Twisted chiral de Rham algebras on $\mathbb{P}^1$

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To Dennis Sullivan on his 60-th birthday

§1. Introduction

1.1. Let G be a semisimple complex Lie group, $B \subset G$ a Borel subgroup, $X = G/B$ the flag space. Let $\mathfrak{g} = Lie(G)$ and $\widehat{\mathfrak{g}}$ the corresponding Kac-Moody Lie algebra. B.Feigin and E.Frenkel [FF1]-[FF3] have constructed (generalizing the previous construction by M.Wakimoto [W]) a family of representations of $\widehat{\mathfrak{g}}$, called *Wakimoto modules*, which are "loop space" analogs of contragradient Verma modules over $\mathfrak{g}$, realized in the spaces of differential operators over an open Schubert cell in X.

Let us recall this construction. From now on we restrict ourselves to the case $G = SL(2)$, so $\mathfrak{g} = sl(2)$, $X = \mathbb{P}^1$.

Let M be a vertex superalgebra generated by even fields $a(z), b(z)$ of conformal weight 1 and 0 respectively and odd fields $\phi(z), \psi(z)$ of conformal weight 0 and 1 respectively, with nontrivial OPE's

$$a(z)b(w) \sim \frac{1}{z-w};\ \psi(z)\phi(w) \sim \frac{1}{z-w} \tag{1.1.1}$$

(this vertex algebra is what is classically called a "ϕ, ψ, a, b-system"). As a vector space, M is a polynomial algebra in infinite set of variables,

$$M = \mathbb{C}[a_{-i-1}\mathbf{1}, b_{-i}\mathbf{1}, \phi_{-i}\mathbf{1}, \psi_{-i-1}\mathbf{1}]_{i\geq 0} \tag{1.1.2}$$

(where $\mathbf{1}$ denotes the vacuum vector). It is $\mathbb{Z}$-graded by "fermion number": $M = \oplus_{i\in\mathbb{Z}} M^i$ where $M^i \subset M$ is the subspace generated by all monomials having the number of letters ϕ - the number of letters ψ equal to i.

Let e, f, h be the standard generators of $\mathfrak{g}$. Let $V_{0,k}$ denote the vacuum module over $\widehat{\mathfrak{g}}$ of level k. This is a vertex algebra generated by the fields $e(z), h(z), f(z)$ subject to standard relations, cf. [K], (2.5.6) and Example 4.9(b).

Let α be a complex number. A correspondence

$$e(z) \mapsto -a(z),\ h(z) \mapsto -2 : a(z)b(z) : -2\alpha : \phi(z)\psi(z) :,$$

$$f(z) \mapsto : b(z)^2 a(z) : +2\alpha : b(z)\phi(z)\psi(z) : -2(1-\alpha^2)b(z)' \tag{1.1.3}$$

defines a morphism of vertex algebras

$$V_{0,2\alpha^2-2} \longrightarrow M \tag{1.1.4}$$

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and makes M (in fact, all M^i) a $\widehat{\mathfrak{g}}$-module of central charge $2\alpha^2-2$. In particular, the fermion number zero component M^0 is a Wakimoto module to be denoted $W_{0,2\alpha^2-2}$.

The above construction coincides with the one from [FF1], except for one modification: we have replaced the "Cartan" boson from *op. cit.* by the product of two fermions $\phi\psi$ using the boson-fermion correspondence (cf. for example [K], 5.2).

1.2. As was mentioned above, the classical analogs of Wakimoto modules are contragradient Verma modules over $\mathfrak{g}$. According to Beilinson-Bernstein, they may be realized as spaces of sections of certain "standard" $\mathcal{D}$-modules over X corresponding to Schubert cells - in our case they correspond to open cells.

The first aim of the present note is to "localize" the definition from 1.1, more precisely to construct sheaves of vertex algebras over $X = \mathbb{P}^1$ whose spaces of sections over an open Schubert cell would contain the Wakimoto modules $W_{0,2\alpha^2-2}$.

In the case $\alpha = 1$ this problem is solved by the chiral de Rham complex Ω^{ch}_X introduced in [MSV]. Let us explain this. Choose a coordinate b on $X = \mathbb{P}^1$ and consider the covering $X = U_0 \cup U_1$ by two affine lines, where $U_0 = \{x \in X |\ b(x) \neq \infty\}$, $U_1 = \{x \in X |\ b(x) \neq 0\}$, so $b:\ U_0 \xrightarrow{\sim} \mathbb{A}^1$ and $b^{-1}:\ U_1 \xrightarrow{\sim} \mathbb{A}^1$.

We identify $b = b_0\mathbf{1}$ and consider the space M from 1.1 as a $\mathbb{C}[b] = \mathcal{O}(U_0)$-module in the obvious way. Let $\mathcal{M}_0 = M^\sim$ be the corresponding quasicoherent sheaf over U_0. According to the Zariski localization procedure described in [MSV] (cf. also [GMS1], 1.10), $\mathcal{M}_0$ admits a structure of a sheaf of vertex (super)algebras over U_0. Similarly using the coordinate b^{-1} we get a sheaf of vertex algebras $\mathcal{M}_1$ over U_1. To get a sheaf over X, we want to glue $\mathcal{M}_0$ and $\mathcal{M}_1$ over the intersection $U_{01} = U_0 \cap U_1 \xrightarrow{\sim} \mathbb{G}_m$, i.e. to define an isomorphism $\mathcal{M}_0(U_{01}) \xrightarrow{\sim} \mathcal{M}_1(U_{01})$. This is done by the formulas

$$b(z) \mapsto b(z)^{-1},\ a(z) \mapsto -:b(z)^2a(z): -2:b(z)\phi(z)\psi(z):$$

$$\phi(z) \mapsto :b(z)^{-2}\phi(z):,\ \psi(z) \mapsto :b(z)^2\psi(z): \tag{1.2.1}$$

The meaning of these formulas is very simple. We consider the superalgebra $\mathbb{C}[a, b, \phi, \psi]$ as a superalgebra of derivations of the de Rham algebra $\Omega^{\cdot}_{\mathbb{C}[b]}$ where $a = \partial_b, \phi = db$ and $\psi = \partial_\phi$. Then we make a change of variables $b \mapsto b^{-1}$ and see how the variables a, ϕ, ψ transform. The formulas (1.2.1) are nothing but the same transformation law translated into the universe of fields. The fact that they define an automorphism of the *vertex algebra* $M[b^{-1}] = \mathcal{M}_0(U_{01})$ ("absence of quantum anomaly") is not banal - it is one of the main observations in [MSV].

