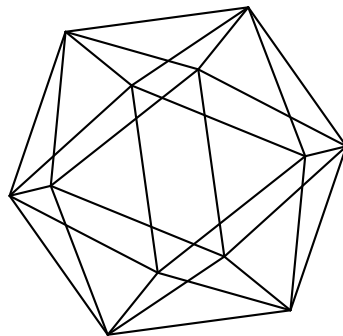


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THE MASS GAP PROBLEM FOR THE YANG–MILLS FIELD

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ABSTRACT. We consider the reduced Hamiltonian of the Yang–Mills field on \mathbb{R}^4 equipped with a Lorentzian metric. We show that the secondary quantized principal term H_0 of the Taylor expansion of this Hamiltonian at the lowest energy point has a mass gap if and only if zero is not a point of the spectrum of the auxiliary self-adjoint operator $\text{curl} = *d$ defined on the space of one-forms ω on \mathbb{R}^3 satisfying the condition $\text{div } \omega = *d*\omega = 0$, where $*$ is the Hodge star operator associated to a metric on \mathbb{R}^3 and d is the exterior differential. In this case the classical lowest energy point of the reduced configuration space is a non-degenerate critical point of the potential energy term of the reduced Hamiltonian of the Yang–Mills field, in the sense of Palais.

INTRODUCTION

The main objective of this paper is the study of the classical and of the quantum low energy behavior of the reduced Hamiltonian of the Yang–Mills field on \mathbb{R}^4 equipped with a non-degenerate Lorentzian metric $g_{\mu\nu}$, $\mu, \nu = 0, 1, 2, 3$ of signature $(+, -, -, -)$ and associated to a compact semisimple Lie group K with Lie algebra \mathfrak{k} .

It is well known that the Yang–Mills field is an example of Hamiltonian systems with first class constraints in Dirac’s terminology (see Sect. 1 and [6]), and hence its effective reduced phase space is obtained by Hamiltonian reduction, namely by the reduction of the cotangent bundle to the space of connections \mathcal{D} on \mathbb{R}^3 with respect to the action induced by the action of the gauge group \mathcal{K} on \mathcal{D} . The Hamiltonian H of the Yang–Mills field is invariant under this action and gives rise to a reduced Hamiltonian H_{red} on the reduced phase space (see Sect. 2, 3). H is expressed in terms of a natural Riemannian metric $\langle \cdot, \cdot \rangle$ on \mathcal{D} induced by the Killing form of \mathfrak{k} and by the metric $g_{\mu\nu}$. This metric is invariant under the gauge action. Thus \mathcal{D}/\mathcal{K} is equipped with a metric as well. For the sake of the Hamiltonian formulation we choose synchronous coordinates on \mathbb{R}^4 , so $g_{00} = 1$, $g_{0i} = 0$, and assume that $-g_{ij} = k_{ij}$ is a positive definite metric on \mathbb{R}^3 independent of $t = x^0$.

The Hamiltonian H_{red} is quadratic in canonical momenta (its kinetic energy term is the square of the canonical momentum with respect to the metric $\langle \cdot, \cdot \rangle$) and its behavior at low energies is essentially guided by the leading term H_0 of its Taylor expansion at the point of the reduced space corresponding to the gauge orbit of the trivial connection. H_0 is quadratic in both canonical coordinates and momenta and can be reduced to a standard form using the standard simultaneous diagonalization of the two quadratic forms defining the kinetic and the potential energy terms in the Hamiltonian H_0 . This amounts to diagonalization of the self-adjoint operator $\text{curl} = *d$ on the space of one-forms ω on \mathbb{R}^3 with values in \mathfrak{k} and satisfying the condition $\text{div } \omega = *d*\omega = 0$, where $*$ is the Hodge star operator associated to the metric k_{ij} on \mathbb{R}^3 and d is the exterior differential.

After that the standard secondary quantization procedure can be applied to H_0 , and we show that the spectrum of the quantized Hamiltonian H_0 has a mass gap m , i.e. the infimum of the spectrum of the secondary quantized Hamiltonian H_0 is m , if and only if zero does not belong to the spectrum of the operator curl . In this case m is the distance from zero to the spectrum of curl .

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The spectrum of curl is closely related to the spectrum of the Hodge-Laplace operators on one-forms and on functions on \mathbb{R}^3 . In particular, if the spectra of both of these operators do not contain zero then the spectrum of the quantized Hamiltonian H_0 has a mass gap.

In the case of the Laplace operator defined on functions a review of the relevant results can be found e.g. in [9, 18]. For instance, if the sectional curvature of the Levi-Civita connection of k_{ij} is asymptotically bounded at infinity by a negative number then the corresponding Laplace operator has a gap in the spectrum separating it from zero (see [8]).

A mass gap can also be achieved even if zero belongs to the spectrum of curl by modifying the reduced phase space. Namely, the Hamiltonians H_{red} and H_0 should be restricted to a certain subspace of the reduced phase space which is defined in such a way that the spectrum of the quantization of the restriction of H_0 to this subspace has a mass gap. The modification amounts to a certain cut-off procedure which presumably should naturally appear as a renormalization procedure in the course of a proper quantization of H_{red} .

If the quantized Hamiltonian H_0 has a mass gap then at the classical level the Hamiltonian H_{red} acquires an interesting property: the gauge orbit of the trivial connection becomes a non-degenerate critical point of the potential energy term of H_{red} in the sense of Palais (see [13]). As one can observe all field theories with a mass gap have this property.

The Morse-Palais lemma asserts now that there is a local coordinate chart in \mathcal{D}/\mathcal{K} containing the gauge orbit of the trivial connection in which the potential energy term of H_{red} is equal to its second differential regarded as a quadratic form in coordinates on \mathcal{D} . Thus in the corresponding chart of the reduced phase space H_{red} is reduced to the sum of the square of the momentum with respect to the Riemannian metric on \mathcal{D}/\mathcal{K} and of a quadratic potential term. On this chart one can consider H_{red} as a deformation of H_0 by means of a deformation of a constant metric on \mathcal{D}/\mathcal{K} in terms of which H_0 is expressed, to the metric $\langle \cdot, \cdot \rangle$.

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1. THE YANG-MILLS FIELD IN THE HAMILTONIAN FORMULATION

In this section we recall the Lagrangian and the Hamiltonian formalism for the Yang-Mills field in presence of external gravity (see e.g. [12]). The canonical variables and the Hamiltonian will be obtained via the Legendre transform starting from the Lagrangian formulation. The general framework for the Hamiltonian approach in field theory is due to Dirac (see [6, 7]). We follow the approach developed in [10] for the Yang-Mills field.

