Quasi-isometric maps and Floyd boundaries of relatively hyperbolic groups.

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Abstract

We describe the kernel of the canonical map from the Floyd boundary of a relatively hyperbolic group to its Bowditch boundary.

Using these methods we then prove that a finitely generated group H admitting a quasi-isometric map φ into a relatively hyperbolic group G is relatively hyperbolic with respect to a system of subgroups whose image under φ is situated within a uniformly bounded distance from the parabolic subgroups of G.

1 Introduction.

We study the actions of discrete groups by homeomorphisms of compact Hausdorff spaces such that

(a) the induced action on the space of triples of distinct points is properly discontinuous, and
(b) the induced action on the space of pairs of distinct points is cocompact.

Let T be a compact Hausdorff topological space (compactum). Denote by $\Theta^n T$ the space of subsets of cardinality n of T endowed with the natural product topology.

Recall that an action of a group G by homeomorphisms on T is called *convergence* if it has the property (a) above: the induced action on $\Theta^3 T$ is properly discontinuous [Bo2], [Tu2]. We also say in this case that the action of G on T is *3-discontinuous.*

An action of G on T is called *cocompact on pairs* or *2-cocompact* if $\Theta^2 T/G$ is compact.

It is shown in [Ge1] that an action with the properties (a), (b) is geometrically finite that is every limit point is either conical or bounded parabolic. From the other hand it follows from [Tu3, Theorem 1.C] that any minimal geometrically finite action on a metrizable compactum has the property (b).

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An action of G on T is said to be *parabolic* if it has a unique fixed point. The existence of a non-parabolic geometrically finite action of a finitely generated group G on some compactum T is equivalent to the relative hyperbolicity of G with respect to proper subgroups [Bo1], [Ya].

So the conditions (a) and (b) above provide a topological characterization of relatively hyperbolic groups which we adopt as a definition of the relative hyperbolicity.

W. Floyd [F] introduced the notion of a boundary of a finitely generated group as follows. The word metric of the Cayley graph of G is scaled by a "conformal factor" f (see the next Section). The Cauchy completion of the space with the scaled metric is called *Floyd completion* and is denoted by $\overline{G}_f$. The *Floyd boundary* is the subspace $\partial_f G = \overline{G}_f \setminus G$. The action of G on itself by left multiplication extends to a convergence action of G on $\overline{G}_f$ [Ka].

It is shown in [F] that for any geometrically finite discrete subgroup $G < \mathrm{Isom}\mathbb{H}^3$ of the isometry group of the hyperbolic space $\mathbb{H}^3$, and for a quadratic scaling function f, there exists a continuous G-equivariant map F from $\partial_f G$ to the limit set $T = \Lambda(G)$ (*Floyd map*). The preimage $F^{-1}(p)$ of a limit point p is not a single point if and only if p is a parabolic point of rank 1 in which case it is a pair of points [F]. P. Tukia generalized Floyd's Theorem to geometrically finite discrete subgroups of $\mathrm{Isom}\mathbb{H}^n$ [Tu1].

If an action of a finitely generated group G on a compactum T has the properties (a), (b) then, for an exponential function f (and hence for every polynomial one), there exists a continuous equivariant map $F : \partial_f G \to T$ [Ge2].

For a subset H of G denote by ∂H its topological boundary in the space $G \cup \partial_f G$. Let $\mathrm{Stab}_G p$ denote the stabilizer of a point $p \in T$ in G. Since the action of G on the Floyd completion $\overline{G}_f$ has the convergence property, the boundary $\partial\mathrm{Stab}_G p$ coincides with the limit set $\Lambda(\mathrm{Stab}_G p)$ for the action of $\mathrm{Stab}_G p$ on $\overline{G}_f$.

Our first result describes the kernel of the map F:

Theorem A. *Let G be a finitely generated group acting on a compactum T 3-discontinuously and 2-cocompactly. Let $F : \partial_f G \to T$ be a G-equivariant continuous map. Then*

$$F^{-1}(p) = \partial(\mathrm{Stab}_G p) \tag{1}$$

for any parabolic point $p \in T$. Furthermore, $F(a) = F(b) = p$ if and only if either $a = b$ or p is parabolic.

□

Note that the subgroup inclusion does not necessarily induce an embedding of Floyd boundaries so we have:

Question 1.1. *Let a finitely generated group G act 3-discontinuously and 2-cocompactly on a compactum T. Let $F : \partial_f G \to T$ be a continuous G-equivariant map. For a parabolic point $p \in T$ is it true that*

$$\partial(\mathrm{Stab}_G p) = \partial_{f_1}\mathrm{Stab}_G p$$

for some scaling function f_1?

□

Our next result describes the quasi-isometric (large-scale Lipschitz in the sense of Gromov) maps into relatively hyperbolic groups:

Theorem B. *Let a finitely generated group G act 3-discontinuously and 2-cocompactly on a compactum T. Let $\varphi : H \to G$ be a quasi-isometric map of a finitely generated group H.*

Then there exist a compactum S, a 3-discontinuous 2-cocompact action of H on S, and a continuous map $\varphi_* : S \to T$ such that for every H-parabolic point $p \in S$ the point $\varphi_* p$ is G-parabolic, and $\varphi(\mathsf{Stab}_H p)$ is contained in a uniformly bounded neighborhood of $\mathsf{Stab}_G(\varphi_* p)$.

Using known facts about the relative hyperbolicity Theorem B can be reformulated as follows:

Corollary 1.2. *Let G be a finitely generated relatively hyperbolic group with respect to a collection of subgroups P_j $(j = 1, ..., n)$. Let H be a finitely generated group and let $\varphi : H \to G$ be a quasi-isometric map. Then H is relatively hyperbolic with respect to a collection Q_i such that φ maps each Q_i into a uniform neighborhood of a conjugate of some P_j (the case $Q_i = H$ is allowed).*

The following particular cases of Corollary 1.2 are already known:

1) when the map $\varphi : H \to G$ admits a quasi-isometric inverse map $\psi : G \to H$ such that $d(\mathrm{id_H}, \psi \circ \varphi) \leq \mathrm{const}$ [Dr, Theorem 1.2].

2) when the group H is not relatively hyperbolic with respect to proper subgroups (in this case $\varphi(H)$ is contained in a bounded neighborhood of a conjugate of some P_i) [BDM, Theorem 4.1].

The proof of Theorem B does not use [Dr] and [BDM] and considers directly the general case.

We now outline the content of the paper. In Section 2, we provide some preliminaries on the convergence actions and the Floyd completions. We prove here several technical lemmas.