By definition, the chiral de Rham algebra Ω^{ch}_X is a sheaf of vertex superalgebras over X obtained by the above gluing. By a general fact (cf. [MS1], Part III, Corollary 1.3) the Kac-Moody algebra $\widehat{\mathfrak{g}}$ acts on Ω^{ch}_X by derivations, with central charge zero. The space of sections of the fermionic zero component $\Omega^{ch,0}_X$ is the Wakimoto module $W_{0,0}$. A description of the spaces $\Gamma(X, \Omega^{ch,i}_X)$ as $\widehat{\mathfrak{g}}$-modules is given in *op. cit.*, Part. III, 2.2 (cf. Proposition 3.3 below).

In order to obtain the other Wakimoto modules $W_{2\alpha^2-2}$ we want to "twist" the definition of Ω^{ch}_X. More precisely, in Ω^{ch}_X the fields ϕ (resp. ψ) are sections of the sheaf of one-forms $\Omega^1_X = \mathcal{O}_{\mathbb{P}^1}(-2)$ (resp. of the tangent sheaf $\mathcal{T}_X = \mathcal{O}_{\mathbb{P}^1}(2)$).

Assume that $2\alpha \in \mathbb{Z}$. In a "twisted" de Rham algebra $\Omega^{ch}_{X,(\alpha)}$ they will be sections of $\mathcal{O}_{\mathbb{P}^1}(-2\alpha)$ and $\mathcal{O}_{\mathbb{P}^1}(2\alpha)$ respectively. In fact, $\Omega^{ch}_{X,(\alpha)}$ is an example of a twisted sheaf of chiral differential operators appeared in [GMS2], cf. 2.5 below.

The Lie algebra $\widehat{\mathfrak{g}}$ acts on $\Omega^{ch}_{X,(\alpha)}$ with central charge $k = 2\alpha^2 - 2$. Similarly to the case $\alpha = 1$ these sheaves are $\mathbb{Z}$-graded by fermionic number. We describe the spaces $\Gamma(X, \Omega^{ch,i}_{X,(\alpha)})$ as extensions of Weyl modules.

We also introduce, following the pattern of [MS2], a sort of *chiral Serre duality* on the category of $\Omega^{ch}_{X,(\alpha)}$-modules and use it to get a similar description of $H^1(X, \Omega^{ch,i}_{X,(\alpha)})$. This allows to compute their characters.

1.3. The next topic of this note will be a "free field resolution" of the vertex superalgebra $H^*(X, \Omega^{ch}_{X,(\alpha)})$, i.e. a description of the last algebra (which is a complicated object) as a cohomology of a simple "free field" algebra with respect to certain square zero odd derivation (which is a Fourier component of a field, and called "a screening charge"). At this point we use the fact that $\mathbb{P}^1$ is a toric variety. In [B] L.Borisov constructs for an arbitrary smooth toric variety Y a remarkable free field resolution of $H^*(Y, \Omega^{ch}_Y)$. We show how to twist the Borisov construction to obtain a free field resolution of $H^*(X, \Omega^{ch}_{X,(\alpha)})$.

It turns out that this "twisted" Borisov resolution essentially coincides with a resolution discussed by B. Feigin in an unpublished manuscript [F]. Introduced in [F] are several free field resolutions of the format from [FF4]. According to the philosophy of *op. cit.*, these should be considered as "quantizations" of standard Bernstein-Gelfand-Gelfand complexes of a maximal nilpotent Lie subalgebra $\mathfrak{n}'$ in a semisimple Lie algebra $\mathfrak{g}'$ (with coefficients in a Heisenberg module). "Quantization" means that instead of $\mathfrak{g}'$ one deals with corresponding quantum group. B. Feigin has discovered that in the case at hand the quantum group that governs the resolution corresponds to the Lie superalgebra $sl(2,1)$, $\mathfrak{n}'$ being generated by 2 odd elements. In our text this manifests in that the operators $D, \bar{D}$ appearing in (5.9.3) generate an algebra isomorphic to $\mathfrak{n}'$, and the complex (5.9.3) is exactly the Bernstein-Gelfand-Gelfand complex associated with V^0_L.

To be more precise, Feigin discusses the analog of this resolution in the case of a *generic* α (and here the quantum group arises). In our picture, the full sheaf $\Omega^{ch}_{X,(\alpha)}$ makes no sense for generic α; however, its component of *fermionic charge* 0 does (and has a vacuum $\widehat{sl}_2$-module as the space of global sections). So we can argue that the present construction gives a natural extension of Feigin's resolution in the case of non-generic α.

The "chiral Serre duality" makes it clear why Feigin's resolution is indeed a resolution and exactly what higher cohomology appear for non-generic α. This is probably the only real advantage of our approach over Feigin's.

The "linear algebra data" underlying our resolution is a 3-dimensional lattice L equipped with a one-parameter family of "hyperbolic reflections" ϕ_α, cf. *infra* (5.7.1)[2].

[2]These data are in some sense associated with the root system of $sl(2)$. Can one define similar data for other root systems?

Remark. One can define similar twisted de Rham algebras which localize the Wakimoto modules for an arbitrary semisimple Lie group G. However, we do not know the analogs of Feigin-Borisov resolutions for an arbitrary G — this is a very interesting open question.

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§2. The sheaf $\Omega^{ch}_{X,(\alpha)}$

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We use all the notations of the Introduction.

