Conformally flat metrics on 4-manifolds

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Abstract

We prove that for each closed smooth spin 4-manifold M there exists a closed smooth 4-manifold N such that $M\#N$ admits a conformally flat Riemannian metric.

Contents

1 Introduction

The goal of this paper is to prove

Theorem 1.1. *Let M^4 be an closed connected smooth spin 4-manifold. Then there exists a closed orientable 4-manifold N such that $M\#N$ admits a conformally flat Riemannian metric. The manifold N is (in principle) computable in terms of triangulation of M.*

Recall that there are many closed 4-dimensional spin manifold which admit no flat conformal structure; for instance, simply-connected manifolds (not diffeomorphic to S^4), manifolds with simple infinite fundamental group. (First examples of first 3-manifolds not admitting flat conformal structure were constructed by W. Goldman in [1].) The above theorem shows that if M admits a flat conformal structure, it does not imply that all components of its connected sum decomposition are also conformally flat.

Our motivation comes from the following theorem of C. Taubes [12]:

Theorem 1.2. *Let M be a smooth closed oriented 4-manifold. Then there exists a number k so that the connected sum of M with k copies of $\overline{\mathbb{CP}^2}$ admits a half-conformally flat structure.*

Here $\overline{\mathbb{CP}^2}$ is the complex-projective plane with the reversed orientation. Recall that a Riemannian metric g on M is *anti self-dual* (or half-conformally flat) if the self-dual part W_+ of the Weyl tensor vanishes. Vanishing of both self dual and anti self-dual parts of the Weyl tensor (i.e., vanishing of the entire Weyl tensor) is equivalent to local conformal flatness of the metric g.

Note that the assumption that M^4 is spin is equivalent to vanishing of all Stiefel-Whitney classes, which is equivalent to triviality of the tangent bundle of $M' = M \setminus \{p\}$. According to the Hirsch-Smale theory (see for instance [7, Theorem 4.7] or [11]), $M' := M \setminus \{p\}$ is parallelizable iff M' admits an immersion into $\mathbb{R}^4$. Thus, by taking M to be simply-connected with nontrivial 2-nd Stiefel-Whitney class, one sees that $M \# N$ does not admit a flat conformal structure for *any* N: Otherwise the developing map would immerse M' into S^4. Therefore the vanishing condition is, to some extent, necessary. Note also that (unlike in Taubes' theorem) one cannot expect N to be simply-connected since the only closed conformally flat simply-connected Riemannian manifold is the sphere with the standard conformal structure.

Sonjong Hwang in his thesis [5], has verified that for 3-manifolds an analogue of Theorem 1.1 holds, moreover, one can use a connected sum of Haken manifolds as the manifold N. Similar arguments can be used to prove an analogous theorem in the context of locally spherical CR structures on 3-manifolds.

The arguments in both 3-dimensional and 4-dimensional cases, in spirit (although, not in the technique), are parallel to Taubes': We start with a singular conformally-flat metric on M, where the singularity is localized in a ball $B \subset M$. The singular metric is obtained by pull-back of the standard metric on the 4-sphere under a branched covering $M \to S^4$. Then we would like to "resolve the singularity". To do so we remove an open tobular neighborhood U of the singular locus. Afterwards, use the "orbifold trick" (cf. for instance [2]) to eliminate the boundary of $M \setminus U$: Introduce a Moebius reflection orbifold structure on $M \setminus U$ to get a closed Moebius orbifold O which is a connected sum of M with an orbifold. After passing to an appropriate finite manifold cover over O we get a conformally-flat manifold which has M as a connected summand.

Lack of reflection groups in higher-dimensional hyperbolic spaces limits this strategy to low dimensions. Using arguments somewaht similar to the *strict hyperbolization* of Charney and Davis (see [3]) one can generalize Theorem 1.1 to higher dimensional almost parallelizable manifolds. We will discuss this issue elsewhere.