In Section 3 we study the compactified space $\widetilde{T} = G \sqcup T$. We introduce here the notion (borrowed from $\mathbb{H}^n$) of a *convex hull* of a subset of $\widetilde{T}$. Then we prove (see the Main Lemma) that the boundary of any set is equal to that of its convex hull. This fact implies several properties of *horospheres*, *horocycles* and the stabilizers of parabolic points. The section 3 ends with the proof of Theorem A.

In Section 4 we prove Theorem B. Using the Floyd map F and the quasi-isometric map φ we construct the space S as a quotient of the Floyd boundary of H determined by the kernel of the map $F \circ \varphi$. The group H acts 3-discontinuously on S, and we show (Proposition 4.1) that the action is 2-cocompact. The construction of S yields a continuous map $\varphi_* : S \to T$. Using the results of Section 3 we prove the last part of the statement.

As an application of our methods we give in Section 5 a short proof of the fact that the existence of a 3-discontinuous and 3-cocompact action on a compactum without isolated points implies that the group is word-hyperbolic [Bo3].

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2 Preliminaries.

2.1 Convergence actions.

By *compactum* we mean a compact Hausdorff space. Let $\mathsf{S}^n T$ denote the quotient of the product $T^n = \underbrace{T\times\ldots\times T}_{n \text{ times}}$ by the action of the permutation group on n symbols. The elements of $\mathsf{S}^n T$ are the generalized non-ordered n-tuples. We endow $\mathsf{S}^n T$ with the quotient topology inherited from T^n. Let $\Theta^n T$ be the open subset of $\mathsf{S}^n T$ consisting of the non-ordered n-tuples whose components are distinct. Put $\boldsymbol{\Delta}^n T = \mathsf{S}^n T \setminus \Theta^n T$. So $\boldsymbol{\Delta}^2 T$ is the image of the diagonal of T^2.

Convention. If the opposite is not stated all group actions on compacta are assumed to have the convergence property. We will also assume that $|T| > 2$.

Recall a few common definitions (see e.g. [Bo2], [GM], [Fr], [Tu2]). The *discontinuity domain* $\Omega(G)$ is the set of the points of T where G acts properly discontinuously. The set $\Lambda(G) = T\setminus\Omega(G)$ is the *limit set* and the points of $\Lambda(G)$ are called *limit points.* A convergence action of G on T is called *minimal* if $\Lambda(G) = T$.

It is known that $|\Lambda(G)| \in \{0, 1, 2, \mathfrak{c}\}$ [Tu2]. An action (or group) is called *elementary* if its limit set is finite.

A point $p \in T$ is called parabolic if $|\Lambda(\mathrm{Stab}_G p)| = 1$.

A limit point $x \in \Lambda(G)$ is called *conical* if there exists an infinite sequence of distinct elements $g_n \in G$ and distinct points $a, b \in T$ such that

$$\forall y \in T \setminus \{x\} \ : \ g_n(y) \to a \in T \ \wedge g_n(x) \to b.$$

Denote by $\mathsf{Nc}T$ the set of non-conical points of T.

A parabolic point $p \in \Lambda(G)$ is called *bounded parabolic* if the quotient space $(\Lambda(G)\setminus\{p\})/\mathrm{Stab}_G p$ is compact.

An action of G on T is called *geometrically finite* if every non-conical limit point is bounded parabolic.

A subset N of the set M acted upon by G is called *G-finite* if its image in M/G is finite.

Lemma 2.1. *[Ge1, Main Theorem] If the action of G on T is 3-discontinuous and 2-cocompact then*

a. The set $\mathsf{Nc}T$ is G-finite.

b. For every $p \in \mathsf{Nc}T$ the quotient $(T \setminus p)/\mathrm{Stab_p G}$ is compact. □

It follows from Lemma 2.1 that for a 2-cocompact action of G on T a non-conical point $p \in \mathsf{Nc}T$ is isolated in T if and only if its stabilizer $\mathrm{Stab}_p G$ is finite. Hence, a non-conical point with infinite stabilizer is bounded parabolic.

2.2 Quasigeodesics and Floyd completion of graphs.

Recall that a *(c-)quasi-isometric map* $\varphi : X \to Y$ between two metric spaces X and Y is a map such that :

$$\frac{1}{c}d_X(x,y) - c < d_Y(\varphi(x),\varphi(y)) \le cd_X(x,y) + c, \tag{2}$$

$d_X,\ d_Y$ denote the metrics of X and Y respectively.

Remark. A quasi-isometric map can in general be multivalued. This more general case can be easily reduced to the case of a one-valued map. The notion of a quasi-isometric map coincides with the definitions of *large-scale Lipschitz map* [Gr1] or *quasi-isometric embedding* [BH].

A *path* is a distance-nonincreasing map $\gamma : I \to \Gamma$ from a nonempty convex subset I of $\mathbb{Z}$. The *length* of γ is the diameter of I in $\mathbb{Z}$. A *subpath* is a path which is a restriction of γ.

A path $\gamma : I \to \Gamma$ is called *c-quasigeodesic* if it is a c-quasi-isometric map. In the case when γ is an isometry, a quasigeodesic is a *geodesic*.

A (c-quasi-)geodesic path $\gamma : I \to \Gamma$ defined on a half-infinite subset I of $\mathbb{Z}$ is called *(c-quasi-)geodesic ray*; a (quasi-)geodesic path defined on the whole $\mathbb{Z}$ is called *(c-quasi-)geodesic line.*

Let $d(,)$ be the canonical shortest path distance function on Γ. We denote by $\mathsf{N}_D M$ the D-neighborhood of a set $M\subset\Gamma$.

We now briefly recall the construction of the Floyd completion of a graph Γ due to W. Floyd [F]. Let Γ be a locally finite connected graph endowed with a basepoint $v \in \Gamma^0$. Let $f : \mathbb{Z}_{>0} \to \mathbb{R}_{>0}$ be a function satisfying the following conditions:

$$\exists\ K > 0\ \forall n \in \mathbb{N}\ :\ 1 \le \tfrac{f(n)}{f(n+1)} \le K \tag{3}$$

$$\sum_{n\in\mathbb{N}} f(n) < \infty. \tag{4}$$

For convenience we extend the function f to $\mathbb{Z}_{\ge 0}$ by putting $f(0) := f(1)$.

Define the Floyd length of an edge joining vertices x and y as $f(n)$ where $n = d(v, \{x, y\})$. Then the length $L_{f,v}$ of a path is the sum of the lengths of its edges. The Floyd distance $\delta_v = \delta_{f,v}(a, b)$ is the shortest path distance:

$$\delta_v(a,b) = \inf_{\alpha} L_{f,v}(\alpha), \tag{5}$$

where the infimum is taken over all paths between a and b.

