On a projectively minimal hypersurface in the unimodular affine space

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INTRODUCTION

It was W. Blaschke who developed systematically the affine differential geometry of surfaces in the affine space A^3 . His book [1] is still one of standard references. There the third order invariant called Fubini-Pick's cubic form played one of fundamental roles. This invariant was found in the projective study of surfaces. See G. Fubini - E. Čech [7] and G. Bol [2]. Note also that this form is used successfully by A.V. Pogorelov and E. Calabi to solve the Minkowski problem and related problems. In §67 of [1], there was defined the projective area functional using this cubic form. A projectively minimal surface is by definition a critical point of this functional. It is known that every affine hypersphere is projectively minimal. This is due to H. Behnke. Later G. Thomsen [12] derived a differential equation defining a projectively minimal surface in the projective category. In chapter VIII of [2] one can find several interesting charactersations of these surfaces of hyperbolic type due to Thomsen, Meyer, Su, Godeaux ... .

The purpose of this paper is to derive a differential equation defining a projecitvely minimal hypersurface in the affine space A^{n+1}. Making use of this we can obtain non-trivial examples of projectively minimal hypersurfaces which are closed in A^{n+1} (Theorem 4.5). Also we prove that only an ellipsoid is a projectively minimal surface in A^3 which is compact and strongly convex (Theorem 4.2). This is a converse to Behnke's theorem and can be seen a projective analogue to the affine Bernstein problem (see [4]). The generalisation in higher dimension remains open.

In §1 we shall recall the definition of the Fubini-Pick form and define the projective area functional for a locally strongly convex hypersurface in a real projective space P^{n+1}. §2 is a preliminary to calculate in §3 the variation of this functional for a hypersurface in A^{n+1}. The variational equation is elliptic and of order 6. The equation for a hypersurface that is in P^{n+1} shall be remarked. In §4 we prove theorems mentioned above.

§1 PROJECTIVE AREA OF A HYPERSURFACE

We recall in this section some projective invariants of a hypersurface to define the projective minimality. We adopt here the method of moving frames. A detailed description by this method is given in [11]. Refer also [5] on the affine treatment of a hypersurface.

Let P^{n+1} denote the real projective space of dimension $n+1$ and $G = PGL(n+1,\mathbf{R})$ the real projective transformation group acting canonically on P^{n+1}. The group G is a principal bundle over P^{n+1} with the isotropy subgroup as the fibre group. Let $e^0 = (e_0^0, e_1^0, \ldots, e_{n+1}^0)$ be a fixed basis of $\mathbf{R}^{n+2}$ with $\det(e_0^0, \ldots, e_{n+1}^0) = 1$. Then for every $g \in G$, $e_g = ge^0 = (e_0, e_1, \ldots, e_{n+1})$ is a new basis and satisfies

$$(1.1) \qquad \det(e_0, e_1, \ldots, e_{n+1}) = 1 \ .$$

We call this basis a projective frame or simply a frame. Let ω denote the Maurer-Cartan form of G and $\omega_\alpha{}^\beta$ be components of ω, then

$$de_\alpha = \omega_\alpha{}^\beta e_\beta$$

and the structure equation is

$$(1.2) \qquad d\omega_\alpha{}^\beta = \omega_\alpha{}^\gamma \wedge \omega_\gamma{}^\beta \ .$$

Here and throughout this paper, the summation convention is used and the index ranges are agreed on

$$(1.3) \qquad 0 < \alpha, \beta, \ldots < n+1 \quad \text{and} \quad 1 < i, j, \ldots < n \quad .$$

From (1.1) it holds

$$(1.4) \qquad \omega_\alpha{}^\alpha = 0 \quad .$$

Let now M be a hypersurface in P^{n+1}. Here we restrict ourselves to frames satisfying

$$(1.5) \qquad \omega_0{}^{n+1} = 0 \quad \text{and}$$

$$\omega_0{}^1, \ldots, \omega_0{}^n \quad \text{are linearly independent}$$

on M. We write $\omega^\alpha = \omega_0{}^\alpha$ for short. By the exterior differentiation of (1.5),

$$0 = d\omega^{n+1} = \omega^i \wedge \omega_i{}^{n+1} \quad .$$

Hence we may put

$$(1.6) \qquad \omega_i{}^{n+1} = h_{ij}\omega^j \; ; \; h_{ij} = h_{ji} \quad ,$$

and define

$$(1.7) \qquad \varphi_2 = h_{ij}\omega^i\omega^j \; , \; h = (h_{ij}) \quad \text{and} \quad H = \det h \quad .$$

Once one fixes a frame satisfying (1.5), then another frame is written as $\tilde{e} = ge$, g being a unimodular matrix of the form

$$g = \begin{pmatrix} \lambda & 0 & 0 \\ b & a & 0 \\ \mu & c & \nu \end{pmatrix}$$

where

$a = (a_i^j)$ is a $n \times n$ matrix,
$b = (b_i)$, $c = (c^j)$ are vectors and
$\lambda, \mu, \nu \in \mathbf{R}$.