2.1. Consider the vertex algebra $M[b^{-1}] = \mathcal{M}_0(U_{01})$. It is generated by the fields $b(z), b(z)^{-1}, a(z), \phi(z), \psi(z)$. By composing the Wakimoto map (1.1.4) with the canonical localisation map $M \longrightarrow M[b^{-1}]$ we get a morphism of vertex algebras

$$V_{0,2\alpha^2-2} \longrightarrow M[b^{-1}] \tag{2.1.1}$$

Each vacuum module $V_{0,k}$ affords an involution defined by

$$e(z) \mapsto f(z),\ \ f(z) \mapsto e(z) \tag{2.1.2}$$

(note that $V_{0,k}$ is generated by two fields $e(z), f(z)$).

On the classical level, if we realize $\mathfrak{g}$ by means of vector fields acting on X and identify e with $-\partial_b$ and f with $b^2\partial_b$ then the interchage of e with f corresponds to the coordinate change $b \mapsto b^{-1}$. The next lemma chiralizes this automorphism.

2.2. Lemma. *Assume that $\alpha \in \frac{1}{2}\mathbb{Z}$. The assignment*

$$b(z) \mapsto b(z)^{-1},\ \ a(z) \mapsto -:b(z)^2 a(z): -2\alpha :b(z)\phi(z)\psi(z): -2(1-\alpha^2)b(z)',$$

$$\phi(z) \mapsto :b(z)^{-2\alpha}\phi(z):,\ \ \psi(z) \mapsto :b(z)^{2\alpha}\psi(z): \tag{2.2.1}$$

defines an automorphism of $M[b^{-1}]$ which is compatible with the involution (2.1.2) *via the map* (2.1.1).

If α is not necessarily a half-integer, the involution (2.2.1) *is still well defined on the fermionic number zero component $M[b^{-1}]^0$.* $\triangle$

2.3. *Remark.* Talking about fields is equivalent to talking about the states. In the language of states (2.2.1) is equivalent to

$$b_0\mathbf{1} \mapsto b_0^{-1}\mathbf{1},\ \ a_{-1}\mathbf{1} \mapsto -b_0^2 a_{-1}\mathbf{1} - 2\alpha b_0\phi_0\psi_{-1}\mathbf{1} - 2(1-\alpha^2)b_{-1}\mathbf{1}$$

$$\phi_0\mathbf{1} \mapsto b_0^{-2\alpha}\phi_0\mathbf{1},\ \ \psi_{-1}\mathbf{1} \mapsto b_0^{2\alpha}\psi_{-1}\mathbf{1} \tag{2.3.1}$$

2.4. On the neighbourhood of infinity U_1 with coordinate $\bar{b} = b^{-1}$ we have the sheaf $\mathcal{M}_1$ generated by the fields $\bar{a}(z), \bar{b}(z), \bar{\phi}(z), \bar{\psi}(z)$. By sending these barred fields to the corresponding right hand side expressions from (2.2.1) we get an isomorphism

$$g:\ \mathcal{M}_1|_{U_{01}} \stackrel{\sim}{\longrightarrow} \mathcal{M}_0|_{U_{01}} \tag{2.4.1}$$

By the way, to get g^{-1} one simply puts the bars over the fields in the rhs of (2.2.1).

Gluing $\mathcal{M}_0$ with $\mathcal{M}_1$ over U_{01} by means of g we get a sheaf of vertex superalgebras on X, to be denoted by $\Omega^{ch}_{X,(\alpha)}$. This sheaf is defined if $\alpha \in \frac{1}{2}\mathbb{Z}$. For $\alpha = 1$ we get the old chiral de Rham algebra $\Omega^{ch}_{X,(1)} = \Omega^{ch}_X$.

For an arbirtary α the whole sheaf is not defined but the fermion number zero component $\Omega^{ch,0}_{X,(\alpha)}$ is.

2.5. One can easily explain the meaning of the above sheaves in the framework of [GMS2]. Namely, consider a supervariety $X_{\mathcal{O}(-2\alpha)}$ with the same underlying space as X and with the structure sheaf $\Omega_{X,(\alpha)} = \mathcal{O}_X \oplus \mathcal{O}_X(-2\alpha)$ being the symmetric (in the super sense) algebra of the line bundle $\mathcal{O}_X(-2\alpha)$ where this line bundle is equipped the odd parity. So this superalgebra is a twisted analog of the de Rham algebra.

The sheaf $\Omega^{ch}_{X,(\alpha)}$ is nothing but *the* sheaf of chiral differential operators on $X_{\mathcal{O}(-2\alpha)}$. Note that since X is a curve, such a sheaf is unique up to a unique isomorphism, by the general cohomological formalism of *op. cit.*

2.6. By definition, $\Omega^{ch}_{X,(\alpha)}$ is a sheaf of $\widehat{\mathfrak{g}}$-modules of level $k = 2\alpha^2 - 2$. Therefore it carries a conformal structure where the Virasoro field is provided by the Sugawara construction. Locally on U_0 it is given by

$$L(z) = \frac{1}{k+2}\{: e(z)f(z) : + : f(z)e(z) : + \frac{1}{2} : h(z)^2 :\} =$$

$$=: b(z)'a(z) : + \frac{1-\alpha}{2\alpha} : \phi(z)\psi(z)' : + \frac{1+\alpha}{2\alpha} : \phi(z)'\psi(z) : \tag{2.6.1}$$

The grading given by the operator L_0 of this field differs from the obvious one: e.g.,

$$L_0(\phi_0\mathbf{1}) = \frac{\alpha-1}{2\alpha}\phi_0\mathbf{1},\ L_0(\psi_{-1}\mathbf{1}) = -\frac{\alpha+1}{2\alpha}\psi_{-1}\mathbf{1} \tag{2.6.2}$$

However, the old grading (but not the old conformal structure) which assigns to $x_i\mathbf{1}$ the degree $-i$, still makes sense for the whole sheaf $\Omega^{ch}_{X,(\alpha)}$. Below when we speak about "conformal weight" we mean always this *old* grading. Thus, the conformal weight zero component $\Omega^{ch}_{X,(\alpha)0} = \mathcal{O}_X \oplus \mathcal{O}_X(-2\alpha)$, $\phi_0\mathbf{1}$ being a local section of $\mathcal{O}_X(-2\alpha)$. Similarly, $\Omega^{ch}_{X,(\alpha)1}$ admits $\mathcal{O}_X(2\alpha)$ as a canonical quotient, $\psi_{-1}\mathbf{1}$ being its local generator, etc.

