \\ R_N \downarrow & & \downarrow h_* \\ \mathcal{A}(\Theta,\zeta) & \longrightarrow & \mathbb{G}\text{-}mod \end{array}$$

where each of the horizontal arrows is an equivalences of categories from Proposition 4. Thus, the left adjoint to h_* functor, $h^* := \mathbb{F}\otimes_{\mathbb{G}}\bullet$ (cf. Proposition 6.8.3), induces a left adjoint to R_N functor L_N. ■

6.8.5. Some details about the functor L_N. Let $\mathbb{F}$ denote, as in 6.8.4, the monad $(\theta^{\bullet},{}'\xi)$ and $\mathbb{G}$ the monad $(\Theta^{\bullet},\zeta')$. Clearly

$$\theta^{\bullet} = \bigoplus_{0\le i<N} \theta^{i}\Theta^{\bullet}. \tag{1}$$

Let $\theta_{N,\xi}$ be a functor from $\mathbb{G}$-*mod* to $\mathbb{F}$-*mod* which assigns to a $\mathbb{G}$-module $\mathbb{V} = (V,\upsilon)$ the $\mathbb{F}$-module $(\bigoplus_{0\le i<N} \theta^{i}(V),\xi_N(\mathbb{V}))$, where the $\mathbb{F}$-module structure $\xi_N(\mathbb{V})$ is defined by

$${}'\xi(V)\colon \theta^{j}\theta^{i}(V) \longrightarrow \theta^{i+j}(V) \quad \text{if} \quad 1 \le i + j < N$$

(cf. 3), and

$$\upsilon\colon \theta^{j}\theta^{i}(V) \longrightarrow V \quad \text{if} \quad i + j = N.$$

It follows from (1) that the canonical epimorphism

$$\mathbb{F} := (\theta^{\bullet},{}'\xi) \longrightarrow \mathbb{F}\otimes_{\mathbb{G}}$$

can be decomposed as

$$\mathbb{F} \longrightarrow \theta_{N,\xi} \longrightarrow \mathbb{F}\otimes_{\mathbb{G}}, \tag{2}$$

Moreover, one can see that the arrow $\theta_{N,\xi} \longrightarrow \mathbb{F}\otimes_{\mathbb{G}}$ in (2) is an isomorphism.

6.9. From *Spec*($\mathbb{G}$-*mod*) **to** *Spec*($\mathbb{F}$-*mod*)**.** The functor θ defines an auto-equivalence, $\theta_{\mathbb{G}}$, of the category $\mathbb{G}$-*mod* as follows.

There is a canonical functor isomorphism $\sigma\colon \mathbb{G}\circ\theta \longrightarrow \theta\circ\mathbb{G}$ which is defined by

$$id\colon \Theta^{i}\circ\theta = \theta^{iN}\circ\theta \longrightarrow \theta\circ\theta^{iN} = \theta\circ\Theta^{i} \quad \text{if} \quad i \ge 0,$$

and by the composition of the isomorphism

$$\lambda^{\wedge}\circ\lambda : \theta^{\wedge}\circ\theta \longrightarrow \theta\circ\theta^{\wedge}$$

(here $\lambda : \theta^{\wedge}\circ\theta \longrightarrow Id_{\mathcal{A}}$ and $\lambda^{\wedge} : Id_{\mathcal{A}} \longrightarrow \theta\circ\theta^{\wedge}$ are, as before, the adjunction isomorphisms) when i is negative.

Now we define $\theta_{\mathbb{G}}$ by $\theta_{\mathbb{G}}(V,\upsilon) = (\theta(V), \theta\upsilon \circ \sigma(V))$, and $\theta_{\mathbb{G}}(f) = \theta f$ for any $\mathbb{G}$-module (V,υ) and any $\mathbb{G}$-module morphism f.

6.9.1. Theorem. *Let $P \in Spec\mathcal{A}$ be such that $\theta^N(P) \simeq P$, but $\theta^i(P)$ is not equivalent to P if $1 \le i < N$. Let (V,υ) be an object of $Spec(\mathbb{G}\text{-}mod)$ such that $<P> \in Ass(V)$.*

(a) Suppose that $\xi\theta^i(P) \neq 0$ for all $1 \le i < N$. Then every nonzero submodule of $\mathbb{F}\otimes_{\mathbb{G}}(V,\upsilon)$ contains $\mathbb{F}\otimes_{\mathbb{G}}(W,\mathfrak{w})$ for some $\mathbb{G}$-submodule $(W,\mathfrak{w})$ of (V,υ).

In particular, $\mathbb{F}\otimes_{\mathbb{G}}(V,\upsilon)$ belongs to $Spec(\mathbb{F}\text{-}mod)$, and it is simple if (V,υ) is simple.

(b) Suppose that $\xi\theta^i(P) = 0$ for some i; and let l, m be such integers that $1 \le l \le m < N$, $\xi\theta^l(P) = 0$, $\xi\theta^m(P) = 0$; but, $\xi\theta^i(P) \neq 0$ if $0 \le i < l$, or $m < i < N$. Then

(b1) $\bigoplus_{l \le i < N} \theta^i(V)$ is an $\mathbb{F}$-submodule of $\mathbb{F}\otimes_{\mathbb{G}}(V,\upsilon) \simeq (\bigoplus_{0 \le i < N} \theta^i(V), \xi_N)$, and the quotient module, $\mathbb{V}_{1,l} \simeq (\bigoplus_{0 \le i < l} \theta^i(V), \mu)$, belongs to the spectrum.

(b2) $\bigoplus_{m \le i < N} \theta^i(V)$ is a submodule of $\mathbb{F}\otimes_{\mathbb{G}}(V,\upsilon)$ which also is in $Spec\mathcal{A}[\theta,\xi]$.

(b3) If (V,υ) is a simple $\mathbb{G}$-submodule, then the $\mathbb{F}$-module $\mathbb{V}_{1,l}$ and the submodule of (b2) are simple.

Proof. According to 6.8.5,

$$\mathbb{F}\otimes_{\mathbb{G}}(V,\upsilon) \simeq \theta_{N,\xi}(V,\upsilon) := (\bigoplus_{0 \le i < N} \theta^i(V), \xi_N(\mathbb{V})),$$

where ξ_N is an 'adjustment' of 'ξ (cf. 6.8.5).

(a) The proof of the assertion *(a)* is similar to the proof of Theorem 6.4 when $\xi\theta^i(P) \neq 0$ for all i, and is the same, as the proof of the first assertion of Theorem 6.1 when $\xi(P) = 0$.

(b) The proof of the assertion *(b)* follows the argument which proves the first assertion of Theorem 6.1. The details are left to the reader. ■

6.9.2. Remark. Set $\mathbf{Spec}_{<P>}\mathbb{G}\text{-}mod := \{<(V,\upsilon)> \in \mathbf{Spec}\mathbb{G}\text{-}mod \mid <P> \in Ass(V)\}$. The set $\mathbf{Spec}_{<P>}\mathbb{G}\text{-}mod$ depends only on whether ξ has a zero on the orbit of $<P>$ or not.