Let K be a compact semisimple Lie group, \mathfrak{k} its Lie algebra and \mathfrak{g} the complexification of \mathfrak{k} . We denote by (\cdot, \cdot) the Killing form of \mathfrak{g} . Recall that the restriction of this form to \mathfrak{k} is non-degenerate and negatively defined. We shall consider the affine space of smooth connections in the trivial K -bundle, associated to the adjoint representation of K , over the four-dimensional space-time $M = \mathbb{R}^4$ with the standard coordinates (x^0, x^1, x^2, x^3) . Fixing the standard trivialization of the trivial K -bundle and the trivial connection as an origin in the affine space of connections we can identify this space with the space $\Omega^1(M, \mathfrak{k})$ of \mathfrak{k} -valued 1-forms on M . Let $\mathcal{A} \in \Omega^1(M, \mathfrak{k})$ be such a connection which is also called sometimes the Yang-Mills field. Denote by \mathcal{F} the curvature 2-form of this connection, $\mathcal{F} = d\mathcal{A} + \frac{1}{2}[\mathcal{A} \wedge \mathcal{A}]$. Here as usual we denote by $[\mathcal{A} \wedge \mathcal{A}]$ the operation which takes the exterior product of \mathfrak{k} -valued 1-forms and the commutator of their values in \mathfrak{k} . We assume that M is equipped with a non-degenerate metric $g_{\mu\nu}$, $\mu, \nu = 0, 1, 2, 3$ of signature $(+, -, -, -)$. We assume that the spatial part g_{ij} , $i, j = 1, 2, 3$ of the metric is negatively defined and denote $k_{ij} = -g_{ij}$, so that k_{ij} is

positively defined. Denote $g = |\det g_{\mu\nu}|$ and $k = |\det k_{ij}|$. Note that

$$g^{00} = \frac{\det k_{ij}}{\det g_{\mu\nu}} = \frac{k}{g} > 0$$

as both $\det k_{ij}$ and $\det g_{\mu\nu}$ are negative.

We shall consider the Yang–Mills action functional YM which is defined by the formula

$$(1.1) \quad YM(\mathcal{A}) = \frac{1}{2} \int_M (\mathcal{F} \wedge, * \mathcal{F}),$$

where $*$ stands for the Hodge star operation associated to the metric $g_{\mu\nu}$, and we evaluate the Killing form on the values of \mathcal{F} and $*\mathcal{F}$ and also take their exterior product.

Next we pass from the Lagrangian to the Hamiltonian formulation for the Yang–Mills field. To this end one should use the modified action YM' ,

$$(1.2) \quad YM' = \int_M \left(\left(d\mathcal{A} + \frac{1}{2} [\mathcal{A} \wedge \mathcal{A}] - \frac{1}{2} \mathcal{F} \right) \wedge, * \mathcal{F} \right),$$

where \mathcal{A} and \mathcal{F} should be regarded as independent variables. The equations of motion obtained from the action functional YM' are equivalent to those derived from action (1.1). Indeed, the equation for \mathcal{F} following from (1.2) is just the definition of the curvature, and the other equation becomes the equation derived from action (1.1) after expressing \mathcal{F} in terms of \mathcal{A} .

We denote by A the “three–dimensional Euclidean part” of \mathcal{A} , $A = A_k dx^k$, where $A_k = A_k$ for $k = 1, 2, 3$. We also introduce the “electric” field E and the “magnetic” field B associated to \mathcal{F} as follows:

$$E^k = \mathcal{F}^{k0} \sqrt{g},$$

$$B = *^3 F, \quad F = d_3 A + \frac{1}{2} [A \wedge A],$$

where $*^3$ is the Hodge star operation with respect to the metric k_{ij} on \mathbb{R}^3 and d_3 is the exterior differential for exterior forms on \mathbb{R}^3 regarded as the subspace of M with coordinates (x^1, x^2, x^3) . Thus F is the “three–dimensional” spatial part of \mathcal{F} .

Using this notation action (1.2) can be rewritten, after integration by parts and up to the integral of a divergence, in the following form:

$$(1.3) \quad \begin{aligned} YM' = \int_M & \left(-(\partial_0 A_i, \sqrt{g} \mathcal{F}^{i0}) - \frac{1}{2} (\mathcal{F}_{i0}, \mathcal{F}^{i0} \sqrt{g}) + \frac{1}{4} (\mathcal{F}_{ij}, \mathcal{F}^{ij} \sqrt{g}) - \right. \\ & \left. - (\mathcal{A}_0, \partial_i (\sqrt{g} \mathcal{F}^{i0}) + [A_i, \sqrt{g} \mathcal{F}^{i0}]) \right) d^4 x, \end{aligned}$$

where, as usual, $\partial_\mu = \frac{\partial}{\partial x^\mu}$.

Denote $C = C(A, E) = \partial_i (E^i) + [A_i, E^i]$, and introduce an orthonormal basis T_a , $a = 1, \dots, \dim \mathfrak{k}$ in \mathfrak{k} with respect to the Killing form, $(T_a, T_b) = -\delta_{ab}$, and the components of A , E , \mathcal{A}_0 and C associated to this basis, $A_k = A_k^a T_a$, $E^k = E^{k,a} T_a$, $\mathcal{A}_0 = \mathcal{A}_0^a T_a$, $C = C^a T_a$. After a straightforward and tedious calculation action (1.3) can be rewritten in terms of these components as follows

$$(1.4) \quad YM' = \int_M \left(E^{i,a} \partial_0 A_i^a - h(A, E) + \mathcal{A}_0^a C^a \right) d^4 x,$$

where

$$h(A, E) = \frac{1}{\sqrt{g^{00}}} \left(\frac{1}{2} \frac{1}{\sqrt{k}} k_{ij} E^{i,a} E^{j,a} + \frac{1}{4} F_{ij}^a F_{kl}^a k^{ik} k^{jl} \sqrt{k} \right) + g^{0j} F_{ij}^a E^{i,a},$$

and

$$k^{ij} = -g^{ij} + \frac{g^{i0} g^{j0}}{g^{00}}$$

is the inverse matrix to k_{ij} . From formula (1.4) it is clear that A_k^a and $E^{k,a}$ are canonical conjugate coordinates and momenta for the Yang–Mills field, $h(A, E)$ is the Hamiltonian density, \mathcal{A}_0^a are Lagrange multipliers and $C^a = 0$ are constraints imposed on the canonical variables.

The equations obtained from action (1.1) become Hamiltonian with respect to the canonical Poisson structure

$$(1.5) \quad \{E^{k,a}(x), A_l^b(y)\} = \delta_l^k \delta^{ab} \delta(x - y),$$

and all the other Poisson brackets of the components of E and A vanish. One can also check that

$$(1.6) \quad \{C^a(x), C^b(y)\} = \sum_c t^{abc} C^c(x) \delta(x - y),$$

where t^{abc} are the structure constants of Lie algebra \mathfrak{k} with respect to the basis T_a , $[T_a, T_b] = \sum_c t^{abc} T_c$, and that

$$(1.7) \quad \{H(A, E), C^a(x)\} = 0,$$

where

$$H(A, E) = \int_{\mathbb{R}^3} h(A, E) d^3x$$

is the Hamiltonian. This means that the Yang–Mills field is a generalized Hamiltonian system with first class constraints according to Dirac’s classification (see [6]).