The fermionic charge grading on $\Omega^{ch}_{X,(\alpha)}$ is defined as in the untwisted case. Our algebra admits a canonical filtration whose graded quotient is given by (cf. [BL], [GMS2], 7.6)

$$[\mathrm{Gr}\Omega^{ch}_{X,(\alpha)}] = \prod_{n\geq 0}[S_{q^n}\Omega^1_X][S_{q^{n+1}}\mathcal{T}_X][\Lambda_{yq^n}\mathcal{O}_X(-2\alpha)][\Lambda_{y^{-1}q^{n+1}}\mathcal{O}_X(2\alpha)] \tag{2.6.3}$$

Here the square brackets denote the class in the K-theory; the power at q (resp. y) counts the conformal weight (resp. fermionic charge). In the right hand side we use the notation

$$[S_x(E)] = \sum_{i=0}^{\infty} [S^i_{\mathcal{O}_X}(E)]x^i \in K(X)[[x]],\ [\Lambda_x(E)] = \sum_{i=0}^{\infty} [\Lambda^i(E)]x^i \in K(X)[x] \tag{2.6.4}$$

for a vector bundle E over X.

§3. Global sections

3.1. In this section we give a description of $\Gamma(X, \Omega^{ch}_{X,(\alpha)})$ as a $\widehat{\mathfrak{g}}$-module and compute its character.

Let m be a nonnegative integer and k an arbitrary complex number. Let L_m denote the $(m+1)$-dimensional irreducible $\mathfrak{g}$-module. The Kac-Moody algebra $\widehat{\mathfrak{g}} = \mathfrak{g}[t,t^{-1}] \oplus \mathbb{C}c$ contains the Lie subalgebra $\widehat{\mathfrak{g}}_+ = \mathfrak{g}[t] \oplus \mathbb{C}c$ equipped with an obvious "evaluation at 0" projection $p:\ \widehat{\mathfrak{g}}_+ \longrightarrow \mathfrak{g} \oplus \mathbb{C}c$. Let $L_{m,k}$ denote a $\mathfrak{g} \oplus \mathbb{C}c$-module equal to L_m as a $\mathfrak{g}$-module, where c acts as multiplication by k; using p we can regard $L_{m,k}$ as a $\widehat{\mathfrak{g}}_+$-module. By definition, the *Weyl module* $V_{m,k}$ is the induced $\widehat{\mathfrak{g}}$-module

$$V_{m,k} = \mathrm{Ind}^{\widehat{\mathfrak{g}}}_{\widehat{\mathfrak{g}}_+} L_{m,k} \tag{3.1.1}$$

3.2. Let $\Omega^{ch,i}_{X,(\alpha)} \subset \Omega^{ch}_{X,(\alpha)}$ denote the component of fermionic charge $i \in \mathbb{Z}$. By construction,

$$\Gamma(U_0, \Omega^{ch,0}_{X,(\alpha)}) = W_{0,2\alpha^2-2} \tag{3.2.1}$$

Similarly,

$$\Gamma(U_0, \Omega^{ch,i}_{X,(\alpha)}) = W_{-2\alpha i,2\alpha^2-2} \tag{3.2.2}$$

where in the right hand side we have the Wakimoto module having the highest weight vector on which h acts as $2\alpha i$.

It is clear that just as in the untwisted ($\alpha = 1$) case discussed in [MS1], Part III, 2, we have

$$\Gamma(X, \Omega^{ch}_{X,(\alpha)}) = \Gamma(U_0, \Omega^{ch}_{X,(\alpha)})^{\mathfrak{g}-\mathrm{int}} \tag{3.2.3}$$

where the superscript in the right hand side denotes the maximal $\mathfrak{g}$-integrable submodule.

The diagram on p. 301 of [FF3] says that

3.3. Proposition. (i) *If $\alpha \in \mathbb{Z}_{\geq 0}, i \leq 0$, then $\Gamma(X, \Omega^{ch,i}_{X,(\alpha)})$ admits a filtration with quotients equal to Weyl modules $V_{-2i\alpha+4j\alpha^2,2\alpha^2-2}$, $j = 0,1,2,\ldots$;*

(ii) *if $\alpha \in \mathbb{Z}_{\geq 0}, i > 0$, then $\Gamma(X, \Omega^{ch,i}_{X,(\alpha)})$ admits a filtration with quotients equal to dual Weyl modules $V^*_{2i\alpha-2+4(j+1)\alpha^2,2\alpha^2-2}$, $j = 0,1,2,\ldots$;*

(iii) *if α is irrational then $\Omega^{ch}_{X,(\alpha)}$ does not exist but $\Omega^{ch,0}_{X,(\alpha)}$ does, and we have*

$$\Gamma(X, \Omega^{ch,0}_{X,(\alpha)}) = W^{\mathfrak{g}-\mathrm{int}}_{0,2\alpha^2-2} = V_{0,2\alpha^2-2} \tag{3.3.1}$$

△

3.4. Let us write down the character of our module. First consider the case $\alpha = 1$. We define

$$ch\ \Gamma(X, \Omega^{ch}_X) := \sum_{i\geq 0} \dim \Gamma(X, \Omega^{ch}_X)_i \cdot q^i \tag{3.4.1}$$

where the subscript $_i$ denotes the component of conformal weight i.

Due to 3.3 (i), (ii) we have

$$ch\,\Gamma(X, \Omega^{ch}_X) = \{\sum_{m=0}^{\infty} (4m+1)(2m+1)q^{(2m+1)m} + \sum_{m=0}^{\infty} (4m+3)(2m+1)q^{(2m+1)(m+1)}\}\cdot$$

$$\cdot \prod_{n=1}^{\infty} (1-q^n)^{-3} = \sum_{m\in\mathbb{Z}} (4m+1)(2m+1)q^{(2m+1)m} \cdot \prod_{n=1}^{\infty} (1-q^n)^{-3} \tag{3.4.2}$$

We want to transform this to a more familiar expression.