In fact, if $\xi\theta^i(P) \neq 0$ for all i, then $\zeta(P) \neq 0$ which implies, according to 6.7 that $\mathbf{Spec}_{<P>}\mathbb{G}\text{-}mod$ is isomorphic to the "$<P>$-part" of the spectrum of the Laurent category corresponding to the auto-equivalence $\Theta := \theta^N$:

$$\mathbf{Spec}_{<P>}\mathbb{G}\text{-}mod \simeq \mathbf{Spec}_{<P>}\mathcal{A}[\Theta]/\mathcal{A}.$$

Here

$$\mathbf{Spec}_{<P>}\mathcal{A}[\Theta]/\mathcal{A} := \{<(M,u)> \in \mathbf{Spec}\mathcal{A}[\Theta]/\mathcal{A} \mid <P> \in Ass(M)\}.$$

If $\xi\theta^i(P) = 0$ for some i, then $\zeta(P) = 0$. Therefore

$$\mathbf{Spec}_{<P>}\mathbb{G}\text{-}mod \simeq \mathbf{Spec}_{<P>}\mathcal{A}[\Theta] \coprod_{\{<P>\}} \mathbf{Spec}\mathcal{A}_{<P>}[\Theta^\wedge]$$

(cf. 6.7). ■

6.10. The whole picture. Thus, $Spec\mathcal{A}$ can be represented as the union:

$$Spec\mathcal{A} = Spec_\infty\mathcal{A} \cup \Big(\bigcup_{n\geq 1} Spec_n\mathcal{A}\Big),$$

where

$Spec_\infty\mathcal{A}$ consists of all $P \in Spec\mathcal{A}$ such that $\theta^i(P)$ is not equivalent to P if $i \neq 0$; or, equivalently, the orbit

$$\{<\theta^i(P)> \mid i \in \mathbb{Z}\} \quad (1)$$

is infinite;

$Spec_n\mathcal{A}$, $n \geq 1$, consists of all $P \in Spec\mathcal{A}$ such that the orbit (1) has exactly n points; i.e. $\theta^i(P)$ is not equivalent to P if $1 \leq i \leq n-1$, and $\theta^n(P) \approx P$.

Let $Spec_\xi\mathcal{A}$ consists of all $P \in Spec\mathcal{A}$ such that $\xi\theta^i(P) = 0$ for some i. Set

$$Spec_{\infty,\xi}\mathcal{A} = Spec_{\infty,}\mathcal{A} \cap Spec_\xi\mathcal{A}, \quad Spec_{n,\xi}\mathcal{A} = Spec_n\mathcal{A} \cap Spec_\xi\mathcal{A},$$

and

$$Spec_{\infty,*}\mathcal{A} = Spec_{\infty,}\mathcal{A} - Spec_\xi\mathcal{A}, \quad Spec_{n,*}\mathcal{A} = Spec_n\mathcal{A} - Spec_\xi\mathcal{A},$$

for all $n \geq 1$.

Thus, we have the following decomposition:

$$Spec\mathcal{A} = Spec_{\infty,\xi}\mathcal{A} \cup Spec_{\infty,*}\mathcal{A} \cup \Big(\bigcup_{n\geq 1} (Spec_{n,*}\mathcal{A} \cup Spec_{n,\xi}\mathcal{A})\Big). \quad (2)$$

Now, consider the part, $Spec'\mathcal{A}\{\theta,\xi\}$, of $Spec\mathcal{A}\{\theta,\xi\}$ which consists of all objects $(\gamma,M,\eta) \in Spec\mathcal{A}\{\theta,\xi\}$ such that the set $Ass(M)$ is not empty. Clearly any decomposition of $Spec\mathcal{A}$ induces a decomposition of $Spec'\mathcal{A}\{\theta,\xi\}$. In particular, we have:

$$Spec'\mathcal{A}\{\theta,\xi\} = Spec'_{\infty,\xi}\mathcal{A}\{\theta,\xi\} \cup Spec'_{\infty,*}\mathcal{A}\{\theta,\xi\} \cup$$

$$\Big(\bigcup_{n\geq 1} (Spec'_{n,*}\mathcal{A}\{\theta,\xi\} \cup Spec'_{n,\xi}\mathcal{A}\{\theta,\xi\})\Big).$$

where $Spec'_{\alpha,\beta}\mathcal{A}\{\theta,\xi\}$ consists of all $(\gamma,M,\eta) \in Spec'\mathcal{A}\{\theta,\xi\}$ for which

$$Ass(M) \cap Spec_{\alpha,\beta}\mathcal{A} \neq \emptyset.$$

(a) According to Therem 6.4, the map

$$\mathfrak{I}'_{\infty,*}: P \longmapsto \mathbb{F}(P) = (g(P),\theta^\bullet(P),h(P)) \quad (3)$$

takes values in $Spec_{\infty,*}\mathcal{A}/\theta,\xi/$; hence it induces a map

$$\mathfrak{I}_{\infty,*}: \mathbf{Spec}_{\infty,*}\mathcal{A} \longrightarrow \mathbf{Spec}_{\infty,*}\mathcal{A}/\theta,\xi/.$$

The map $\mathfrak{I}_{\infty,*}$ *is surjective, and the preimage of each point* (γ,M,η) *in* $\mathbf{Spec}_{\infty,*}\mathcal{A}/\theta,\xi/$ *coincides with the orbit,*

$$/<\theta^i(P)>|\ n \in \mathbb{Z}/$$

of any $<P> \in Ass(M)$.

(cf. Theorem 6.4).

(b) Let now $P \in Spec_{\infty,\xi}\mathcal{A}$. Define a map, $\mathfrak{I}'_{\infty,\xi}$, as follows:

If $\xi\theta^i(P) \neq 0$ for all $i \leq 0$, then set

$$\mathfrak{I}'_{\infty,\xi}(P) = \mathfrak{B}^\wedge(\theta^{\nu-1}(P)), \tag{4}$$

where ν is the least positive number such that $\xi\theta^\nu(P) = 0$.

If $\xi\theta^i(P) \neq 0$ for $i \geq 1$, then set

$$\mathfrak{I}'_{\infty,\xi}(P) = \mathfrak{B}(\theta^{\nu^\wedge}(P)), \tag{5}$$

where $\nu^\wedge$ is the biggest (necessarily) non-positive number such that $\xi\theta^{\nu^\wedge}(P)$ is equal to zero.

If $\xi\theta^i(P) = 0$ for some positive and non-positive values of i, then set

$$\mathfrak{I}'_{\infty,\xi}(P) = \Psi_{\xi,\nu^\wedge+\nu}(\theta^{\nu^\wedge}(P)), \tag{6}$$

where $\nu^\wedge$ and ν are defined above.

By Theorem 6.3, the map $\mathfrak{I}'_{\infty,\xi}$ given by the formulas (4), (5), (6) induces a surjective map $\mathbf{Spec}_{\infty,\xi}\mathcal{A} \longrightarrow \mathbf{Spec}_{\infty,\xi}\mathcal{A}/\theta,\xi/$.

Note that the preimage of a point $<(\gamma,M,\eta)> \in \mathbf{Spec}_{\infty,\xi}\mathcal{A}/\theta,\xi/$ is the ray $/<\theta^i(P)>|\ i \leq \nu - 1/$ in the first case (cf. (4)), the ray $/<\theta^i(P)>|\ i \geq \nu^\wedge/$ in the second case (cf. (5)), and the interval $/<\theta^i(P)>|\ \nu^\wedge \leq i \leq \nu - 1/$ in the third case, where $<P>$ is an associate point of M.

(c) Fix a positive integer n, and consider the set $\mathfrak{X}_{n,*}$ of all pairs $(<P>,<\mathbb{V}>)$, where $<P> \in \mathbf{Spec}_{n,*}\mathcal{A}$ and $\mathbb{V} = (\mathfrak{u},V,\mathfrak{v})$ an object of $Spec\mathcal{A}/\theta^n,\xi_n/$ such that $<P> \in Ass(V)$. Clearly the group $\mathbb{Z}/n\mathbb{Z}$ acts effectively on $\mathfrak{X}_{n,*}$:

$$m\bullet(<P>,<\mathbb{V}>) = (<\theta^m(P)>,<\theta_n{}^m(\mathbb{V})>),$$

where θ_n the induced by θ action on $Spec\mathcal{A}/\theta^n,\xi_n/$.