2. THE STRUCTURE OF THE PHASE SPACE OF THE YANG–MILLS FIELD

In this section we collect some facts on the Poisson geometry of the phase space of the Yang–Mills field and related gauge actions. These results are well known (see [2, 10, 11, 16, 17, 19]).

Consider the affine space of smooth connections in the trivial K -bundle over \mathbb{R}^3 associated to the adjoint representation of K . As in Section 1 we fix the standard trivialization of this bundle and the trivial connection as an origin in the affine space of connections and identify this space with the space $\Omega^1(\mathbb{R}^3, \mathfrak{k})$ of \mathfrak{k} -valued 1-forms on \mathbb{R}^3 . Let \mathcal{D} be the space of W_2^1 -Sobolev connections (see [11], §3 or [19], Appendix B). \mathcal{D} is the space of \mathfrak{k} -valued one-forms on \mathbb{R}^3 with components in $W_2^1(\mathbb{R}^3) \otimes \mathfrak{k}$, where in the definition of the Sobolev space the components are differentiated with the help of the covariant derivative of the Levi-Civita connection corresponding to the metric k_{ij} .

The tangent space $T_A \mathcal{D}$ to \mathcal{D} at point $A \in \mathcal{D}$ is isomorphic to \mathcal{D} . We define a scalar product on this space by

$$(2.1) \quad \langle \omega_1, \omega_2 \rangle = - \int_{\mathbb{R}^3} (\omega_1 \wedge *^3 \omega_2),$$

where $\omega_{1,2} \in T_A \mathcal{D}$, $*^3$ stands for the Hodge star operation associated to the metric k_{ij} on \mathbb{R}^3 , and we evaluate the Killing form on the values of ω_1 and $*\omega_2$ and also take their exterior product. Thus the space \mathcal{D} is equipped with a Riemannian metric.

Let $T_A^* \mathcal{D}$ be the space of W_2^1 -Sobolev \mathfrak{k} -valued contravariant densities on \mathbb{R}^3 of weight one with respect to the metric k_{ij} . Elements of this space are of the form $\sqrt{k}V$, where V is \mathfrak{k} -valued vector field on \mathbb{R}^3 with components in $W_2^1(\mathbb{R}^3) \otimes \mathfrak{k}$.

There is a natural pairing between this space and the space $T_A \mathcal{D}$ which we denote by the same symbol as the scalar product on $T_A \mathcal{D}$,

$$(2.2) \quad \langle E, A \rangle = \int_M E^{i,a} A_i^a d^3x, \quad E = E^{i,a} T_a \frac{\partial}{\partial x^i} \in T_A^* \mathcal{D}, \quad A = A_i^a T_a dx^i \in T_A \mathcal{D}.$$

Using this pairing one can define an induced scalar product on $T_A^*\mathcal{D}$ which we denote again by the same symbol as the scalar product on $T_A\mathcal{D}$,

$$\langle Z, W \rangle = \int_M \frac{1}{\sqrt{k}} k_{ij} Z^{i,a} W^{j,a} d^3x, \quad Z = Z^{i,a} T_a \frac{\partial}{\partial x^i}, \quad W = W^{i,a} T_a \frac{\partial}{\partial x^i} \in T_A^*\mathcal{D}.$$

Pairing (2.2) gives rise to an embedding of the space $T_A^*\mathcal{D}$ into the space dual to $T_A\mathcal{D}$. Therefore the bundle $T^*\mathcal{D}$ over \mathcal{D} the fiber of which at point $A \in \mathcal{D}$ is $T_A^*\mathcal{D}$ naturally becomes a subbundle of the cotangent bundle to \mathcal{D} .

Using the last imbedding the bundle $T^*\mathcal{D}$ can be equipped with the natural structure of a Poisson manifold induced by the canonical symplectic structure of the cotangent bundle to \mathcal{D} .

Assume from now on that the coordinates x^μ are synchronous for the metric $g_{\mu\nu}$, i.e. $g_{00} = g^{00} = 1$, $g^{0i} = g_{0i} = 0$, and hence $g = k$. We shall also always suppose that k_{ij} is a positive definite metric on \mathbb{R}^3 independent of $t = x^0$. In this case Poisson structure (1.5) has a natural geometric interpretation in terms of the Poisson structure of $T^*\mathcal{D}$.

Indeed, for each A the “electric” field E belongs to the space of \mathfrak{k} -valued contravariant densities on \mathbb{R}^3 of weight one with respect to the metric k_{ij} which contains $T_A^*\mathcal{D}$ as a subspace. The Poisson structure (1.5) on the space $T^*\mathcal{D}$ can be identified with the natural Poisson structure on it induced by the canonical symplectic structure of the cotangent bundle to \mathcal{D} .

Now let us discuss the meaning of the constrains. First of all we note that the constrains $C(A, E)$ infinitesimally generate the gauge action on the phase space $T^*\mathcal{D}$. More precisely, consider the group \mathcal{K} of K -valued W_2^2 -Sobolev maps $h : \mathbb{R}^3 \rightarrow K$ satisfying the condition $\lim_{x \rightarrow \infty} h(x) = I$, where I is the identity element of K (for the definition of \mathcal{K} see [19], Appendix A, [11], §3 or [17] for the case of a non-compact base). \mathcal{K} is a Lie group called the gauge group.

The gauge group \mathcal{K} continuously acts on the space of connections \mathcal{D} (see e.g. [19], Lemma A.6, [17], Theorem 1) by

$$(2.3) \quad \begin{aligned} \mathcal{K} \times \mathcal{D} &\rightarrow \mathcal{D}, \\ h \times A &\mapsto h \circ A = -dh h^{-1} + h A h^{-1}, \end{aligned}$$

where we denote $h A h^{-1} = \text{Ad}h(A)$.

The action (2.3) of \mathcal{K} on the space of connections \mathcal{D} induces an action

$$(2.4) \quad \begin{aligned} \mathcal{K} \times T^*\mathcal{D} &\rightarrow T^*\mathcal{D}, \\ h \times (A, E) &\mapsto (-dh h^{-1} + h A h^{-1}, h E h^{-1}), \end{aligned}$$

where we write $h E h^{-1} = \text{Ad}h(E)$. This action gives rise to an action of the Lie algebra of the gauge group \mathcal{K} on $T^*\mathcal{D}$ by vector fields. If X is an element of the Lie algebra of \mathcal{K} then the corresponding vector field $V_X(A, E)$ is given by

$$(2.5) \quad V_X(A, E) = (-dX + [X, A], [X, E]), \quad E \in T_A^*\mathcal{D}, A \in \mathcal{D} \simeq T_A\mathcal{D}.$$

Here we, of course, identify $T_{(A,E)}T^*\mathcal{D} \simeq T_A\mathcal{D} \times T_A^*\mathcal{D}$.