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2 Definitions and notation

We let $\text{Mob}(S^4)$ denote the full group of Moebius transformations of S^4, i.e. the group generated by inversions in round spheres. Equivalently, $\text{Mob}(S^4)$ is the restriction of the full group of isometries $Isom(\mathbb{H}^5)$ to the 4-sphere S^4 which is the ideal boundary of $\mathbb{H}^5$. We will regard S^4 as 1-point compactification $\mathbb{R}^4 \cup \{\infty\}$ of then Euclidean 4-space.

Definition 2.1. *Let Q be a unit cube in $\mathbb{R}^4$. We define the* PL inversion *J in the boundary of Q as follows. Let $h : S^4 \to S^4$ be a PL homeomorphism which sends $\Sigma = \partial Q$ onto the round sphere $S^3 \subset \mathbb{R}^4$ and $h(\infty) = \infty$. Let $j : S^4 \to S^4$ be the ordinary inversion in S^3. Then $J := h^{-1} \circ j \circ h$.*

Definition 2.2. *A* Moebius *or a* flat conformal structure *on a smooth 4-manifold M is an atlas $\{(V_\alpha, \varphi_\alpha), \alpha \in A\}$ which consist of diffeomorphisms $\varphi_\alpha : V_\alpha \to U_\alpha \subset S^4$ so that the transition mappings $\varphi_\alpha \circ \varphi_\beta^{-1}$ are restrictions of Moebius transformations.*

Equivalently, one can describe Moebius structures on M are conformal classes of conformally-Euclidean Riemannian metrics on M. Each conformal structure on M gives rise to a local conformal diffeomorphism, called a *developing map*, $d : \tilde{M} \to S^4$, where $\tilde{M}$ is the universal cover of M. If M is connected, the mapping d is equivariant with respect to a *holonomy representation* $\rho : \pi_1(M) \to \text{Mob}(S^4)$, where $\pi_1(M)$ acts on $\tilde{M}$ as the group of deck-transformations. Given a pair (d, ρ), where ρ is a representation of $\pi_1(M)$ into $\text{Mob}(S^4)$ and d is a ρ-equivariant local diffeomorphism from $\tilde{M}$ to S^4, one constructs the corresponding Moebius structure on M by taking a pull-back of the standard flat conformal structure on S^4 to $\tilde{M}$ via d and then projecting the structure to M.

Analogously, one defines a *complex-projective structures* on complex 3-manifold Z, as a $\mathbb{CP}^3$-valued holomorphic atlas on Z so that the transition mappings belong to $PGL(3, \mathbb{C})$.

The concept of Moebius structure generalizes naturally to the category of orbifolds:

A 4-dimensional *Moebius orbifold O* is a pair $(X, \mathcal{A})$, where X is a Hausdorff topological space, the *underlying space of the orbifold*, $\mathcal{A}$ is a family of local parameterizations $\psi_\alpha : U_\alpha \to U_\alpha/\Gamma_\alpha = V_\alpha$, where $\{V_\alpha, \alpha \in A\}$ is an open covering of X, U_α are open subsets in S^4, Γ_α are finite groups of Moebius automorphisms of U_α and the mappings ψ_α satisfy the usual compatibility conditions:

If $V_\alpha \to V_\beta$ is the inclusion map then we have a Moebius embedding $U_\alpha \to U_\beta$ which is equivariant with respect to a monomorphism $\Gamma_\alpha \to \Gamma_\beta$, so that the diagram

$$\begin{array}{ccc} U_\alpha & \to & U_\beta \\ \downarrow & & \downarrow \\ V_\alpha & \to & V_\beta \end{array}$$

is commutative. The groups Γ_α are the *local fundamental groups* of the orbifold O.

For a Moebius orbifold one defines a developing mapping $d : \tilde{O} \to S^4$ (which is a local homeomorphism from the universal cover $\tilde{O}$ of O) and, if O is connected, a holonomy homeomorphism $\rho : \pi_1(O) \to \mathrm{Mob}(S^4)$, which satisfy the same equivariance condition as in the manifold case. Again, given a pair (d, ρ), where d is a ρ-equivariant homeomorphism, one defines the corresponding Moebius structure via pull-back.