The formula $\mathfrak{I}_{n,*}(<P>,<\mathbb{V}>) := <\mathbb{F}\otimes_{\mathbb{G}_n}\mathbb{V}>$ defines a surjection

$$\mathfrak{I}_{n,*} : \mathfrak{X}_{n,*} \longrightarrow \mathbf{Spec}_{n,*}\mathcal{A}/\theta,\xi/$$

such that the preimage of any point $<\mathbb{F}\otimes_{\mathbb{G}_n}\mathbb{V}> \in \mathbf{Spec}_{n,*}\mathcal{A}/\theta,\xi/$ is the corresponding to $<\mathbb{V}>$ $\mathbb{Z}/n\mathbb{Z}$-orbit in $\mathfrak{X}_{n,*}$. Here we use the fact that $\mathbb{V}$ defines the orbit uniquely. Still it is convenient to consider the set of pairs $\mathfrak{X}_{n,*}$ and keep in mind the diagram

$$\begin{array}{ccccc} & \mathfrak{X}_{n,*} & \xrightarrow{\quad \mathfrak{J}_{n,*} \quad} & \mathbf{Spec}'_{n,*}\mathcal{A}(\theta,\xi) & \\ \pi \swarrow & & \searrow \pi' & \nearrow & \mathbb{F}\otimes_{\mathbb{G}_n} \\ \mathbf{Spec}_{n,*}\mathcal{A} & & \mathbf{Spec}_{n,*}\mathcal{A}(\theta_n,\xi_n) & & \end{array}$$

where π and π' are projections. Note that, according to Remark 6.9.2, $\mathbf{Spec}_{n,*}\mathcal{A}(\theta_n,\xi_n)$ is isomorphic to the corresponding part of the spectrum of the Laurent category $\mathcal{A}[\theta^n]/\mathcal{A}$.

(d) Now fix a positive integer n and consider the set $\mathfrak{X}_{n,\xi}$ of pairs $(<P>,<\mathbb{V}>)$, where $<P> \in \mathbf{Spec}_{n,\xi}\mathcal{A}$ and $\mathbb{V} = (\mathfrak{u},V,\mathfrak{v})$ is an object of $Spec_{<P>}\mathcal{A}(\theta_n,\xi_n)$; i.e. $<P> \in Ass(V)$. Recall that, according to Remark 6.9.2,

$$\mathbf{Spec}_{<P>}\mathbb{G}\text{-}mod \simeq \mathbf{Spec}_{<P>}\mathcal{A}[\theta_n] \coprod_{(<P>)} \mathbf{Spec}\mathcal{A}_{<P>}[\theta_n{}^{\wedge}].$$

There is a natural action of $\mathbb{Z}/n\mathbb{Z}$ on $\mathfrak{X}_{n,\xi}$ defined by

$$1\bullet(<P>,<\mathbb{V}>) = (<\theta(P)>,<\theta_n(\mathbb{V})>).$$

Let $\mathfrak{J}_{n,\xi}$ denote the map from $\mathfrak{X}_{n,\xi}$ to $\mathbf{Spec}_{n,\xi}\mathcal{A}(\theta,\xi)$ that assigns to any element $(<P>,<\mathbb{V}>)$ of $\mathfrak{X}_{n,\xi}$, the element $<\mathbb{V}_{1,l}> = <(\bigoplus_{0\le i<l} \theta^i(V),\mu)>$ (cf. Theorem 6.9.1).

According to Theorem 6.9.1, the map $\mathfrak{J}_{n,\xi}$ is surjective, and the preimage of the element $<\mathbb{V}_{1,l}>$ is the "interval"

$$\{<\theta^i(P),<\theta_n{}^i\mathbb{V}>\mid 0 \le i \le l\text{-}1\}.$$

6.10.1. Remark. The definition of the subsets $\mathbf{Spec}_{\alpha,\beta}\mathcal{A}(\theta,\xi)$ of the spectrum is very cautious:

$$\mathbf{Spec}_{\alpha,\beta}\mathcal{A}(\theta,\xi) := \{<(\gamma,M,\eta)> \in \mathbf{Spec}\mathcal{A}(\theta,\xi) \mid Ass(M) \cap \mathbf{Spec}_{\alpha,\beta}\mathcal{A} \neq \varnothing\}.$$

It follows, however, from the description above that

$$\mathbf{Spec}_{\alpha,\beta}\mathcal{A}(\theta,\xi) := \{<(\gamma,M,\eta)> \in \mathbf{Spec}\mathcal{A}(\theta,\xi) \mid Ass(M) \subset \mathbf{Spec}_{\alpha,\beta}\mathcal{A}\}.$$

In particular,

$$\mathbf{Spec}_{\alpha,\beta}\mathcal{A}(\theta,\xi) \cap \mathbf{Spec}_{\alpha',\beta'}\mathcal{A}(\theta,\xi) = \varnothing, \quad \text{if } (\alpha,\beta)\neq(\alpha',\beta');$$

i.e. $\mathbf{Spec}'\mathcal{A}(\theta,\xi)$ is the disjoint union of the subsets $\mathbf{Spec}_{\alpha,\beta}\mathcal{A}(\theta,\xi)$. ■

COMPLEMENTARY FACTS AND EXAMPLES.

C1. HYPERBOLIC CATEGORIES OF HIGHER RANK.

C1.1. Iterated hyperbolic categories. Let $\mathcal{A}$ be a category, and ϑ, θ two auto-equivalences of $\mathcal{A}$ which quasi-commute; i.e. there exists an isomorphism φ from $\theta\circ\vartheta$ to $\vartheta\circ\theta$. Let ξ and ζ be endomorphisms of the identical functor $Id_{\mathcal{A}}$ such that $\theta\xi = \xi\theta$ and $\vartheta\zeta = \zeta\vartheta$.

C1.1.1. Lemma. *Under the above conditions, the functor θ defines an auto-equivalence, Θ, of the hyperbolic category $\mathcal{A}\{\vartheta,\xi\}$ and ζ defines an endomorphism of the identical functor of the category $\mathcal{A}\{\vartheta,\xi\}$.*

Proof. Denote by Θ the map which assigns to any object (γ,M,η) the object $(\varphi(M)\circ\theta\gamma,\theta(M),\theta\eta\circ\varphi^{-1}(M))$ and to any morphism $f\colon (\gamma,M,\eta) \longrightarrow (\gamma',M',\eta')$ the morphism θf. The map Θ is a functor $\mathcal{A}\{\vartheta,\xi\} \longrightarrow \mathcal{A}\{\vartheta,\xi\}$.

In fact, by condition we have:

$$\theta\eta\circ\varphi^{-1}(M)\varphi(M)\circ\theta\gamma = \theta\eta\circ\theta\gamma = \theta\xi = \xi\theta,$$

and

$$(\varphi(M)\circ\theta\gamma)\circ(\theta\eta\circ\varphi^{-1}(M)) = \varphi(M)\circ\theta\eta\circ\theta\gamma\circ\varphi^{-1}(M) =$$

$$\varphi(M)\circ\theta\xi\vartheta\circ\varphi^{-1}(M) = \varphi(M)\circ\xi\theta\vartheta\circ\varphi^{-1}(M) = \xi\vartheta\theta.$$

Clearly Θ is an auto-equivalence of the category $\mathcal{A}\{\vartheta,\xi\}$.

Note that, for any object (γ,M,η) of the category $\mathcal{A}\{\vartheta,\xi\}$, the morphism $\zeta(M)$ is, thanks to the condition $\zeta\vartheta = \vartheta\zeta$, an endomorphism of (γ,M,η):

$$\gamma\circ\zeta(M) = \zeta\vartheta(M)\circ\gamma = \vartheta\zeta(M)\circ\gamma, \text{ and } \zeta(M)\circ\eta = \eta\circ\zeta\vartheta = \psi\circ\vartheta\zeta.$$

This implies immediately that the map ζ' which assigns to any object (γ,M,η) of the category $\mathcal{A}\{\vartheta,\xi\}$ the arrow $\xi(M)$ is an endomorphism of the identical functor. ∎

So, one can define the iterated hyperbolic category $\mathcal{B}\{\Theta,\zeta'\}$, where $\mathcal{B}:= \mathcal{A}\{\vartheta,\xi\}$.