The gauge action (2.3) is free, so that the quotient \mathcal{D}/\mathcal{K} is a smooth manifold (see [16]). Indeed, if X belongs to the Lie algebra of the stabilizer of a point A then $dX + [A, X] = 0$, and by the invariance of the Killing form we have

$$d(X, X) = (dX + [A, X], X) + (X, dX + [A, X]) = 0,$$

i.e. (X, X) is constant, and hence $\lim_{x \rightarrow \infty} X(x) \neq 0$, i.e. X is not in the Lie algebra of \mathcal{K} .

The action (2.5) is generated by the constraint $C(A, E)$ in the sense that if X is an element of the Lie algebra of \mathcal{K} and $(A, E) \in T^*\mathcal{D}$ then we have

$$\left\{ \int_{\mathbb{R}^3} (C(A, E), X) d^3x, A(x) \right\} = -dX(x) + [X(x), A(x)],$$

and

$$\left\{ \int_{\mathbb{R}^3} (C(A, E), X) d^3x, E(x) \right\} = [X(x), E(x)].$$

Using the language of Poisson geometry and taking into account formula (1.6) for the Poisson brackets of the constraints one can say that $\mathcal{K} \times T^*\mathcal{D} \rightarrow T^*\mathcal{D}$ is a Hamiltonian group action, and the map

$$(2.6) \quad \mu(A, E) = C(A, E)$$

is a moment map for this action. In particular, action (2.4) preserves the Poisson structure of $T^*\mathcal{D}$.

We note that action (2.3) also preserves the Riemannian structure of the configuration space \mathcal{D} . This follows from the fact that the Killing form on \mathfrak{k} is invariant with respect to the adjoint action of K .

The properties of the phase space of the Yang–Mills field and of the gauge action discussed above are formulated in the following proposition.

Proposition 2.1. *Let \mathcal{D} be the space of W_2^1 -Sobolev K -connections on \mathbb{R}^3 , \mathcal{K} the corresponding group of W_2^2 -Sobolev gauge transformations vanishing at infinity. Then*

(i) *The space \mathcal{D} is an infinite dimensional Riemannian manifold equipped with metric (2.1). The bundle $T^*\mathcal{D}$ can be equipped with the natural structure of a Poisson manifold induced by the canonical symplectic structure of the cotangent bundle to \mathcal{D} ;*

(ii) *The gauge action $\mathcal{K} \times \mathcal{D} \rightarrow \mathcal{D}$ preserves Riemannian metric (2.1) and gives rise to a Hamiltonian group action $\mathcal{K} \times T^*\mathcal{D} \rightarrow T^*\mathcal{D}$ with the moment map*

$$\mu(A, E) = C(A, E), \quad (A, E) \in T^*\mathcal{D};$$

(iii) *The action of the gauge group \mathcal{K} on the spaces \mathcal{D} and $T^*\mathcal{D}$ is free, and the reduced phase space $\mu^{-1}(0)/\mathcal{K}$ is a smooth manifold.*

Finally we make a few remarks on the structure of the Hamiltonian of the Yang–Mills field.

In terms of the above defined scalar products, and under the assumption that the coordinates x^μ are synchronous for the metric $g_{\mu\nu}$, the Hamiltonian $H(A, E)$ takes a very simple form,

$$(2.7) \quad H(A, E) = \frac{1}{2}(\langle E, E \rangle + \langle B, B \rangle).$$

Since the Hamiltonian $H(A, E)$ of the Yang–Mills field is invariant under gauge action (2.4) (this fact can be checked directly and also follows from formula (1.7)) the generalized Hamiltonian dynamics described by this Hamiltonian together with the constraints $C(A, E) = 0$ is equivalent to the usual one on the reduced phase space $\mu^{-1}(0)/\mathcal{K}$ (see [1, 6]).

The Hamiltonian (2.7) itself has a very standard structure; $H(A, E)$ is equal to the sum of a half of the square of the canonical momentum, $\frac{1}{2} \langle E, E \rangle$, and of a potential $U(A)$, $U(A) = \frac{1}{2} \langle B, B \rangle$. The potential $U(A)$ is, in turn, equal to a half of the square of the vector field $B \in \Gamma(T\mathcal{D})$. By definition the vector field B is invariant with respect to the gauge action of \mathcal{K} , $B(g \circ A) = gB(A)g^{-1}$.

Let $\Omega^n(\mathbb{R}^3, \mathfrak{k})$ be the space of \mathfrak{k} -valued differential n -forms on \mathbb{R}^3 . We recall that the covariant derivative $d_A : \Omega^n(\mathbb{R}^3, \mathfrak{k}) \rightarrow \Omega^{n+1}(\mathbb{R}^3, \mathfrak{k})$ associated to a connection $A \in \Omega^1(\mathbb{R}^3, \mathfrak{k})$ is defined by $d_A \omega = d_3 \omega + [A \wedge \omega]$, and the operator $d_A^* : \Omega^{n+1}(\mathbb{R}^3, \mathfrak{k}) \rightarrow \Omega^n(\mathbb{R}^3, \mathfrak{k})$ formally adjoint to d_A with respect to the scalar product on $\Omega^*(\mathbb{R}^3, \mathfrak{k})$ given by formula (2.1) is equal to $d_A^* = (-1)^{3(n+1)} *^3 d_A *^3$.

We denote by div_A the part of this operator acting from $\Omega^1(\mathbb{R}^3, \mathfrak{k})$ to $\Omega^0(\mathbb{R}^3, \mathfrak{k})$, with the opposite sign,

$$\operatorname{div}_A = *^3 d_A *^3 : \Omega^1(\mathbb{R}^3, \mathfrak{k}) \rightarrow \Omega^0(\mathbb{R}^3, \mathfrak{k}).$$

Note that under the isomorphism

$$T_A^* \mathcal{D} \simeq T_A \mathcal{D}, \quad E \mapsto E'$$

induced by pairing (2.2) the operator div_A corresponds to $C(A, \cdot)$. More precisely, if $E = \sqrt{k} E^i \frac{\partial}{\partial x^i} \in T_A^* \mathcal{D}$ then the corresponding element of $T_A \mathcal{D}$ is $E' = E^i k_{ij} dx^j$, and $C(A, E) = \sqrt{k} \operatorname{div}_A E'$.

The value of the vector field B at each point $A \in \mathcal{D}$ belongs to the kernel of the operator div_A , $\operatorname{div}_A B(A) = 0$ for all $A \in \mathcal{D}$. Indeed, from the Bianci identity $d_A F = 0$, the definition of $B = *^3 F$ and the formula $*^3 *^3 = id$ it follows that

$$\operatorname{div}_A B = *^3 d_A *^3 *^3 F = *^3 d_A F = 0.$$

Now we summarize the properties of the Hamiltonian of the Yang–Mills field.