3.5. Introduce the standard notation

$$\eta(q) = q^{1/24} \prod_{n=1}^{\infty} (1-q^n) \tag{3.5.1}$$

$$G_2(q) = -\frac{1}{24} + \sum_{n=1}^{\infty} \sigma_1(n)q^n,\ \sigma_1(n) := \sum_{d|n} d \tag{3.5.2}$$

and (with D. Zagier, [Z]) a less standard one

$$\theta(q) = \sum_{m=0}^{\infty} q^{(2m+1)^2/8} \tag{3.5.3}$$

for the generating function counting the triangular numbers.

We have the Gauß formula (cf. [K2], Exercise 12.4, p. 176)

$$\phi(q^2)^2/\phi(q) = \sum_{m\in\mathbb{Z}} q^{m(2m+1)} \tag{3.5.3}$$

where

$$\phi(q) = \prod_{n=1}^{\infty} (1-q^n) \tag{3.5.4}$$

from which one deduces easily, using the identity

$$m(2m+1) = \frac{(4m+1)^2-1}{8} \tag{3.5.5}$$

that

$$\theta(q)=\eta(q^2)^2/\eta(q) \tag{3.5.6}$$

Similarly, using the Jacobi formula (cf. *loc. cit*)

$$\phi(q)^3=\sum_{m\in\mathbb{Z}}(4m+1)q^{m(2m+1)} \tag{3.5.7}$$

one deduces that

$$\eta(q)^3=\sum_{m\in\mathbb{Z},m=1(\mathrm{mod}4)} mq^{m^2/8} \tag{3.5.8}$$

where we have used that

$$\sum_{m\ \mathrm{odd},m>0} m^i q^{m^2/8}=\sum_{m=1(\mathrm{mod}4),\ m\in\mathbb{Z}} m^i q^{m^2/8} \tag{3.5.9}$$

We also have

$$q\partial_q\ \log\ \eta(q)=-G_2(q) \tag{3.5.10}$$

(see [HBJ], Appendix I, Lemma 4.9).

3.6. It follows from (3.5.6) and (3.5.10) that

$$q\partial_q\ \log\ \theta(q)=G_2(q)-2G_2(q^2) \tag{3.6.1}$$

On the other hand, using again (3.5.9),

$$q\partial_q\ \theta(q)=\frac{1}{8}\sum_{m\in\mathbb{Z},m=1(\mathrm{mod}4)} m^2q^{m^2/8} \tag{3.6.2}$$

Returning to our character (3.4.2) we deduce (using (3.5.5))

$$ch\ \Gamma(X,\Omega_X^{ch})=q^{-1/8}\sum_{n=1(\mathrm{mod}4),\ n\in\mathbb{Z}}\frac{n(n+1)}{2}q^{n^2/8}=$$

$$=\frac{1}{2}\{8q\partial_q\ \theta(q)+\eta^3(q)\}\eta(q)^{-3}=\frac{1}{2}+\frac{4\theta(q)}{\eta(q)^3}\{G_2(q)-4G_2(q^2)\} \tag{3.6.3}$$

Note that

$$(\theta(q)/\eta(q)^3)^4=\eta(q^2)^8/\eta(q)^{16}=(16\epsilon(q))^{-1} \tag{3.6.4}$$

where $\epsilon=(e_1-e_2)(e_1-e_3)$ is a standard modular form of weight 4 with respect to $\Gamma_0(2)$, cf. [HBJ], Appendix I, Cor. 4.11.

3.7. Now let us consider the case of an arbitrary nonnegative half-integer α. From 3.3 we have

$$ch\ H^0(X,\Omega_{X,(\alpha)}^{ch})=\sum_{(m,j)\in Q_+\cup -Q_-}(4j\alpha^2+2m\alpha+1)q^{(m\alpha+2j\alpha^2)(1+m\alpha+2j\alpha^2)/2\alpha^2}\cdot\phi(q)^{-3}=$$

where $Q_+ = \{(m,j)\in\mathbb{Z}^2|m\geq 0,\ j\geq 0\};\ Q_- = \{(m,j)\in\mathbb{Z}^2|m<0,\ j<0\}$; now make change of variables $m+2\alpha j = d+2\alpha k,\ 0\leq d<2\alpha$,

$$= q^{-1/8\alpha^2}\sum_{d=0}^{2\alpha-1}\sum_{k\in\mathbb{Z}}(k+1)(2\alpha(2\alpha k+d)+1)q^{2\alpha^2(k+\frac{2\alpha d+1}{4\alpha^2})^2}\cdot\phi(q)^{-3} =$$

$$= q^{-1/8\alpha^2}\sum_{d=0}^{2\alpha-1}\sum_{l\in\mathbb{Z}+\frac{2\alpha d+1}{4\alpha^2}}\{4\alpha^2 l+4\alpha^2-2\alpha d-1\}lq^{2\alpha^2 l^2}\cdot\phi(q)^{-3} \tag{3.7.1}$$

Introduce the Jacobi theta functions

$$\theta_{n,m}(\tau,z) = \sum_{l\in\mathbb{Z}+n/2m} e^{2\pi i m(l^2\tau - lz)},\ m\in\frac{1}{2}\mathbb{Z},\ m\geq 0,\ n\in\mathbb{Z}/2m\mathbb{Z} \tag{3.7.2}$$

cf. [K2], Example 13.3; as usually $q=e^{2\pi i\tau}$.

We see that the numerator in (3.7.1) is a finite linear combination of the values $\partial_z\theta_{2\alpha d+1,2\alpha^2}(\tau,0)$ and $\partial_\tau\theta_{2\alpha d+1,2\alpha^2}(\tau,0)$. This implies that it is a quasimodular form with respect to a finite index subgroup $\Gamma\subset SL(2;\mathbb{Z})$, cf. [KZ].

§4. The first cohomology and a chiral Serre duality

In this section we give a similar description $H^1(X,\Omega^{ch}_{X,(\alpha)})$ as a $\widehat{\mathfrak{g}}$-module, the tool of trade being the observation that the chiral Serre duality [MS2] carries over to the twisted case word for word. Let us explain the last point very briefly.

