ON A POTENTIAL FUNCTION FOR THE WEIL-PETERSSON METRIC ON TEICHMÜLLER SPACE

by

A.J. Tromba

Max-Planck-Institut
für Mathematik
Gottfried-Claren-Str. 26
D-5300 Bonn 3

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§0 Introduction

In 1956 Weil suggested a Riemannian metric on Teichmüller space and in [1] Ahlfors proved it was Kähler, Somewhat later he showed that it had negative Ricci and holomorphic sectional curvature. In [7] the author showed that the sectional curvature is negative. In 1982 we proved the existence of a potential function for this metric. In the ensueing years this result has been used by several authors [5],[8]. Recently [6] it was used in Jost's own computation of the curvature of Teichmüller space, and was rediscovered by Wolf [8] in his 1986 thesis. The growing interest in this result makes it worthwhile to have a proof in the literature.

§1 Preliminaries

Let M be an oriented compact, $\partial M = \phi$* and let $\mathcal{M}_{-1}$ be the Tame Frechêt manifold [2] of Riemannian metrics of constant negative curvature on M. The tangent space of $\mathcal{M}_{-1}$ at a metric, $g, T_g\mathcal{M}_{-1}$ consists of those $(0,2)$ tensors h on M satisfying the equation

$$(1.1) \qquad -\Delta(\mathrm{tr}_g h) + \delta_g\delta_g h + \frac{1}{2}(\mathrm{tr}_g h) = 0$$

where $\mathrm{tr}_g h = g^{ij}h_{ij}$ is the trace of h w.r.t. the metric tensor g_{ij}, $\delta_g\delta_g h$ is the double covariant divergence of h w.r.t. g and Δ is the Laplace-Beltrami operator on functions. For example see [2] for details.

Let $\mathcal{D}_0$ be the Tame Frechêt Lie group [2] of diffeomorphisms of M which are homotopic to the identity. Then $\mathcal{D}_0$ acts on

* the case with boundary follows analogously

$\mathcal{M}_{-1}$ by pull back, i.e. $f \longrightarrow f^*g$. Teichmüller space is then defined as

(1.2) $\mathcal{T}(M) = \mathcal{M}_{-1}/\mathcal{D}_0$.

In [2],[5] we show that $\mathcal{T}(M)$ is a C^∞ finite dimensional manifold diffeomorphic to $\mathbb{R}^q$, $q = 6\,(\text{genus } M) - 6$. The L_2-metric on $\mathcal{M}_{-1}$ is given by the inner product.

(1.3) $\langle\langle h,k \rangle\rangle_g = \frac{1}{2} \int_M \text{trace } (HK)\, d\mu_g$

where $H = g^{-1}h$, $K = g^{-1}k$ are the (1:1) tensors on M obtained from h and k via the metric g, or "by raising an index", i.e.

$$H^i_j = g^{ik} h_{kj}$$

and similarly for K. Finally μ_g is the volume element induced on M by g and the given orientation.

The inner product (1.3) is $\mathcal{D}_0$ invariant. Thus $\mathcal{D}_0$ acts smoothly on $\mathcal{M}_{-1}$ as a group of isometries with respect to this metric, and consequently we have an induced metric on $\mathcal{T}(M)$ in such a way that the projective map $\pi : \mathcal{M}_{-1} \longrightarrow \mathcal{M}_{-1}|\mathcal{D}_0$ becomes a Riemannian submersion [2]. In [3] it is shown that this induced metric is precisely the metric originally introduced by Weil.

Let $\langle , \rangle$ be the induced metric on $\mathcal{T}(M)$. We can characterize $\langle , \rangle$ as follows. From [2] we can show that given $g \in \mathcal{M}_{-1}$ every $h \in T_g\mathcal{M}_{-1}$ can be uniquely written as

(1.4) $h = h^{TT} + L_X g$

where $L_X g$ is the Lie derivative of g w.r.t. some (unique X) and h^{TT} is a trace free, divergence free, symmetric tensor. Moreover the decomposition (1.4) is L_2-orthogonal. Recall that a conformal coordinate system (where $g_{ij} = \lambda \delta_{ij}$, λ some smooth positive function) is also a complex holomorphic coordinate system. In this system

$$h^{TT} = \text{Re}(\xi(z)\,dz^2)$$

where Re is "real part" and $\xi(z)dz^2$ is a holomorphic quadratic differential. In fact trace free, divergence free symmetric two tensors are precisely the real parts of holomorphic quadratic differentials.