Proposition 2.2. (i) *The generalized Hamiltonian system on the Poisson manifold $T^* \mathcal{D}$ with the Hamiltonian $H(A, E)$, $H(A, E) = \frac{1}{2}(\langle E, E \rangle + \langle B, B \rangle)$, $B = *^3 F$, $F = d_3 A + \frac{1}{2}[A \wedge A]$, and the constrains $C(A, E) = 0$ describes the Yang–Mills dynamics on $T^* \mathcal{D}$.*

(ii) *The Hamiltonian $H(A, E)$ is invariant under the gauge action $\mathcal{K} \times T^* \mathcal{D} \rightarrow T^* \mathcal{D}$ and the generalized Hamiltonian dynamics described by this Hamiltonian together with the constrains $C(A, E) = 0$ is equivalent to the usual one on the reduced phase space $\mu^{-1}(0)/\mathcal{K}$.*

(iii) *The vector field B is invariant with respect to the gauge action of \mathcal{K} , $B(g \circ A) = gB(A)g^{-1}$. The value of this field at each point $A \in \mathcal{D}$ belongs to the kernel of the operator div_A , $\operatorname{div}_A B(A) = 0$ for all $A \in \mathcal{D}$.*

3. THE STRUCTURE OF THE REDUCED PHASE SPACE: A MODEL CASE

In Propositions 2.1 and 2.2 we formulated all properties of the Yang–Mills field which are important for our further consideration. In this section we study an arbitrary Hamiltonian system satisfying these properties. For simplicity, in this section we only consider the finite-dimensional case.

First we consider a phase space equipped with a Lie group action of the type described in Proposition 2.1. Actually the Riemannian metric introduced in that proposition is only important for the definition of the Hamiltonian of the Yang–Mills field. This metric is not relevant to Poisson geometry. We used this metric in the description of the phase space in order to avoid analytic difficulties arising in the infinite-dimensional case. Now let us forget about the metric for a moment and discuss the geometry of the reduced space.

The Poisson structure described in Proposition 2.1 is an example of the canonical Poisson structure on the cotangent bundle, and the group action on this bundle is induced by a group action on the base manifold. Thus we start with a manifold \mathcal{M} and a Lie group G freely acting on \mathcal{M} . The canonical symplectic structure on $T^* \mathcal{M}$ can be defined as follows (see [1]).

Denote by $\pi : T^* \mathcal{M} \rightarrow \mathcal{M}$ the canonical projection, and define a 1-form θ on $T^* \mathcal{M}$ by $\theta(v) = p(d\pi v)$, where $p \in T_x^* \mathcal{M}$ and $v \in T_{(x,p)}(T^* \mathcal{M})$. Then the canonical symplectic form on $T^* \mathcal{M}$ is equal to $d\theta$.

Recall that the induced Lie group action $G \times T^* \mathcal{M} \rightarrow T^* \mathcal{M}$ is a Hamiltonian group action with a moment map $\mu : T^* \mathcal{M} \rightarrow \mathfrak{g}^*$, where \mathfrak{g}^* is the dual space to the Lie algebra \mathfrak{g} of G . The moment map μ is uniquely determined by the formula (see [14], Theorem 1.5.2)

$$(3.1) \quad (\mu(x, p), X) = \theta(\widehat{X})(x, p) = p(\widehat{X}(x)),$$

where \widehat{X} is the vector field on \mathcal{M} generated by an arbitrary element $X \in \mathfrak{g}$, \widehat{X} is the induced vector field on $T^*\mathcal{M}$ and (\cdot, \cdot) stands for the canonical pairing between \mathfrak{g} and \mathfrak{g}^* .

Formula (3.1) implies that for any $x \in \mathcal{M}$ the map $\mu(x, p)$ is linear in p . We denote this linear map by $m(x)$, $m(x) : T_x^*\mathcal{M} \rightarrow \mathfrak{g}^*$,

$$(3.2) \quad m(x)p = \mu(x, p).$$

Next, following [1], Appendix 5, with some modifications of the proofs suitable for our purposes, we describe the structure of the reduced space $\mu^{-1}(0)/G$. We start with a simple lemma.

Lemma 3.1. *The annihilator $T_x\mathcal{O}^\perp \in T_x^*\mathcal{M}$ of the tangent space $T_x\mathcal{O}$ to the G -orbit $\mathcal{O} \subset \mathcal{M}$ at point x is isomorphic to $\text{Ker } m(x)$, $T_x\mathcal{O}^\perp = \text{Ker } m(x)$.*

Proof. First we note that the space $T_x\mathcal{O}^\perp$ is spanned by the differentials of G -invariant functions on \mathcal{M} . But from the definitions of the moment map and of the Poisson structure on $T^*\mathcal{M}$ we have

$$(3.3) \quad L_{\widehat{X}}f(x) = \{(X, \mu), f\}(x) = (X, m(x)df(x)),$$

where \widehat{X} is the vector field on \mathcal{M} generated by element $X \in \mathfrak{g}$, $f \in C^\infty(\mathcal{M})$, and (\cdot, \cdot) stands for the canonical pairing between \mathfrak{g} and \mathfrak{g}^* .

Formula (3.3) implies that f is G -invariant if and only if $df(x) \in \text{Ker } m(x)$. This completes the proof. □

Proposition 3.2. *The action of the group G on $T^*\mathcal{M}$ is free, and the reduced phase space $\mu^{-1}(0)/G$ is a smooth manifold. Moreover, we have an isomorphism of symplectic manifolds, $\mu^{-1}(0)/G \simeq T^*(\mathcal{M}/G)$, where $T^*(\mathcal{M}/G)$ is equipped with the canonical symplectic structure. Under this isomorphism $T_{\mathcal{O}_x}^*(\mathcal{M}/G) \simeq T_x\mathcal{O}_x^\perp$, where \mathcal{O}_x is the G -orbit of x .*

Proof. Let \mathcal{O}_x be the G -orbit of point $x \in \mathcal{M}$ and $\pi : \mathcal{M} \rightarrow \mathcal{M}/G$ the canonical projection, $\pi(x) = \mathcal{O}_x$. Denote by Ξ the foliation of the space \mathcal{M} by the subspaces $T_x\mathcal{O}_x^\perp$. Since the foliation Ξ is G -invariant and $\text{Ker } d\pi|_{T_x\mathcal{M}} = T_x\mathcal{O}_x$ we can identify the subspace $T_x\mathcal{O}_x^\perp$ with the space $T_{\mathcal{O}_x}^*(\mathcal{M}/G)$ by means of the dual map to the differential of the projection π . But the definition of the moment map μ and Lemma 3.1 imply that $\mu^{-1}(0) = \{(x, p) \in T^*\mathcal{M} : p \in T_x\mathcal{O}_x^\perp\}$. Therefore the quotient $\mu^{-1}(0)/G$ is diffeomorphic to $T^*(\mathcal{M}/G)$, the diffeomorphism being induced by the canonical projection π .