Since the value of the Maurer-Cartan form at $\tilde{e}$, which we denote by $\tilde{\omega}$, is given by

$$(1.8) \qquad \tilde{\omega} = dg \cdot g^{-1} + g\omega g^{-1} ,$$

some computation yields

$$(1.9) \qquad \tilde{h}_{ij} = (\lambda\nu)^{-1} a_i^k h_{k\ell} a_j^\ell , \; \tilde{H} = (\det a)^{n+2} H ,$$

and

$$(1.10) \qquad \begin{aligned} &\frac{1}{n+2} \, d \log \tilde{H} + \tilde{\omega}_0{}^0 + \tilde{\omega}_{n+1}{}^{n+1} = \\ &= \frac{1}{n+2} \, d \log H + \omega_0{}^0 + \omega_{n+1}{}^{n+1} + \nu^{-1} c^j h_{jk} \omega^k - b_i A_k^i \omega^k . \end{aligned}$$

Now we assume

ASSUMPTION 1.1. The matrix h is positive definite everywhere on M. The equality (1.9) implies this property does not depend on the choice of frames. In fact this is equivalent to the fact that the hypersurface is locally strongly convex.

Under this assumption it is able to choose a frame $\tilde{e}$ so that

$$\frac{1}{n+2}\, d \log \tilde{H} + \tilde{\omega}_0{}^0 + \omega_{n+1}{}^{n+1} = 0 \tag{1.11}$$

and, moreover, we may assume

$$\tilde{H} = 1 \tag{1.12}$$

due to the fact (1.10) and (1.9). Limiting ourselves to such frames and assuming the frame e itself satisfies this condition from the beginning, we can see that a frame change must satisfy

$$c^j h_{jk} = \nu b_i A^i_k \ , \quad \lambda\nu = 1 \quad \text{and} \quad \det a = 1 \ , \tag{1.13}$$

$$\omega_0{}^0 + \omega_{n+1}{}^{n+1} = 0 \ . \tag{1.14}$$

Let us next take a derivation of (1.6). Then, by the structure equation (1.2), $\left(dh_{ij} - h_{ik}\omega^k_j - h_{kj}\omega^k_i\right) \wedge \omega^j = 0$. Therefore we can put

$$dh_{ij} - h_{ik}\omega^k_j - h_{kj}\omega^k_i = h_{ijk}\omega^k \tag{1.15}$$

using symmetric h_{ijk} . From the assumption (1.12),

$$0 = d \log H = h^{ij} dh_{ij} = h^{ij} h_{ijk} \omega^k + 2\omega_j^j ,$$

(h^{ij}) being the inverse of (h_{ij}) . Since $\omega_j^j = 0$ by (1.4) and (1.14), this yields

$$(1.16) \qquad h^{ij} h_{ijk} = 0 .$$

This identity has been called the <u>apolarity</u> condition. Now a calculation by use of the relation (1.8) and the definition (1.15) shows an important relation

$$(1.17) \qquad \lambda \tilde{h}_{ijk} = h_{pqr} a_i^p a_j^q a_k^r .$$

In view of this relation, we put

$$(1.18) \qquad \varphi_3 = h_{ijk} \omega^i \omega^j \omega^k \quad \text{and} \quad F = h_{ijk} h_{pqr} h^{ip} h^{jq} h^{kr} .$$

This symmetric form φ_3 is called the <u>Fubini-Pick cubic form</u>. It is known that it vanishes for a quadratic hypersurface and measures the difference of a hypersurface from quadratic hypersurfaces (see f. ex. [6]). The scalar F , which is the square of the norm of φ_3 with respect to h , is called the Fubini-Pick invariant. This is nonnegative by Assumption 1.1. The fact (1.9) implies the following invariance under the group action.

$$(1.19) \qquad \tilde{\varphi}_2 = \lambda^2 \varphi_2 , \ \tilde{\varphi}_3 = \lambda^2 \varphi_3 \quad \text{and} \quad \tilde{F} = \lambda^{-2} F .$$

Hence the 2-form $F\varphi_2$ is independent of the choice of frames. We call this the projective metric of a hypersurface, though it is totally degenerate where $F = 0$. The formula (1.19) also implies that the form φ_2 defines a conformal structure on the hypersurface. As for this structure see [11]. We put

$$dA = H^{1/2}\omega^1 \wedge \ldots \wedge \omega^n \quad . \tag{1.20}$$

Then the volume form of the projective metric is $F^{\frac{n}{2}} dA$, which we call the <u>projective area element</u>. Recall that the ratio φ_3/φ_2 is called the projective line element in [7].

Now we put

$$P(C) = \int_C F^{\frac{n}{2}} dA$$

for any compact oriented region C in the hypersurface M.

<u>DEFINITION 1.2</u>. A hypersurface is called projective minimal if this functional P is critical for every region C.