Example 2.3. *Let $G \subset \mathrm{Mob}(S^4)$ be a subgroup acting properly discontinuously on an open subset $\Omega \subset S^4$. Then quotient space Ω/G has a natural Moebius orbifold structure. The local charts ϕ_α appear in this case as restrictions of the projection $p : \Omega \to \Omega/G$ to open subsets with finite stabilizers.*

In particular, suppose that G is a finite subgroup of $\mathrm{Mob}(S^4)$ generated by reflections, the quotient $Q := \Omega/G$ can be identified with the intersection of a fundamental domain of G with Ω. The Moebius structures on 4-dimensional manifolds and orbifolds constructed in this paper definitely do not arise this way. A more interesting example is obtained by taking a manifold M and a local homeomorphism $h : M \to S^4$, so that $Q \subset h(M)$. Then we can pull-back the Moebius orbifold structure on Q to an appropriate subset X of M, to get a 4-dimensional Moebius orbifold. As another example of a pull-back construction, let O be a Moebius orbifold and $M \to O$ be an orbifold cover such that M is a manifold. Then one can pull-back the Moebius orbifold structure from O to an ordinary Moebius structure on M.

We refer the reader to [9] for a more detailed discussion of geometric structures on orbifolds.

3 Reflection groups in S^4 with prescribed combinatorics of the fundamental domains

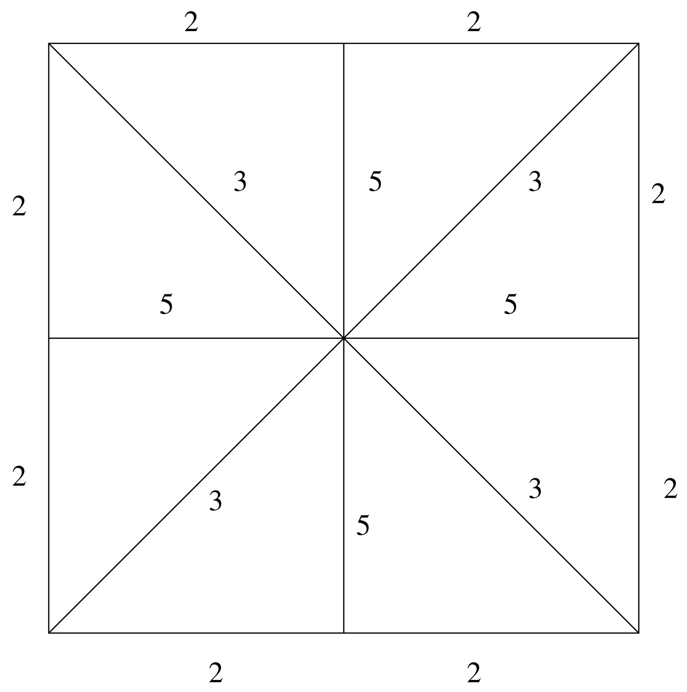

Figure 1: *Barycentric subdivision of a square.*

Consider the standard cubulation $\mathcal{Q}$ of $\mathbb{R}^4$ by the Euclidean cubes with the edges of length 2 and let X denote the 2-skeleton of this cubulation. Given a collection of round balls $\{B_i, i \in I\}$ in $\mathbb{R}^4$, with the nerve $\mathcal{N}$, we define the canonical simplicial mapping $f : \mathcal{N} \to \mathbb{R}^4$ by sending each vertex of $\mathcal{N}$ to the center of the corresponding ball and extending f linearly to the simplices of $\mathcal{N}$. For a subcomplex $K \subset \mathcal{Q}$ define its *barycentric subdivision* $\beta(K)$ to be the following simplicial complex. Subdivide each edge of K by its midpoint. Then inductively subdivide each k-cube Q in K by coning off the barycentric subdivision of ∂Q from the center of Q.