From the definitions of the Poisson structures on $T^*(\mathcal{M}/G)$ and on the reduced space $\mu^{-1}(0)/G$ it follows that the diffeomorphism $\mu^{-1}(0)/G \simeq T^*(\mathcal{M}/G)$ is actually an isomorphism of symplectic manifolds. □

Using the last proposition one can easily describe the space $\Gamma T^*(\mathcal{M}/G)$ of covector fields on \mathcal{M}/G .

Corollary 3.3. *The space $\Gamma T^*(\mathcal{M}/G)$ is isomorphic to the space of G -invariant sections $V \in \Gamma T^*\mathcal{M}$ such that $V(x) \in T_x\mathcal{O}_x^\perp$ for any $x \in \mathcal{M}$. Such covector fields will be called horizontal G -invariant covector fields on \mathcal{M} . We denote this space by $\Gamma_G^\perp T^*\mathcal{M}$, $\Gamma_G^\perp T^*\mathcal{M} \simeq \Gamma T^*(\mathcal{M}/G)$.*

Now we discuss the class of Hamiltonians on $T^*\mathcal{M}$ we are interested in. First, recalling Proposition 2.1 we equip the manifold \mathcal{M} with a Riemannian metric \langle, \rangle and assume that the action of G on \mathcal{M} preserves this metric. Using this metric we can establish an isomorphism of G -manifolds, $T\mathcal{M} \simeq T^*\mathcal{M}$. We shall always identify the tangent and the cotangent bundle of \mathcal{M} and the spaces of vector and covector fields on \mathcal{M} by means of this isomorphism. The tangent bundle $T\mathcal{M}$ will be regarded as a symplectic manifold with the induced symplectic structure. Under the identification $T\mathcal{M} \simeq T^*\mathcal{M}$ the subspace $T_x\mathcal{O}^\perp \subset T_x\mathcal{M}$ is isomorphic to the orthogonal complement of the tangent

space $T_x\mathcal{O}$ in $T_x\mathcal{M}$. Note also that since $T_{\mathcal{O}_x}^*(\mathcal{M}/G) \simeq T_x\mathcal{O}_x^\perp$, and the metric on \mathcal{M} is G -invariant, $T_{\mathcal{O}_x}^*(\mathcal{M}/G)$ has a scalar product induced from $T_x\mathcal{O}_x^\perp$, i.e. \mathcal{M}/G naturally becomes a Riemannian manifold. We shall also identify $T^*(\mathcal{M}/G) \simeq T(\mathcal{M}/G)$ by means of the metric. Denote by $\Gamma_{\frac{1}{G}}T\mathcal{M}$ the space of G -invariant horizontal vector fields on \mathcal{M} , i.e. vector fields $V(x)$ satisfying the condition $V(x) \in T_x\mathcal{O}_x^\perp$ for any $x \in \mathcal{M}$. By Corollary 3.3 we have an isomorphism, $\Gamma_{\frac{1}{G}}T\mathcal{M} \simeq \Gamma T(\mathcal{M}/G)$.

On the symplectic manifold $T\mathcal{M}$ we define a Hamiltonian H of the type described in Proposition 2.2. In order to do that we fix a G -invariant horizontal vector field V on \mathcal{M} . Then we put

$$H(x, p) = \frac{1}{2}(\langle p, p \rangle + \langle V(x), V(x) \rangle), \quad p \in T_x\mathcal{M}.$$

This Hamiltonian is obviously G -invariant and gives rise to a reduced Hamiltonian H_{red} on the reduced space $\mu^{-1}(0)/G \simeq T^*(\mathcal{M}/G)$. Since by Corollary 3.3 V can be regarded as a (co)vector field on \mathcal{M}/G we have

$$(3.4) \quad H_{red}(\mathcal{O}_x, p_\perp) = \frac{1}{2}(\langle p_\perp, p_\perp \rangle + \langle V(x), V(x) \rangle), \quad p_\perp \in T_x\mathcal{O}_x^\perp \simeq T_{\mathcal{O}_x}^*(\mathcal{M}/G).$$

4. LOW ENERGY BEHAVIOR OF THE YANG–MILLS HAMILTONIAN ON THE REDUCED PHASE SPACE AND ITS QUANTIZATION

In this section we describe a low energy approximation H_0 to the Hamiltonian of the Yang–Mills field on the reduced phase space (see Proposition 2.2) and obtain a necessary and sufficient condition under which the spectrum of the secondary quantized Hamiltonian H_0 has a mass gap. This condition implies that the gauge orbit of the trivial connection is a non-degenerate critical point for the potential $U(A)$ of the Hamiltonian of the Yang–Mills field on the space \mathcal{D}/\mathcal{K} .

First observe that according to Proposition 2.2 the vector field $B(A)$ on the space \mathcal{D} is \mathcal{K} -invariant and horizontal. Therefore from the last observation made in the previous section we infer that Hamiltonian (2.7) gives rise to the reduced Hamiltonian

$$(4.5) \quad H_{red}(\mathcal{O}_A, E_\perp) = \frac{1}{2}(\langle E_\perp, E_\perp \rangle + \langle B(A), B(A) \rangle), \quad C(A, E_\perp) = 0$$

on the reduced space $\mu^{-1}(0)/\mathcal{K}$.

As we observed above under the isomorphism $T_A^*\mathcal{D} \simeq T_A\mathcal{D}$, $E \mapsto E'$ induced by pairing (2.2) the operator div_A corresponds to $C(A, \cdot)$, and hence the space $T_{\mathcal{O}_A}\mathcal{D}/\mathcal{K}$ is isomorphic to the kernel of the operator div_A in $T_A\mathcal{D}$. The metric (2.1) induces a Riemannian metric on \mathcal{D}/\mathcal{K} which we denote by the same symbol.

Consider the scalar product on the space $\Omega_c^*(\mathbb{R}^3, \mathfrak{g})$ of all compactly supported \mathfrak{g} -valued forms on \mathbb{R}^3 given by the formula

$$(4.6) \quad \langle \omega_1, \omega_2 \rangle = - \int_{\mathbb{R}^3} (\omega_1 \wedge * \bar{\omega}_2), \quad \omega_{1,2} \in \Omega^*(\mathbb{R}^3, \mathfrak{g}),$$

where $\bar{\omega}_2$ is the complex conjugate of ω_2 with respect to the complex structure induced by the decomposition $\mathfrak{g} = \mathfrak{k} + i\mathfrak{k}$. Note that $\mathfrak{k} \subset \mathfrak{g}$ is a real subspace with respect to this complex structure.

Let \mathcal{H}^1 be the completion of the space $\Omega_c^1(\mathbb{R}^3, \mathfrak{g})$ of compactly supported \mathfrak{g} -valued 1-forms on \mathbb{R}^3 with respect to scalar product (4.6). We also denote by \mathcal{H}^0 the completion of the space $\Omega_c^0(\mathbb{R}^3, \mathfrak{g})$ with respect to scalar product (4.6). Then, clearly, the operator div_A is naturally extended to a linear operator $\text{div}_A : \Omega_c^1(\mathbb{R}^3, \mathfrak{g}) \rightarrow \Omega_c^0(\mathbb{R}^3, \mathfrak{g})$ and the closure of this operator is a well-defined closed operator $\text{div}_A : \mathcal{H}^1 \rightarrow \mathcal{H}^0$. Denote by $P_A : \mathcal{H}^1 \rightarrow \text{Ker } \text{div}_A$ the orthogonal projection operator onto the kernel of this operator. Note that both div_A and P_A preserve the natural real structure on \mathcal{H}^1 , i.e. they are real operators.