4.1. Recall that we have defined in 2.5 a sheaf $\Omega_{X,(\alpha)} = \mathcal{O}_X\oplus\mathcal{O}_X(-2\alpha)$ of supercommutative $\mathcal{O}_X$-algebras. Let $D_{\Omega_{X,(\alpha)}}$ be the algebra of superdifferential operators acting on $\Omega_{X,(\alpha)}$.

Consider the category of left $D_{\Omega_{X,(\alpha)}}$-modules, to be denoted $D_{\Omega_{X,(\alpha)}}-\mathrm{Mod}^l$, and of *restricted* left $\Omega^{ch}_{X,(\alpha)}$-modules, to be denoted $\Omega^{ch}_{X,(\alpha)}-\mathrm{Mod}^l$; restricted means that the subsheaves of negative conformal weights are equal to 0.

To compare $D_{\Omega_{X,(\alpha)}}-\mathrm{Mod}^l$ with $\Omega^{ch}_{X,(\alpha)}-\mathrm{Mod}^l$ introduce, by analogy with [MSV], 6.11, or [MS1], I.4.5., the *Weyl* functor

$$\mathbb{W}^l : D_{\Omega_{X,(\alpha)}}-\mathrm{Mod}^l\to\Omega^{ch}_{X,(\alpha)}-\mathrm{Mod}^l \tag{4.1.1}$$

to be the left adjoint to the functor of taking the conformal weight zero component.

4.2. *Example.* $\mathbb{W}^l(\Omega_{X,(\alpha)}) = \Omega^{ch}_{X,(\alpha)}$.

4.3. Proposition. $\mathbb{W}^l$ *is an equivalence of categories.* $\triangle$

4.4. Now we want to play a similar game with *right* modules. Consider the category of right $D_{\Omega_{X,(\alpha)}}$-modules, to be denoted $D_{\Omega_{X,(\alpha)}}-\mathrm{Mod}^r$, and of restricted right $\Omega^{ch}_{X,(\alpha)}$-modules, to be denoted $\Omega^{ch}_{X,(\alpha)}-\mathrm{Mod}^r$; here restricted means that the subsheaves of positive conformal weights are equal to 0. (We leave the definition of a right module over a vertex algebra to the reader.)

Define the right Weyl functor

$$\mathbb{W}^r : D_{\Omega_{X,(\alpha)}} - \mathrm{Mod}^r \to \Omega^{ch}_{X,(\alpha)} - \mathrm{Mod}^r \tag{4.4.1}$$

to be the left adjoint to the functor of taking the conformal weight zero component.

4.5. Proposition. *$\mathbb{W}^r$ is an equivalence of categories.* $\triangle$

4.6. Let us call a $D_{\Omega_{X,(\alpha)}}$-module *smooth* if it is a locally free $\mathcal{O}_X$-module of finite rank. Given a smooth $D_{\Omega_{X,(\alpha)}}$-module N, set

$$N^\vee = \mathcal{H}om_{\mathcal{O}_X}(N, \Omega^1_X) \tag{4.6.1}$$

In virtue of the Serre duality

$$H^1(X, N) \xrightarrow{\sim} H^0(X, N^\vee)^* \tag{4.6.2}$$

The $\mathcal{O}_X$-module $N^\vee$ admits a canonical structure of a right $D_{\Omega_{X,(\alpha)}}$-module.

Property (4.6.2) chiralizes as follows (c.f. [MS2] formula (11.6)).

4.7. Proposition. *If N is a smooth $D_{\Omega_{X,(\alpha)}}$-module, then*

$$H^1(X, \mathbb{W}(N)) \xrightarrow{\sim} H^0(X, \mathbb{W}^r(N^\vee))^* \tag{4.7.1}$$

$\triangle$

4.8. Denote $\Omega^{ch^\vee}_{X,(\alpha)} = \mathbb{W}^r(\Omega^\vee_{X,(\alpha)})$. Example 4.2 and Proposition 4.7 combined give

$$H^1(X, \Omega^{ch}_{X,(\alpha)}) = H^0(X, \Omega^{ch\vee}_{X,(\alpha)})^* \tag{4.8.1}$$

It is clear that the dual sheaf $\Omega^{ch\vee}_{X,(\alpha)}$ is very similar to $\Omega^{ch}_{X,(\alpha)}$ because both are defined as left adjoint functors. $\Omega^{ch\vee}_{X,(\alpha)}$ inherits the fermion number grading from $\Omega^{ch}_{X,(\alpha)}$ if we place Ω^1_X in, say, the fermion number 1 component:

$$\Omega^{ch\vee}_{X,(\alpha)} = \oplus_{i\in\mathbb{Z}}\, \Omega^{ch\vee,i}_{X,(\alpha)} \tag{4.8.2}$$

Further, $\Omega^{ch\vee}_{X,(\alpha)}$ carries a right action of $\widehat{sl}_2$, and this action precomposed with a canonical antiinvolution of $\widehat{sl}_2$ makes $\Omega^{ch\vee}_{X,(\alpha)}$ a sheaf of $\widehat{sl}_2$-modules of level $2\alpha^2-2$.

The analogs of (3.2.1-3.2.3) are as follows:

$$\Gamma(U_0, \Omega^{ch\vee,1}_{X,(\alpha)}) = W_{-2,2\alpha^2-2}, \tag{4.8.3}$$

$$\Gamma(U_0, \Omega^{ch\vee,i}_{X,(\alpha)}) = W_{-2(i-1)\alpha i-2,2\alpha^2-2}, \tag{4.8.4}$$

$$\Gamma(X, \Omega^{ch\vee}_{X,(\alpha)}) = \Gamma(U_0, \Omega^{ch\vee}_{X,(\alpha)})^{\mathfrak{g}-\mathrm{int}} \tag{4.8.5}$$

Comparing duality (4.8.1) with (4.8.3-5) we arrive at the following analog of Proposition 3.3.

