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by

Pieter Moree
Sumaia Saad Eddin
Alisa Sedunova
Yuta Suzuki

Max-Planck-Institut für Mathematik
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Pieter Moree
Sumaia Saad Eddin
Alisa Sedunova
Yuta Suzuki

Max-Planck-Institut für Mathematik
Vivatsgasse 7
53111 Bonn
Germany

Institute of Financial Mathematics and Applied Number Theory
JKU Linz
Altenbergerstrasse 69
4040 Linz
Austria

Graduate School of Mathematics
Nagoya University
Furo-cho, Chikusa-ku
Nagoya
Japan

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JORDAN TOTIENT QUOTIENTS

PIETER MOREE, SUMAIA SAAD EDDIN, ALISA SEDUNOVA, AND YUTA SUZUKI

ABSTRACT. The Jordan totient $J_k(n)$ can be defined by $J_k(n) = n^k \prod_{p|n}(1 - p^{-k})$. In this paper, we study the average behavior of fractions P/Q of two products P and Q of Jordan totients, which we call Jordan totient quotients. To this end, we prepare some general and ready-to-use methods to deal with a wider class of totient functions first by an elementary method, and then by using an advanced method due to Balakrishnan and Pétermann. As an application, we determine the average behavior of the Jordan totient quotient, the k^{th} normalized derivative of the n^{th} cyclotomic polynomial $\Phi_n(z)$ at $z = 1$, the second normalized derivative of the n^{th} cyclotomic polynomial $\Phi_n(z)$ at $z = -1$, and the average order of the Schwarzian derivative of $\Phi_n(z)$ at $z = 1$.

1. INTRODUCTION

Jordan totient quotients. Let $k \geq 1$ be an integer. The k^{th} *Jordan totient function* $J_k(n)$ is the number of k-tuples chosen from a complete residue system modulo n such that the greatest common divisor of each set is coprime to n. It is not difficult to show that

$$J_k(n) = n^k \prod_{p|n} \left(1 - \frac{1}{p^k}\right), \tag{1}$$

where p here, and indeed in the whole paper, denotes a prime number. The Jordan function first showed up in the work of Camille Jordan in 1870 in formulas for the order of finite matrix groups (such as $\mathrm{GL}(m, \mathbb{Z}/n\mathbb{Z})$). For an introduction to Jordan totients see Section 2.

Definition. Let $r \geq 1$ be an integer and $\boldsymbol{e} = (e_1, \ldots, e_r)$ be a vector with integer entries. Put $w = \sum_i ie_i$. An arithmetic function $J_{\boldsymbol{e}}$ of the form

$$J_{\boldsymbol{e}}(n) = \prod_{i=1}^{r} J_i^{e_i}(n) = n^w \prod_{p|n} \prod_{i=1}^{r} \left(1 - \frac{1}{p^i}\right)^{e_i}, \tag{2}$$

is said to be a *Jordan totient quotient of* weight w. If $w = 0$, then we say that it is a *balanced Jordan totient quotient.* If the weight is different from 0 we call it *unbalanced.*

Note that if $J_{\boldsymbol{e}}$ is balanced, then $J_{\boldsymbol{e}}(n)$ depends only on the square-free kernel of n. A famous (unbalanced) Jordan totient quotient is *the Dedekind* Ψ*-function* defined by

$$\Psi(n) = n \prod_{p|n} \left(1 + \frac{1}{p}\right) = \frac{J_2(n)}{J_1(n)},$$

which showed up in the work of Dedekind on modular forms.

In this paper we study the average behavior of Jordan totient quotients. In the remainder of the introduction we describe our main results, including an application to the study of the average of the normalized derivative of cyclotomic polynomials.

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Our first result gives an asymptotic formula for the summatory function of any balanced Jordan totient quotient $J_{\boldsymbol{e}}(n)$, which implies that $J_{\boldsymbol{e}}(n)$ is constant on average.

Theorem 1. *Let $r \in \mathbb{N}$, $\boldsymbol{e} = (e_1, \ldots, e_r) \in \mathbb{Z}^r$ be a vector of integers, and $J_{\boldsymbol{e}}$ be a Jordan totient quotient of weight $w = \sum_i ie_i = 0$. Then asymptotically*

$$\sum_{n\le x} J_{\boldsymbol{e}}(n) = \mathfrak{S}_{\boldsymbol{e}}x + \sum_{r=1}^{|e_1|} C_{\boldsymbol{e},r}(\log x)^r + O_{\boldsymbol{e}}((\log x)^{2|e_1|/3}(\log\log x)^{4|e_1|/3}),$$

where the constant

$$\mathfrak{S}_{\boldsymbol{e}} = \prod_p \left(1 + \frac{J_{\boldsymbol{e}}(p)p^{-w} - 1}{p}\right). \tag{3}$$

is positive and the $C_{\boldsymbol{e},r}$ are some constants[1].

Note that the convergence of $\mathfrak{S}_{\boldsymbol{e}}$ is ensured since

$$1 + \frac{J_{\boldsymbol{e}}(p)p^{-w} - 1}{p} = 1 + O(p^{-2}).$$

As $J_{\boldsymbol{e}}(p)p^{-w} > 0 > 1 - p$, we have $\mathfrak{S}_{\boldsymbol{e}} > 0$. This constant can be expanded as a product of partial zeta values, see Moree and Niklasch [7, 8]. As partial zeta values can be easily evaluated up to high precision (say, with thousand decimals), this then allows one to do the same for $\mathfrak{S}_{\boldsymbol{e}}$.

In case $\boldsymbol{e} = (0)$ is the zero vector, then $J_{\boldsymbol{e}}(n) = 1$ for every $n \ge 1$, $\mathfrak{S}_{\boldsymbol{e}} = 1$ and Theorem 1 merely states that $\sum_{n\le x} 1 = x + O(1)$.

We consider not only the balanced Jordan totient quotients, but also a more general class of totient functions (see Section 3 for the definitions). This class is similar to the one earlier studied by Kaczorowski [5] in the context of inverse theorems for the Selberg class. An analog of Theorem 1 for non-zero weight can be easily established on invoking Lemma 6 and partial summation. As this is a long and rather inelegant result, we leave it to the interested reader to write it down.

The proof of Theorem 1 uses the method of Balakrishnan and Pétermann [2], but before applying it (in Section 4), we develop a simpler argument (see Section 3), which actually applies to a wider class of totients. This method allows us to get the main term of Theorem 1, however only with a weaker error term. Just as Theorem 1, Theorem 2 can be established by elementary means for non-zero weight also (see Proposition 1).