Now $L_X g$ is always tangent to the orbit of $\mathcal{D}_0$ through g. We say that $L_X g$ is the <u>vertical</u> part of h in decomposition 1.4. Similarly we say that h^{TT} represents the <u>horizontal</u> part of H. Given $h,k \in T_{[g]}\mathcal{T}(M)$ there are unique horizontal vectors $\tilde{h},\tilde{k} \in T_g\mathcal{M}_{-1}$ such that $d\pi(g)\tilde{h} = h$ and $d\pi(g)\tilde{k} = k$. Then

$$\langle h,k\rangle_{[g]} = \langle\langle\tilde{h},\tilde{k}\rangle\rangle_g \quad .$$

Suppose now that $g_0 \in \mathcal{M}_{-1}$ is fixed and that $s!(M,g) \longrightarrow (M,g_0)$ is a smooth C^1 map homotopic to the identity and is viewed as a map from M with some arbitrary metric $g \in \mathcal{M}_{-1}$ to M with its g_0 metric.

Define the Dirichlet energy of s by the formula

$$(1.5) \qquad E_g(s) = \frac{1}{2}\int_M |ds|^2 \, d\mu_g$$

where $|ds|^2$ = trace ds^*ds depends on both g and g_0.

By the embedding theorem of Nash-Moser we may assume that (M,g_0) is isometrically embedded in some Euclidean $\mathbb{R}^K$. Thus we can think of $s : (M,g) \longrightarrow (M,g_0)$ as a map into $\mathbb{R}^K$ and Dirichlet's functional takes the equivalent form

$$(1.6) \qquad E_g(s) = \frac{1}{2}\sum_{i=1}^{k} \int g(x) \, \langle\nabla_g s^i(x),\nabla_g s^i(x)\rangle d\mu_g \quad .$$

There is another, equivalent, and useful way to express (1.5) and (1.6) using local conformal cordinate systems $g_{ij} = \sigma\delta_{ij}$ and $(g_0)_{ij} = \rho\delta_{ij}$ on (M,g) and (M,g_0) respectively, namely

$$(1.7) \qquad E_g(s) = \frac{1}{4}\int_M [\rho(s(z))|s_z|^2 + \rho(s(z))|s_{\bar{z}}|^2]dzd\bar{z}$$

For fixed g, the critical points of E are there said to be <u>harmonic</u> <u>maps</u>. The follwing result is due to Schoen-Yau [9].

<u>Theorem.</u> Given metrics g and g_0 there exists a unique harmonic map $s(g) : (M,g) \longrightarrow (M,g_0)$ which is homotopic to the identity. Moreover $s(g)$ depends differentially on g in any H^r topology,

$r > 2$, and is a C^∞ diffeomorphism.

Consider now the function

$$g \longrightarrow E_g(s(g)) \quad .$$

This function on $\mathcal{M}_{-1}$ is $\mathcal{D}$-invariant and thus can be viewed as a function on Teichmüller space.

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This function on $\mathcal{M}_{-1}$ is $\mathcal{D}$-invariant and thus can be viewed as a function on Teichmüller space. To see this one must show that

$$E_{f^*g}(s(f^*(g))) = E_g(s(g)) \quad .$$

Let $c(g)$ be the complex structure associated to g, and induced by a conformal coordinate system for g. For $f \in \mathcal{D}_0$, $f : (M,f^*c(g)) \longrightarrow (M,c(g))$ is holomorphic and consequently since the composition of harmonic maps and holomorphic maps is still harmonic we may conclude, by uniqueness that

$$s(f^*g) = s(g) \circ f \quad .$$

Since Dirichlet's functional is invariant under complex holomorphic changes of coordinates it follows immediately that

$$E_{f^*(g)}(s(g) \circ f) = E_g(s(g)) \quad .$$

Consequently for $[g] \in \mathcal{M}_{-1}|\mathcal{D}_0$ define the C^∞ smooth function

$$\tilde{E} : \mathcal{M}_{-1}|\mathcal{D}_0 \longrightarrow R$$

by

$$\tilde{E}[g] = E_g(s(g)) \ .$$

§2 The Main Result

Theorem 2.1 $[g_0]$ is the only critical point of $\tilde{E}$. The Hessian of $\tilde{E}$ at $[g_0]$ is given by

$$d^2\tilde{E}[g_0](h,k) = 2\langle h,k\rangle$$

$h,k \in T_{[g_0]}\mathcal{T}(M)$. That is, the second variation of Dirichlet's energy function is (up to a positive constant) Weil-Petersson metric.