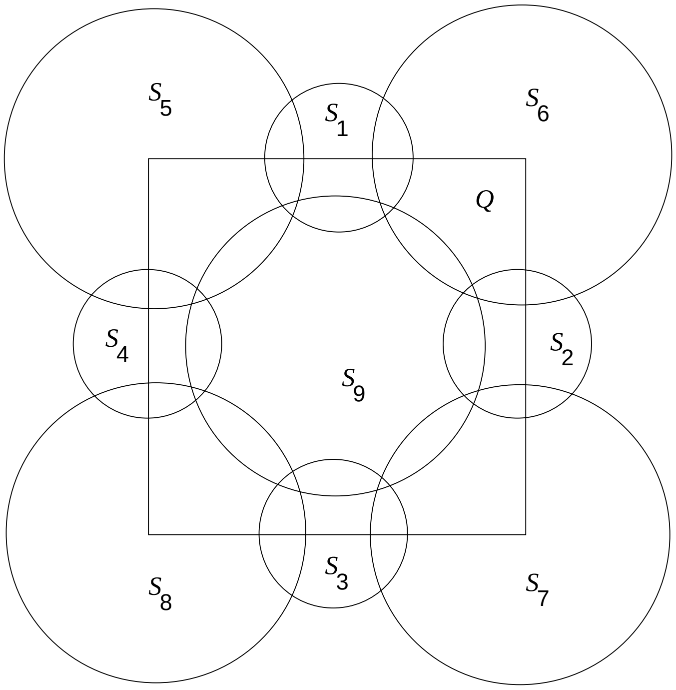

Figure 2:

Proposition 3.1. *Suppose that $K \subset X$ is a 2-dimensional compact subcomplex such that each vertex belongs to a 2-cell. Then there exists a collection of open round 4-balls B_i, $i = 1, ..., k$, centered at the vertices of $\beta(K)$, so that:*

(1) The Moebius inversions R_i in the round spheres $S_i = \partial B_i$ generate a discrete reflection group $G \subset \mathrm{Mob}(S^4)$.

(2) The complement $S^4 \setminus \cup_{i=1}^k B_i$ is a fundamental domain Φ of G.

(3) The canonical mapping from the nerve of $\{B_i, i = 1, ..., k\}$ to $\mathbb{R}^4$ is a simplicial isomorphism onto $\beta(K)$.

Proof: We begin by constructing the family of spheres $S_i, i \in \mathbb{N}$ centered at certain points of X. For each square Q in X we pick 9 points $x_1, ..., x_9$: $x_5, ..., x_8$ are the vertices of Q, $x_1, ..., x_4$ are midpoints of the edges of Q and x_9 is the center of Q. Pick radii r, R, ρ so that:

(a) The spheres $S_1 = S(x_1, r), S_5 = S(x_5, R)$ are orthogonal.

(b) The (exterior) angle of intersection between the spheres $S(x_1, r), S_9 = S(x_9, \rho)$ equals $\pi/3$.

(c) The (exterior) angle of intersection between the spheres $S(x_5, R), S(x_9, \rho)$ equals $\pi/5$. (Actually the latter angle can be taken $\pi/4$ as well, but $\pi/3$ would not suffice.)

Then (by doing MAPLE calculations) for the radii r, R, ρ we get:

$$R \approx 0.8534646790, r \approx 0.5211506901, \rho \approx 0.6317819089$$

In particular, r and ρ are both less than $1/\sqrt{2} \approx 0.7071067810$.