It follows from (3) that every two metrics δ_{v_1} and δ_{v_2} are bilipschitz equivalent with a Lipschitz constant depending on $d(v_1, v_2)$. The Cauchy completion $\overline{\Gamma}_f$ of the metric space (Γ^0, δ_v) is called *Floyd completion.* It is compact and does not depend on the choice of the basepoint v. Denote by $\partial_f\Gamma$ the set $\overline{\Gamma}_f \setminus \Gamma$ and call it *Floyd boundary.* The distance δ_v extends naturally to $\overline{\Gamma}_f$.

The following Lemma shows that the Floyd length of a far quasigeodesic is small.

Lemma 2.2. (Karlsson Lemma) *For every $\varepsilon > 0$ and every $c > 0$, there exists a finite set $D \subset \Gamma$ such that δ_v-length of every c-quasigeodesic $\gamma \subset \Gamma$ that does not meet D is less than ε.* □

Remark. A. Karlsson [Ka] proved it for geodesics in the Cayley graphs of finitely generated groups. The proof of [Ka] does not use the group action and is also valid for quasigeodesics.

Consider a set S of paths of the form $\alpha : [0, n[\to \Gamma$ with unbounded length starting at the same point $a = \alpha(0)$. Every path $\alpha \in S$ can be considered as an element of the product $\prod_{i\in I} \mathsf{N}_i(a)$. The space $\prod_{i\in I} \mathsf{N}_i(a)$ is compact in the Tikhonov topology. It is a common fact that S possesses an infinite "limit path" $\delta : [0, +\infty) \to \Gamma$ whose initial segments are initial segments of paths in S. Note that the infinite limit path exists in a more general case when $\gamma(0)$ is not a point but a fixed finite set.

Definition 2.3. For a c-quasigeodesic ray $r : [0, \infty[\to \Gamma$ we say that r *converges to a point* in $\partial_f\Gamma$ if the sequence $(r(n))_n$ is a Cauchy sequence for the δ_f-metric. We also say in this case that r joins the points $r(0)$ and $x = \lim_{n\to\infty} r(n) \in \partial_f\Gamma$.

Proposition 2.4. *Let Γ be a locally finite connected graph. Then*

a. *For each $c > 0$ every c-quasigeodesic ray in Γ converges to a point in $\partial_f\Gamma$.*

b. *For every $p\in\partial_f\Gamma$ and every $a \in \Gamma$ there exists a geodesic ray joining a and p.*

c. *Every two distinct points in $\partial_f\Gamma$ can be joined by a geodesic line.*

Proof. a: Let $r : [0, \infty[\to \Gamma$ be a c-quasigeodesic ray. Put $x_n = r(n)$ and $r_n = r([n, \infty[)$. For any vertex $v \in \Gamma^0$ we have $d(v, r(n)) \to \infty$. It follows from Karlsson Lemma that $L_{f,v}(r_n) \to 0$.

b: Let $B_f(p, R)$ denote the ball in the Floyd metric at $p \in \partial_f\Gamma$ of radius R. For $n\geqslant 1$, choose $a_n \in B_f(p, \frac{1}{n})$ and join a with a_n by a geodesic segment γ_n. Let γ be the limit path for the family $S = \{\gamma_n : n>0\}$. By (a) γ converges to a point $q\in\partial_f\Gamma$. If $p \neq q$ set $3\delta = \delta_1(p, q) > 0$. Let n be an integer for which $L_{f,1}(\gamma|_{[n,\infty[}) \leq \delta$. For $m \geqslant n$ we can choose k such that $\gamma_k|_{[0,m]} = \gamma|_{[0,m]}$ and $\delta_1(a_k, p)\leqslant\delta$. So $L_{f,1}(\gamma_k|_{[m,k]}) \geq \delta$. Since the distance $d(1, a_n)$ is unbounded, by Karlsson Lemma the quantity $L_{f,1}(\gamma_k|_{[m,k]})$ should tend to zero. This contradiction shows that $p = q$.

c: Let $p, q\in\partial_f\Gamma$ and $p\neq q$. By (b) there exist geodesic rays $\alpha, \beta : [0, \infty[\to \Gamma$ such that $\alpha(0) = \beta(0) = a$ and $\alpha(\infty) = p$, $\beta(\infty) = q$. Let $3\delta = \delta_1(p, q)$. By Karlsson Lemma every geodesic segment joining a point in $B_f(p, \delta)$ with a point in $B_f(q, \delta)$ intersects a finite set $B \subset \Gamma$. There exists an infinite sequence of geodesic segments γ_n passing through a point $b \in B$ whose endpoints converge to the pair $\{p, q\}$. A limit path for such sequence is a geodesic line in question. □

Let Γ_i $(i = 1, 2)$ be locally finite connected graphs with basepoints $1 \in \Gamma_1, 1 \in \Gamma_2$. Denote by $\overline{\Gamma}_i$ their completions with respect to the scaling functions f_1 and f_2 satisfying (3) with constants E and K respectively. The following Lemma gives a sufficient condition to extend a quasi-isometric map between the completions of the graphs.

Lemma 2.5. *Let $\varphi : \Gamma_1 \to \Gamma_2$ be a c-quasi-isometric map for some $c \in \mathbb{N}$. Suppose that there exists a constant $D > 0$ such that*

$$\frac{f_2(n)}{f_1(cn)} < D \ (n \in \mathbb{N}) \tag{6),}$$

Then the map φ extends to a uniformly continuous map $\overline{\Gamma}_1 \to \overline{\Gamma}_2$.

Proof. Denote by d and δ the canonical distances and the Floyd distances with respect to the chosen basepoints in both graphs Γ_i $(i = 1, 2)$. We first prove that φ is a Lipschitz map with respect to the Floyd metric, i.e.

$$\forall\, x, y \in \Gamma_1 \ : \ \delta(x, y) \geq \varepsilon\delta(\varphi x, \varphi y) \tag{7}$$

for some $\varepsilon > 0$.

It suffices to prove the statement for the case when $d(x, y) = 1$.

By (2) we have $d(\varphi x, \varphi y) < cd(x, y) + c = 2c$. Let $\gamma : [0, n] \to \Gamma_2$ be a geodesic realizing the distance $d(\varphi x, \varphi y)$, and let $a_i = \gamma(i)$ $(i = 0, ..., n)$ be its vertices where $a_0 = \varphi x, a_n = \varphi y$. We have $\delta(x, y) = f_1(d(1, \{x, y\}))$. Assume that $d(1, \{x, y\}) = d(1, x)$. Then

$$d(1, a_i) \geq d(\varphi(1), \varphi x) - d(\varphi x, a_i) - d(1, \varphi(1)) \geq d(\varphi(1), \varphi x) - 2c - d(1, \varphi(1)) = d(\varphi(1), \varphi x) - n_0,$$

where $n_0 = 2c + d(1, \varphi(1))$.