C1.2. A generalization: categories $\mathcal{A}\{\Xi\}$. Let ϑ_i, $i \in J$, be a quasi-commuting family of auto-equivalences of the category $\mathcal{A}$; moreover, for every pair $i,j \in J$, a pair of isomorphisms,

$$\psi_{ij}, \varphi_{ij}\colon \vartheta_i\circ\vartheta_j \longrightarrow \vartheta_j\circ\vartheta_i$$

is given such that

$$\varphi_{ii} = \psi_{ii} = id, \text{ and } \varphi_{ij}\circ\varphi_{ji} = id = \psi_{ij}\circ\psi_{ji}.$$

And let ξ_i, $i \in J$, be a family of endomorphisms of $Id_{\mathcal{A}}$ such that

$\vartheta_i\xi_j = \xi_j\vartheta_i$ if $i \neq j$. Denote this data, $\{\vartheta_i, \psi_{ij}, \varphi_{ij}, \xi_i | i,j \in J\}$, by Ξ, and consider the category $\mathcal{A}[\Xi]$ the objects of which are

$$(\gamma_i, M, \eta_i | i \in J),$$

where $M \in Ob\mathcal{A}$, and

(a) (γ_i, M, η_i) is an object of the category $\mathcal{A}(\vartheta_i, \xi_i)$ for every $i \in J$; i.e. γ_i and η_i are arrows $M \longrightarrow \vartheta_i(M)$ and $\vartheta_i(M) \longrightarrow M$ respectively such that $\eta_i \circ \gamma_i = \xi_i(M)$, and $\vartheta_i(\eta_i \circ \gamma_i) = \xi_i \vartheta_i(M)$.

(b) The diagrams

$$\begin{array}{ccccc}
\vartheta_j(M) & \xrightarrow{\vartheta_j\gamma_i} & \vartheta_j\vartheta_i(M) & \xrightarrow{\varphi_{ji}} & \vartheta_i\vartheta_j(M) \\
\eta_j \downarrow & & & & \downarrow \vartheta_i\eta_j \\
M & & \xrightarrow{\quad\gamma_i\quad} & & \vartheta_i(M)
\end{array}$$

are commutative for any $i, j \in J$ such that $i \neq j$.

(c) The diagrams

$$\begin{array}{ccccc}
\vartheta_j(M) & \xrightarrow{\vartheta_j\gamma_i} & \vartheta_j\vartheta_i(M) & \xrightarrow{\psi_{ji}} & \vartheta_i\vartheta_j(M) \\
\gamma_j \uparrow & & & & \uparrow \vartheta_i\gamma_j \\
M & & \xrightarrow{\quad\gamma_i\quad} & & \vartheta_i(M)
\end{array}$$

and

$$\begin{array}{ccccc}
\vartheta_j(M) & \xleftarrow{\vartheta_j\eta_i} & \vartheta_j\vartheta_i(M) & \xrightarrow{\psi_{ji}} & \vartheta_i\vartheta_j(M) \\
\eta_j \downarrow & & & & \downarrow \vartheta_i\eta_j \\
M & & \xleftarrow{\quad\eta_i\quad} & & \vartheta_i(M)
\end{array}$$

are commutative for every $i, j \in J$.

Arrows in $\mathcal{A}[\Xi]$ are defined in an obvious way:

an arrow from $(\gamma_i, M, \eta_i | i \in J)$ to $('\gamma_i, 'M, '\eta_i | i \in J)$ is a morphism

$$f: M \longrightarrow 'M$$

such that $\gamma_i \circ f = \vartheta_i(f) \circ \gamma_i$, and $f \circ \eta_i = \eta_i \circ \vartheta_i(f)$ for all $i \in J$.

It follows from Lemma C1.1.1 that in case when $J = [1,n]$, this data provides an n-th iterated hyperbolic category, $\mathcal{A}_n((\vartheta_i),(\xi_i))$.

C1.3. Example: modules over iterated hyperbolic rings. Let R be a commutative ring, $\{\theta_i \mid i \in J\}$ a family of pairwise commuting automorphisms of R, and $\{\xi_i \mid i \in J\}$ elements in R. Denote by $R\{(\theta_i),(\xi_i)\}$ the ring generated by R and indeterminates x_i, y_i, $i \in J$, which satisfy the relations:

$$x_i r = \theta_i(r)x_i, \quad ry_i = y_i\theta_i(r)$$
$$x_i y_i = \xi_i$$
$$x_i y_j = y_j x_i, \quad x_i x_j = x_j x_i, \quad y_i y_j = y_j y_i$$

for every $r \in R$ and $1 \le i,j \le n$, $i \neq j$.

The category $R\{(\theta_i),(\xi_i)\}$-*mod* of $R\{(\theta_i),(\xi_i)\}$-modules is equivalent to the n-th hyperbolic category $\mathcal{A}_n\{(\Theta_i),(\xi_i')\}$, where $\mathcal{A} = R\text{-}mod$, Θ_i the induced by θ_i auto-equivalence of $R\text{-}mod$, ξ_i' the corresponding to the element ξ_i endomorphism of the identical functor $Id_{R\text{-}mod}$.

C1.4. A special case. Let $\{\vartheta_i \mid i \in J\}$ be a family of commuting automorphisms of a commutative ring, A; let u_i be arbitrary and ρ_i invertible elements in A, $i \in J$.

Denote by $A[\{\vartheta_i\},\{u_i\},\{\rho_i\}]$ the ring generated by A and by the indeterminates x_i, y_i, $i \in J$, subject to the relations:

$$x_i a = \vartheta_i(a)x_i, \quad ay_i = y_i\vartheta_i(a)$$
$$x_i y_i - \rho_i y_i x_i = \delta_{ii} u_i$$
$$x_i y_j = y_j x_i, \quad x_i x_j = x_j x_i, \quad y_i y_j = y_j y_i$$

for every $a \in A$ and $i,j \in J$, $i \neq j$.

C1.5. Lemma. *The ring $A[(\vartheta_i),(u_i),(\rho_i)]$ is isomorphic to the ring $R\{(\theta_i),(\xi_i)\}$, where $R = A[\xi_i \mid i \in J]$, and θ_i is, for every $i \in J$, the extension of ϑ_i to an automorphism of the ring R given by*

$$\theta_i(\xi_i) = \vartheta_i(\rho_i)\xi_i + \vartheta(u_i),$$

and

$$\vartheta_i(\xi_j) = \xi_j \quad \text{if} \quad i \neq j.$$

Proof. The isomorphism in question is given by

$$x_i \longmapsto x_i, \quad y_i \longmapsto y_i, \quad x_i y_i \longmapsto \xi_i$$

for every i, $i \in J$. ∎

It follows from Lemma C1.2.1 that if $J = [1,n]$, then $\mathcal{A}\{\Xi\}$ is the *n-th iterated hyperbolic category*. Therefore our results about the spectrum of the hyperbolic categories $\mathcal{A}\{\theta,\xi\}$ provide a step by step description of the spectrum of the forgetting functor $\mathfrak{F} : \mathcal{A}\{\Xi\} \longrightarrow \mathcal{A}$; i.e. the subset

$\{<(\gamma_i, M, \eta_i)> \in \mathbf{Spec}\mathcal{A}[\Xi] \mid Ass(M) \neq \emptyset\}.$

Basically, the problem is reduced on each step to the search of θ-stable points (cf. C1.1).

C2. Rings and categories of Heisenberg and Weyl types.

C2.1. Rings of the Heisenberg type. Let R be an associative ring, $\Xi := \{\theta_i, \xi_i \mid i \in J\}$, where $\{\theta_i \mid i \in J\}$ is a family of pairwise commuting automorphisms of an associative ring R, and $\{\xi_i \mid i \in J\}$ elements of the center of R.

Call the corresponding to the data Ξ hyperbolic ring $R\{\Theta\}$ (cf. 0.3 or C1.2.3) *of Heisenberg type* if

$$(\theta_i + \theta_i^{-1})(\xi_i) = (\rho_i + \rho_i^{-1})\xi_i \tag{1}$$

for some invertible θ_i-stable elements ρ_i in R and for all i.

C2.1.1. Lemma. *If $R\{\Xi\}$ is of Heisenberg type, then the element*

$$c_i := (\xi_i - \rho_i\theta_i^{-1}(\xi_i))(\xi_i - \rho_i^{-1}\theta_i^{-1}(\xi_i))$$

is central for all i.

Proof. Since c_i belongs to the center of R, it suffices to show that c_i is θ_j-stable for all $j \in J$. Clearly $\theta_j c_i = c_i$ for all $j \in J-\{i\}$, since both ξ_i and ρ_i have this property.