Since $P(M) = 0$ for a quadratic hypersurface M, we have a trivial

<u>EXAMPLE 1.3</u>. Every strongly convex quadratic hypersurface is projectively minimal.

§2 RECALL OF THE AFFINE METRIC OF A HYPERSURFACE

In the calculation of the projective area it is better to choose a frame which enables computation as far as simple. A frame which meets our argument is a affine normal frame. See Remark 3.2. In this section we recall this frame and the related formulas. So we consider a hypersurface M in the affine space A^{n+1} of dimension $n+1$ and deal with the unimodular affine group instead of the projective transformation group.

Let x denote a point of M. We say a basis $(e_1,\dots,e_{n+1})$ of A^{n+1} is a unimodular affine frame (an affine frame shortly) when $\det(e_1,\dots,e_{n+1}) = 1$ and $dx = \Sigma\omega^i e_i$ $(1 < i < n)$. Then the normalization of affine frames relative to the unimodular affine group is carried out similarly as in the projective case. If we denote by $\omega_\alpha{}^\beta (0 < \alpha < n+1,\ 1 < \beta < n+1)$ the Maurer-Cartan form of this group, then it is possible to choose an affine frame so that

$$(2.1) \qquad \omega_0{}^{n+1} = \omega_{n+1}{}^{n+1} = 0 \quad \text{and} \quad \det h_{ij} = 1 ,$$

where we define h_{ij} by $\omega_i^{n+1} = h_{ij}\omega^j$. We assume as before Assumption 1.1. This normalized frame is called an <u>affine normal frame</u>. The last vector e_{n+1} is called the affine normal of the hypersurface. The form φ_2 defines now a Riemannian metric, which is called the <u>affine metric</u>. The invariants φ_3 and F can be defined by the same formula.

Since the conformal class φ_2 is now fixed, we can apply Riemannian geometry to the hypersurface. For this purpose we introduce another invariant. Take an exterior derivative of the equation $\omega_{n+1}{}^{n+1} = 0$ to get

(2.2) $$\omega_{n+1}{}^{i} \wedge \omega_{i}{}^{n+1} = 0 \quad .$$

Since $\{\omega_i{}^{n+1}\}$ is linearly independent by Assumption 1.1, it is possible to write

(2.3) $$\omega_{n+1}{}^{i} = \ell^{ij}\omega_j{}^{n+1} \quad , \quad \ell^{ij} = \ell^{ji} \quad .$$

The tensor $\ell_{ij} := h_{ik}\ell^{km}h_{mj}$ and the arithmetic mean of this trace

(2.4) $$\ell = \frac{1}{n}\,\ell_{ij}h^{ij}$$

are called the affine mean curvature tensor and the affine mean curvature respectively. The quadratic form

(2.5) $$\psi = \ell_{ij}\omega^i\omega^j$$

is invariant under the unimodular affine group, though not projectively invariant. The forms φ_3 and ψ are fundamental affine invariants of a hypersurface.

An affine minimal (or maximal in a strict sense [4]) hypersurface is by definition a hypersurface with $\ell = 0$.

See [1], [5], [4]. A hypersurface which satisfies the equation $\ell_{ij} = kh_{ij}$ everywhere for some function k is called an affine hypersphere. It turns out that k is constant and equal to ℓ. According as $-\ell > 0$, $= 0$ or < 0, the surface is called of elliptic, parabolic or hyperbolic type. The geometrical meaning of this notion is described in §43 of [1]. Every quadratic hypersurface is known to be an affine hypersphere. The next theorem is one of converses to this fact.

THEOREM ([3], [9]). Let M be an affine hypersphere which is strongly convex and closed in the affine space. Assume M is of elliptic or parabolic type. Then it is necessarily a quadratic hypersurface.

An affine hypersphere of parabolic type is by definition affinely minimal. When $n = 2$, we know of a global result due to Calabi [4] which states that every affine minimal surface in A^3 which is closed and strongly convex is a paraboloid provided that the affine metric is complete.

Next we recall structure equations. The connection form associated to the Riemannian metric φ_2 which we denote by $\pi_i{}^j$, is determined by the requirements $d\omega^i = \omega^j \wedge \pi_j{}^i$ and $dh_{ij} = h_{ik}\pi_j^k + h_{jk}\pi_i^k$. That is,

$$\pi_i{}^j = \omega_i{}^j + \frac{1}{2}h^j{}_{ik}\,\omega^k \quad . \tag{2.6}$$

Here and from now on we use the lowering or raising indices by the metric tensor h_{ij} . The curvature tensor $R_{ijk\ell}$ is given by the formula

(2.7) $$R_{ijk\ell} = \frac{1}{2}\left(h_{ik}\ell_{j\ell} - h_{jk}\ell_{i\ell} - h_{j\ell}\ell_{ik} - h_{i\ell}\ell_{jk}\right) + \\ + \frac{1}{4}\left(h_{i\ell m}h_{jk}{}^{m} - h_{ikm}h_{j\ell}{}^{m}\right) .$$