Assume that $x \in \Gamma_1 \setminus B(1, r_0)$ where $B(1, r_0)$ is the ball centered at 1 of a radius r_0 such that $d(1, \varphi(x)) > n_0$ and $d(1, x) > c^2$. Then using the monotonicity of f_2 and condition (3) we obtain

$$\delta(\varphi x, \varphi y) = \sum_{i=0}^{n-1} f_2(d(1, \{a_i, a_{i+1}\})) \leq \sum_{i=0}^{n-1} f_2(d(\varphi(1), \varphi x) - n_0) \leq K^{n_0} f_2(d(\varphi(1), \varphi(x))).$$

The last term can be estimated using (2) and (6):

$$f_2(d(\varphi(1), \varphi(x)) \leq Df_1(c(d(\varphi(1), \varphi(x))) \leq Df_1(c \cdot d(1, x)/c - c^2) \leq D \cdot E^{c^2} \cdot f_1(d(1, x)).$$

Summing all up we conclude

$$\delta(\varphi x, \varphi y) \leq DK^{n_0}E^{c^2} \cdot f_1(d(1, x)).$$

So (7) is true for the constant $\varepsilon = (DK^{n_0}E^{c^2})^{-1}$ outside of the ball $B(1, r_0)$. By decreasing the constant ε we obtain the inequality (7) everywhere on Γ_1.

The map $\varphi : (\Gamma_1, \delta) \to (\Gamma_2, \delta)$ being Lipschitz extends to an equicontinuous map $\overline{\Gamma}_1 \to \overline{\Gamma}_2$. □

Remark. If, for a function f the value $\frac{f(n)}{f(2n)}$ is bounded from above then one can take the same scaling function $f_1 = f_2 = f$ for both graphs Γ_1 and Γ_2 independently of c.

If the scaling function for the graph Γ_2 is $f_2(n) = \alpha^n$ $(\alpha \in]0, 1[)$ then to satisfy (6) we can take $f_1(n) = \beta^n$ as the scaling function for the group H where $\beta = \alpha^{1/c}$.

Let G be a finitely generated group and let S be a finite generating set for G. Denote by d the word metric. Let $\overline{G}_f$ denote the Floyd completion $G \sqcup \partial_f G$ of the Cayley graph of G with respect to S corresponding to a function f satisfying (3-4). Condition (3) implies equicontinuity of the G-action by left multiplication on G, so it extends to a G-action on $\overline{G}_f$ by homeomorphisms. The Floyd metric δ_g is the g-shift of δ_1 (where 1 is the identity element of G):

$$\delta_g(x, y) = \delta_1(g^{-1}x, g^{-1}y), \quad x, y \in \overline{G}_f, g \in G.$$

On the space $\overline{G}_f$ we also consider the following *shortcut* pseudometrics. Let ω be a closed G-invariant equivalence relation on $\overline{G}_f$. Then there is an induced G-action on the quotient space $\overline{G}_f/\omega$. A shortcut pseudometric $\overline{\delta}_g$ is the maximal element in the set of symmetric functions $\varrho : \overline{G}_f \times \overline{G}_f \to \mathbb{R}_{\geqslant 0}$ that vanish on ω and satisfy the triangle inequality, and the inequality $\varrho \leqslant \delta_g$.

For $p, q \in \overline{G}_f$ the value $\overline{\delta}_g(p, q)$ is the infimum of the finite sums $\sum_{i=1}^{n} \delta_g(p_i, q_i)$ such that $p = p_1$, $q = q_n$ and $\langle q_i, p_{i+1} \rangle \in \omega$ $(i = 1, \ldots, n-1)$ [BBI, pp 77]. Obviously, the shortcut pseudometric $\overline{\delta}_g$ is the g-shift of $\overline{\delta}_1$. The metrics $\overline{\delta}_{g_1}$, $\overline{\delta}_{g_2}$ are bilipschitz equivalent with the same constant as for δ_{g_1}, δ_{g_2}. Furthermore, the pseudometric $\overline{\delta}_g$ induces a pseudometric on the quotient space $\overline{G}_f/\omega$. We denote this induced pseudometric by the same symbol $\overline{\delta}_g$.

Lemma 2.6. *[Ge2] Let G be a finitely generated group acting 3-discontinuously and 2-cocompactly on a compactum T ($|T| > 2$). There exists a continuous G-equivariant map $F : \partial_f G \to T$, where $f(n) = \alpha^n$ for some $\alpha \in]0, 1[$ sufficiently close to 1. Furthermore $\Lambda(G) = F(\partial_f G)$.* □

The map F given by Lemma 2.6 is called *Floyd map.*

3 The orbit compactification space $\widetilde{T}$ and its convex subsets.

In this Section we fix a 3-discontinuous and 2-cocompact action by homeomorphisms of a finitely generated group G on a compactum T containing at least 3 points.

3.1 The space $\widetilde{T}$.

Let $F : \partial_f G \to T$ be the Floyd map. We extend F over $\overline{G}_f = G \sqcup \partial_f G$ to the disjoint union $\widetilde{\widetilde{T}} = T \sqcup G$ by the identity map $\mathrm{id} : G \to G$. We keep the notation F for this extension. The maps $T \xrightarrow{\mathrm{id}} \widetilde{\widetilde{T}} \xleftarrow{F} \overline{G}_f$ determine on $\widetilde{\widetilde{T}}$ the pushout topology: a set $S \subset \widetilde{\widetilde{T}}$ is open if and only if $S \cap T$ is open in T and $F^{-1}S$ is open in $\overline{G}_f$. The space $\widetilde{\widetilde{T}}$ being the union of two compact spaces T and $F(\overline{G}_f)$ is a compactum.

By Lemma 2.1 the set $T \setminus \Lambda(G)$ is G-finite. Denote by $\widetilde{T}$ the subspace $\Lambda(G) \sqcup G$ of $\widetilde{\widetilde{T}}$.

Remark. We need to introduce $\widetilde{\widetilde{T}}$ before $\widetilde{T}$ in order to include the exceptional case of 2-ended groups. In this case $\Lambda(G)$ consists of 2 points and we need at least one more point to apply Lemma 2.6.