It remains to check θ_i-stability of c_i. Set

$$u_i := \xi_i - \rho_i\theta_i^{-1}(\xi_i), \quad u_i^\wedge := \xi_i - \rho_i^{-1}\theta_i^{-1}(\xi_i).$$

It follows from (1) that:

$$\theta_i(u_i) = \theta_i(\xi_i) - \rho_i\xi_i = \rho_i^{-1}\xi_i - \theta_i^{-1}(\xi_i) = \rho_i^{-1}u_i. \tag{2}$$

Since ρ_i and ρ_i^{-1} enter into (1) symmetrically, we have the relation

$$\theta_i(u_i^\wedge) = \rho_i u_i^\wedge \tag{3}$$

for free. The formulas (2) and (3) imply that $\theta_i(c_i) = c_i$. ■

Denote by z_i the element $y_i u_i = y_i(\xi_i - \rho_i\theta_i^{-1}(\xi_i))$. Then we have:

$$x_i z_i - \rho_i^{-2} z_i x_i = c_i$$

for every i. In other words, the map which is identical on x_i, $i \in J$, and on the ring R, and sends z_i into $y_i u_i$ for every $i \in J$, defines a morphism of the ring $R<(\theta_i),(c_i),(\rho_i^{-2})>$ given by the relations

$$x_i r = \theta_i(r)x_i, \quad r z_i = z_i\theta_i(r)$$

$$x_iz_i - \rho_i^{-2}z_ix_i = \delta_{ii}c_i$$

$$x_iz_j = z_jx_i, \quad x_ix_j = x_jx_i, \quad z_iz_j = z_jz_i$$

for every $r \in R$ and $i,j \in J$, $i \neq j$ (cf. C1.2.4) into $R[\Xi]$.

Note that if the elements c_i are invertible, then this morphism is an isomorphism.

C2.2. Rings of the Weil type. Let $R[\Xi]$ be the Heisenberg type ring corresponding to the data $[\Xi,\rho] = [(\theta_i), (\xi_i), (\rho_i)]$. Define the *Weyl type ring* $W(\Xi,\rho)$ as the quotient of the ring $R[\Xi]$ by the ideal generated by the (central) elements c_i-1, $i \in J$. Clearly the ring $W[\Xi,\rho]$ is also hyperbolic. And Hayashi's Weyl algebras belong to this class.

It follows from C2.1 that the Weyl type ring $W[\Xi,\rho]$ is isomorphic to the ring $A_J<[\theta_i,\rho_i^{-2},1]>$. In particular, it contains a subring $A_J((\theta_i),\rho^{-2})$ which is generated over the subring of (θ_i)-stable $(=\ \theta_i$-stable for all $i)$ elements of R by the indeterminates x_i, z_i subject to the relations

$$x_iz_i - \rho_i^{-2}z_ix_i = 1$$

for every $i \in J$.

C2.3. The categories of Heisenberg type. A hyperbolic category $\mathcal{A}[\theta,\xi]$ will be called of *Heisenberg type* if there exists a θ-stable automorphism r of the identical functor $Id_{\mathcal{A}}$ such that

$$\theta^2\xi + \xi\theta^2 = (r\theta^2 + r^{-1}\theta^2)\circ\theta\xi\theta. \tag{1}$$

Here θ-*stable* means that $\theta r = r\theta$.

We call r *the weight* of the category $\mathcal{A}[\theta,\xi]$, and will write $\mathcal{A}[\theta,\xi;r]$ in case we need to indicate the weight.

C2.3.1. Remark. Note that the relation (1) is equivalent to any of the relations

$$\theta\xi + \xi^\wedge\theta = \theta(r + r^{-1})\circ\theta\xi^\wedge \tag{2}$$

$$\theta^\wedge\xi^\wedge + \xi\theta^\wedge = \theta^\wedge(r + r^{-1})\circ\theta^\wedge\xi \tag{3}$$

$$\xi + \sigma\circ\theta^\wedge\xi^\wedge\theta\circ\sigma^{-1} = (r + r^{-1})\circ\xi^\wedge, \tag{2'}$$

$$\xi^\wedge + \varepsilon^{-1}\circ\theta\xi\theta^\wedge\circ\varepsilon = (r + r^{-1})\circ\xi, \tag{3'}$$

where $\xi^\wedge := \sigma\circ\theta^\wedge\xi\theta\circ\sigma^{-1}$; and

$$\sigma:\ \theta^\wedge\theta \longrightarrow Id_{\mathcal{A}} \quad\text{and}\quad \varepsilon:\ Id_{\mathcal{A}} \longrightarrow \theta\theta^\wedge$$

are the adjunction isomorphisms.

It follows from the equivalence of (2) and (3) that the adjoint to

$\mathcal{A}\{\theta,\xi;r\}$ hyperbolic category is also of Heisenberg type with the same weight r. ■

Fix a Heisenberg type category $\mathcal{A}(\theta,\xi;r)$. Let u and u^ denote the endomorphisms of $Id_{\mathcal{A}}$ defined by the relations:

$$u = \xi - r\circ\xi\wedge \quad \text{and} \quad u\wedge = \xi - r^{-1}\circ\xi\wedge. \tag{4}$$

C2.4. Lemma. *There are the following equalities:*

$$\theta(u) = r^{-1}\theta\circ u\theta, \quad \textit{and} \quad \theta(u\wedge) = r\theta\circ u\wedge\theta.$$

In particular, the endomorphism $c := u\wedge\circ u$ *of* $Id_{\mathcal{A}}$ *is* θ*-invariant.*

Proof. It follows from (2) that:

$$\theta(u) := \theta\xi - \theta r\circ\theta\xi\wedge = \theta(r + r^{-1})\circ\theta\xi\wedge - \xi\wedge\theta - \theta r\circ\theta\xi\wedge =$$

$$\theta r^{-1}\circ\theta\xi\wedge - \xi\wedge\theta = r^{-1}\theta\circ(\theta\xi\wedge - r\theta\circ\xi\wedge\theta) = r^{-1}\theta\circ(\xi - r\circ\xi\wedge)\theta := r^{-1}\theta\circ u\theta.$$

Since automorphisms r and r^{-1} enter into the condition (1) symmetrically, we obtain the other equality, $\theta u\wedge = r\theta\circ u\wedge\theta$, for free. ■

C2.4.1. Remark. Moreover, one can see that the condition (1) is equivalent to the equality $\theta(u) = r^{-1}\theta\circ u\theta$. ■

C2.5. The spectrum of a category of Heisenberg type. Fix a category $\mathcal{A}(\theta,\xi;r)$ of Heisenberg type.

C2.5.1. Lemma. *For every nonnegative integer* n,

$$\xi\theta^n = \theta^n(r^{-n+1}\circ\xi\wedge - (\sum_{0\le i\le n} r^{2i-n})\circ u), \tag{5}$$

$$\xi\wedge\theta\wedge^n = \theta\wedge^n(r^{n-1}\circ\xi - (\sum_{0\le i\le n} r^{-2i+n})\circ u\wedge). \tag{6}$$

Proof. If $n = 0$, then the equality holds by obvious reason.