Then the Ricci tensor is

(2.8) $$R_{ij} = -\frac{1}{2}(n-2)\ell_{ij} - \frac{1}{2}n\ell h_{ij} + \frac{1}{4}h_{ik\ell}h_{j}{}^{k\ell} .$$

The Codazzi-Minardi equations are

(2.9) $$h_{ijk,m} - h_{ijm,k} = h_{jk}\ell_{jm} - h_{ik}\ell_{im} - h_{jm}\ell_{ik} - h_{im}\ell_{jk}$$

and

(2.10) $$\ell_{jk,m} - \ell_{jm,k} = \frac{1}{2}\left(\ell_{ik}h^{i}{}_{jm} - \ell_{im}h^{i}{}_{jk}\right) .$$

In particular we have

(2.11) $$h_{ijk,}{}^{k} = -n\left(\ell_{ij} - \ell h_{ij}\right)$$

(2.12) $$h_{i}{}^{jk}\ell_{jk} = -2\left(\ell_{ij,}{}^{j} - n\ell_{,i}\right) .$$

For the induction of these formulas see [6], [4], [11].

§3 VARIATIONAL FORMULA OF P

Using notations in §2, we shall derive a variational formula of the functional P . Let M be a locally strongly convex hypersurface in the affine space A^{n+1} and let x generally represent a point of M . We fix an affine normal frame $\{e_1,\ldots,e_{n+1}\}$ at each x , varying smoothly with x . To compute the variation of P , we consider a deformation of the hypersurface M . Let M_t be a family of nearby hypersurfaces: its point x_t has a representation

$$(3.1) \qquad x_t = x + a^i(t,x)e_i(x) + \nu(t,x)e_{n+1}(x) \quad ,$$

where t is a continuous parameter. We assume $M_0 = M$ and M_t depends smoothly on t : a^i and ν are smooth functions and

$$(3.2) \qquad a^i(0,x) = \nu(0,x) = 0 \quad .$$

We also assume that these functions are compactly supported and sufficiently near to zero so that the deformed hypersurface remains locally strongly convex.

In order to compute $P(M_t)$, we first look for an affine normal frame of M_t . We use the usual tensorial notation to denote the covariant derivatives. Recall the Riemannian connection is $\pi_i{}^j$ defined in (2.6). From (3.1)

$$(3.3) \qquad dx_t = \bar{\omega}^i e_i + \left(a^j\omega_j{}^{n+1} + \nu_i\omega^i\right) e_{n+1}$$

where

$$\bar{\omega}^i = \omega^i + da^i + a^j\omega_j{}^i + \nu\omega^i_{n+1}$$

$$d\nu = \nu_i\omega^i \quad .$$

We put

$$(3.4)\qquad \begin{aligned} C_{ij} &= h_{ij} + a_{i,j} - \tfrac{1}{2}a_k h^k{}_{ij} + \nu\ell_{ij} \ ; \ a_i = h_{ij}a^j \quad , \\ b_i &= h_{ij}C^{jk}(a_k + \nu_k) \ ; \ (C^{ij}) = (C_{ij})^{-1} \quad , \end{aligned}$$

and define $\bar{e}_i = e_i + b_i e_{n+1}$. Then (3.3) is rewritten as

$$dx_t = \bar{\omega}^i e_i \ , \ \bar{\omega}^i = h^{ij}\ell_{jk}\omega^k \quad .$$

Introducing a form D_i by

$$(3.5)\qquad \begin{aligned} D_i &= D_{ij}\omega^j = db_i + \omega_i{}^{n+1} - b_j(\omega_i{}^j + b_i\omega^j_{n+1}) \quad , \\ D_{ij} &= b_{i,j} + \tfrac{1}{2}b_k h^k{}_{ij} + h_{ij} - b_i b_k \ell^k{}_j \quad , \end{aligned}$$

we define a new frame $\{\tilde{e}_1, \ldots, \tilde{e}_{n+1}\}$ at x_t by the formula

$$(3.6)\qquad \tilde{e}_i = f\bar{e}_i = f(e_i + b_i e_{n+1}) \ , \ e_{n+1} = f^{-n}e_{n+1} + p^i e_i$$

where

$$f = (D/C)^{-1/n(n+2)} \ ; \ C = \det C_{ij} \ , \ D = \det D_{ij} \ ,$$

(3.7)

$$p^i = nf^{-n-2}D^{ij}f_j \ ; \ df = f_i\omega^i \ , (D^{ij}) = (D_{ij})^{-1} \ .$$

Then a calculation verifies that this frame is an affine normal frame field along M_t. The associated Riemannian tensor $\tilde{h}_{ij}$ and the coframe $\tilde{\omega}_i{}^j$ are given by

$$\text{(3.8)} \qquad \begin{aligned} \tilde{h}_{ij} &= f^{n+2}D_{ik}C^k \ h_{\ell j} \\ \tilde{\omega}^i &= f^{-1}h^{ij}C_{jk}\omega^k \\ \tilde{\omega}_i{}^j &= d \log f\delta_i{}^j + \omega_i{}^j + b_i\omega^j_{n+1} - f^{n+1}p^jD_i \ . \end{aligned}$$

Hence the area form $d\tilde{A}$ of M_t is

$$\text{(3.9)} \qquad d\tilde{A} = f^{-n}C\,dA \ .$$

In [4] Calabi has shown that it is enough to consider only a deformation with the property $b_i = 0$. But we do not assume here this property because we would like to consider later the deformation along e_i and the deformation along e_{n+1} separately and because it is easier to carry out an analogous calculation in the projective case (see Remark 3.2).