Lemma 3.1. *The induced action G on $\widetilde{T}$ is 3-discontinuous and 2-cocompact.*

We start with the following Proposition:

Proposition 3.2. *Let G act on compacta X and Y and let $\psi : X \to Y$ be a G-equivariant continuous surjective map. If the action of G on X is 3-discontinuous, then the action of G on Y is 3-discontinuous.*

Proof of the Proposition. The map ψ induces a proper G-equivariant continuous surjective map $\mathsf{S}^3X \to \mathsf{S}^3Y$. Let K and L be compact subsets of Θ^3Y. Since Y is Hausdorff the preimage of every compact in $\Theta^3Y \subset \mathsf{S}^3Y$ is compact in Θ^3X. Thus $K_1 = \psi^{-1}(K)$ and $L_1 = \psi^{-1}(L)$ are compact subsets of Θ^3X. The action on X is discontinuous so the set $\{g \in G \mid gK_1 \cap L_1 \neq \emptyset\}$ is finite. By the equivariance of ψ the set $\{g \in G \mid gK \cap L \neq \emptyset\}$ is finite too. □

Proof of the Lemma. By [Ka] the group G acts 3-discontinuously on $\overline{G}_f = G \sqcup \partial_f G$. The Floyd map $F : \overline{G}_f \to \widetilde{T}$ is G-equivariant and continuous. So Proposition 3.2 implies that the action on $\widetilde{T}$ is 3-discontinuous.

If K is a compact fundamental set for the action of G on $\Theta^2(T)$ then $K_1 = K \cup \{1\} \times (\widetilde{T} \setminus \{1\})$ is a compact fundamental set for the action of G on $\Theta^2\widetilde{T}$. □

Let ω be the kernel of the Floyd map $F : \overline{G}_f \to \widetilde{T}$, i.e. $(x, y) \in \omega$ if and only if $F(x) = F(y)$. It determines the shortcut metrics $\overline{\delta}_g$ $(g \in G)$ (see Subsection 2.2). It is shown in [Ge2] that every $\overline{\delta}_g$ is a metric on $\widetilde{T}$, i.e.

$$\forall p, q \in \widetilde{T} \ : \ \overline{\delta}_g(p, q) = 0 \Longrightarrow \ p = q. \tag{8}$$

Moreover, F transfers the shortcut pseudometric on $\partial_f G$ into the shortcut metric on T isometrically:

$$\forall x, y \in \partial_f G \quad : \ \overline{\delta}_g(x, y) = \overline{\delta}_g(F(x), F(y)). \tag{8'}$$

Any metric $\overline{\delta}_g$ determines the topology of $\widetilde{T}$.

Lemma 3.3. *Let H be the stabilizer of a parabolic point p. Every H-invariant set $M \subset G$ closed in $\widetilde{T} \setminus \{p\}$ is H-finite.*

Proof. By Lemma 3.1 the action of G on $\widetilde{T}$ is 3-discontinuous and 2-cocompact, so by Lemma 2.1 the space $(\widetilde{T} \setminus \{p\})/H$ is compact. Since $G \subset \widetilde{T}$ is an orbit of isolated points, the closed subset M/H of $(\widetilde{T} \setminus \{p\})/H$ consists of isolated points. Since $(T \setminus p)/H$ is compact the set M/H is finite. □

3.2 Horocycles and horospheres

By Proposition 2.4.1 every c-quasigeodesic ray $\gamma : \mathbb{N} \to G$ converges to a point $p \in X$. We call the point p *target* of γ and denote it by $\gamma(\infty)$. The path $F \circ \gamma$ converges to $F(p) \in \widetilde{T}$ which we also call *target*. In other words a c-quasigeodesic ray extends to a continuous map from $\overline{\mathbb{N}} = \mathbb{N} \cup \{+\infty\}$ to $\widetilde{T}$. The target is necessarily a limit point for the action $G \curvearrowright \widetilde{T}$.

A bi-infinite c-quasigeodesic $\gamma : \mathbb{Z} \to G$ extends to a continuous map of $\overline{\mathbb{Z}} = \mathbb{Z} \cup \{-\infty, +\infty\}$ with $\gamma(\pm\infty) \subset T$. So $\gamma(\pm\infty)$ is either a pair of limit points or a single limit point.

Definition 3.4. A bi-infinite c-quasigeodesic $\gamma : \mathbb{Z} \to G$ is called *c-horocycle* at $p \in T$ if $\gamma(+\infty) = \gamma(-\infty) = p$.

Definition 3.5. The *c-hull* $\mathsf{H}_c M$ of a set $M \subset \widetilde{T}$ is the union of all c-quasigeodesics (finite or infinite) having the endpoints in M:

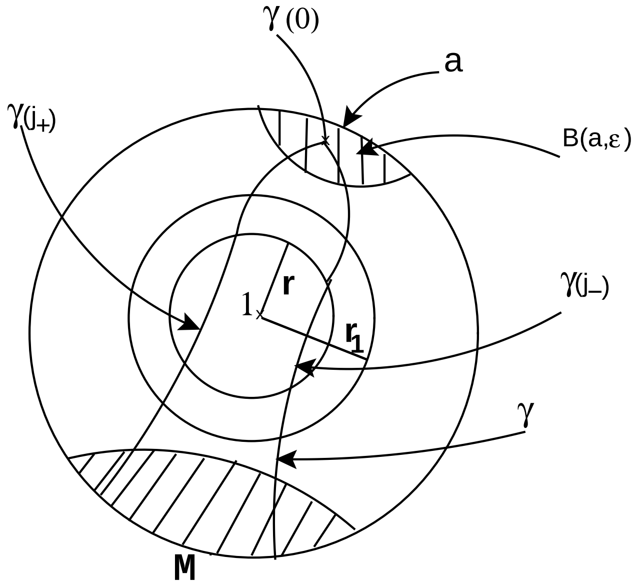
$$\mathsf{H}_c M = \cup\{\gamma(I) \mid \gamma : I \to G \text{ is a c} - \text{quasigeodesic}, \ I \subset \mathbb{Z}, \text{ and } \gamma(\partial I) \subset M\}.$$

The c-hull $\mathsf{H}_c p$ of a single-point set $\{p\} \in T$ is called *c-horosphere at p.*

By $\overline{M}$ we denote the closure of M in $\widetilde{T}$.

Main Lemma. *$T \cap \overline{M} = T \cap \overline{\mathsf{H}_c M}$ for every $M \subset \widetilde{T}$ and $c > 0$.*

Proof. Suppose by contradiction that there exists a counterexample $\langle M, c\rangle$ and let
$a \in T \cap \overline{\mathsf{H}_\alpha M} \setminus \overline{M}$. By Karlsson Lemma there exists r such that the δ_1-length of every c-quasigeodesic outside of the ball $\mathsf{N}_r(1) \subset G$ is less than $\varepsilon = \overline{\delta}_1(\overline{M}, a)/2 > 0$.