If (5) holds for some $n \ge 0$, then we have:

$$\xi\theta^{n+1} = \theta^n(r^{-n+1}\circ\xi\wedge - (\sum_{0\le i\le n} r^{2i-n})\circ u)\theta =$$

$$\theta^n(r^{-n+1}\circ\xi\wedge\theta - (\sum_{0\le i\le n} r^{2i-n})\circ u\theta) =$$

$$\theta^n(\theta r^{-n+1}\circ\theta(r^{-1}\circ\xi\wedge - u) - \theta(\sum_{0\le i\le n} r^{2i-n})\circ r\circ u) =$$

$$\theta^{n+1}(r^{-n}\circ\xi^\wedge - (\sum_{0\le i\le n} r^{2i-n-1})\circ u)$$

which proves (5). The formula (6) is exactly the formula (5) for the category $\mathcal{A}/\theta^\wedge,\xi^\wedge;r^{-1}/$. ∎

C2.5.2. Decompositions. Let f be an endomorphism of $Id_{\mathcal{A}}$. Denote by $\mathcal{A}(f\,|\,0)$ the full subcategory of the category $\mathcal{A}$ generated by all objects M such that $f(M) = 0$. One can check that $\mathcal{A}(f\,|\,0)$ is a thick subcategory, and the quotient category $\mathcal{A}/\mathcal{A}(f\,|\,0)$ is equivalent to the category $(f)^{-1}\mathcal{A}$ obtained by inversing all arrows $\{f(M)\,|\, M \in Ob\mathcal{A}\}$. Besides,

C2.5.2.1. Lemma. *If* $P \in Spec\mathcal{A}$, *and* $f(P) \neq 0$, *then* $f(P)$ *is a monomorphism.*

Proof. Note that $f(Ker(f(P))) = 0$. If $Ker(f(P)) \neq 0$, then $Ker(f(P)) \succ P$; therefore the equality $f(Ker(f(P))) = 0$ implies that $f(P) = 0$. ∎

Now take $f = u$. Thanks to the property $\theta(u) = (r^{-1}\circ u)\theta$, the subcategory $\mathcal{A}(u\,|\,0)$ is θ-stable. Thus, we can associate with u two hyperbolic categories - the subcategory $\mathcal{A}(u\,|\,0)/\theta',\xi'/$ and the quotient category $\mathcal{A}(u\,|\,*)/\theta^-,\xi^-/$, where $\mathcal{A}(u\,|\,*) := \mathcal{A}/\mathcal{A}(u\,|\,0)$.

Note that $\mathcal{A}(u\,|\,0)/\theta',\xi'/$ is a thick subcategory of $\mathcal{A}/\theta,\xi/$, and the corresponding quotient category $\mathcal{A}/\theta,\xi/ / \mathcal{A}(u\,|\,0)/\theta',\xi'/$ is naturally equivalent to the category $\mathcal{A}(u\,|\,*)/\theta^-,\xi^-/$.

Consider the related to the thick subcategory $\mathcal{A}(u\,|\,0)/\theta',\xi'/$ decomposition of the spectrum into Zariski open and closed subsets: $\mathbf{Spec}\mathcal{A}/\theta,\xi/ \simeq \mathbf{V}(u) \cup \mathbf{U}(u)$, where

$$\mathbf{V}(u) = \{<(\gamma,M,\eta) \in \mathbf{Spec}\mathcal{A}/\theta,\xi/\,|\, u(M) = 0\} = \mathbf{Spec}\mathcal{A}(u\,|\,0)/\theta',\xi'/$$

(the last equality follows from the fact that $\mathcal{A}(u\,|\,0)/\theta',\xi'/$ is a thick subcategory of $\mathcal{A}/\theta,\xi/$), and $\mathbf{U}(u) = \mathbf{Spec}\mathcal{A}/\theta,\xi/ - \mathbf{V}(u)$.

Since the localization at the subcategory $\mathcal{A}(u\,|\,0)/\theta',\xi'/$ induces an embedding of the open subset $\mathbf{U}(u)$ into the spectrum of $\mathcal{A}(u\,|\,*)/\theta^-,\xi^-/$, the study of the spectrum of $\mathcal{A}/\theta,\xi/$ splits into two cases: the case when $u \equiv 0$ and the case when u is invertible. Consider each of them.

(a) Let $u \equiv 0$. This equality is equivalent to the equality

$$\theta\xi = (r\circ\xi)\theta. \qquad (1)$$

Therefore we can repeat the same argument, but this time for ξ, and obtain, as a result, the decomposition $\mathbf{Spec}\mathcal{A}/\theta,\xi/ = \mathbf{V}(\xi) \cup \mathbf{U}(\xi)$, where

$$V(\xi) = \mathbf{Spec}\mathcal{A}(\xi|0)/[\theta',0], \quad \text{and} \quad U(\xi) \subseteq \mathbf{Spec}\mathcal{A}(\xi|*)/[\theta^-,\xi^-].$$

This time, we know the answers:

$$\mathbf{Spec}\mathcal{A}(\xi|0)/[\theta',0] \simeq \mathbf{Spec}\mathcal{A}(\xi|0)/[\theta'] \coprod_{\mathbf{Spec}\mathcal{A}(\xi|0)} \mathbf{Spec}\mathcal{A}(\xi|0)/[\theta'^{\wedge}]$$

(cf. 6.5), and

$$\mathbf{Spec}\mathcal{A}(\xi|*)/[\theta^-,\xi^-] = \mathbf{Spec}\mathcal{A}(\xi|*)/[\theta']/\mathcal{A}(\xi|*)$$

(cf. Lemma 5.2). I.e. the problem is reduced to the investigation of the spectrum of skew polynomial and skew Laurent categories.

(b) Suppose now that u and $u^{\wedge}$ are invertible.

Consider the decomposition of $\mathbf{Spec}\mathcal{A}/[\theta,\xi]$ with respect to the θ-stable cenrtal endomorphism $\lambda = id - r^2$. Again, we have the splitting into two cases: $\lambda = 0$, and *"λ is invertible".*

(b0) Let $\lambda = 0$. Then the formulas

$$\xi\theta^n = \theta^n\big(r^{-n+1}\circ\xi^{\wedge} - (\sum_{0\leq i\leq n} r^{2i-n})\circ u\big), \tag{2}$$

$$\xi^{\wedge}\theta^{\wedge n} = \theta^{\wedge n}\big(r^{n-1}\circ\xi - (\sum_{0\leq i\leq n} r^{-2i+n})\circ u^{\wedge}\big). \tag{3}$$

from Lemma C2.5.1 can be rewritten as

$$\xi\theta^n = \theta^n\big(r^{-n+1}\circ\xi^{\wedge} - (n+1)r^{-n})\circ u\big) = \theta^n\big(r^{-n}\circ(r\circ\xi^{\wedge} - (n+1)\circ u)\big) \tag{4}$$

and

$$\xi^{\wedge}\theta^{\wedge n} = \theta^{\wedge n}\big(r^{n-1}\circ(\xi - (n+1)\circ r\circ u^{\wedge})\big) \tag{5}$$

respectively.

Let, for every object M of $\mathcal{A}$, $\mathfrak{z}(\mathcal{A}|M)$ denote the image of the center $\mathfrak{z}(\mathcal{A}) := End(Id_{\mathcal{A}})$ of the category $\mathcal{A}$ in $\mathcal{A}(M,M)$.

Applying Theorem 6.1 and Theorem 6.4 to the formulas (4) and (5), we obtain the following assertion:

C2.5.2.2. Proposition. *Let $\mathcal{A}/[\theta,\xi]$ be a Heisenberg type category with the weight r such that $r^2 = id$ and the morphisms u, $u^{\wedge}$ are invertible.*

*(a) Let $<P> \in$ **Spec**$\mathcal{A}$, and $\xi(P) = 0$.*

(i) If $Char(\mathfrak{z}(\mathcal{A}|P)) = p > 0$, then $\Psi_{\xi,p}(P) \in Spec\mathcal{A}/[\theta,\xi]$.

(ii) If $Char(\mathfrak{z}(\mathcal{A}|M)) = 0$, then $\mathfrak{B}(P)$ and $\mathfrak{B}^{\wedge}(P)$ are objects of $Spec\mathcal{A}/[\theta,\xi]$.

Any object $(\gamma,M,\eta) \in Spec\mathcal{A}/[\theta,\xi]$ such that $<P> \in Ass(M)$ is equivalent either to $\Psi_{\xi,p}(P)$ (if $Char(\mathfrak{z}(\mathcal{A}|M)) = p \geq 2$), or to $\mathfrak{B}(P)$, or to $\mathfrak{B}^{\wedge}(P)$ (when $Char(\mathfrak{z}(\mathcal{A}|M)) = 0$).

(b) Let $P \in Spec\mathcal{A}$ be such that $\theta^n(P) \succ P$ only if $n = 0$, and

$$(r\circ\xi^{\wedge} - nu)(P) \neq 0, \quad (\xi - nr\circ u^{\wedge})(P) \neq 0$$

for all $n \geq 0$. *Then* $\theta^{\bullet}(P) \in Spec\mathcal{A}/\theta,\xi/$.