We now compute the variation at $t = 0$. For a given quantity Q depending on a^i and ν, the notation δQ means the derivative of Q with respect to t, evaluated at $t = 0$. The initial condition (3.2) implies

$$C_{ij} = D_{ij} = \tilde{h}_{ij} = h_{ij},\ b_i = p_i = 0,\ f = 1 \tag{3.10}$$

at $t = 0$. Hence we have

$$\begin{aligned}
\delta b_i &= \delta a_i + \delta\nu_i \\
\delta C_{ij} &= \delta a_{i,j} - \tfrac{1}{2}\delta a_k h^k{}_{ij} + \ell_{ij}\delta\nu \\
\delta D_{ij} &= \delta b_{i,j} + \tfrac{1}{2}\delta b_k h^k{}_{ij} \\
\delta\tilde{h}_{ij} &= (n+2)\delta f h_{ij} + \delta E_{ij} \\
\delta\tilde{\omega}^i &= -\delta f\omega^i + h^{ij}\delta C_{jk}\omega^k \\
\delta\tilde{\omega}_i{}^j &= \delta df\delta_i{}^j + \delta b_i\omega^i_{n+1} - \delta p^j D_i \quad ,
\end{aligned} \tag{3.11}$$

where we have set

$$\delta E_{ij} = \delta D_{ij} - \delta C_{ij} \quad . \tag{3.12}$$

Although the explicit computation of the Fubini-Pick invariant of M_t is complicated, its variation can be calculated as follows. Recall the defining equation of the invariant $\tilde{h}_{ijk}$ of M_t : $\tilde{h}_{ijk}\tilde{\omega}^k = d\tilde{h}_{ij} - \tilde{h}_{ik}\tilde{\omega}_j{}^k - \tilde{h}_{kj}\tilde{\omega}_i{}^k$. Then taking derivatives with respect to t and making use of formulas in (3.11), we have

$$\delta h_{ijk} = (n+3)h_{ijk}\delta f + nh_{ij}\delta f_k - h_{ij}{}^{\ell}\delta C_{\ell k} +$$
$$+ \delta E_{ij,k} + \frac{1}{2}\left(\delta E_{i\ell}h_{jk}{}^{\ell} + \delta E_{j\ell}h_{ik}{}^{\ell}\right) -$$
$$- \left(\ell_{ik}\delta b_j + \ell_{jk}\delta b_i\right) + \left(h_{i\ell}h_{jk} + h_{j\ell}h_{ik}\right)\delta P^{\ell} \quad .$$

Hence

$$h^{ijk}\delta h_{ijk} = (n+3)F\delta f - K^{ij}\delta C_{ij} + h^{ijk}\delta E_{ij,k} +$$
$$+ K^{ij}\delta E_{ij} - 2h^{ijk}\ell_{jk}\delta b_i \quad .$$

Here we have introduced

$$K_{ij} = h_{ik\ell}h_j{}^{k\ell} \quad . \tag{3.13}$$

Since $\delta F = 2h^{ijk}\delta h_{ijk} + 3K_{ij}\delta h^{ij}$, we get

$$\delta F = -nF\delta f - 2K^{ij}\delta C_{ij} - K^{ij}\delta E_{ij} + 2h^{ijk}\delta E_{ij,k} - 4h^{ijk}\ell_{jk}\delta b_i \quad .$$

On the other hand, from (3.9), $\delta dA = (-n\delta f + \delta C)dA$. Therefor the variation of P is given by

$$\delta P = \int Q \, dA \quad ,$$

$$Q = F^{\frac{n}{2}}\left(\delta C - \frac{1}{2}n(n+2)\delta f\right) + \tag{3.14}$$
$$+ \frac{n}{2} F^{\frac{n}{2}-1}\left(2h^{ijk}\delta E_{ij,k} - 2K^{ij}\delta C_{ij} - K^{ij}\delta E_{ij} - 4h^{ijk}\ell_{jk}\delta b_i\right) \quad .$$