Locally the reduced configuration space \mathcal{D}/\mathcal{K} can be described as follows. For every point $A \in \mathcal{D}$ there exists an open neighborhood such that each \mathcal{K} -orbit locally intersects $\text{Ker div}_A \cap \mathcal{D}$ at a unique point (see, e.g., [16]). Therefore $\text{Ker div}_A \cap \mathcal{D}$ plays the role a local model (coordinate chart) for \mathcal{D}/\mathcal{K} in a neighborhood of the orbit \mathcal{O}_A of A .

We are primarily interested in the behavior of $H_{red}(\mathcal{O}_A, E_\perp)$ in a neighborhood of the orbit \mathcal{O}_0 of the trivial connection. Let $\mathcal{C} = \text{Ker div}_0|_{\mathcal{D}}$ be the coordinate chart for this neighborhood. For brevity we shall also write $\text{div}_0 = \text{div}$. $T^*\mathcal{C} = \{(A, E), A \in \mathcal{C}, E \in T_0^*(\mathcal{D}), C(0, E) = 0\} \simeq \{(A, E'), A, E' \in \mathcal{C}\}$ is a coordinate chart for $T^*\mathcal{D}/\mathcal{K}$ near the orbit \mathcal{O}_0 of the trivial connection.

Observe that P_A is the orthogonal projection operator onto Ker div_A , and $\text{Ker div}_A|_{\mathcal{D}} \simeq T_{\mathcal{O}_A}\mathcal{D}/\mathcal{K}$ as linear spaces equipped with scalar product induced by (2.1). Therefore in an open neighborhood of the orbit \mathcal{O}_0 and in terms of the local chart $T^*\mathcal{C}$ for this neighborhood the Hamiltonian H_{red} takes the form

$$(4.7) \quad H_{red}(A, E') = \frac{1}{2}(\langle P_A E', P_A E' \rangle + \langle B(A), B(A) \rangle), A \in W \cap \mathcal{C}, E' \in \mathcal{C},$$

where $W \subset \mathcal{C}$ is an open subset containing $A = 0$ (see [2], Sect. 4 for more details on the metric in terms of the coordinate chart $T^*\mathcal{C}$). Since $B(A) = *dA + *[A \wedge A]$, the first term of the Taylor expansion at $A = 0$ of Hamiltonian (4.7) is

$$(4.8) \quad H_0(A, E') = \frac{1}{2}(\langle E', E' \rangle + \langle *dA, *dA \rangle), A, E' \in \mathcal{C}.$$

To study this Hamiltonian we have to simultaneously diagonalize the quadratic forms $\langle E', E' \rangle$ and $\langle *dA, *dA \rangle$. This is achieved by observing that $\text{curl} = *d$ gives rise to a self-adjoint operator $\text{curl} : \text{Ker div} \rightarrow \text{Ker div}$ with the domain $\text{Ker div} \cap \mathcal{D}_{\mathbb{C}}$, where $\mathcal{D}_{\mathbb{C}}$ is the complexification of \mathcal{D} (see e.g. [15], Section 3.6).

Let

$$\Psi : \text{Ker div} \rightarrow \int_{\mathbb{R}} H(\lambda) d\mu(\lambda) = \mathcal{H}$$

be the corresponding generalized Fourier transform, where $\int_{\mathbb{R}} H(\lambda) d\mu(\lambda)$ is a direct Hilbert space integral and μ is a measure on \mathbb{R} . Note that integration in the formula above is actually taken over the support $\text{supp } \mu$ of μ which coincides with the spectrum $\sigma(\text{curl})$ of the operator $\text{curl} = *d$ on Ker div . Note that $\Psi \text{curl} \Psi^{-1} = M(\lambda)$ is the operator of multiplication by λ .

Let $e_i(\lambda)$, $i \in \mathbb{N}$ be an orthonormal basis in the Hilbert space $H(\lambda)$, so every element $f(\lambda) \in H(\lambda)$ is uniquely represented as the sum of a converging series $f(\lambda) = \sum_{i=1}^{\infty} f_i(\lambda) e_i(\lambda)$, $f_i(\lambda) \in \mathbb{C}$.

Now using the generalized Fourier transform Ψ one can introduce other canonical coordinates on $T^*\mathcal{C}$,

$$(4.9) \quad q_{\lambda, i}(A) = \Psi(A)_i(\lambda),$$

$$(4.10) \quad p_{\lambda, i}(E) = \Psi(E')_i(\lambda),$$

where we denote by $\Psi(\cdot)_i(\lambda)$ the components of the Fourier transform Ψ with respect to the basis $e_i(\lambda)$, $i \in \mathbb{N}$. They have the following canonical Poisson brackets

$$\{p_{\lambda, i}, q_{\lambda', j}\} = \delta_{ij} \delta_{\mu}(\lambda - \lambda'),$$

where $\delta_{\mu}(\lambda - \lambda')$ is the delta function associated to the measure μ .

In terms of coordinates (4.9), (4.10) Hamiltonian (4.8) can be rewritten in the following form

$$H_0 = \int_{\text{supp } \mu} \sum_{i=1}^{\infty} (|p_{\lambda, i}|^2 + |\lambda|^2 |q_{\lambda, i}|^2) d\mu(\lambda).$$

Introducing other new coordinates

$$a_{\lambda,j} = \frac{1}{\sqrt{2|\lambda|}}(|\lambda|q_{\lambda,j} + ip_{\lambda,j}), \quad a_{\lambda,j}^* = \frac{1}{\sqrt{2|\lambda|}}(|\lambda|\bar{q}_{\lambda,j} - i\bar{p}_{\lambda,j})$$

with Poisson brackets

$$\{a_{\lambda,k}, a_{\lambda',j}^*\} = i\delta_{kj}\delta_{\mu}(\lambda - \lambda')$$

we can rewrite H_0 in the form

$$(4.11) \quad H_0 = \int_{\text{supp } \mu} \sum_{i=1}^{\infty} |\lambda| a_{\lambda,i}^* a_{\lambda,i} d\mu(\lambda).$$

After applying the secondary quantization procedure $a_{\lambda,i}^*$, $a_{\lambda,i}$, $\lambda \in \text{supp } \mu$ become the creation and the annihilation operators in the Fock space generated by the ground state $|0\rangle$ satisfying the conditions

$$a_{\lambda,i}|0\rangle = 0 \text{ for all } \lambda \in \text{supp } \mu, \quad i \in \mathbb{N}.$$