Remark 3.2. *Here is a proof that $\rho < 1/\sqrt{2}$ without help of MAPLE. We have system of equations:*

$$\begin{cases} R^2 + r^2 = 1, \\ R^2 + \rho^2 + R\rho = 2, \\ r^2 + \rho^2 + \sqrt{2}\rho r = 1 \end{cases}$$

Observe that $R = R(\rho)$ and $r = r(\rho)$ are both decreasing functions of ρ, for $\rho > 0$. Let's estimate $R^2(\rho) + r^2(\rho)$ with $\rho = 1/\sqrt{2}$. We get:

$$R = \frac{\sqrt{13} - 1}{2\sqrt{2}}, r = \frac{\sqrt{3} - 1}{2}.$$

Then $R^2 + r^2 < 1$. Thus, by monotonicity, $R(\rho)^2 + r(\rho)^2 < 1$ for all $\rho \geq 1/\sqrt{2}$. Hence each triple of solutions of our system satisfies: $\rho < 1/\sqrt{2}$. From the equation $R^2 + r^2 = 1$ we also see that $R < 1 < \sqrt{2}$.

Similar calculation shows that $r < 1/\sqrt{2}$.

We then consider the collection of round balls $B(x_9, \rho)$, $B(x_i, r)$, $i = 1, ..., 4$ and $B(x_i, R)$, $i = 5, ..., 8$; see Figure 2. The condition $r < \sqrt{2}/2$ and our choice of the angles of intersection between the spheres imply that the nerve of the above collection of balls is the barycentric subdivision of Q. See Figure 1 for the Coxeter graph of the Coxeter group generated by inversions in the spheres $S_1, ..., S_9$.

Suppose now that Q^4 is a 4-cube in the cubulation $\mathcal{Q}$, apply the above construction to each 2-face of Q^4. Then the condition $\rho, r < \sqrt{2}/2$ implies that the the nerve $\mathcal{N}_{Q^4}$ of the resulting collection of balls $\{B_i\}$ is such that the canonical mapping $\mathcal{N}_{Q^4} \to \beta((Q^4)^{(2)})$ is a simplicial isomorphism.

Now we are ready to construct the covering $\{B_i : i = 1, ..., k\}$ of the 2-complex K. For each 2-face Q of K introduce the family of nine round spheres S_i constructed above, consider the inversions R_i is these spheres; the spheres S_i bound balls $\{B_i : i = 1, ..., k\}$. The fact that for each 4-cube Q^4 the mapping $\mathcal{N}_{Q^4} \to \beta((Q^4)^{(2)})$ is a simplicial isomorphism, implies that the mapping from the nerve of the covering $\{B_i : i = 1, ..., k\}$ to K is a simplicial isomorphism as well. Thus the exterior angles of intersections between the spheres equal $\frac{\pi}{2}$ and $\frac{\pi}{3}$, thus we can apply Poincare's fundamental polyhedron theorem [10] to ensure that the intersection of the complements to the balls B_i is a fundamental domain for the Moebius group G generated by the above reflections. $\square$

Remark 3.3. *Instead of collections of round balls based on a cubulation of $\mathbb{R}^4$ one could use a periodic triangulation of $\mathbb{R}^4$, however in this case the construction of a collection of balls covering the 2-skeleton of a 4-simplex would be more complicated.*