This formula shows that the deformation is infinitesimally a composite of the deformation in case $\delta\nu = 0$ and the deformation in case $\delta a^i = 0$. So we consider separately. From now on we forget raising indices for simplicity of writing. We first treat the case $\delta\nu = 0$. In this case, from (3.11)

$$\begin{aligned} \delta C_{ij} &= \delta a_{i,j} - \tfrac{1}{2}\delta a_k h_{ijk} , \quad \delta C = \delta a_{i,i} \\ \delta D_{ij} &= \delta a_{i,j} + \tfrac{1}{2}\delta a_k h_{ijk} , \quad \delta E_{ij} = \delta a_k h_{ijk} , \\ \delta b_i &= \delta a_i , \quad \delta f = 0 . \end{aligned} \tag{3.15}$$

Then the repeated use of Stoke's formula implies

$$\delta P = \int F^{\frac{n}{2}-1} Q_i' \delta a^i \, dA ,$$

$$Q_i' = K_{ij,j} - \tfrac{1}{2}F_i - h_{ijk}h_{jk\ell,\ell} - 2h_{ijk}\ell_{jk} .$$

Now apply the Codazzi-Minardi equation (2.9). Namely

$$\begin{aligned} K_{ij,j} &= (h_{ik\ell}h_{jk\ell})_{,j} \\ &= h_{ik\ell}h_{jk\ell,j} + \left(h_{ik\ell,j} - h_{jk\ell,i}\right)h_{jk\ell} + h_{jk\ell,i}h_{jk\ell} \\ &= h_{ik\ell}h_{jk\ell,j} + \frac{1}{2}F_i + \\ &\quad + h_{jk\ell}\left(\ell_{jk}h_{i\ell} + \ell_{j\ell}h_{ik} - \ell_{ik}h_{j\ell} - \ell_{i\ell}h_{jk}\right) \\ &= \frac{1}{2}F_i + h_{ik\ell}h_{jk\ell,j} + 2h_{ijk}\ell_{jk} . \end{aligned} \tag{3.16}$$

Hence $Q_i' = 0$ always, i.e. the deformation along e_i is infinitesimally trivial.

We next treat the case $\delta a^i = 0$, $1 \le i \le n$. In this case

$$(3.17)\qquad \begin{aligned} &\delta C_{ij} = \ell_{ij}\delta\nu , \quad \delta C = n\ell\delta\nu , \quad \delta b_k = \delta\nu_k \\ &\delta D_{ij} = \delta\nu_{i,j} + \tfrac{1}{2}h_{ijk}\delta\nu_k \quad . \end{aligned}$$

Then

$$Q = \frac{1}{2}F^{\frac{n}{2}}(\Delta(\delta\nu) + n\ell\delta\nu) - nF^{\frac{n}{2}-1}K_{ij}\ell_{ij}\delta\nu - 2nF^{\frac{n}{2}-1}h_{ijk}\ell_{jk}\delta\nu_i$$

$$-\frac{n}{2}F^{\frac{n}{2}-1}K_{ij}\delta E_{ij} + nF^{\frac{n}{2}-1}h_{ijk}\delta E_{ij,k} \quad .$$

In order to apply Stoke's formula, we assume $\operatorname{supp}\delta\nu \cap \{F = 0\} = \emptyset$. From (2.11) and (3.16), we have

$$(3.18)\qquad \frac{1}{2}F_i = (n-2)h_{ijk}\ell_{jk} + K_{ij,j} \quad .$$

Hence

$$(3.19)\qquad \frac{1}{2}\Delta\left(F^{\frac{n}{2}}\right) = \frac{n}{2}\left(F^{\frac{n}{2}-1}K_{ij,i}\right)_i + \frac{n(n-2)}{2}\left(F^{\frac{n}{2}-1}h_{ijk}\ell_{jk}\right)_i \quad .$$

Now the repeated use of Stoke's formula yields

$$\delta P = -\int \frac{n}{2}\, p\, \delta\nu \, dA$$

where

$$(3.20)\quad \begin{aligned} p = & -\ell F^{\frac{n}{2}} + F^{\frac{n}{2}-1} K_{ij}\ell_{ij} + \left(\left(F^{\frac{n}{2}-1}\right)_{j} K_{ij}\right)_{,i} \\ & - \left(\left(F^{\frac{n}{2}-1} h_{ijk}\right)_{,k} h_{ij\ell}\right)_{,\ell} - (n+2)\left(F^{\frac{n}{2}-1} h_{ijk}\ell_{jk}\right)_{,i} \\ & - 2\left(F^{\frac{n}{2}-1} h_{ijk}\right)_{,k} \ell_{ij} - \frac{1}{2}\left(F^{\frac{n}{2}-1} K_{ij}h_{ijk}\right)_{,k} \\ & + 2\left(F^{\frac{n}{2}-1} h_{ijk}\right)_{,kji} . \end{aligned}$$

Since the definition of F implies that h_{ijk} and F_k are of order $F^{1/2}$, each term of p is at least of order $F^{\frac{n}{2}-2}$. Hence, except $n = 3$, p is finite even where $F = 0$. Therefore we have proved.

<u>THEOREM 3.1</u>. Let M be a locally strongly convex hypersurface in the affine space A^{n+1}. Suppose $F \neq 0$ in case $n = 3$. Then M is projectively minimal if and only if it satisfies the differential equation $p = 0$.