(b)* Suppose now that $\lambda = id - r^2$ is invertible. Then the formulas (2) and (3) can be rewritten as

$$\xi\theta^n = \theta^n(r^{-n}(r\circ\xi\wedge - \lambda^{-1}(id - r^{2(n+1)})\circ u)) \quad (6)$$

$$\xi\wedge\theta\wedge^n = \theta\wedge^n(r^{n-1}(\xi - \lambda^{-1}(id - r^{-2(n+1)})\circ r\circ u\wedge)) \quad (7)$$

Given an endomorphism, φ, of the identical functor $Id_{\mathcal{A}}$, define the φ-*characteristic*, $ch(M,\varphi)$, of an object M of the category $\mathcal{A}$ as zero, if $\varphi^n(M) \neq id$ for all $n \neq 0$; otherwise, as a minimal positive n such that $\varphi^n(M) = id$. One can easily check that $ch(M,\varphi)$ depends only on the equivalence class of the object M: $ch(M,\varphi) = ch(<M),\varphi)$. In particular, we have a well defined notion of the φ-characteristic of a point of the spectrum of $\mathcal{A}$.

As a consequence of the formulas (6), (7), Theorem 6.1, and Theorem 6.4, we obtain the following 'multiplicative' version of Proposition C2.5.2.2:

C2.5.2.3. Proposition. *Let* $\mathcal{A}/\theta,\xi/$ *be a Heisenberg type category with the weight* r *such that the morphisms* u, $u\wedge$, *and* $id - r^2$ *are invertible.*

(a) Let $<P> \in$ **Spec**$\mathcal{A}$, *and* $\xi(P) = 0$.

(i) If $ch(P,r^2) = p$, *then* $\Psi_{\xi,p}(P) \in Spec\mathcal{A}/\theta,\xi/$.

(ii) If $ch(P,r^2) = 0$, *then* $\mathcal{B}(P)$ *and* $\mathcal{B}\wedge(P)$ *are objects of* $Spec\mathcal{A}/\theta,\xi/$.

Any object $(\gamma,M,\eta) \in Spec\mathcal{A}/\theta,\xi/$ *such that* $<P> \in Ass(M)$ *is equivalent either to* $\Psi_{\xi,p}(P)$ *(if* $ch(P,r^2) = p \geq 1$*), or to* $\mathcal{B}(P)$*, or to* $\mathcal{B}\wedge(P)$ *(when* $ch(P,r^2) = 0$*).*

(b) Let $P \in Spec\mathcal{A}$ *be such that* $\theta^n(P) \succ P$ *only if* $n = 0$, *and*

$$(r\circ\xi\wedge - \lambda^{-1}\circ(id - r^{2(n+1)})\circ u)(P) \neq 0,$$

$$(\xi - \lambda^{-1}\circ(id - r^{-2(n+1)})\circ u\wedge)(P) \neq 0$$

for all $n \geq 0$. *Then* $\theta^{\bullet}(P) \in Spec\mathcal{A}/\theta,\xi/$.

C2.6. The categories of Weyl type. Let a category of Heisenberg type $\mathcal{A}/\theta,\xi;r/$ be given. Denote by c the central endomorphism $u\wedge\circ u$ (cf. Lemma C2.4); and consider the full subcategory $\mathcal{A}(c|1)$ of the category $\mathcal{A}$ generated by all objects M of $\mathcal{A}$ such that $c(M) = id_M$. One can check that $\mathcal{A}(c|1)$ is a thick subcategory of the category $\mathcal{A}$. It follows from the θ-stability of c that the subcategory $\mathcal{A}(c|1)$ is θ-stable. Therefore the restriction onto $\mathcal{A}(c|1)$ of the data (θ,ξ,r) defines a Heisenberg type category with the property: $c \equiv id$.

Define a *Weyl type category* as a Heisenberg type category, $\mathcal{A}/\theta,\xi;r/$, such that $c \equiv id$.

Since in that case, both u and $u^\wedge$ are invertible, we can apply to Weyl type categories Propositions C2.5.2.2 and C2.5.2.3.

C2.7. Weyl type rings and quantized enveloping algebras. Consider a special case of the Weyl rings. Namely, we assume that $J = [1,n]$, $\rho_i = \rho$ for every $i \in J$, and that the element $\rho - \rho^{-1}$ is invertible.

Define new variables, $\{e_i, f_i, h_i | \ 1 \le i \le n-1\}$ by the formulas:

$$e_i := x_i y_{i+1}, \quad f_i := x_{i+1} y_i, \quad h_i := u_i^{-1} u_{i+1} \quad (1 \le i \le n-1),$$

where

$$u_i := \xi_i - \rho^{-1}\theta_i^{-1}(\xi_i); \tag{1}$$

hence

$$u_i^{-1} = \xi_i - \rho\theta_i^{-1}(\xi_i). \tag{2}$$

We shall need the inverse formulas:

$$\xi_i = \tau(\rho u_i - \rho^{-1} u_i^{-1}), \tag{3}$$

$$\theta_i^{-1}(\xi_i) = \tau(u_i - u_i^{-1}), \tag{4}$$

where $\tau := (\rho - \rho^{-1})^{-1}$.

We have:

$$e_i f_i - f_i e_i = x_i y_{i+1} x_{i+1} y_i - x_{i+1} y_i x_i y_{i+1} = \xi_i \theta_{i+1}^{-1}(\xi_{i+1}) - \xi_{i+1}\theta_i^{-1}(\xi_i) =$$

$$\tau(\rho u_i - \rho^{-1}u_i^{-1})\tau(u_{i+1} - u_{i+1}^{-1}) - \tau(\rho u_{i+1} - \rho^{-1}u_{i+1}^{-1})\tau(u_i - u_i^{-1}) =$$

$$\tau^2(\rho u_i u_{i+1} + \rho^{-1}u_i^{-1}u_{i+1}^{-1} - \rho u_i u_{i+1}^{-1} - \rho^{-1}u_i^{-1}u_{i+1}$$

$$- \rho u_i u_{i+1} - \rho^{-1}u_i^{-1}u_{i+1}^{-1} + \rho u_i^{-1}u_{i+1} + \rho^{-1}u_i u_{i+1}^{-1}) = \tau(h_i - h_i^{-1}).$$

Clearly

$$e_i e_j := x_i y_{i+1} x_j y_{j+1} = e_j e_i$$

if $j \ne i+1$. If $j = i+1$, we have:

$$e_i e_i e_{i+1} - (\rho+\rho^{-1}) e_i e_{i+1} e_i + e_{i+1} e_i e_i =$$

$$x_i y_{i+1} x_i y_{i+1} x_{i+1} y_{i+2} - (\rho+\rho^{-1}) x_i y_{i+1} x_{i+1} y_{i+2} x_i y_{i+1} + x_{i+1} y_{i+2} x_i y_{i+1} x_i y_{i+1} =$$

$$y_{i+2} x_i x_i y_{i+1}(\theta_{i+1}^{-1}(\xi_{i+1}) - (\rho+\rho^{-1})\xi_{i+1} + \xi_{i+1}) = 0.$$

Similarly,

$$f_i f_j = f_j f_i$$

if $j \neq i+1$, and

$$f_i f_i f_{i+1} - (\rho+\rho^{-1}) f_i f_{i+1} f_i + f_{i+1} f_i f_i = 0.$$

This means that the subring generated by the elements e_i, f_i, h_i, $1 \leq i \leq n-1$, and the subring R^{Θ} of Θ-invariant elements of R is a quotient ring of the quantized enveloping algebra $U_{\rho}(A_{n-1};R^{\Theta})$ over the ring R^{Θ} corresponding to the Cartan matrix A_{n-1}.