Proof. We begin by computing the first derivative $d\tilde{E}[g_0]$. We again view a map $S : (M,g) \longrightarrow (M,g_0)$ as a map into $\mathbb{R}^k$. Consider the two form

$$\xi(z)dz^2 = \sum_{i=1}^{k} (s_z^i)^2 dz^2 = \sum_{i=1}^{k} \left(\frac{\partial s^i}{\partial z}\right)^2 dz^2 .$$

We start by proving

Proposition 2.2. If $s : (M,g) \longrightarrow (M,g_0)$ is harmonic the form $\xi(z)dz^2$ is a holomorphic quadratic differential on the complex curve $(M,c(g_0))$, and thus $\mathrm{Re}\ \xi(z)dz^2$ represents a trace free, divergence free symmetric two tensor on (M,g_0). Hence $\mathrm{Re}\ \xi(z)dz^2$ is a horizontal tangent vector to $\mathcal{M}_{-1}$ at g_0. Finally

$$(2.3) \qquad d\tilde{E}[g_0]h = -\,\mathrm{Re}\langle\langle \xi(z)dz^2, \tilde{h}\rangle\rangle_{g_0}$$

where $\tilde{h}$ is the horizontal left of $h \in T_{(g_0)}\mathcal{T}(M)$.

Proof (of 2.2)

We have Dirichlet's functional

$$E(g,s) = \frac{1}{2} \sum_{i=1}^{k} \int_M g(x)\ (\nabla_g s^i, \nabla_g s^i) d\mu_g \ .$$

Suppose s is harmonic. Let Ω denote the second fundamental form of $(M,g_0) \subset \mathbb{R}^k$. Thus for each $p \in M$, $\Omega(p) : T_pM \times T_pM \longrightarrow T_pM^{\perp}$. Let Δ denote the (non-linear) Laplacian of maps from (M,g) to (M,g_0) and Δ_β denote the Laplace-Betrami operator on functions. Then if s is harmonic we have

(2.4) $$0 = \Delta s = \Delta_\beta s + \sum_{j=1}^{2} \Omega(s)(ds(e_j), ds(e_j))$$

$e_1(p)$, $e_2(p)$ on orthonormal basis for T_pM with respect to g. $\xi(z)dz^2$ will be holomorphic of

$$\frac{\partial}{\partial \bar{z}}\Big(\sum_{i=1}^{k} \frac{\partial s^i}{\partial z} \cdot \frac{\partial s^i}{\partial z}\Big) = 0 \quad .$$

But this is equal to

$$\frac{2}{\sigma} \sum_{i=1}^{k} \Delta_\beta s^i \cdot \frac{\partial s^i}{\partial z}$$

where in conformal coordinates $g_{ij} = \sigma\delta_{ij}$. By (2.4) we see that this, in time, is equal to

$$-\frac{2}{\sigma} \sum_{i=1}^{k} \sum_{j=1}^{2} \Omega^i(s)(ds(e_j), ds(e_j)) \cdot \frac{\partial s^i}{\partial z}$$

$$= -\frac{2}{\sigma} \sum_{j=1}^{2} \Big\{ \sum \Omega(s)(ds(e_j), ds(e_j)) \cdot \frac{\partial s}{\partial x} + i\Omega(s)(ds(e_j), ds(e_i)) \cdot \frac{\partial s}{\partial y} \Big\} \quad .$$

Since $\Omega(p)$ takes values in $T_pM^\perp$ it follows that both the real and imaginary parts of the expression vanish. Thus $\xi(z)dz^2$ is holomorphic.