We shall examine the local property of p. Assume the hypersurface is locally given by the equation $x^{n+1} = \psi(x^1,\dots,x^n)$ near $x = 0$, the function ψ being strictly convex, i.e. the matrix $(\psi_{ij} = \partial^2\psi/\partial x^i \partial x^j) > 0$. Put $c = (\det \psi_{ij})^{-1/n(n+2)}$. Then one can prove the next formulas:

$$h_{ij} = c^{n+2}\psi_{ij} ,$$

$$h_{ijk} = nc^{n+2}(c_i\psi_{jk} + c_j\psi_{ki} + c_k\psi_{ij}) + c^{n+3}\psi_{ijk} ,$$

$$\ell_{ij} = n(n+1)c_i c_j - nc(c_k\psi^{k\ell})_i\psi_{j\ell} .$$

Here subindices mean derivatives with respect to (x^i) and $(\psi^{ij}) = (\psi_{ij})^{-1}$. From these formulas we know that the operator p is of degree 6 and the principal part comes from the term $2\left(F^{\frac{n}{2}-1} h_{ijk}\right)_{,kji}$. For simplicity assume $\psi_{ij}(0) = \delta_{ij}$. Then it is easy to see that the principal part at 0 is

$$(3.21) \qquad 2(n-2)F^{\frac{n}{2}-2}\psi_{ijkpqr}h_{ijk}h_{pqr} + 2\frac{n-1}{n+2} F^{\frac{n}{2}-1}\psi_{iijjkk} .$$

Hence the operator p is elliptic. When $n = 2$, it is semi-linear and the principal part is a triply harmonic operator, while, when $n \geq 3$, it is highly nonlinear.

REMARK 3.2. The above calculation is carried out for a hypersurface in A^{n+1} and the formula of p contains an affine invariant ℓ_{ij} which is not projectively invariant. We remark here that the calculation for a hypersurface in P^{n+1} can be similarly carried out once we define projective invariants. The result is as follows. By differentiating (1.14), we see first $(h_{ij}\omega^j_{n+1} - \omega_i{}^0) \wedge \omega^i = 0$. Hence we can put $h_{ij}\omega^j_{n+1} - \omega_i{}^0 = L_{ij}\omega^j$; $L_{ij} = L_{ji}$. This is the projective analogue of ℓ_{ij}. We next normalize a frame further by requiring $h^{ij}L_{ij} = 0$. Moreover, where $F \neq 0$, there exists a frame with $F = 1$ and, then, φ_2 turns out to be a Riemannian metric which is now projectively invariant. See [11] for these matters. We put $h_{ij}\omega^j_{n+1} + \omega_i{}^0 = U_{ij}\omega^j$.

This U_{ij} is symmetric. The resulting equation is

(3.22)
$$h_{ijk,kji} - h_{ijk,k}U_{ij} - (h_{ijk}U_{ij})_{,k} + \\ + \frac{1}{2} L_{ij}K_{ij} - \frac{1}{4}(K_{ij}h_{ijk})_{,k} = 0 \quad .$$

REMARK 3.3. We assumed the local strong convexity (Assumption 1.1). But in the calculation it is sufficient to assume that h is non-degenerate. In this case we define the projective area element by $|F|^{\frac{n}{2}}|H|^{\frac{1}{2}}\omega^1 \wedge \ldots \wedge \omega^n$.

§4 APPLICATIONS

We shall find special solutions of the differential equation $p = 0$ and prove theorems stated in the introduction. We first consider the case

1. $n=2$. In case $n = 2$, the differential equation turns out to be extremely simple. This is due to the fact

LEMMA 4.1. $K_{ij} = \frac{F}{2} h_{ij}$ in case $n = 2$.

PROOF. Assume $h_{ij} = \delta_{ij}$ for simplicity. Then the apolarity condition is $h_{111} + h_{122} = h_{112} + h_{222} = 0$. This implies
$K_{11} = h_{111}^2 + h_{112}^2 + h_{121}^2 + h_{122}^2 = 2(h_{111}^2 + h_{222}^2)$,
$K_{12} = (h_{111} + h_{122})h_{112} + (h_{112} + h_{222})h_{122} = 0$ and similarly
$K_{22} = 2(h_{111}^2 + h_{222}^2)$. Hence $F = K_{ii} = 4(h_{111}^2 + h_{222}^2)$ and the result follows.

From this lemma

$$K_{ij}h_{ijk} = 0 \quad , \quad K_{ij}\ell_{ij} = \ell F \quad . \tag{4.1}$$

By using these identities, we have

$$p = -\frac{1}{2}\left(h_{ijk,k}h_{ij}\right)_{,\ell} - 2\left(h_{ijk}\ell_{jk}\right)_{,i} - h_{ijk,k}\ell_{ij} + h_{ijk,kji} \; .$$

Then the Codazzi-Minardi equations (2.11) and (2.12) imply

$$p = -2\left\{\Delta \ell - \|\ell_{ij} - \ell h_{ij}\|^2\right\} .$$

Namely the differential equation defining a projectively minimal surface is

$$\Delta \ell = \|\ell_{ij} - \ell h_{ij}\|^2 . \tag{4.2}$$

THEOREM 4.2. (1) (H. Behnke) Every affine sphere in A^3 is projectively minimal. (2) If the surface is compact, strongly convex and projectively minimal, then it is a quadric.