4 Proof of Theorem 1.1

Recall that the manifold M is almost parallelizable, i.e $M^\circ = M \setminus \{p\}$ is parallelizable; hence, by [7], there exists an immersion $f : M^\circ \to \mathbb{R}^4$. Let B denote a small open round ball centered at p and let M' denote the complement $M \setminus B$. We retain the notation f for the restriction $f|M'$. We next convert to the piecewise-cubical setting: Consider the pull-back of the standard cubulation $\mathcal{Q}$ of $\mathbb{R}^4$ to M' via f. The image $f(M')$ is compact; hence, after replacing if necessary $\mathcal{Q}$ by its sufficiently fine cubical subdivision, we may assume that for each point $x \in M'$ there exists a cube $Q_x \in \mathcal{Q}$ and a component $\tilde{Q}_x \subset M^\circ$ of $f^{-1}(Q)$ so that $f|\tilde{Q}_x : \tilde{Q}_x \to Q_x$ is a homeomorphism. In particular, if $C \subset M'$ denotes the union of the cubes $\tilde{Q}_x, x \in M'$, then C is a compact cubical complex. After replacing B by a slightly larger open ball B' we get: $\partial B' \subset int(C)$. Let $M^\bullet$ denote the complement $M \setminus B'$. Now consider the 2-nd cubical subdivision C'' of C and the regular neighorhood $N := N(Fr_M(C))$ in C'' of the frontier $Fr_M(C)$: The frontier of N in C is a 3-dimensional submanifold $Y \subset B'$ which is contained in the 3-skeleton of C''. Thus $f(Y)$ is also contained in the 3-skeleton of $\mathcal{Q}''$. Let M'' denote the closure of the component of $M \setminus Y$ which is disjoint from p. Clearly, M'' is a compact cubulated manifold with the boundary Y. We retain the notation f for the restriction $f|M''$. We now double M'' across its boundary Y, the result is a closed cubulated manifold DM, let $\tau : DM \to DM$ denote the involution fixing Y pointwise; the mapping f extends to $DM \setminus M''$ by $f \circ \tau$. Thus we get a globally defined pieciewise-linear map $F : DM \to \mathbb{R}^4$, which is a homeomorphism on each 4-cube in DM and is a local diffeomorphism on $M^\bullet$. By cutting DM along the sphere $\partial M^\bullet$ we get a connected sum decomposition $DM = M \# W$. We orient DM so that F preserves the orientation on $M^\bullet$.

We now borrow the standard arguments from the proof of Alexander's theorem which states that each closed n-dimensional PL manifold is a branched cover over the n-sphere, see e.g. [4]. For each cube $Q' \subset DM$ such that $F|Q'$ is orientation-reversing we replace $F|Q'$ with the composition $J \circ F|Q'$, where J is the PL *inversion* in the boundary of the unit cube $F(\partial Q')$ (see Definition 2.1). The resulting mapping $h : DM \to S^4$ has the property that it is a local PL homeomorphism away from a 2-dimensional subcomplex $L \subset DM \setminus M''$, which is therefore disjoint from $M^\bullet$. (Note that L has dimension 2 near every point: Each vertex in L belongs to a 2-cube in L.) Thus the mapping h is a branched covering over S^4 with the singular locus L contained in $DM \setminus M^\bullet$, the branch-locus of h is the compact subcomplex $K = h(L) \subset \mathbb{R}^4$. The branched covering h has the property that for each point $x \in K$ there exists a neighborhood $U(x) \subset \mathbb{R}^4$ such that $h^{-1}(U(x))$ is a disjoint union of balls $V(y), y \in h^{-1}(x) \in DM \setminus M''$, (whose interiors contain y), so that for each $y \in h^{-1}(x)$, the restriction $h|V(y)$ is a branched covering onto $U(x)$. Moreover, each branched covering $h|V(y)$ is obtained by coning off a branched covering from the 3-sphere $\partial V(y)$ to the 3-sphere $U(x)$.

Question 4.1. *Given a local diffeomorphism $f : M^\circ \to \mathbb{R}^4$, is it possible to modify f within the ball B to make it a branched cover $M \to S^4$ which is ramified over a smooth surface in S^4? This can be easily arranged in the case of 3-manifolds. In dimension 4 this would be is a relative version of a recent theorem of Iori and Piergallini [8].*

Let T denote a regular neighborhood of K in $\mathbb{R}^4$, so that $U(x) \subset T$ for each

$x \in K$. Next, subdivide the cubulation of $\mathbb{R}^4$ and scale the subdivision up to the standard cubulation $\mathcal{Q}$ by regular cubes with edges of length 2, so that the discrete group G and the collection of balls $\{B_j, j = 1, ..., k\}$ associated with the subcomplex K in section 3 have the properties:

1. $T \subset \cup_{i=1}^{k} B_k$.