Theorem 2. *Let $r \in \mathbb{N}$, $\boldsymbol{e} = (e_1, \ldots, e_r) \in \mathbb{Z}^r$ be a vector of integers, and $J_{\boldsymbol{e}}$ be a Jordan totient quotient of weight $w = \sum_i ie_i = 0$. Then asymptotically*

$$\sum_{n\le x} J_{\boldsymbol{e}}(n) = \mathfrak{S}_{\boldsymbol{e}}x + O_{\boldsymbol{e}}((\log x)^{|e_1|}),$$

where the constant $\mathfrak{S}_{\boldsymbol{e}}$ is positive and given by (3).

It is an open problem to obtain a result at least as strong as Theorem 1 by more elementary methods than used by Balakrishnan and Pétermann.

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[1] Work in progress by the fourth author [13] suggests that the exponent $4|e_1|/3$ of $\log\log x$ in the error term can be decreased to $|e_1|/3$.

Applications. In Section 5 of the present paper, we consider normalized higher derivatives of cyclotomic polynomials at 1. Our main result shows that they are constant on average. We use the standard notation Φ_n and B_n for the n^{th} cyclotomic polynomial and n^{th} Bernoulli number, respectively (cf. Section 5.1).

Theorem 3. *Let $k \geq 1$. There exist a computable constant $\mathfrak{S}_k(\Phi)$ and constants $C_0, \ldots, C_r$ such that asymptotically*

$$\sum_{1<n\leq x} \frac{1}{\varphi(n)^k} \frac{\Phi_n^{(k)}(1)}{\Phi_n(1)} = \mathfrak{S}_k(\Phi)x + \sum_{r=1}^{k} C_r(\log x)^r + O_k((\log x)^{2k/3}(\log\log x)^{4k/3}),$$

where the constant $\mathfrak{S}_k(\Phi)$ is defined by

$$\mathfrak{S}_k(\Phi) = (-1)^k k! \sum_{(*)} \prod_{i=1}^{k} \frac{1}{\lambda_i!} \left(\frac{B_i}{i! \cdot i}\right)^{\lambda_i} \mathfrak{S}_{\boldsymbol{e}(\boldsymbol{\lambda})} \tag{4}$$

with the summation $\sum_{(*)}$ over all non-negative $\lambda_1, \ldots, \lambda_k \geq 0$ with $\lambda_1 + 2\lambda_2 + \ldots + k\lambda_k = k$, and with the indices $\boldsymbol{e}(\boldsymbol{\lambda}) = \boldsymbol{e}(\lambda_1, \ldots, \lambda_k)$ defined by

$$\boldsymbol{e}(\boldsymbol{\lambda}) = (e_i(\boldsymbol{\lambda}))_{i=1}^{\infty}, \qquad e_i(\boldsymbol{\lambda}) = \begin{cases} \lambda_1 - k, & i = 1, \\ \lambda_i, & 2 \leq i \leq k, \\ 0, & i > k. \end{cases} \tag{5}$$

Note that the vectors $\boldsymbol{e}(\boldsymbol{\lambda})$ appearing as summands in (4) are all balanced. Neither can we predict the sign of $\mathfrak{S}_k(\Phi)$, nor can we exclude that $\mathfrak{S}_k(\Phi) = 0$.

Although some part of the sum $\sum_{r=1}^{k} C_r(\log x)^r$ can be swamped by the error term, it turns out to be easier to work with this full series rather than an appropriately truncated one.

In case $k = 1$, we have by (17)

$$\sum_{1<n\leq x} \frac{1}{\varphi(n)} \frac{\Phi_n'(1)}{\Phi_n(1)} = \sum_{1<n\leq x} \frac{1}{2} = \frac{x}{2} + O(1), \tag{6}$$

improving Theorem 3. However, as our method of proof naturally includes the case $k = 1$, we have not excluded it from our formulation of Theorem 3.

Theorem 3 is a simple consequence of Lemma 8 and Theorem 1. We expect that an analogous result can be obtained with 1 replaced by any primitive root of unity of order m, and that this would involve averages of generalized Jordan totients (introduced in Bzdęga et al. [3]) of the form

$$J_k(\chi; n) = \sum_{d|n} \mu(n/d)\chi(d)d^k,$$

with χ a Dirichlet character of modulus m. We will see such a result for -1 in case $k = 2$ in the proof of Theorem 4, which is due to Herrera-Poyatos and the first author [4]. Finally, in Theorem 5, we determine the average of the Schwarzian derivative of $\Phi_n(z)$ evaluated at $z = 1$.

2. The totient functions

Let $k \geq 1$ be an integer and $J_k(n)$ be the k^{th} Jordan totient function. This is one of many generalizations of Euler's totient function (the case $k = 1$), see Sivaramakrishnan [11]. It is easy to see, cf. [12, p. 91], that

$$n^k = \sum_{d|n} J_k(d),$$

which, by Möbius inversion, yields

$$J_k(n) = \sum_{d|n} \mu(d) \left(\frac{n}{d}\right)^k. \tag{7}$$

Thus J_k is a Dirichlet convolution of two multiplicative functions and hence is itself multiplicative. By the Euler product formula, it then follows from (7) that (1) holds true.

Given a Jordan totient quotient function of weight $w = \sum_i ie_i$ as in (2), we normalize it by dividing by n^w, resulting in

$$\frac{J_{\boldsymbol{e}}(n)}{n^w} = \prod_{p|n} \prod_{i=1}^{r} \left(1 - \frac{1}{p^i}\right)^{e_r} = \prod_{p|n} \left(1 - \frac{e_1}{p} + O\left(\frac{1}{p^2}\right)\right). \tag{8}$$

Although our focus is the study of this particular function, our methods easily allow a more general class of totients to be dealt with.

Definition (General totient)**.** Let θ_n be a complex valued multiplicative function supported on square-free numbers. Define *the θ-totient* $\phi_\theta(n)$ by

$$\phi_\theta(n) = \prod_{p|n}(1 + \theta_p) = \sum_{d|n} \theta_d.$$

It is easy to see that any arithmetic function f that only depends on the square-free kernel of n for every $n \geq 1$, is of the form ϕ_θ for some θ.

We next describe the conditions we impose on θ throughout the paper.