PROOF. (1) follows from the definition of affine hyperspheres (§2) in view of (4.2). (2) is seen by integrating (4.2) over the surface. In fact we have $\ell_{ij} - \ell h_{ij} = 0$ and the structure theorem in §2 proves the result.

The converse of (1) of this theorem is not true due to the fact that a projective transform of an affine hypersphere is not generally an affine hypersphere, while the projective minimality is preserved by every projective transformation. For example, consider the surface $S_1 = \left\{xyz = (1+ax)^3\right\}$ $(a \neq 0)$ in A^3 which is a projective image of $S_2 = \left\{xyz = 1\right\}$. The surface S_2 is an affine sphere of hyperbolic type, but the surface S_1 is not.

Suppose the surface is both affinely and projectively minimal. Then (4.2) implies $\ell_{ij} = 0$. Hence this surface is an affine sphere of parabolic type. In particular by Theorem in §2 we have

THEOREM 4.3. Assume the surface is strongly convex and closed in A^3 . If it is both affinely and projectively minimal, then it is a paraboloid.

In this theorem, the closedness is necessary.

EXAMPLE 4.4. Consider the surface given by

$$\left\{(x,y,z) \in A^3 \;\middle|\; k(y-x^2)^3 = (z+2x^3-3xy)^2\right\} \quad (k \neq -4) \quad ,$$

which is found by Enriques ([8]). Outside the singular set $\{y-x^2=0\}$, the function z of x and y satisfies the equation $z_{xx}z_{yy} - z_{xy}^2 = -9 - 9k/4$. Hence the surface is an affine sphere of parabolic type ([3]) and it is definte or indefinite according as $k < -4$ or $k > -4$. In case $k = 0$, the surface is known to be affinely homogeneous.

2. HIGHER DIMENSIONAL EXAMPLES. We next generalize the first part of Theorem 4.2 in general dimension. We assume the hypersurface is an affine hypersphere. Since this means $\ell_{ij} = \ell h_{ij}$ and ℓ is constant, the equations (2.11) and (2.12) imply

(4.3) $$h_{ijk,k} = h_{ijk}\ell_{jk} = 0 \quad .$$

Then, from (3.25), we have

(4.4) $$p = 2\left(F^{\frac{n}{2}-1} h_{ijk}\right)_{,kji} - \frac{1}{2}\left(F^{\frac{n}{2}-1} K_{ij}h_{ijk}\right)_{,k} \quad .$$

THEOREM 4.5. Let M be an affine hypersphere in A^{n+1}. Assume it is homogeneous under the affine transformation group preserving M and the affine metric is Einstein. Then M is projectively minimal.

PROOF. From the equation (2.8) of Ricci-tensor and by the assumption, $K_{ij} = kh_{ij}$ for some constant k. So $K_{ij}h_{ijk} = 0$ by the apolarity condition. Hence $p = 0$ because F is constant by the affine homogeneity.

EXAMPLE 4.6. Let V be a non-degenerate convex cone in A^{n+1} and let χ denote the characteristic function of V defined by $\chi(x) = \int_{V^*} e^{-\langle x,\xi\rangle} d\xi$, where V^* is the dual cone of V and $\langle , \rangle$ is the pairing ([13]). If V is affinely homogeneous, then the hypersurface $\{\chi = 1\}$ is an affine hypersphere of hyperbolic type ([10]).

(1) When V is the quadrant cone $\left\{x^1 > 0, \ldots, x^{n+1} > 0\right\}$, then $\chi = (x^1, \ldots, x^{n+1})^{-1}$ and the abelian group $\mathbf{R}^n$ acts on the space $\{\chi = 1\}$ by $(x^1, \ldots, x^{n+1}) \longrightarrow (e^{-a_1}x^1, \ldots, e^{-a_n}x^n, e^{\Sigma a_i}x^{n+1})$ transitively. So the affine

metric is flat and Einstein. Hence it is projectively minimal by Theorem 4.5.

(2) When V is an irreducible self-dual cone, i.e. one of the circular cone and the cone of positive-definite hermitian symmetric matrices $H^{+}(n,\mathbf{K})$ for $\mathbf{K} = \mathbf{R}, \mathbf{C}, \mathbf{H}$ or $Ca(n = 3)$, it is known that the space $\{\chi = 1\}$ is a symmetric space ([10]). Hence it is projectively minimal.

QUESTION 4.7. The part (2) of Theorem 4.2. can be thought of a solution of the Bernstein problem in the projective case. It is an interesting problem to see whether this can be generalized also in higher dimensional case.

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