Recall that s is harmonic iff $\frac{\partial E}{\partial s}(g,s) = 0$. We now compute $\frac{\partial E}{\partial g}$. If we have local coordinates represented by $(x,y) \in W$, then in this coordinate system

$$E(g,s) = \frac{1}{2} \sum_{\ell=1}^{k} \int_M g(x) \langle G^{-1}\nabla s^\ell, \nabla s^\ell \rangle_{\mathbb{R}^2} \sqrt{\det G}\, dxdy$$

where ∇s^ℓ is the vector $(\frac{\partial s^\ell}{\partial x}, \frac{\partial s^\ell}{\partial y})$, G is the matrix $\{g_{ij}\}$ of g and $\langle , \rangle_{\mathbb{R}^2}$ denotes the ordinary $\mathbb{R}^2$ inner product and $\sqrt{\det G}\, dxdy$ is the local representation of $d\mu_g$. In the following computation we adopt the convention, that summations over the index ℓ will be understood.

(2.5) $$\frac{\partial E}{\partial g}(g_0,s)\tilde{h} = -\int \langle G_0^{-1} H G_0 \nabla s^\ell, \nabla s^\ell \rangle \sqrt{\det G_0}\, dxdy + \frac{1}{2} \int \langle G_0^{-1} \nabla s^\ell, \nabla s^\ell \rangle \frac{\operatorname{trace} H}{\sqrt{\det G_0}}\, dxdy$$

where $H = \{\tilde{h}_{ij}\}$ is the matrix of the symmetries tensor h in these coordinates. Here we use the fact that the derivative of $G \longrightarrow G^{-1}$ is $H \longrightarrow G^{-1}HG^{-1}$. Suppose we look at this first derivative in conformal coordinates $(g_0)_{ij} = \lambda\delta_{ij}$. Then if $\tilde{h}$ is horizontal the second term in (2.5) vanishes (h is trace free) and

$$\frac{\partial E}{\partial g}(g_0,s)\tilde{h} = -\int \frac{1}{\lambda} \langle \nabla s^{\ell}, \nabla s^{\ell} \rangle_{\mathbb{R}^2} \, dxdy$$

$$= -\int \frac{1}{\lambda} \left\{ \tilde{h}_{11}\left(\frac{\partial s^{\ell}}{\partial x}\right)^2 + 2\tilde{h}_{12}\left(\frac{\partial s^{\ell}}{\partial x}\right)\left(\frac{\partial s^{\ell}}{\partial y}\right) + \tilde{h}_{22}\left(\frac{\partial s^{\ell}}{\partial y}\right)^2 \right\} dxdy \quad .$$

Since $h_{11} = -h_{22}$ this is equal to

$$(2.6) \qquad -\int \frac{1}{\lambda}\left\{ \tilde{h}_{11}\left[\left(\frac{\partial s^{\ell}}{\partial x}\right)^2 - \left(\frac{\partial s^{\ell}}{\partial x}\right)^2 + 2\tilde{h}_{12}\left(\frac{\partial s^{\ell}}{\partial x}\right)\left(\frac{\partial s^{\ell}}{\partial y}\right) \right\} dxdy \quad .$$

Now

$$\left(\frac{\partial s^{\ell}}{\partial x} - i\frac{\partial s^{\ell}}{\partial y}\right)(dx + dy)^2 = \xi(z)dz^2$$

is a quadratic differential. But

$$\mathrm{Re}(\xi(z)dz^2) = \left[\left(\frac{\partial s^{\ell}}{\partial x}\right)^2 - \left(\frac{\partial s^{\ell}}{\partial y}\right)^2\right]dx^2 + \left[\left(\frac{\partial s^{\ell}}{\partial y}\right)^2 - \left(\frac{\partial s^{\ell}}{\partial x}\right)^2\right]dy^2 + 4\left(\frac{\partial s^{\ell}}{\partial x}\right)\left(\frac{\partial s^{\ell}}{\partial y}\right)dxdy .$$

If s is harmonic $\mathrm{Re}(s(z)dz^2)$ is a trace free divergence free tensor. Let us compute

$$\langle\langle \mathrm{Re}\ \xi(z)dz^2, \tilde{h} \rangle\rangle_{g_0} \quad .$$