By the assumption there exists a c-quasigeodesic $\gamma : I = [i_-, i_+] \to G$ such that $0 \in I$, $\gamma(\partial I) \subset M$, and $\gamma(0)$ is arbitrarily close to a. So we can assume that $\overline{\delta}_1(a, \gamma(0)) < \varepsilon$ and $\gamma(0) \notin N_{r_1}(1)$ where $r_1 = r + cr + \frac{c}{2}$. Let $\gamma_+ = \gamma|_{I\cap\mathbb{N}}$, $\gamma_- = \gamma|_{I\cap(-\mathbb{N})}$. We have

$$L_1(\gamma_\pm) \geq \overline{\delta}_1(\gamma(i_\pm), \gamma(0)) \geq \overline{\delta}_1(\gamma(i_\pm), a) - \overline{\delta}_1(a, \gamma(0)) \geq \overline{\delta}_1(\overline{M}, a) - \overline{\delta}_1(a, \gamma(0)) \geq 2\varepsilon - \varepsilon = \varepsilon.$$

So there exists a subseqment $J = [j_-, j_+]$ of I such that $0 \in J$ and $d(1, \gamma(j_\pm)) \leqslant r$.

We obtain $d(\gamma(j_-), \gamma(j_+)) = \mathsf{diam}(\gamma(\partial J)) \leqslant 2r$ and $\mathsf{length}(\gamma|_J) \leqslant c(2r + 1)$. Thus $d(1, \gamma(0)) \leq d(1, \gamma(\partial J)) + d(\gamma(0), \gamma(\partial J)) \leq r_1 = r + \frac{c}{2}(2r + 1)$. So $\gamma(0) \in N_{r_1}(1)$. A contradiction. □

We finish the subsection by obtaining several corollaries of the Main Lemma.

Lemma 3.6. *There is no c-horocycle at conical points.*

Proof. Let $p \in T$ be a conical point. There exist distinct points $a, q \in T$ and a sequence $(g_n) \subset G$ such that $g_n(p) \to q$ and $g_n(x) \to a$ for all $x \in T \setminus \{p\}$. Suppose by contradiction that γ is an c-horocycle at p.

Let Q be a closed neighborhood of q such that $a \notin Q$. We can assume that $q_n = g_n p \in Q$ for all n. So $g_n\gamma(0) \in \mathsf{H}_c Q$. By Lemma 3.1 the action $G \curvearrowright \widetilde{T}$ has the convergence property. So $g_n\gamma(0) \to a$. It follows that $a \in (\overline{\mathsf{H}_c Q} \cap T) \setminus (Q \cap T)$ contradicting to the Main Lemma. □

Until the end of this Subsection we fix a parabolic fixed point p, and denote by H the stabilizer of p in G.

Lemma 3.7. *For every c the set $(G \cap \mathsf{H}_c p)/H$ is finite.*

Proof. The set $\mathsf{H}_c p$ is H-invariant. By the Main Lemma it is closed in $\widetilde{T}$ and p is its unique limit point. Thus the set $G \cap \mathsf{H}_c p$ is a closed H-invariant subset of $\widetilde{T}\backslash p$. The result follows from Lemma 3.3. □

Lemma 3.8. *The closure in $\widetilde{T}$ of any H-finite subset M of G is $M \cup \{p\}$.*

Proof. It suffices to consider the case when M is an H-orbit. As $d(M, \mathsf{H}_c p)$ is bounded, the Floyd distance $\delta_1(m, \mathsf{H}_c p)$ tends to zero while $m \in M$ tends to T. □

Corollary 3.9. *There exists a constant C_0 such that the stabilizer of every parabolic point is C_0-quasiconvex.*

Proof. Let $\mathsf{H}_c p$ be a c-horosphere at p. By Lemmas 3.7 and 3.8 the set $M = \mathsf{H}_c p \cap G$ is H-finite and $\overline{M} = M \cup \{p\}$. By the Main Lemma $\overline{\mathsf{H}_c M} = \mathsf{H}_c M \cup \{p\}$. So $\mathsf{H}_c M$ is closed in $\widetilde{T} \setminus \{p\}$ and by Lemma 3.3 it is also H-finite.

Let $\gamma : I \to G$ be a geodesic segment with endpoints in M. Then γ and M are both subsets of the H-finite set $\mathsf{H}_c M$. Hence for any $a \in \gamma(I)$ there exist $h_i \in H$ $(i = 1, 2)$ and $b \in M$ such that $d(h_1(a), h_2(b)) \leq \text{const}$. Since M is H-invariant we have $h_1^{-1}h_2(b) \in M$ and so $d(a, M) \leq \text{const}$. Thus H is quasiconvex.

By Lemma 2.1 the set of parabolic points is G-finite so there exists a uniform constant C_0 such that every stabilizer of a parabolic point is C_0-quasiconvex. □

3.3 The kernel of the Floyd map.

Theorem A. *Let G be a finitely generated group acting on a compactum T 3-discontinuously and 2-cocompactly. Let $F : \partial_f G \to T$ be a G-equivariant continuous map. Then*

$$F^{-1}(p) = \partial(\mathrm{Stab}_G p) \qquad (1)$$

for any parabolic point $p \in T$. Furthermore, $F(a) = F(b) = p$ if and only if either $a = b$ or p is parabolic. □

Proof. Denote $H = \mathrm{Stab}_G p$. Let $x \in F^{-1}(p)$. We will show that $x \in \partial H$. Let $y \in \partial H$. If $y = x$ then there is nothing to prove. If not, then by Proposition 2.4.c there exists a bi-infinite geodesic γ joining x and y. It is a horocycle in $\widetilde{T}$, so $\gamma(\mathbb{Z}) \subset \mathsf{H}_c p$. By Lemma 3.7, $\gamma(\mathbb{Z})$ is contained in $Hg_1 \cup \ldots \cup Hg_l$. By Lemma 3.8 the boundary of each H-coset is $\{p\}$. So $x = \lim\limits_{n\to\infty} g_i h_n$ where $i \in \{1, ..., l\}$ and $h_n \in H$. It follows that $\delta_1(x, h_n) \to 0$ and $x \in \partial H \subset \partial_f G$.

Assume that $a \neq b$. Then as above join a and b by a bi-infinite geodesic γ. Then γ is an horocycle in $\widetilde{T}$, and by Lemma 3.6 the point $p = F(a) = F(b)$ is parabolic. □

Corollary 3.10. *In the notation of Theorem A the set* $\partial(\mathrm{Stab}_G p)$ *is the quotient of the Floyd boundary* $\partial_{f_1}(\mathrm{Stab}_G p)$ *with respect to some scaling function* f_1.

Proof. Since $H = \mathrm{Stab}_G p$ is undistorted in G [Ge1], the inclusion map $H \hookrightarrow G$ is c-quasi-isometric for some integer c. For a given scaling function f there exists a scaling function f_1 satisfying conditions (3-4) such that $f(n)/f_1(cn)$ is bounded from above. By Lemma 2.5 the inclusion extends to a continuous map $\overline{H}_{f_1} \to \overline{G}_f$. It maps $\partial_{f_1}(\mathrm{Stab}_G p)$ onto $\partial(\mathrm{Stab}_G p)$. □

Remarks. It follows from Karlsson Lemma that the Floyd boundary of a virtually abelian group is either a point or pair of points. In particular it is true for any discrete elementary subgroup of $\mathrm{Isom}\mathbb{H}^n$.