4.9. Proposition. (i) *If* $\alpha \in \mathbb{Z}_{>0}, i \geq 1$, *then* $H^1(X, \Omega^{ch,i}_{X,(\alpha)})$ *admits a filtration with quotients equal to the dual Weyl modules* $V^*_{2i\alpha-2+4j\alpha^2,2\alpha^2-2}$, $j = 0, 1, 2, \ldots$;

(ii) *if* $\alpha \in \mathbb{Z}_{>0}, i < 0$, *then* $H^1(X, \Omega^{ch,i}_{X,(\alpha)})$ *admits a filtration with quotients equal to the Weyl modules* $V_{-2i\alpha+4(j+1)\alpha^2,2\alpha^2-2}$, $j = 0, 1, 2, \ldots$;

(iii) *if* α *is irrational then* $\Omega^{ch}_{X,(\alpha)}$ *does not exist but* $\Omega^{ch,0}_{X,(\alpha)}$ *does, and we have*

$$H^1(X, \Omega^{ch,0}_{X,(\alpha)}) = W^{\mathfrak{g}-\mathrm{int}}_{-2,2\alpha^2-2} = 0. \tag{4.9.1}$$

$\triangle$

§5. Free field realization

In this section we want to write down a free field resolution of the vertex superalgebra $H^*(X, \Omega^{ch}_{X,(\alpha)})$. From now on unless mentioned otherwise *assume that* $\alpha \in \mathbb{Z}$.

5.1. Consider a 3-dimensional integer lattice L with generators μ, β, ρ and an inner product $(\mu, \beta) = (\beta, \mu) = (\rho, \rho) = 1$, the other products of basic elements being zero. One associates with it a lattice vertex superalgebra V_L. Recall (cf. [K], 5.5) that as a vector space

$$V_L = Vac_{\mathfrak{h}} \otimes \mathbb{C}[L] \tag{5.1.1}$$

where $Vac_{\mathfrak{h}}$ is the vacuum module over the Heisenberg Lie algebra $\widehat{\mathfrak{h}} = \mathfrak{h}[t, t^{-1}] \oplus \mathbb{C}c$, $\mathfrak{h} := \mathbb{C} \otimes_{\mathbb{Z}} L$, $\mathbb{C}[L]$ being the group algebra of L whose basic elements are denoted by e^γ, $(\gamma \in L)$. The parity of e^γ is equal to

$$p(e^\gamma) = (\gamma, \gamma) \bmod 2 \tag{5.1.2}$$

the factor $Vac_{\mathfrak{h}}$ is purely even.

According to *loc. cit.*, the definition of V_L depends also on a choice of certain 2-cocycle $\epsilon : L \times L \longrightarrow \mathbb{C}^*$. However, the algebras corresponding to different choices of ϵ are canonically isomorphic. We fix one ϵ and banish it from the notation.

As a vertex algebra V_L is generated by fields $\gamma(z) := (\gamma \otimes 1)(z)$ and $e^\gamma(z) := (1 \otimes e^\gamma)(z)$ $(\gamma \in L)$.

Thus, V_L is $\mathbb{Z}^3$-graded

$$V_L = \oplus_{(i,j,k) \in \mathbb{Z}^3} V_L^{ijk}, \; V_L^{ijk} := Vac_{\mathfrak{h}} \otimes e^{i\beta + j\mu + k\rho} \tag{5.1.3}$$

We will use the notation V_L^γ for V_L^{ijk} where $\gamma = i\beta + j\mu + k\rho$, so

$$V_L = \oplus_{\gamma \in L} V_L^\gamma \tag{5.1.4}$$

We also use the $\mathbb{Z}$-grading

$$V_L = \oplus_{j\in\mathbb{Z}}\, V_L^j,\ V_L^j := \oplus_{(i,k)\in\mathbb{Z}^2}\, V_L^{ijk} \tag{5.1.5}$$

The subspace $Vac_{\mathfrak{h}} \otimes 1 = V_L^{000}$ is purely even vertex subalgebra of V_L, to be denoted simply by $Vac_{\mathfrak{h}}$. Each V_L^γ is a vertex module over $Vac_{\mathfrak{h}}$; it is an irreducible Heisenberg $\widehat{\mathfrak{g}}$-module on whose highest weight $\mathfrak{h}$ acts by means of the character $(\gamma, ?)$.

For $\gamma' = i'\beta + j'\mu + k'\rho \in L$, the field $e^\gamma(z)$ has degree (i', j', k'), that is, all its Fourrier components act from $V_L^\gamma = V_L^{ijk}$ to $V_L^{\gamma+\gamma'} = V_L^{i+i',j+j',k+k'}$.

5.2. *Remark.* The Heisenberg modules V_L^γ make perfect sense for arbitrary $\gamma \in \mathfrak{h} = L_{\mathbb{C}}$. Also one can write down the usual formula (cf. [K] (5.5.9)) for vertex operators $e^{\gamma'}(z)$ for an arbitrary $\gamma' \in \mathfrak{h}$.

These "fields" act also from V_L^γ to $V_L^{\gamma+\gamma'}$. However, they are multivalued: they have "monodromy" (whose logarithm is equal to $2\pi i(\gamma, \gamma')$). (Note that for $\gamma = 0$, or more generally for such γ that $(\gamma, \gamma') \in \mathbb{Z}$ this monodromy disappears). Their composition acquires monodromy as well. These monodromies is a source of appearance of quantum groups in the picture.

5.3. According to [B], the assignment

$$b(z) \mapsto e^\beta(z),\ b(z)^{-1} \mapsto e^{-\beta}(z),$$

$$a(z) \mapsto :(\mu(z) - \rho(z))e^{-\beta}(z):,\ \phi(z) \mapsto e^{\beta+\rho}(z),\ \psi(z) \mapsto e^{-\beta-\rho}(z) \tag{5.3.1}$$

determines an embedding of vertex superalgebras

$$M[b^{-1}] \hookrightarrow V_L \tag{5.3.2}$$

which induces an isomorphism

$$M[b^{-1}] \xrightarrow{\sim} V_L^0 \tag{5.3.3}$$

5.4. Consider an odd (since $(\mu - \rho, \mu - \rho) = 1$) field $e^{\mu-\rho}(z)$ and denote its zeroth component

$$d_M = \int e^{\mu-\rho}(z) \tag{5.4.1}$$

Here and below we will use the notation $\int c(z)$ for the component $c_{(0)}$ of a field $c(z) = \sum\ c_{(i)}z^{-i-1}$.