C2.7.1. Proposition. *There exist ring morphisms defined by the following formulas:*

(a) $\varphi_A: U_{\rho}(A_{n-1};R^{\Theta}) \longrightarrow W[\Xi,\rho]$,

$$e_i \longmapsto x_i y_{i+1},\quad f_i \longmapsto x_{i+1} y_i,\quad h_i \longmapsto u_i^{-1} u_{i+1} \quad (1 \leq i \leq n-1),$$

(c) $\varphi_C: U_{\rho}(C_n;R^{\Theta}) \longrightarrow W[\Xi,\rho]$,

$$e_i \longmapsto x_i y_{i+1},\quad f_i \longmapsto x_{i+1} y_i,\quad h_i \longmapsto u_i^{-1} u_{i+1} \quad (1 \leq i \leq n-1),$$

$$e_n \longmapsto (\rho^2 + \rho^{-2})^{-1} x_n^2,\quad f_n \longmapsto -(\rho^2 + \rho^{-2})^{-1} y_n^2,\quad h_n \longmapsto \rho^{-1} u_n^{-2}.$$

Proof. We have checked *(a)* already. The rest is equally straightforward. ■

The Proposition C2.7.1 is a straightforward generalization of the concerning quantized Weyl algebra part of Theorem 3.2 in [Ha]. The remaining part of Hayashi's theorem deals with quantized Clifford algebras. It is worth a while to include them into our picture.

C2.8. Clifford type rings and quantized enveloping algebras. Let A be a commutative ring; and let $\Theta := \{\theta_i, u_i, \rho_i \mid i \in J\}$, where $\{\theta_i\}$ is a commuting family of automorphisms of A, u_i and ρ_i are invertible elements of the ring A. Denote by $A(\Theta)$, or $A(\Theta,\{x_i, y_i\})$, the ring generated by A and the variables x_i, y_i, $i \in J$, subject to the relations:

$$x_i x_j + x_j x_i = 0 = y_i y_j + y_j y_i \quad \text{for any} \quad i, j \in J, \tag{1}$$

$$x_i y_j + y_j x_i = 0 \quad \text{if} \quad i \neq j, \tag{2}$$

$$x_i y_i + \rho_i^{-1} y_i x_i = u_i^{-1} \tag{3}$$

$$x_i y_i + \rho_i y_i x_i = u_i \tag{4}$$

for any $i \in J$;

$$x_i a = \theta_i(a)x_i, \quad ay_i = y_i\theta_i(a) \quad \text{for any} \quad a \in A; \tag{5}$$

We call the ring $A(\Theta)$ a ring *of Clifford type* if

$$\theta_i(u_i) = \rho_i^{-1}u_i \quad \text{for every} \quad i \in J.$$

C2.8.1. Change of coordinates. In the ring $A(\Theta)$, set $x_iy_i = \xi_i$.

Clearly $\xi_i a = a\xi_i$ for any $i \in J$ and $a \in A$. Besides, for all $i \in J$,

$$\xi_i\xi_i = x_iy_ix_iy_i = u_ix_iy_i = u_i\xi_i, \tag{6}$$

and

$$\xi_i\xi_j = x_iy_ix_jy_j = x_jy_jx_iy_i = \xi_j\xi_i$$

if $i \neq j$.

Denote by R the quotient of the polynomial ring $A[\{\xi_i\}]$ by the relations (6). Clearly R is a free A-module with the basis $\{\xi_{i_1}...\xi_{i_n} \mid 1 \leq i_1 < i_2 < ... < i_n, \ n \in \mathbb{Z}_+\}$. In particular, if J is finite, than the ring R is a finite ($2^{|J|}$-) dimensional A-algebra.

We have:

$$y_i\xi_i = y_ix_iy_i = \rho_i(u_i^{-1} - \xi_i)y_i = \rho_iu_i^{-1}y_i;$$

$$x_i\xi_i = 0, \quad \xi_ix_i = x_i\rho_i(u_i^{-1} - \xi_i) = x_i\rho_iu_i^{-1}$$

$$x_i\xi_j = x_ix_jy_j = \xi_jx_i, \quad \text{and} \quad \xi_jy_i = y_i\xi_j \quad \text{if} \quad i \neq j.$$

These formulas show that the automorphisms θ_i might be extended onto R by

$$\theta_i^{-1}(\xi_i) = \rho_i(u_i^{-1} - \xi_i) = \rho_i^{-1}(u_i - \xi_i),$$

or

$$\theta_i(\xi_i) = \theta_i(u_i) - \rho_i\xi_i;$$

$$\theta_i(\xi_j) = \xi_j \quad \text{for all} \quad i \neq j,$$

we see that the ring $A(\Theta)$ is generated over its commutative subring R by $\{x_i, y_i \mid i \in J\}$ subject to the relations

$$x_ix_j + x_jx_i = 0 = y_iy_j + y_jy_i \quad \text{for any} \quad i, j \in J, \tag{1}$$

$$x_iy_j + y_jx_i = 0 \quad \text{if} \quad i \neq j, \tag{2}$$

$$x_iy_i = \xi_i. \tag{3}$$

$$y_ix_i = \rho_i(u_i^{-1} - \xi_i) = \rho_i^{-1}(u_i - \xi_i) = \theta_i^{-1}(\xi_i) \tag{4}$$

for any $i \in J$;

$$x_ir = \theta_i(r)x_i, \quad ry_i = y_i\theta_i(r) \quad \text{for any} \quad r \in R. \tag{5}$$

Note that

$$(\theta_i - \theta_i^{-1})(\xi_i) = \theta_i(u_i) - \rho_i\xi_i - \rho_i^{-1}(u_i - \xi_i) = -(\rho_i - \rho_i^{-1})\xi_i$$

since, by condition, $\theta_i(u_i) = \rho_i^{-1}u_i$ for each $i \in J$.

A ring $R\{(\theta_i),(\xi_i)\}$ described by the relations (1)-(5) and

$$(\theta_i - \theta_i^{-1})(\xi_i) = -(\rho_i - \rho_i^{-1})\xi_i \tag{6}$$

will be called a ring of *Clifford type.*

Given a ring $R\{(\theta_i),(\xi_i)\}$ of Clifford type, set

$$u_i := \xi_i + \rho_i\theta_i^{-1}(\xi_i).$$

We have:

$$\theta_i(u_i) = \theta_i(\xi_i + \rho_i\theta_i^{-1}(\xi_i)) = \theta_i(\xi_i) + \rho_i(\xi_i) =$$

$$\theta_i^{-1}(\xi_i) - (\rho_i - \rho_i^{-1})\xi_i + \rho_i(\xi_i) =$$

$$\rho_i^{-1}(\xi_i + \rho_i\theta_i^{-1}(\xi_i)) := \rho_i^{-1}u_i.$$

Consider now the case, when $J = [1,n]$, $\rho_i = \rho$ for all i, and the element $\rho - \rho^{-1}$ is invertible. Denote the corresponding Clifford type ring by $R(\{\Xi,\rho\})$.

Now we are able to formulate the natural generalization of the remaining part of Theorem 3.2 in [Ha]:

C2.8.2. Proposition. *There exist ring morphisms defined by the following formulas:*

(a) $\psi_A: U_\rho(A_{n-1};R^\Theta) \longrightarrow R(\{\Xi,\rho\})$

$$e_i \longmapsto x_iy_{i+1},\quad f_i \longmapsto x_{i+1}y_i,\quad h_i \longmapsto (u_{i+1})^{-1}u_i\quad (1 \le i \le n-1);$$

(b) $\psi_B: U_\rho(B_n;R^\Theta) \longrightarrow R(\{\Xi,\rho^2\}),\quad e_i \longmapsto x_iy_{i+1},\quad f_i \longmapsto x_{i+1}y_i,$

$$h_i \longmapsto (u_{i+1})^{-1}u_i\quad (1 \le i \le n-1),$$

$$e_n \longmapsto x_n,\quad f_n \longmapsto y_n,\quad h_n \longmapsto \rho u_n;$$

(d) $\psi_D: U_\rho(D_n;R^\Theta) \longrightarrow R(\{\Xi,\rho\}),$

$$e_i \longmapsto x_iy_{i+1},\quad f_i \longmapsto x_{i+1}y_i,\quad h_i \longmapsto u_i^{-1}u_{i+1}\quad (1 \le i \le n-1),$$

$$e_n \longmapsto x_{n-1}x_n,\quad f_n \longmapsto y_ny_{n-1},\quad h_n \longmapsto \rho u_{n-1}u_n.$$

Proof is a straightforward calculation which is left to the reader. ■

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