Condition $\boldsymbol{\Theta}$1. There exist non-negative constants σ, κ, A with $0 \leq \sigma < 1$ such that for any $x \geq 2$ we have

$$\sum_{p \leq x} \frac{|\theta_p|}{p^\sigma} \leq \kappa \log\log x + A.$$

Condition $\boldsymbol{\Theta}$2. There exist $0 < \lambda < 1/2$ and $\alpha \in \mathbb{R}$ with $|\alpha| \geq 1$ such that for all primes p we have

$$\theta_p = \alpha/p + r_p, \quad r_p = O(p^{-1-\lambda}).$$

Condition $\boldsymbol{\Theta}$3. With respect to p the function $p\theta_p$ is ultimately monotonic[2].

Note that if Condition **$\boldsymbol{\Theta}$2** is satisfied, then so is Condition **$\boldsymbol{\Theta}$1** with $\sigma = 0$ and $\kappa = |\alpha|$. We point out that in order to prove Theorem 2 only Condition **$\boldsymbol{\Theta}$1** is needed, whereas to prove Theorem 1 we shall impose the stronger Condition **$\boldsymbol{\Theta}$2**. Notice that if θ is defined by $J_{\boldsymbol{e}}(n)/n^w = \phi_\theta(n)$, then Condition **$\boldsymbol{\Theta}$2** is satisfied with $\alpha = -e_1$ and $\lambda = 1$, cf. (8).

3. MEAN VALUES OF GENERAL TOTIENTS VIA AN ELEMENTARY METHOD

In this section, we give a simple method to obtain asymptotic formulas for the mean value of multiplicative functions of a certain type. The ideas and techniques are not new, but our aim is to provide a quick way to translate the definition of multiplicative functions to the asymptotic formula of its mean value. As we have seen, our θ-totient is modeled on the normalized Jordan totient quotient (8). Thus we need to introduce a weight factor n^β.

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[2] Condition **$\boldsymbol{\Theta}$3** can be removed at the expense of more technicalities, see [13].

Lemma 1. *Let β be an arbitrary real number. For $x \ge 1$ we have*

$$\sum_{n \le x} n^{\beta} = M_{\beta}(x) + C_0(\beta) + O_{\beta}(x^{\beta}),$$

where $C_0(\beta)$ is a constant depending only on β, $M_{-1}(x) = \log x$ and

$$M_{\beta}(x) = \frac{x^{\beta+1}}{\beta+1}, \text{ if } \beta \ne -1.$$

Proof. Follows from parts (a), (b), and (d) of [1, Theorem 3.2]. □

Lemma 2. *Let ϕ_θ be a θ-totient and β be an arbitrary real number. Assume that θ satisfies Condition* **Θ1**. *We then have*

$$\sum_{n \le x} n^{\beta} \phi_\theta(n) = \mathfrak{S}_\theta M_{\beta}(x) + C(\theta, \beta) + O_{\sigma,\kappa,A,\beta}(x^{\sigma+\beta}(\log x)^{\kappa}),$$

where $\mathfrak{S}_\theta$ is given by the absolutely convergent product

$$\mathfrak{S}_\theta = \prod_p \left(1 + \frac{\theta_p}{p}\right), \tag{9}$$

and $C(\theta, \beta)$ is a constant depending only on θ and β.

Proof. By the definition of θ-quotient, we have

$$\sum_{n \le x} n^{\beta} \phi_\theta(n) = \sum_{n \le x} n^{\beta} \sum_{d \mid n} \theta_d = \sum_{d \le x} d^{\beta} \theta_d \sum_{m \le x/d} m^{\beta}.$$

Thus, by Lemma 1, we have

$$\sum_{n \le x} n^{\beta} \phi_\theta(n) = \frac{x^{\beta+1}}{\beta+1} \sum_{d \le x} \frac{\theta_d}{d} + C_0(\beta) \sum_{d \le x} d^{\beta} \theta_d + O_{\beta}\Big(x^{\beta} \sum_{d \le x} |\theta_d|\Big)$$

if $\beta \ne -1$, and

$$\sum_{n \le x} n^{\beta} \phi_\theta(n) = \sum_{d \le x} \frac{\theta_d}{d} \log \frac{x}{d} + C_0(\beta) \sum_{d \le x} \frac{\theta_d}{d} + O\Big(\frac{1}{x} \sum_{d \le x} |\theta_d|\Big)$$

if $\beta = -1$. Using Condition **Θ1**, we find that

$$\sum_{d \le x} \frac{|\theta_d|}{d^{\sigma}} \le \prod_{p \le x} \Big(1 + \frac{|\theta_p|}{p^{\sigma}}\Big) \le \exp\Big(\sum_{p \le x} \frac{|\theta_p|}{p^{\sigma}}\Big) \ll_{\sigma,\kappa,A} (\log x)^{\kappa}.$$

This implies that for $\beta \ge -\sigma$

$$\sum_{d \le x} d^{\beta} |\theta_d| \le x^{\sigma+\beta} \sum_{d \le x} \frac{|\theta_d|}{d^{\sigma}} \ll_{\sigma,\kappa,A} x^{\sigma+\beta} (\log x)^{\kappa},$$

and that for $\beta < -\sigma$

$$\begin{aligned} \sum_{d > x} d^{\beta} |\theta_d| &\ll_{\sigma,\beta} \sum_{d > x} \frac{|\theta_d|}{d^{\sigma}} \int_d^{\infty} u^{\sigma+\beta-1} du \ll_{\sigma,\beta} \int_x^{\infty} \Big(\sum_{d \le u} \frac{|\theta_d|}{d^{\sigma}}\Big) u^{\sigma+\beta-1} du \\ &\ll_{\sigma,\kappa,A,\beta} x^{\sigma+\beta} (\log x)^{\kappa}. \end{aligned}$$

Hence, in particular,

$$\sum_{d \le x} |\theta_d| \ll x^{\sigma} (\log x)^{\kappa},$$

and

$$\sum_{d\le x}\frac{\theta_d}{d}=\mathfrak{S}_\theta+r(x),\quad r(x)\ll_{\sigma,\kappa,A}x^{\sigma-1}(\log x)^\kappa.$$

By combining the above, we obtain the assertion in case $\beta\neq-1$.

For the case $\beta=-1$, we have to evaluate the main term. We have

$$\begin{aligned}\sum_{d\le x}\frac{\theta_d}{d}\log\frac{x}{d}&=\sum_{d\le x}\frac{\theta_d}{d}\int_d^x\frac{du}{u}=\int_1^x\Big(\sum_{d\le u}\frac{\theta_d}{d}\Big)\frac{du}{u}\\&=\mathfrak{S}_\theta\log x+\int_1^x\frac{r(u)}{u}du=\mathfrak{S}_\theta\log x+\int_1^\infty\frac{r(u)}{u}du-\int_x^\infty\frac{r(u)}{u}du.\end{aligned}$$

The last integral can be estimated as

$$\int_x^\infty\frac{r(u)}{u}du\ll\int_x^\infty u^{\sigma-2}(\log u)^\kappa du\ll x^{\sigma-1}(\log x)^\kappa$$

since $\sigma<1$ by assumption. This completes the proof when $\beta=-1$. $\square$

As a special case we obtain the following result involving the Jordan totient quotient.