We do not understand the proof in [F] that the preimage of a parabolic point $p \in \Lambda(G)$ for a Kleinian group $G < \mathrm{Isom}\mathbb{H}^3$ belongs to the boundary of its stabilizer in $\overline{G}_f$ [F, page 216]. Corollary 3.10 completes the argument for geometrically finite groups in $\mathrm{Isom}\mathbb{H}^n$.

4 Compactification of a quasi-isometric map. Proof of Theorem B.

Let G and H be finitely generated groups with fixed finite generating sets. We denote the corresponding word metrics by the same symbol d. Let $\overline{H}_{f_1}$ and $\overline{G}_{f_2}$ denote the Floyd completions corresponding to the functions f_1 and f_2 respectively.

We choose the functions f_i $(i = 1, 2)$ to satisfy the hypotheses of Lemma 2.5.

To simplify notation we put

$$X = \partial_{f_2} G,\ Y = \partial_{f_1} H,\ \widetilde{X} = X \sqcup G,\ \widetilde{Y} = Y \sqcup H.$$

Let $\varphi : H \to G$ be a c-quasi-isometric map. By Lemma 2.5 it extends to a uniformly continuous map $\widetilde{Y} \to \widetilde{X}$ which we keep denoting by φ. By continuity reason the inequality (7) of Lemma 2.5 remains valid for this extension.
The kernel θ_0 of the composition $\widetilde{Y} \xrightarrow{\varphi} \widetilde{X} \xrightarrow{F} \widetilde{T}$ is a closed equivalence relation on $\widetilde{Y}$. We have

$$(x, y) \in \theta_0 \iff F\varphi(x) = F\varphi(y). \tag{9}$$

The following equivalence on $\widetilde{Y}$ is closed and H-invariant:

$$\theta = \cap\{h\theta_0 : h \in H\}. \tag{10}$$

So $(x, y) \in \theta$ if and only if $(h(x), h(y)) \in \theta_0$ for each $h \in H$.

Let $\widetilde{S} = \widetilde{Y}/\theta$. Denote the quotient map $\widetilde{Y} \to \widetilde{S}$ by π. It is H-equivariant. Since θ is closed the space $\widetilde{S}$ is a compactum [Bourb, Prop I.10.8]. The open subspace $A = \pi(H)$ of $\widetilde{S}$ is an H-orbit of isolated points. The group H acts 3-discontinuously on $\widetilde{Y}$. By Proposition 3.2 the action of H on $\widetilde{S}$ is 3-discontinuous too.

Since φ sends θ into the kernel of F, it induces a continuous map $\varphi_* : \widetilde{S} \to \widetilde{T}$ such that the following diagram is commutative:

$$\begin{array}{ccc} \widetilde{X} & \xrightarrow{F} & \widetilde{T} \\ {\scriptstyle\varphi}\uparrow & & \uparrow{\scriptstyle\varphi_*} \\ \widetilde{Y} & \xrightarrow{\pi} & \widetilde{S} \end{array}$$

The equivalence θ determines the shortcut pseudometrics $\overline{\delta}_g$ on $\widetilde{S}$ such that

$$\forall\ p, q \in \widetilde{Y}, \forall g \in G \quad : \quad \overline{\delta}_g(p, q) = \overline{\delta}_g([p], [q]). \tag{11}$$

Proposition 4.1. *The action of H on $\widetilde{S}$ is 2-cocompact.*

Proof. By Lemma 3.1 it suffices to verify 2-cocompactness on S. Let $[p], [q]$ be distinct θ-classes in S. For some $h{\in}H$ and $p, q \in Y$ we have $\langle hp, hq\rangle{\notin}\theta_0$, so $F\varphi(hp){\neq}F\varphi(hq)$. Denote by K a compact fundamental set for the action of G on $\Theta^2\widetilde{T}$, i.e.

$$\Theta^2\widetilde{T} = \cup\{gK : g{\in}G\}.$$

Let δ be the infimum of the continuous function $\overline{\delta}_1|_K$. It is strictly positive by (8). There exists $g{\in}G$ such that $\{g^{-1}F\varphi(hp), g^{-1}F\varphi(hq)\}{\in}K$. So $\overline{\delta}_g(F\varphi(hp), F\varphi(hq)) \geq \delta$. Let γ be a bi-infinite geodesic in H with $\partial\gamma = \{hp, hq\}$. Since φ is c-quasi-isometric, $\varphi(\gamma)$ is contained in a c-quasigeodesic in G. By Karlsson Lemma there exists $r = r(c, \delta)$ such that $d(g, \varphi(\gamma)){\leqslant}r$. Assume that $d(g, g_0){\leqslant}r$ for $g_0 = \varphi(h_0),\ h_0 = \gamma(0)$.

By the bilipschitz equivalence of the shortcut metrics, $\overline{\delta}_{g_0}(F\varphi(hp), F\varphi(hq) \geq \frac{\delta}{D(r)}$ for a function $D(r)$ depending only on r. By $(8')$, $\overline{\delta}_{g_0}(\varphi(hp), \varphi(hq)) = \overline{\delta}_{g_0}(F\varphi(hp), F\varphi(hq))$. By Lemma 2.5 we obtain

$$\overline{\delta}_1(h_0^{-1}hp, h_0^{-1}hq) = \overline{\delta}_{h_0}(hp, hq) \geq \varepsilon\overline{\delta}_{g_0}(\varphi(hp), \varphi(hq)) \geq \frac{\varepsilon\delta}{D(r)} = \delta_1.$$

Using (11) we conclude that the set $\{\{s_1, s_2\}{\in}\Theta^2 S : \overline{\delta}_1(s_1, s_2) \geq \delta_1\}$ is a compact fundamental set for the action of H on $\Theta^2 S$. □

Theorem B. *Let a finitely generated group G act 3-discontinuously and 2-cocompactly on a compactum T. Let $\varphi : H \to G$ be a quasi-isometric map of a finitely generated group H.*

Then there exist a compactum S, a 3-discontinuous 2-cocompact action of H on S, and a continuous map $\varphi_* : S \to T$ such that for every H-parabolic point $p{\in}S$ the point $\varphi_* p$ is G-parabolic, and $\varphi(\mathsf{Stab}_H p)$ is contained in a uniformly bounded neighborhood of $\mathsf{Stab}_G(\varphi_* p)$.