This inner product is given locally by the expression

$$(2.7) \qquad \frac{1}{2}\int g_0^{ab} g_0^{cd} k_{ac} \tilde{h}_{bd} \, d\mu_g$$

where k_{ac} is the coordinate representative of the two tensor $\xi(z)dz^2$. Therefore

$$k_{11} = \left\{\left(\frac{\partial s^{\ell}}{\partial x}\right)^2 - \left(\frac{\partial s^{\ell}}{\partial y}\right)^2\right\} , \quad k_{12} = k_{21} = 2\left(\frac{\partial s^{\ell}}{\partial x}\right)\left(\frac{\partial s^{\ell}}{\partial y}\right) \quad .$$

Thus in conformal coordinates (2.7) is equal to

$$\int \frac{1}{2\lambda}\{k_{ac}\tilde{h}_{ac}\}dxdy$$

$$= \int \frac{1}{2\lambda}\{k_{11}\tilde{h}_{11} + 2k_{11}\tilde{h}_{12} + k_{22}\tilde{h}_{22}\}dxdy \ .$$

Since $k_{11} = -k_{22}$, $\tilde{h}_{11} = -\tilde{h}_{22}$ this equals

$$\int \frac{1}{\lambda}\{k_{11}\tilde{h}_{11} + k_{12}\tilde{h}_{12}\}dxdy$$

$$= \int \frac{1}{\lambda}\{[(\frac{\partial s^\ell}{\partial x})^2 - (\frac{\partial s^\ell}{\partial y})^2]\tilde{h}_{11} + 2(\frac{\partial s^\ell}{\partial x})(\frac{\partial s^\ell}{\partial y})\tilde{h}_{12}\}dxdy \ .$$

Comparing this with expression (2.6) establishes the formula

$$\frac{\partial E}{\partial g}(g_0,s)\tilde{h} = -\langle\langle \mathrm{Re}\ \xi(z)dz^2,\tilde{h}\rangle\rangle_{g_0} \ .$$

However $\tilde{E}[g] = E(g,s(g))$. Since $s(g)$ is harmonic $\frac{\partial E}{\partial s}(g_0,s(g_0)) = 0$. This immediately implies that

$$\frac{\partial \tilde{E}}{\partial g}[g_0]h = -\langle\langle \mathrm{Re}\ \xi(z)dz^2,\tilde{h}\rangle\rangle_{g_0}$$

which establishes 2.2. We should remark that this formula tells us that the gradient of Dirichlet's function on Teichmüller space is represented as a holomorphic quadratic differential.

To complete theorem 2.1 we need to compute a second derivative. Again working locally and thinking of the map s as now being fixed we see that for $\tilde{h},\tilde{k}$ horizontal

$$\frac{\partial^2 E}{\partial g^2}(g_0,s)(\tilde{h},\tilde{k}) = \int \langle G_0^{-1}KG_0^{-1}HG_0^{-1}\nabla s^\ell,\nabla s^\ell\rangle_{\mathbb{R}^2}\sqrt{\det G_0}\ dxdy$$
$$+ \int \langle G_0^{-1}HG_0^{-1}KG_0^{-1}\nabla s^\ell,\nabla s^\ell\rangle_{\mathbb{R}^2}\sqrt{\det G_0}\ dxdy$$

and in conformal coordinates this is equal to

$$\int \frac{1}{\lambda^2}\langle KH\nabla s^\ell,\nabla s^\ell\rangle_{\mathbb{R}^2}\ dxdy + \int \frac{1}{\lambda^2}\langle HK\nabla s^\ell,\nabla s^\ell\rangle\ dxdy$$
$$\int \frac{2}{\lambda^2}\{\tilde{h}_{11}\tilde{k}_{11} + \tilde{h}_{12}\tilde{k}_{12})\ (\frac{\partial s^\ell}{\partial x})^2 + (\frac{\partial s^\ell}{\partial y})^2]\}dxdy \ .$$

Now at the point g_0, the unique harmonic map s is the identity map of (M,g_0) to itself. Since (M,g_0) is isommetrically immersed in $\mathbb{R}^K$, $s(g_0)^*G_{\mathbb{R}^K} = g_0$; where $G_{\mathbb{R}^K}$ is the Euclidean metric on $\mathbb{R}^K$. But if g_0 is expressed in local conformal coordinates this says exactly that

$$\{(\frac{\partial s^\ell}{\partial x})^2 + (\frac{\partial s^\ell}{\partial y})^2\} = \lambda.$$