Proposition 1. *Let $\boldsymbol{e}=(e_1,\dots,e_r)\in\mathbb{Z}^r$ be a vector of integers and $J_{\boldsymbol{e}}(n)$ be the associated Jordan totient quotient of weight $w=\sum_i ie_i$. For any real number β we have*

$$\sum_{n\le x}J_{\boldsymbol{e}}(n)n^\beta=\mathfrak{S}_{\boldsymbol{e}}M_{\beta+w}(x)+C(\boldsymbol{e},\beta)+O_{\boldsymbol{e},\beta}(x^{\beta+w}(\log x)^{|e_1|}),$$

where $\mathfrak{S}_{\boldsymbol{e}}$ is given by (3) *and $C(\boldsymbol{e},\beta)$ is a constant depending only on $\boldsymbol{e}$ and β.*

Proof. We can regard $J_{\boldsymbol{e}}(n)n^{-w}$ as a general totient $\phi_\theta(n)$ with components

$$\theta_p=-e_1/p+O(p^{-2}).$$

This gives

$$\sum_{p\le x}|\theta_p|=\sum_{p\le x}\frac{|e_1|}{p}+O(1)=|e_1|\log\log x+O(1),$$

i.e. θ satisfies Condition **Θ1** with $\sigma=0$, $\kappa=|e_1|$. Note that

$$\theta_p=J_{\boldsymbol{e}}(p)p^{-w}-1,$$

and so the comparison of (9) and (3) yields $\mathfrak{S}_\theta=\mathfrak{S}_{\boldsymbol{e}}$. Under the above setting, we can rewrite the left-hand side of the assertion as

$$\sum_{n\le x}J_{\boldsymbol{e}}(n)n^\beta=\sum_{n\le x}n^{\beta+w}\Big(\frac{J_{\boldsymbol{e}}(n)}{n^w}\Big)=\sum_{n\le x}n^{\beta+w}\phi_\theta(n),$$

and the proposition follows by Lemma 2. $\square$

Corollary 1. *For $k\ge1$ we have*

$$\sum_{n\le x}\frac{n^{k-1}}{\varphi(n)^k}=\mathfrak{S}_{(-k)}\log x+C_k+O_k\Big(\frac{(\log x)^k}{x}\Big),$$

where $\mathfrak{S}_{(-k)}$ is given by (3) *and C_k is a constant depending on k.*

Proof. Apply Proposition 1 with $J_{\boldsymbol{e}}(n)=\varphi(n)^{-k}$, $w=-k$ and $\beta=k-1$. $\square$

4. MEAN VALUES OF GENERAL TOTIENTS BY BALAKRISHNAN-PÉTERMANN

In this section we use the method of Balakrishnan and Pétermann [2] in order to prove Theorem 1. It consists of Propositions 2 and 3 below and yields an asymptotic formula for the mean value of θ-totients, provided some condition stronger than Condition **Θ1** is satisfied.

Proposition 2 (Balakrishan and Pétermann [2, Theorem 1])**.** *Let*

$$f(s)=\sum_{n=1}^{\infty}\frac{b_n}{n^s}$$

be a Dirichlet series that converges absolutely for $\sigma>1-\lambda$, with λ a positive real number. Define two arithmetic functions a_n and v_n by

$$\sum_{n=1}^{\infty}\frac{a_n}{n^s}=\zeta(s)\zeta(s+1)^{\alpha}f(s+1),\qquad \sum_{n=1}^{\infty}\frac{v_n}{n^s}=\zeta(s)^{\alpha}f(s),$$

where $\sigma>1$, α is an arbitrary real number and the branch of $\zeta(s+1)^{\alpha}$ is taken by the one for which $\arg\zeta(s+1)$ *equals zero on the positive real line. Then we have*

$$\sum_{n\le x}a_n=\zeta(2)^{\alpha}f(2)x+\sum_{r=0}^{[\alpha]}A_r(\log x)^{\alpha-r}+R(x)+o(1)$$

as $x\to\infty$, where the coefficients A_r are computable from the Laurent expansion of $\zeta(s)^{\alpha}f(s)$ at $s=1$, the remainder term $R(x)$ is given by

$$R(x)=\sum_{n\le y}\frac{v_n}{n}\psi\left(\frac{x}{n}\right),$$

with $y=x\exp(-(\log x)^{1/6})$, and $\psi(x)=\{x\}-1/2$. The implicit constant in the error term might depend on all the input data.

Remark. There are several results in [2] that depend on the specific zero-free region for the Riemann zeta function being used. Throughout this paper, we will use the zero-free region

$$\operatorname{Re}s\ge 1-\frac{1}{(\log t)^{4/5}}\qquad\text{and}\qquad \operatorname{Im}s\ge t_0,$$

where t_0 is some large constant, see [14, Eq. (6.15.1)]. This zero-free region enables us to take $b=1/6$ in [2], leading to $y=x\exp(-(\log x)^{1/6})$ in Proposition 2, see [2, Subsection 1.4, Lemmas 3 and 5].

Lemma 3 (Balakrishnan and Pétermann [2, Lemma 3])**.** *In the notation of Proposition 2 we have*

$$\sum_{n\le x}\frac{v_n}{n}=\sum_{0\le r\le(\log x)^{1/6}}V_r(\log x)^{\alpha-r}+O(\exp(-(\log x)^{1/6})),$$

with $|V_r|\le(cr)^r$ for every $r\ge1$ and $c\ge1$ a constant possibly depending on v.

Now we prove Theorem 1. As already mentioned, we need to assume that θ satisfies a stronger condition than Condition **Θ1**. In this section, we use Conditions **Θ2** and **Θ3**, and hence all implicit constants in this section will depend on the constants α,λ and the implicit constant appearing in Condition **Θ2**.

Lemma 4. *Let ϕ_θ be a θ-totient with θ satisfying Condition* $\mathbf{\Theta 2}$. *Consider the formal Dirichlet series*

$$f(s+1) = \sum_{n=1}^{\infty} \frac{b_n}{n^{s+1}} = \zeta(s)^{-1}\zeta(s+1)^{-\alpha} \sum_{n=1}^{\infty} \frac{\phi_\theta(n)}{n^s},$$

where α is the same one as in Condition $\mathbf{\Theta 2}$. *Then $f(s)$ converges absolutely for* $\operatorname{Re} s > 1-\lambda$.