Proof. The space S and the map φ_* are already constructed. We are going to prove that φ_* maps H-parabolic points to G-parabolic points. Let p be a parabolic point for the action of H on $\widetilde{S}$ and let Q be its stabilizer. Since Q is infinite there exists a bi-infinite geodesic $\gamma : \mathbb{Z} \to Q$. By [Ge1, Main Theorem, d] Q is finitely generated and undistorted in H. So the embedding $Q \hookrightarrow H$ is quasi-isometric. Thus γ is a c-quasigeodesic in H for some c. Moreover since the set of non-conical points $\mathsf{Nc}S$ is H-finite the constant c can be chosen uniformly for all p.

By Proposition 4.1 the action of H on $\widetilde{S}$ is 2-cocompact. By the Main Lemma the boundary of Q in $\widetilde{S}$ is $\{p\}$. In particular, γ is a c-horocycle. Since φ is quasi-isometric, the path $\varphi \circ \gamma : \mathbb{Z} \to \widetilde{T}$ is a l-quasigeodesic for some uniform constant $l > 0$. The continuity of φ_* and the commutativity of the above diagram imply that $\lim\limits_{n \to \pm\infty} \varphi_*(\gamma(n)) = \varphi_*(p)$. Thus $\varphi_* \circ \gamma$ is a l-horocycle at the point $\varphi_* p$. It follows from Lemma 3.6 that $\varphi_* p$ is parabolic for the action of G on T.

Every $h \in Q$ belongs to a bi-infinite geodesic in Q. By the above argument we have $\varphi(Q) \subset \mathsf{H}_l(\varphi_* p)$. By Lemma 3.7 the set $G \cap \mathsf{H}_l(\varphi_* p)$ is $\mathsf{Stab}_G(\varphi_* p)$-finite. Since $\mathsf{Nc}T$ is G-finite $\varphi(Q)$ is contained in a uniformly bounded neighborhood of $\mathsf{Stab}_G(\varphi_* p)$. □

Proof of the Corollary 1.2: Suppose that G is a finitely generated relatively hyperbolic group with respect to parabolic subgroups P_i $(i = 1, ...n)$ in the strong sens of Farb [Fa]. Then by [Bo1] (see also [Hr]) the group G possesses a geometrically finite 3-discontinuous action on a compact metrizable space X. It follows from [Tu3, Theorem 1.C] that the space $\Theta^2 X/G$ is compact. Let S be a compactum as in Theorem B on which the group H acts 3-discontinuously and 2-cocompactly. By [Ge1] this action is geometrically finite, the set of parabolic points is H-finite, and their stabilizers are all finitely generated. Thus it follows from [Ya] that H is relatively hyperbolic with respect to the stabilizers of H-non-equivalent parabolic points. By Theorem B the image of every parabolic subgroup of H by φ is contained in a uniform neighborhood of a parabolic subgroup of G. □

5 Appendix: a short proof that 3-cocompactness of an action implies word-hyperbolicity of the group.

As an application of our method we give a short proof of the following theorem of B. Bowditch:

Theorem [Bo3]. *Let G be a group acting 3-discontinuously and 3-cocompactly on a compactum T without isolated points. Then G is word-hyperbolic.*

The following lemma requires some additional information from [Ge1].

Lemma 5.1. *Let a group G act 3-discontinuously on a compactum T. Let p, q be distinct non-conical points in T and let K be a compact subset of $T^2 \backslash \Delta^2 T$. Then the set $S = \{g \in G : (gp, gq) \in K\}$ is finite.*

Proof. Assume that S is infinite. The compact K can be covered by finitely many closed sub-products of the form $A \times B$ with $A \cap B = \varnothing$. So we can assume that $K = A \times B$ where A, B are closed disjoint sets. The set $\Lambda_{\mathsf{rep}} S$ of the repellers of the limit crosses for S (see [Ge1, subsection 18]) is nonempty. It is contained in $\{p, q\}$ since otherwise, for some $g \in S$, the pair $\{gp, gq\}$ becomes arbitrarily small.

So S contains an infinite subset S_1 with $\Lambda_{\mathsf{rep}} S_1$ being a single point. Without loss of generality we can assume that $S{=}S_1$ and $\Lambda_{\mathsf{rep}} S{=}\{p\}$. The set $\Lambda_{\mathsf{attr}} S$ of the attractors of the limit crosses is contained in B. Let B_1 be a closed neighborhood of B disjoint from A. Thus, for $a \in T\setminus\{p\}$, the set $\{g \in S : ga \notin B_1\}$ is finite since it possesses no limit crosses. Hence $\{\{gp, ga\} : g \in S\}$ is contained in a compact subset of $\Theta^2 T$. So p is conical by of [Ge1, Definition 3]. □

Remark. With an additional assumption that T metrisable the Lemma can be also proved using Gehring-Martin's definition of the convergence property.

Corollary. *If G acts 3-discontinuously and 3-cocompactly on a compactum T without isolated points then every point of T is conical.*

Proof. Clearly, the 3-discontinuity implies the 2-discontinuity hence every nonconical point is either isolated or parabolic [Ge1]. By the assumption the discontinuity domain is empty. Assume that parabolic points do exist. Hence there exist at least two parabolic points since otherwise we must have a discontinuity domain.

Let p, q be distinct parabolic points and let L be a compact fundamental set for the action of G on $\Theta^3 T$. We can assume that L has the form $\bigcup_{i=1}^{n} A_i \times B_i \times C_i$ where A_i, B_i, C_i are closed disjoint subsets of T. For every $a \in T\setminus\{p, q\}$ there exist $g_a \in G$ such that $g_a(p, q, a) \in L$. By Lemma 5.1 the set $\{g_a : a \in T\setminus\{p, q\}\}$ is finite. If $g_a(p, q, a) \in A_{i(a)} \times B_{i(a)} \times C_{i(a)}$ then $T\setminus\{p, q\}$ is a union of finitely many closed sets $g_a^{-1} C_{i(a)}$. Thus $\{p, q\}$ is open and p and q are isolated contradicting the assumption. □

Proof of the Theorem. Assume that G is not hyperbolic. There exists a sequence of geodesic triangles with the sides $\{l_n, m_n, k_n\}$ so that $d(x_n, m_n \cup k_n) \to \infty$ for $x_n \in l_n$. Using the G-action we can make x_n equal to 1 for all n. By Karlsson Lemma the Floyd length of $m_n \cup k_n$ tends to zero and so $\delta_1(y_n, z_n) \to 0$ where $\partial l_n = \{y_n, z_n\}$. Since all l_n pass through the same point 1 we can choose a subsequence converging to a geodesic horocycle l. By Lemma 3.6 the target of l is not conical contradicting the above Corollary. □

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