Thus at the point g_0, we see that

$$\frac{\partial^2 E}{\partial g^2}(g_0,\mathrm{id})(\tilde{h},\tilde{k}) = \int \frac{2}{\lambda}(\tilde{h}_{11}\tilde{k}_{11} + \tilde{h}_{12}\tilde{k}_{12})\,dxdy$$

Since $\tilde{k}_{11} = -\tilde{k}_{22}$, $\tilde{h}_{11} = -\tilde{h}_{22}$, applying formula (2.7) for the Weil-Petersson metric we see that

$$(2.8)\qquad \frac{\partial^2 E}{\partial g^2}(g_0,\mathrm{id})(\tilde{h},\tilde{k}) = 2\langle\langle\tilde{h},\tilde{k}\rangle\rangle \quad .$$

However we are interested in the map

$$\tilde{E}[g] = E(g,s(g)).$$

Clearly

$$\frac{\partial\tilde{E}}{\partial g}[g]h = \frac{\partial E}{\partial g}(g,s(g))\tilde{h} + \frac{\partial E}{\partial s}(g,s(g))\cdot Ds(g)\tilde{h}$$

where $Ds(g)$ represents the derivative of s with respect to g. However the second term is identically zero since $s(g)$ is harmonic implies $\frac{\partial E}{\partial s}(g,s(g)) \equiv 0$. Therefore

$$\frac{\partial^2\tilde{E}}{\partial g^2}[g_0](h,k) = \frac{\partial^2 E}{\partial g^2}(g_0,\mathrm{id})(\tilde{h},\tilde{k}) + \frac{\partial^2 E}{\partial g\partial s}(g_0,\mathrm{id})(\tilde{h},Ds(g_0)\tilde{k})$$

and by 2.8

$$= 2\langle\langle\tilde{h},\tilde{k}\rangle\rangle + \frac{\partial^2 E}{\partial g\partial s}(g_0,\mathrm{id})(\tilde{h},Ds(g_0)\tilde{k}).$$

Theorem 2.1 will now follow immediately from the following.

<u>Proposition 2.9.</u> $Ds(g_0)\tilde{h} = 0$, if $\tilde{h}$ is trace free divergence free.

Proof. In order to compute this derivative we write down the general equation of a harmonic map from a Riemanian manifold (M,g) to a Riemannian manifold $(N,\mathcal{g})$. Namely $f : (M,g) \to (N,\mathcal{g})$ is harmonic if in local coordinates, $f = (f',\ldots,f^n)$, $n = \dim N$

$$(2.10) \qquad \frac{1}{\sqrt{g}} \frac{\partial}{\partial x_j} g^{ij}\sqrt{g} \frac{\partial}{\partial x_i} f^{\alpha}\} + \Gamma^{\alpha}_{\gamma\beta} \frac{\partial f^{\gamma}}{\partial x_i} \frac{\partial f^{\beta}}{\partial x_j} g^{ij} = 0$$

where $\Gamma^{\alpha}_{\gamma\beta}$ are the Christofel symbol of the metric $\mathcal{g}$.

If $\dim N = 2 = \dim M$ and we express (2.10) in local conformal coordinates $g_{ij} = \lambda\delta_{ij}$ and $\mathcal{g}_{ij} = \rho\delta_{ij}$ we see that (2.10) is equivalent to

$$(2.11) \qquad f_{z\bar{z}} + (\log \rho)_f f_z f_{\bar{z}} = 0$$

where $(\log \rho)_f = \frac{\rho(f)}{\rho'(f)}$.

In the case under consideration g is the fixed metric g_0 on M. We now think of f^{α} as depending on g, and let $w^{\alpha} = Df^{\alpha}(\tilde{h})$ be the linearization of f^{α} in the direction $\tilde{h}$. We now differentiate equation (2.10) w.r.t. g in the direction $\tilde{h}$. We first make three important observations. The Christofel symbol Γ^{α}_{β} are fixed and do not depend on g. Second the derivative of $\sqrt{g}$ in a direction $\tilde{h}$ is given by $\tilde{h} \longrightarrow tr_g h/\sqrt{g}$

If $\tilde{h}$ is trace free this derivative vanishes. Thirdly, the derivative of $g^{ij}\sqrt{g}$ in the direction $\tilde{h}$ is $\tilde{h} \longrightarrow -\tilde{h}^{ij}$.