Proof. By the definition of ϕ_θ we have

$$\sum_{n=1}^{\infty} \frac{\theta_n}{n^s} = \zeta(s+1)^{\alpha} f(s+1). \tag{10}$$

If we consider the Dirichlet series given by

$$\zeta(s)^{-\alpha} = \sum_{n=1}^{\infty} \frac{\tau_{-\alpha}(n)}{n^s},$$

then, using (10) for the coefficients of $f(s)$, we obtain

$$b_n = \sum_{dm=n} \tau_{-\alpha}(d)\theta_m m. \tag{11}$$

Using the Euler product expansion and the generalized binomial formula, we see that

$$\zeta(s)^{-\alpha} = \prod_p \left(1 - \frac{1}{p^s}\right)^{\alpha} = \prod_p \left(1 + \sum_{\nu=1}^{\infty} (-1)^{\nu} \binom{\alpha}{\nu} \frac{1}{p^{\nu s}}\right) := \prod_p \left(1 + H_\alpha(p^{-s})\right), \tag{12}$$

where

$$\binom{\alpha}{\nu} = \frac{1}{\nu!} \prod_{\ell=0}^{\nu-1} (\alpha - \ell),$$

is a *generalized binomial coefficient.* Since

$$|H_\alpha(p^{-s})| = \frac{|\alpha|}{p^\sigma} + O_\alpha\left(\frac{1}{p^{2\sigma}}\right),$$

the Euler product (12) is absolutely convergent for $\sigma = \operatorname{Re} s > 1$ and

$$\tau_{-\alpha}(p^\nu) = (-1)^\nu \binom{\alpha}{\nu}.$$

Note that

$$|\tau_{-\alpha}(p^\nu)| \le \left|\binom{\alpha}{\nu}\right| \le \frac{1}{\nu!} \prod_{\ell=1}^{\nu} (|\alpha| + \ell - 1) \le \prod_{\ell=1}^{\nu} \left(1 + \frac{|\alpha|}{\ell}\right) \le \exp\left(\sum_{\ell=1}^{\nu} \frac{|\alpha|}{\ell}\right) \ll \nu^{|\alpha|} \ll_\varepsilon p^{\nu\varepsilon}$$

for every $\varepsilon > 0$. Substituting $n = p$ and $n = p^\nu$ into (11) and using Condition $\mathbf{\Theta 2}$, we find that

$$b_p = \tau_{-\alpha}(p) + p\theta_p = -\alpha + \alpha + pr_p = O(p^{-\lambda}),$$

respectively, $b_{p^\nu} \ll |\tau_{-\alpha}(p^\nu)| \ll_\varepsilon p^{\nu\varepsilon}$ for $\nu \ge 2$ and every $\varepsilon > 0$. As

$$\sum_{n=1}^{\infty} \frac{|b_n|}{n^\sigma} = \prod_p \left(1 + \frac{|b_p|}{p^\sigma} + \sum_{\nu=2}^{\infty} \frac{|b_{p^\nu}|}{p^{\nu\sigma}}\right)$$

is bounded when both $\sigma + \lambda > 1$ and $2\sigma > 1$, the result follows since $\lambda < 1/2$. ☐

Lemma 5. *Let ϕ_θ be a θ-totient with θ satisfying Condition* **Θ2**. *Then we have*

$$\sum_{n\le x}\phi_\theta(n)=\mathfrak{S}_\theta x+\sum_{r=0}^{[\alpha]}C_r(\theta)(\log x)^{\alpha-r}+R(x)+o(1),$$

where $\mathfrak{S}_\theta$ is given by (9)*, $R(x)$ by*

$$R(x)=\sum_{n\le y}\theta_n\psi\left(\frac{x}{n}\right),$$

and $y=x\exp(-(\log x)^{1/6})$.

Proof. With the choice $a_n=\phi_\theta(n)$, we are in the scope of Proposition 2 by Lemma 4, and on applying it and noting that $v_n=n\theta_n$, the proof is completed. □

We next estimate the error term in Proposition 2. For this purpose, we need Theorem 1 of Pétermann [10], which we state below[3]. Note that the parameter α in [10] corresponds to $|\alpha|-1$ in Proposition 2. In order to avoid possible confusion caused by this clash of notation, we replace α in [10] by α_1.

Proposition 3 (Pétermann [10, Theorem 1])**.** *Let v_n be a real-valued multiplicative function. Assume that there exist real numbers $\alpha_1,\beta\ge 0$, and a sequence of real numbers $\{V_r\}_{r=0}^{\infty}$, such that for every integer $B>0$ and real number $x\ge 4$, we have*

$$\textbf{(h1)}\qquad \sum_{n\le x}|v_n|=x\sum_{r=0}^{B+[\alpha_1]}V_r(\log x)^{\alpha_1-r}+O_B(x(\log x)^{-B}),$$

$$\textbf{(h2)}\qquad \sum_{n\le x}|v_n|^2\ll x(\log x)^{\beta},$$

(h3) *v_p is ultimately monotonic with respect to p,*
v_{p^ν} is bounded as p^ν runs over the prime powers.

Then, for $x\ge 4$, we have

$$\sum_{n\le y}\frac{v_n}{n}\psi\left(\frac{x}{n}\right)\ll(\log x)^{2(\alpha_1+1)/3}(\log\log x)^{4(\alpha_1+1)/3},$$

where $y=x\exp(-(\log x)^{\frac16})$ and the implicit constant depends on the constants in Conditions (**h1**)*,* (**h2**) *and* (**h3**)*.*

We now apply Proposition 3 to our setting. For this purpose, we need Lemma 3 (which can, in principle, also be proven via the Selberg–Delange method).