2. Each ball B_j, $j = 1, ..., k$, (centered at $x_j \in K$) is contained in the neighborhood $U(x_j)$.

We now use the branched covering h to introduce a Moebius orbifold structure O on the complement X_O to an open tubular neighborhood $\mathcal{N}^0(L)$ of L in M as follows:

For each ball $B_j \subset U(x_j)$ centered at $x_j \in K$ and for each $y_j \in h^{-1}(x_j) \cap L$, such that the restriction $h|V(y_j)$ is *not a homeomorphism onto its image*, we let $\tilde{B}(y_j)$ denote the inverse image $h^{-1}(B_j) \cap V(y_j)$. It follows that each $\tilde{B}(y_j)$ is a polyhedral 4-ball in M and the union of these balls is a tubular neighborhood $\mathcal{N}(L)$ of L. The boundary of $\mathcal{N}(L)$ has a natural partition into subcomplexes: "vertices", "edges", "2-faces" and "3-faces":

- The "vertices" are the points of triple intersections of the 3-spheres $\partial\tilde{B}(y_j)$, $\partial\tilde{B}(y_i)$, $\partial\tilde{B}(y_l)$.
- The "2-faces" are the connected components of the double intersections of the 3-spheres $\partial\tilde{B}(y_j)$, $\partial\tilde{B}(y_i)$.
- The "3-faces" are the connected components of the complements

$$\partial\tilde{B}(y_j) \setminus \cup_{i \neq j} \tilde{B}(y_i).$$

We declare each "3-face" a boundary reflector of the orbifold O. The dihedral angles between the balls B_j define the dihedral angles between the boundary reflectors in O. Since the restriction $h|M \setminus L$ is a local homeomorphism, this construction defines a Moebius orbifold O. The mapping $h|X_O$ is the projection of the developing mapping $\tilde{h} : \tilde{O} \to S^4$ of this Moebius orbifold. Let $O^\bullet$ denote the orbifold with boundary $O \setminus M^\bullet$; let O' be the closed orbifold obtained by attaching 4-disk D^4 to $O^\bullet$ along the boundary sphere S^3.

We next convert back to the smooth category. It is clear from the construction that the orbifold O is obtained by (smooth) gluing of the manifold with boundary $M \setminus B$ and the orbifold with boundary $O^\bullet$. Hence O is diffeomorphic to the connected sum of the manifold M with the orbifold O'.

It remains to construct a finite manifold covering $\hat{M}$ over the orbifold O, so that $M \setminus B$ lifts homeomorphically to $\hat{M}$; the construction is analogous to the one used by M. Davis in [2]. The universal cover $\tilde{O}$ is a manifold since it admits a (locally homeomorphic) developing mapping to S^4. The fundamental group $\pi_1(O)$ is the free product $\pi_1(M) * \pi_1(Q)$. We have holonomy homomorphism

$$\phi : \pi_1(O) \to G,$$

the subgroup $\pi_1(M)$ is contained in the kernel of this homomorphism; by construction, the kernel of ϕ acts freely on $\tilde{O}$. The Coxeter group G is virtually torsion-free, let θ :

$G \to A$ be a homomorphism onto a finite group A, so that $Ker(\theta)$ is torsion-free and orientation-preserving. Then the kernel of the homomorphism $\psi = \theta \circ \phi : \pi_1(O) \to A$ is a finite index subgroup of $\pi_1(O)$, which contains $\pi_1(M)$ and still acts freely on $\tilde{O}$. Let $\hat{M} \to O$ denote the finite orbifold cover corresponding to the subgroup $Ker(\psi)$. Then $\hat{M}$ is a smooth oriented conformally flat manifold, the submanifold $M^\bullet$ lifts diffeomorphically to $M^\bullet \subset \hat{M}$. Thus the connected sum decomposition $O = M\#O'$ also lifts to $\hat{M}$, so that the latter manifold is diffeomorphic to the connected sum of M and a 4-manifold N. $\square$

We observe that the proof of Theorem 1.1 can be modified to prove the following:

Theorem 4.2. *Suppose that M is a closed smooth 4-manifold whose orientable 2-fold cover is Spin. Then there exists a closed smooth 4-manifold N so that $\hat{M} = M\#N$ admits a conformally-Euclidean Riemannian metric.*