Taking the derivative of (2.10) w.r.t. g in the direction $\tilde{h}$, evaluating it in conformal coordinates $(g_0)_{ij} = \lambda\delta_{ij}$ at $f = id$, and using formula 2.12 for the complex form of $w = w + iw_2$ we see that

$$(2.12) \qquad w_{z\bar{z}} + (\log \lambda)_z w_{\bar{z}} = + \frac{1}{\lambda} \frac{\partial}{\partial x_j} \{\tilde{h}^{\alpha j}\} + \frac{\Gamma^{\alpha}_{ij}\tilde{h}_{ij}}{\lambda^2} \qquad .$$

Lemma 2.13 If $\tilde{h}$ is trace free and divergence free, the expression

$$(2.14) \qquad \frac{1}{\lambda} \frac{\partial}{\partial x_j} \{\tilde{h}^{\alpha j}\} + \frac{1}{\lambda^2} \Gamma^{\alpha}_{ij} \tilde{h}_{ij} = 0.$$

Before proving 2.13 let us see how it implies proposition 2.9.

Using 2.12 we see that the linearization $w = Ds(g_0)\tilde{h}$ satisfies

$$w_{z\bar{z}} + (\log\lambda)_z w_{\bar{z}} = 0$$

or

$$\frac{\partial}{\partial z}(\lambda w_{\bar{z}}) = 0 \ .$$

Now this implies that

$$\int \frac{\partial}{\partial z}(\lambda w_{\bar{z}})\bar{w}\, dz \wedge d\bar{z} = 0 \ .$$

Integrating by parts we further see that

$$\int \lambda |w_{\bar{z}}|^2 dz \wedge d\bar{z} = 0 \ .$$

Therefore $w_{\bar{z}} = 0$ and consequently w is a holomorphic vector field on $(M, c(g_0))$. Since (genus M) > 1 this clearly implies that $w \equiv 0$ concluding 2.9.

To prove lemma 2.13 we note that

$$\Gamma^{\alpha}_{ij} = \frac{1}{2\lambda}\{\frac{\partial\lambda}{\partial x_j}\delta_{i\alpha} + \frac{\partial\lambda}{\partial x_i}\delta_{j\alpha} - \frac{\partial\lambda}{\partial x_\alpha}\delta_{ij}\}$$

and that $\tilde{h}^{\alpha j} = \frac{1}{\lambda}\tilde{h}_{\alpha j}$. Since $\tilde{h}$ is divergence free $\frac{\partial}{\partial x_j}\tilde{h}_{\alpha j} = 0$ and so

$$\frac{1}{\lambda}\frac{\partial}{\partial x_j}(\tilde{h}^{\alpha j}) = -\frac{1}{\lambda^3}\tilde{h}_{\alpha j}\frac{\partial\lambda}{\partial x_j} \ .$$

Therefore expression 2.14 equals

$$-\frac{1}{\lambda^3}\frac{\partial\lambda}{\partial x_j}\tilde{h}_{\alpha j} + \frac{1}{2\lambda^3}\{\frac{\partial\lambda}{\partial x_j}\delta_{i\alpha} + \frac{\partial\lambda}{\partial x_i}\delta_{j\alpha} - \frac{\partial\lambda}{\partial x_\alpha}\delta_{ij}\}\tilde{h}_{ij}$$

$$= -\frac{1}{\lambda^3}\frac{\partial\lambda}{\partial x_j}\tilde{h}_{\alpha j} + \frac{1}{2\lambda^3}\frac{\partial\lambda}{\partial x_j}\tilde{h}_{\alpha j} + \frac{1}{2\lambda^3}\frac{\partial\lambda}{\partial x_i}\tilde{h}_{i\alpha} - \frac{1}{2\lambda^3}\frac{\partial\lambda}{\partial x_\alpha}\tilde{h}_{ii} \ .$$

Clearly the sum of the first three terms is zero and since $\tilde{h}$ is trace free the fourth also vanishes. This completes lemma 2.13 and this paper.

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