Lemma 6. *Let ϕ_θ be a θ-totient. Assume that θ satisfies Conditions* **Θ2** *and* **Θ3**. *Then*

$$\sum_{n\le x}\phi_\theta(n)=\mathfrak{S}_\theta x+\sum_{r=0}^{[\alpha]}C_r(\theta)(\log x)^{\alpha-r}+O((\log x)^{2|\alpha|/3}(\log\log x)^{4|\alpha|/3}),$$

where $\mathfrak{S}_\theta$ is given by (9)*. Furthermore, for β real,*

$$\sum_{n\le x}n^\beta\phi_\theta(n)=\mathfrak{S}_\theta M_\beta(x)+C(\theta,\beta)+\sum_{r=0}^{[\alpha]}C_r(\theta,\beta)x^\beta(\log x)^{\alpha-r}+E(x;\beta),$$

---

[3] Note that [2, Theorem 2] contains an error. See the errata of [2] and [10].

where $M_\beta(x)$ is defined in Lemma 1, and

$$E(x;\beta) \ll x^{\beta}(\log x)^{2|\alpha|/3}(\log\log x)^{4|\alpha|/3}.$$

Proof. By Lemma 5, it is sufficient to show that $R(x) = O((\log x)^{2|\alpha|/3}(\log\log x)^{4|\alpha|/3})$, which we do via Proposition 3. Hence, we need to check that Conditions (**h1**), (**h2**) and (**h3**) are all satisfied. Since θ_n satisfies Condition $\mathbf{\Theta 2}$, $|\theta_n|$ also satisfies Condition $\mathbf{\Theta 2}$, but with $|\alpha|$ instead of α. Thus, we can apply Lemma 4 with $|\theta_n|$ instead of θ_n. Then, as $v_n = n\theta_n$, we can replace v_n in Proposition 2 by $|v_n|$.

We start with Condition (**h1**). We apply Lemma 3 and obtain

$$\sum_{n\le x}\frac{|v_n|}{n} = \sum_{0\le r\le(\log x)^{1/6}} V_r(\log x)^{|\alpha|-r} + O(\exp(-(\log x)^{1/6})),$$

where the V_r are some constants satisfying $|V_r| \le (cr)^r$ with some $c \ge 1$. Let $B > 0$ be an integer that is kept fixed. Then it is easy to see that for x larger than some constant depending on B and α the consecutive terms of the sequence

$$V_r(\log x)^{|\alpha|-r} \quad \text{with} \quad B + |\alpha| < r \le (\log x)^{1/6}$$

have the ratio $\le 1/2$ and so their sum is bounded by the first term as

$$\sum_{B+|\alpha|<r\le(\log x)^{1/6}} V_r(\log x)^{|\alpha|-r} \ll V_{B+[|\alpha|]+1}(\log x)^{|\alpha|-(B+[|\alpha|]+1)} \ll_B (\log x)^{-B}.$$

This enables us to truncate the sum over r to obtain

$$S(x) := \sum_{n\le x}\frac{|v_n|}{n} = \sum_{0\le r\le B+|\alpha|} V_r(\log x)^{|\alpha|-r} + R_S(x), \quad R_S(x) \ll_B (\log x)^{-B}.$$

By partial summation,

$$\begin{aligned}\sum_{n\le x}|v_n| &= \int_2^x u\,dS(u) + O(1)\\ &= \sum_{0\le r\le B+|\alpha|}(|\alpha|-r)V_r\int_2^x(\log u)^{|\alpha|-r-1}du + \int_2^x u\,dR_S(u) + O(1).\end{aligned}$$

The main terms can be evaluated using integration by parts as

$$\int_2^x(\log u)^{|\alpha|-r-1}\,du = \sum_{0\le m\le B+|\alpha|-r-1} C_m x(\log x)^{|\alpha|-r-1-m} + O_B(x(\log x)^{-B}),$$

with some constants C_m depends on α and r. The error term can be estimated as

$$\int_2^x u\,dR_S(u) \ll_B x(\log x)^{-B} + \int_2^x(\log u)^{-B}du \ll_B x(\log x)^{-B}.$$

By combining the above estimates, we arrive at

$$\sum_{n\le x}|v_n| = x\sum_{r=0}^{B+[|\alpha|-1]}\tilde{V}_r(\log x)^{|\alpha|-1-r} + O(x(\log x)^{-B}),$$

where the $\tilde{V}_r$ are constants. By Condition $\mathbf{\Theta 2}$, we have $|\alpha| \ge 1$. Hence, Condition (**h1**) of Proposition 3 is satisfied with $\alpha_1 = |\alpha| - 1 \ge 0$.

As to Condition (**h2**), we start with the string of estimates

$$\sum_{n\le x}|v_n|^2=\sum_{n\le x}n^2|\theta_n|^2\le x\sum_{n\le x}n|\theta_n|^2\le x\prod_{p\le x}\left(1+p|\theta_p|^2\right)\le x\exp\Big(\sum_{p\le x}p|\theta_p|^2\Big). \tag{13}$$

Now Condition $\mathbf{\Theta 2}$ implies that

$$\sum_{p\le x}p|\theta_p|^2\ll\sum_{p\le x}\frac{1}{p}\ll\log\log x. \tag{14}$$

By combining (13) and (14), we see that Condition (**h2**) is satisfied as well.

The remaining Condition (**h3**) follows immediately from our setting and Condition $\mathbf{\Theta 3}$.

Thus Conditions (**h1**), (**h2**) and (**h3**) are satisfied and we get the claimed upper bound for $R(x)$, which on insertion in Lemma 5 yields the first assertion of the lemma. The second claim now follows by partial summation. □

Proof of Theorem 1. Consider the θ-quotient $\phi_\theta(n)$ defined by $\phi_\theta(n)=J_{\boldsymbol{e}}(n)n^{-w}$. Note that θ satisfies Condition $\mathbf{\Theta 2}$ with $\alpha=-e_1$ and $\lambda=1$ and, moreover, satisfies Condition $\mathbf{\Theta 3}$. Thus, in case $e_1\ne 0$, Theorem 1 follows immediately from Lemma 6. The case $e_1=0$ is just a corollary of Theorem 2. □

5. Applications

Definition. Let $f(X)\in\mathbb{Z}[X]$ be a polynomial and let $\deg f$ denote its degree with respect to X. For any complex number z such that $f(z)\ne 0$, we define

$$F^{(k)}(z)=\frac{1}{(\deg f)^k}\frac{f^{(k)}(z)}{f(z)}$$

as *the normalized* k^{th} *derivative of* f *at* z.

In case $f(X)\in\mathbb{Z}_{\ge 0}[X]$, $z\ge 1$ is real, and $f(z)\ne 0$, it is easy to show that $F^{(k)}(z)\le 1$. This observation leads to the following problem.