They obey the commutation relations

$$[a_{\lambda,i}, a_{\lambda',j}^*] = \delta_{ij}\delta_{\mu}(\lambda - \lambda').$$

The generalized eigenvectors of the quantized Hamiltonian H_0 are of the form

$$a_{\lambda_1, i_1}^* \dots a_{\lambda_n, i_n}^* |0\rangle, \quad n \in \mathbb{N}, \lambda_i \in \sigma(\text{curl}) = \text{supp } \mu,$$

and they have eigenvalues

$$(4.12) \quad |\lambda_1| + \dots + |\lambda_n|.$$

However, the spectrum $\sigma(\text{curl})$ depends on the metric $g_{\mu\nu}$. From (4.12) it follows that if this metric is such that $\sigma(\text{curl})$ does not contain point zero then the spectrum of the secondary quantized Hamiltonian H_0 has a mass gap equal to the distance m from the point zero to the set $\sigma(\text{curl})$. Note that various types of the spectrum of this Hamiltonian may occur depending on $\sigma(\text{curl})$, e.g. in addition to the absolutely continuous spectrum a point spectrum corresponding to bound states or a singular spectrum may occur.

One can study the spectrum of curl by considering the selfadjoint operator T ,

$$(4.13) \quad T = \begin{pmatrix} \text{curl} & -d \\ \text{div} & 0 \end{pmatrix},$$

acting in the space $\mathcal{H}^1 \dot{+} \mathcal{H}^0$ with the natural domain

$$\mathfrak{D}(T) = \{(\omega, u) \in \mathcal{H}^1 \dot{+} \mathcal{H}^0 : \text{curl } \omega, du \in \mathcal{H}^1, \text{div } \omega \in \mathcal{H}^0\}$$

(see [4, 5]). One has

$$(4.14) \quad T^2 = \begin{pmatrix} \widehat{\Delta} & 0 \\ 0 & \Delta \end{pmatrix},$$

where $\Delta = -\text{div } d$ is the usual Laplace operator and $\widehat{\Delta} = \text{curl } \text{curl} - d_3 \text{div}$ is the Laplace-Hodge operator on \mathcal{H}^1 . $\text{Ker } \text{div} \subset \mathcal{H}^1$ is an invariant subspace for T and $T|_{\text{Ker } \text{div}} = \text{curl}$. Therefore $(\sigma(\text{curl}))^2 \subset \sigma(T^2)$. Thus if $0 \notin \sigma(T^2)$ then $0 \notin \sigma(\text{curl})$, and the problem of the mass gap is reduced to the study of the spectra of the operators Δ and $\widehat{\Delta}$. In particular, if $0 \notin \sigma(\Delta) \cup \sigma(\widehat{\Delta}) = \sigma(T^2)$ then the quantized Hamiltonian H_0 has a mass gap.

If $0 \in \sigma(\text{curl})$ then one can still get a mass gap m by quantizing the restriction of H_0 to the subspace $T^*\mathcal{C}_m$,

$$\mathcal{C}_m = \{x \in \mathcal{C} : \Psi(x) \in \mathcal{H}_m\},$$

where

$$\mathcal{H}_m = \{f \in \mathcal{H} : f(\lambda) = 0 \text{ for } |\lambda| < m\},$$

and $m > 0$ is a parameter. Formula similar to (4.11) will still hold, where the integration is performed over $\text{supp } \mu \setminus (-m, m)$.

At the classical level the condition $0 \notin \sigma(\text{curl})$ implies that the operator $M(\lambda)$ is a bounded isomorphism of its domain

$$\mathfrak{D}(M(\lambda)) = \{f(\lambda) \in \mathcal{H} : \int_{\text{supp } \mu} \lambda^2 |f(\lambda)|^2 d\mu(\lambda) < \infty\}$$

with the norm given by

$$\|f(\lambda)\| + \|\lambda f(\lambda)\|,$$

where $\|\cdot\|$ is the norm in \mathcal{H} , onto \mathcal{H} . Indeed, the bounded inverse operator is given by the multiplication by $\frac{1}{\lambda}$ since this function is bounded measurable on $\text{supp } \mu$. Recalling that $\text{Ker div} \cap \mathcal{D}_{\mathbb{C}}$ is the domain of the real operator curl isomorphic to $\mathfrak{D}(M(\lambda))$ under Ψ and that $\mathcal{H} \simeq \mathcal{H}^*$, since \mathcal{H} is a Hilbert space, we deduce that the bilinear form $\langle \text{curl } \omega, \text{curl } \omega' \rangle, \omega, \omega' \in \mathcal{C}$ equal to the second differential of $\frac{1}{2} \langle B(A), B(A) \rangle$ at $A = 0 \in \mathcal{C}$ gives rise to an isomorphism

$$\mathcal{C} \simeq \mathcal{C}^*, \omega \mapsto \langle \text{curl } \omega, \text{curl } \cdot \rangle.$$

Therefore $A = 0 \in \mathcal{C}$ is a non-degenerate critical point of $U(A) = \frac{1}{2} \langle B(A), B(A) \rangle$. Observe that, as a function of A , $U(A)$ is a continuous polynomial function on \mathcal{C} . Therefore it is differentiable infinitely many times and according to the Morse-Palais lemma (see [13]) there is a neighborhood V of $A = 0 \in \mathcal{C}$ and a C^∞ -diffeomorphism $\Phi : V \rightarrow V$ such that, if we denote $Q = \Phi(A)$, then

$$U(\Phi^{-1}(Q)) = \frac{1}{2} \langle \text{curl } Q, \text{curl } Q \rangle.$$

Thus if we write $P = (d\Phi^*)^{-1}E$, $E \in T_0^*(\mathcal{D})$, $C(0, E) = 0$ and denote by $G(\cdot, \cdot)$ the metric on $T\mathcal{C} \simeq T^*\mathcal{C}$ in the corresponding variables $P' \in \mathcal{C}$ then

$$(4.15) \quad H_{red}(Q, P) = \frac{1}{2}(G(P', P') + \langle \text{curl } Q, \text{curl } Q \rangle), Q \in V, P' \in \mathcal{C}.$$

We conclude that H_{red} is a member of a family of Hamiltonians parametrized by Riemannian metrics on \mathcal{C} and with the same potential term. In particular, when G is replaced with the constant metric $\langle \cdot, \cdot \rangle$ we get H_0 .

We summarize the discussion above in the following theorem.

Theorem 4.4. *The spectrum of the secondary quantized Hamiltonian H_0 has a mass gap $m > 0$ if and only if 0 is not a point of the spectrum of the operator curl . The gap m is equal to the distance from zero to the spectrum of the operator curl . In this case $A = 0 \in \mathcal{C}$ is a non-degenerate critical point of the Hamiltonian H_{red} and there is a local coordinate chart in $T\mathcal{D}/\mathcal{K} \simeq T^*\mathcal{D}/\mathcal{K}$ containing the gauge orbit of the trivial connection in which H_{red} takes form (4.15).*

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