Proof: The difference with Theorem 1.1 is that M can be nonorientable. Let $\tilde{M} \to M$ be the orientable double cover with the deck-transformation group $D \cong \mathbb{Z}/2$. Then all Stiefel-Whitney classes of $\tilde{M}$ are trivial. As before, let $p \in M$, $\{p_1, p_2\}$ be the preimage of $\{p\}$ in M. Consider a Euclidean reflection τ in $\mathbb{R}^4$ and an epimorphism $\theta : D \to \langle\tau\rangle$. Then, arguing as in the proof of Hirsch's theorem [7, Theorem 4.7], one gets a θ-equivariant immersion $\tilde{f} : \tilde{M} \setminus \{p_1, p_2\} \to \mathbb{R}^4$. This yields a D-invariant flat conformal structure on $\tilde{M} \setminus \{p_1, p_2\}$ via pull-back of the flat conformal structure from $\mathbb{R}^4$. Let $B_1 \sqcup B_2$ be a D-invariant disjoint union of open balls around the points p_1, p_2. Then the rest of the proof of Theorem 1.1 goes through: Replace the ball B_1 with a manifold with boundary N_1 so that the flat conformal structure on $\tilde{M} \setminus (B_1 \cup B_2)$ extends over N_1. Then glue a copy of N_1 along the boundary of B_2 in D-invariant fashion. Note that the quotient of the manifold $P := (\tilde{M} \setminus (B_1 \cup B_2)) \cup (N_1 \cup N_2)$ by the group D is diffeomorphic to a closed manifold $M\#N$, where N is obtained from N_1 by attaching the 4-ball along the boundary. Finally, project the D-invariant Moebius structure on P to a Moebius structure on the manifold $M\#N$. $\square$

As a corollary of Theorem 1.1 we get:

Corollary 4.3. *Let Γ be a finitely-presented group. Then there exists a 3-dimensional complex manifold Z which admits a complex-projective structure, so that the fundamental group of Z splits as $\Gamma * \Gamma'$.*

Proof: Our argument is similar to the one used to construct (via Taubes' theorem) 3-dimensional complex manifolds with the prescribed finitely-presented fundamental group. We first construct a smooth closed oriented 4-dimensional spin manifold M with the fundamental group Γ. This can be done for instance as follows. Let $\langle x_1, ..., x_n | R_1, ..., R_\ell\rangle$ be a presentation of Γ. Consider a 4-manifold X which is the connected sum of n copies of $S^3 \times S^1$. This manifold is clearly spin. Pick a collection of disjoint embedded smooth loops $\gamma_1, ..., \gamma_\ell$ in X, which represent the conjugacy classes of the words $R_1, ..., R_\ell$ in the free group $\pi_1(X)$. Consider the pair (S^4, γ), where γ is an embedded smooth loop in S^4. For each i pick a diffeomorphism f_i between a tubular neighborhood $T(\gamma)$ of γ in S^4 and a tubular neighborhood $T(\gamma_i)$ of γ_i in X. We can choose f_i so that it matches the spin structures of $T(\gamma)$ and $T(\gamma_i)$. Now, attach n copies of $S^4 \setminus T(\gamma)$ to $X \setminus \cup_i T(\gamma_i)$ via the diffeomorphisms f_i. The result is a smooth spin 4-manifold M with the fundamental group Γ.

Next, by Theorem 1.1 there exists a smooth 4-manifold N (with the fundamental group Γ') such that $\hat{M} = M \# N$ admits a conformally-Euclidean Riemannian metric. Applying the twistor construction to the manifold $\hat{M}$ we get a complex 3-manifold Z which is an S^2-bundle over $\hat{M}$ and the flat conformal structure on $\hat{M}$ lifts to a complex-projective structure on Z, see for instance [6]. Clearly, $\pi_1(Z) \cong \pi_1(\hat{M}) = \Gamma * \Gamma'$. $\square$

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