Problem. *Let z be given. Let $\mathcal{F}$ be an infinite family of polynomials f with $f(z)\ne 0$. Study the average behavior and value distribution of $F^{(k)}(z)$ in the family $\mathcal{F}$.*

Here we consider the family $\mathcal{F}=\{\Phi_n : n\ge 2\}$, where Φ_n denotes the n^{th} cyclotomic polynomial. It can be defined by

$$\Phi_n(X)=\prod_{\substack{1\le j\le n\\(j,n)=1}}(X-\zeta_n^j)=\sum_{k=0}^{\varphi(n)}a_n(k)X^k,$$

with ζ_n any primitive n^{th} root of unity. Note that $\Phi_n(1)\ne 0$ for $n>1$ and that $\Phi_n(-1)\ne 0$ for $n>2$. Theorem 3 shows that

$$\frac{1}{\varphi(n)^k}\frac{\Phi_n^{(k)}(1)}{\Phi_n(1)},$$

the k^{th} normalized derivative of Φ_n at 1, is constant on averaging over n.

5.1. **The k^{th} derivative of Φ_n at 1.** In this section we first recall some known results on $\Phi_n^{(k)}$. For a survey (and some new results) see Herrera-Poyatos and Moree [4].

The *Bernoulli numbers* B_n can be recursively defined by

$$B_n = -\sum_{k=0}^{n-1}\binom{n}{k}\frac{B_k}{n-k+1},$$

with $B_0 = 1$. The coefficients $c(k,j)$ of the polynomial

$$X(X-1)\cdots(X-k+1) = \sum_{j=0}^{k} c(k,j)X^j$$

are called *the signed Stirling numbers of the first kind.*

Lemma 7 (Lehmer [6, Theorems 2 and 3])**.** *For $n > 1$ and $k \geq 1$, we have*

$$\frac{\Phi_n^{(k)}(1)}{\Phi_n(1)} = k!\sum_{(*)}\prod_{i=1}^{k}\frac{(-s_i(n))^{\lambda_i}}{\lambda_i!i^{\lambda_i}} \tag{15}$$

where the summation $\sum\limits_{(*)}$ is as in Theorem 3 and

$$s_i(n) := -\frac{1}{(i-1)!}\sum_{h=1}^{i}(-1)^h\frac{B_h}{h}c(i,h)J_h(n). \tag{16}$$

Remark. Theorem 2 of [6] gives the formula

$$s_i(n) = \frac{(-1)^i}{2}\varphi(n) - \frac{1}{(i-1)!}\sum_{h=1}^{[i/2]}\frac{B_{2h}}{2h}c(i,2h)J_{2h}(n).$$

The expression above is slightly different from (16), but we can simplify this as in Lemma 7 since $B_h = 0$ for odd $h > 1$, $B_1 = -1/2$, and $c(i,1) = (-1)^{i-1}(i-1)!$.

In particular, using Lemma 7 with $k = 1, 2$ for $n > 1$ we obtain

$$\frac{\Phi_n'(1)}{\Phi_n(1)} = \frac{\varphi(n)}{2}, \tag{17}$$

and

$$\frac{\Phi_n''(1)}{\Phi_n(1)} = \frac{\varphi(n)}{4}\left(\varphi(n) + \frac{\Psi(n)}{3} - 2\right).$$

Lemma 8. *For $n > 1$ and $k \geq 1$, we have*

$$\frac{1}{\varphi(n)^k}\frac{\Phi_n^{(k)}(1)}{\Phi_n(1)} = k!\sum_{(*)}\prod_{i=1}^{k}\frac{(-1)^{i\lambda_i}}{\lambda_i!}\left(\frac{B_i}{i!\cdot i}\right)^{\lambda_i}\frac{J_i(n)^{\lambda_i}}{\varphi(n)^k} + O_k\left(\frac{n^{k-1}}{\varphi(n)^k}\right),$$

where the summation $\sum\limits_{(*)}$ is as in Theorem 3.

Proof. Since $J_h(n) \leq n^h$ and $c(i,i) = 1$, it follows from (16) that

$$-s_i(n) = (-1)^i\frac{B_i}{i!}J_i(n) + O_k(n^{i-1}).$$

Hence, by raising this to the λ_i-th power,

$$(-s_i(n))^{\lambda_i} = (-1)^{i\lambda_i}\left(\frac{B_i}{i!}J_i(n)\right)^{\lambda_i} + O_k(n^{i\lambda_i-1}).$$

By substituting this estimate into (15), the proof of the lemma is concluded by taking the product over $1 \le i \le k$ and noting that the error term is $O_k(n^{\sum_{i=1}^k i\lambda_i - 1}) = O_k(n^{k-1})$ for each choice of $\lambda_1, \ldots, \lambda_k$ contributing to the sum $\sum\limits_{(*)}$. ☐

Proof of Theorem 3. By (6), we may assume $k \ge 2$. By Lemma 8 and Corollary 1,

$$\sum_{1<n\le x} \frac{1}{\varphi(n)^k}\frac{\Phi_n^{(k)}(1)}{\Phi_n(1)} = k!\sum_{(*)}\prod_{i=1}^{k}\frac{(-1)^{i\lambda_i}}{\lambda_i!}\left(\frac{B_i}{i!\cdot i}\right)^{\lambda_i}\sum_{n\le x} J_{\boldsymbol{e}(\boldsymbol{\lambda})}(n) + O_k(\log x),$$

where we used the summation $\sum\limits_{(*)}$ and the indices $\boldsymbol{e}(\boldsymbol{\lambda})$ defined in Theorem 3. Note that every index $\boldsymbol{e}(\boldsymbol{\lambda})$ appearing on the right-hand side has weight

$$w = \sum_{i=1}^{\infty} i e_i(\boldsymbol{\lambda}) = \sum_{i=1}^{k} i\lambda_i - k = 0.$$

Trivially $|e_1(\boldsymbol{\lambda})| \le k$ and hence, by applying Theorem 1 and using $k \ge 2$,

$$\sum_{1<n\le x} \frac{1}{\varphi(n)^k}\frac{\Phi_n^{(k)}(1)}{\Phi_n(1)} = \mathfrak{S}_k(\Phi)x + \sum_{r=1}^{k} C_r(\log x)^r + O_k((\log x)^{2k/3}(\log\log x)^{4k/3}),$$

where

$$\mathfrak{S}_k(\Phi) := (-1)^k k!\sum_{(*)}\prod_{i=1}^{k}\frac{1}{\lambda_i!}\left(\frac{B_i}{i!\cdot i}\right)^{\lambda_i}\mathfrak{S}_{\boldsymbol{e}(\boldsymbol{\lambda})}.$$ ☐

5.2. **The second derivative of Φ_n at -1.** We prove an analogous result for the normalized second derivative of Φ_n at -1.

Theorem 4. *We have*

$$\sum_{2<n\le x} \frac{1}{\varphi(n)^2}\frac{\Phi_n''(-1)}{\Phi_n(-1)} = \frac{x}{48}(5\mathfrak{S}_{(-2,1)} + 12) + c_2\log^2 x + O((\log x)^{4/3}(\log\log x)^{8/3}),$$

where c_2 is a constant and $\mathfrak{S}_{(-2,1)}$ computed via (3) *equals*

$$\mathfrak{S}_{(-2,1)} = \prod_p\Big(1 + \frac{2}{p(p-1)}\Big).$$

Proof. By [4, Corollary 22] it follows that for $n \ge 3$ we have

$$\frac{\Phi_n''(-1)}{\Phi_n(-1)} = \frac{\varphi(n)}{4}\left(\varphi(n) + a_n\Psi(n) - 2\right),$$

where

$$a_n = \begin{cases} 1 & \text{if } n \text{ is odd},\\ 1/9 & \text{if } 2 \parallel n,\\ 1/3 & \text{otherwise.}\end{cases}$$

Using the above and Lemma 1, it now follows that

$$\sum_{2<n\le x}\frac{1}{\varphi(n)^2}\frac{\Phi_n''(-1)}{\Phi_n(-1)}=\frac{x}{4}+\frac{1}{4}\sum_{n\le x}a_n\frac{\Psi(n)}{\varphi(n)}+O(\log x). \tag{18}$$

Note that

$$\sum_{n\le x}\left(a_n-\frac{1}{3}\right)\frac{\Psi(n)}{\varphi(n)}=\frac{2}{3}\sum_{\substack{n\le x\\ 2\nmid n}}\frac{\Psi(n)}{\varphi(n)}-\frac{2}{9}\sum_{\substack{n\le x\\ 2\| n}}\frac{\Psi(n)}{\varphi(n)}=\frac{2}{3}\sum_{\substack{n\le x\\ 2\nmid n}}\frac{\Psi(n)}{\varphi(n)}-\frac{2}{3}\sum_{\substack{n\le x/2\\ 2\nmid n}}\frac{\Psi(n)}{\varphi(n)},$$

and so

$$\sum_{n\le x}a_n\frac{\Psi(n)}{\varphi(n)}=\frac{1}{3}\sum_{n\le x}\frac{\Psi(n)}{\varphi(n)}+\frac{2}{3}\sum_{\substack{n\le x\\ 2\nmid n}}\frac{\Psi(n)}{\varphi(n)}-\frac{2}{3}\sum_{\substack{n\le x/2\\ 2\nmid n}}\frac{\Psi(n)}{\varphi(n)}.$$

By Theorem 2, for the first sum, we have

$$\sum_{n\le x}\frac{\Psi(n)}{\varphi(n)}=\mathfrak{S}_{(-2,1)}x+c_1'\log^2 x+O((\log x)^{4/3}(\log\log x)^{8/3}).$$

On noting that

$$1_{2\nmid n}\frac{\Psi(n)}{\varphi(n)}=\prod_{p\mid n}(1+\theta_p),\quad \theta_p=\frac{2}{p-1}\ (p\neq 2),\quad \theta_2=-1,$$

we get on applying Lemma 6,

$$\sum_{\substack{n\le x\\ 2\nmid n}}\frac{\Psi(n)}{\varphi(n)}=\frac{1}{4}\mathfrak{S}_{(-2,1)}x+c_2'\log^2 x+O((\log x)^{4/3}(\log\log x)^{8/3}).$$

Combining the results above we obtain

$$\sum_{n\le x}a_n\frac{\Psi(n)}{\varphi(n)}=\frac{5}{12}\mathfrak{S}_{(-2,1)}x+4c_2\log^2 x+O((\log x)^{4/3}(\log\log x)^{8/3}),$$

which, together with (18), concludes the proof. □

5.3. **Schwarzian derivative of Φ_n at 1.** Given a holomorphic function f of one complex variable z, we define its *Schwarzian derivative*, cf. [9], as

$$S(f(z))=\frac{f'''(z)}{f'(z)}-\frac{3}{2}\left(\frac{f''(z)}{f'(z)}\right)^2.$$

Theorem 5. *We have*

$$\sum_{n\le x}\frac{S(\Phi_n(1))}{\varphi(n)^2}=-\frac{1}{24}(\mathfrak{S}_{(-4,2)}+3)x+c_4\log^4 x+c_3\log^3 x+O((\log x)^{8/3}(\log\log x)^{16/3}),$$

where c_3, c_4 are constants and $\mathfrak{S}_{(-4,2)}$ computed via (3) equals

$$\mathfrak{S}_{(-4,2)}=\prod_p\Big(1+\frac{4}{(p-1)^2}\Big).$$

Proof. By Lemma 7, we have for $n \geq 2$

$$S(\Phi_n(1)) = -\frac{\varphi(n)^2}{8} - \frac{\Psi(n)^2}{24} + \frac{1}{2},$$

and thus

$$\sum_{n \leq x} \frac{S(\Phi_n(1))}{\varphi(n)^2} = -\frac{1}{8} \sum_{n \leq x} 1 - \frac{1}{24} \sum_{n \leq x} \frac{\Psi(n)^2}{\varphi(n)^2} + \frac{1}{2} \sum_{n \leq x} \frac{1}{\varphi(n)^2}.$$

The last sum is bounded by a constant by Proposition 1 with $J_{\boldsymbol{e}}(n) = 1/\varphi(n)^2$ and $\beta = 0$. The result now follows on applying Theorem 1 with $\boldsymbol{e} = (-4, 2)$. $\square$

Remark. On applying the elementary Theorem 2, we obtain Theorems 4 and 5 with error terms $O(\log^2 x)$ and $O(\log^4 x)$, respectively.

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(P. Moree, A. Sedunova) MAX-PLANCK-INSTITUT FÜR MATHEMATIK, VIVATSGASSE 7, D-53111 BONN, GERMANY.
E-mail address, P. Moree: `moree@mpim-bonn.mpg.de`
E-mail address, A. Sedunova: `alisa.sedunova@phystech.edu`

(S. Saad Eddin) INSTITUTE OF FINANCIAL MATHEMATICS AND APPLIED NUMBER THEORY, JKU LINZ, ALTENBERGERSTRASSE 69, 4040 LINZ, AUSTRIA.
E-mail address: `sumaia.saad_eddin@jku.at`

(Y. Suzuki) GRADUATE SCHOOL OF MATHEMATICS, NAGOYA UNIVERSITY, FURO-CHO, CHIKUSA-KU, NAGOYA, JAPAN.
E-mail address: `suzuyu1729@gmail.com`