The operator d_M has degree 1 with respect to the grading (5.1.5), and $d_M^2 = 0$.

5.5. Lemma, [B]. *The sequence*

$$0 \longrightarrow M \longrightarrow V_L^0 \xrightarrow{d_M} V_L^1 \xrightarrow{d_M} \dots \tag{5.5.1}$$

is exact. $\triangle$

Here the first arrow is the composition of the obvious embedding $M \hookrightarrow M[b^{-1}]$ with the isomorphism (5.2.3).

5.6. We can compute the cohomology superalgebra $H^*(X, \Omega^{ch}_{X,(\alpha)})$ by means of the Cech complex of the covering $X = U_0 \cup U_1$:

$$0 \longrightarrow M \oplus \bar{M} \longrightarrow M[b^{-1}] \longrightarrow 0 \tag{5.6.1}$$

Here $\bar{M} = \Gamma(U_1, \Omega^{ch}_{X,(\alpha)})$ is simply another copy of M (whose fields we will denote by barred letters). The differential is the obvious embedding on the first component and (taken with the minus sign) composition of the obvious embedding $\bar{M} \hookrightarrow M[b^{-1}]$ and the automorphism of $M[b^{-1}]$ described in Lemma 2.2.

Now we want to replace $M, \bar{M}$ by their Borisov resolutions (5.5.1) (denote them V_L and $V_{\bar{L}}$) and $M[b^{-1}]$ by the isomorphic algebra V_L^0, cf. (5.2.3).

The only problem is to lift the differential (5.5.1) to a map of resolutions

$$V_L \oplus V_{\bar{L}} \longrightarrow V_L^0 \tag{5.6.2}$$

where we consider V_L^0 as a complex sitting in degree zero.

5.7. Consider an automorphism $\phi_\alpha :\ L \stackrel{\sim}{\longrightarrow} L$ given by

$$\phi_\alpha(\beta) = -\beta,\ \phi_\alpha(\mu) = -\mu - 2(\alpha-1)\rho + 2(\alpha-1)^2\beta,\ \phi_\alpha(\rho) = \rho - 2(\alpha-1)\beta \tag{5.7.1}$$

One checks that ϕ_α respects the inner product defined in 5.1 and that $\phi_\alpha^2 = Id$.

Hence it defines an involution of V_L which we will denote by the same letter

$$\phi_\alpha :\ V_L \stackrel{\sim}{\longrightarrow} V_L \tag{5.7.2}$$

5.8. Lemma. *If we identify $M[b^{-1}]$ with V_L^0 by means of the isomorphism* (4.2.3), *then the involution ϕ_α* (5.7.2) *induces the automorphism* (2.2.1) *of $M[b^{-1}]$.* $\triangle$

5.9. This lemma provides the necessary tool for defining (5.5.2). We have to define a morphism

$$\widehat{d}:\ V_L^0 \oplus V_{\bar{L}}^0 \longrightarrow V_L^0 \tag{5.9.1}$$

It will be the identity on the first component and taken with the minus sign composition of the tautological identification $V_{\bar{L}} = V_L$ and the involution (5.7.2).

Since bicomplex (5.6.2) is a resolution of the Cech complex (5.6.1), the cohomology algebra $H^*(X, \Omega^{ch}_{X,(\alpha)})$ is canonically identified with the cohomology of the complex

$$0 \longrightarrow \mathrm{Ker}\ \widehat{d} \stackrel{d_M \oplus d_{\bar{M}}}{\longrightarrow} V_L^1 \oplus V_{\bar{L}}^1 \stackrel{d_M \oplus d_{\bar{M}}}{\longrightarrow} V_L^2 \oplus V_{\bar{L}}^2 \longrightarrow \dots \tag{5.9.2}$$

Now let us identify $V_{\bar{L}}$ with V_L using the composition of the tautological identification and of (5.6.2). Then (5.8.2) becomes

$$0 \longrightarrow V_L^0 \stackrel{(D,\bar{D})}{\longrightarrow} V_L^1 \oplus V_L^{-1} \stackrel{D \oplus \bar{D}}{\longrightarrow} V_L^2 \oplus V_L^{-2} \longrightarrow \dots \tag{5.9.3}$$

where

$$D = d_M = \int e^{\mu-\rho}(z):\ V_L^i \longrightarrow V_L^{i+1} \tag{5.9.4}$$

cf. (5.3.1), and

$$\bar{D} = \int e^{\bar{\mu}-\bar{\rho}}(z) = \int e^{-\mu+(1-2\alpha)\rho+2\alpha(\alpha-1)\beta}(z) : \ V_L^i \longrightarrow V_L^{i-1} \tag{5.9.5}$$

5.10. It follows from (5.9.3) that

$$H^0(\mathbb{P}^1, \Omega^{ch}_{X,(\alpha)}) = \mathrm{Ker}(D : V_L^0 \longrightarrow V_L^1) \cap \mathrm{Ker}(\bar{D} : V_L^0 \longrightarrow V_L^{-1}) \tag{5.10.1}$$

Thus our space of global sections is the intersection of kernels of two screening operators connected with lattice vectors $\mu - \rho$ and $\bar{\mu} - \bar{\rho}$ respectively. Let us compute the angle between these two vectors:

$$(\mu - \rho, \bar{\mu} - \bar{\rho}) = (\mu - \rho, -\mu + (1 - 2\alpha)\rho + 2\alpha(\alpha - 1)\beta) = 2\alpha^2 - 1 = k + 1 \tag{5.10.2}$$

(Recall that $k = 2\alpha^2 - 2$ is the central charge of $\widehat{\mathfrak{g}}$ acting on our modules.)

5.11. Set $V_L'^0 = \oplus_{i\in\mathbb{Z}} V_L^{i00} \subset V_L^0$. The operators $D, \bar{D}$ make sense on $V_L'^0$ for all $\alpha \in \mathbb{C}$ not necessarily integer, cf. 4.2. Intersection of their kernels is equal to $H^0(X, \Omega^{ch,0}_{X,(\alpha)})$, which due to Proposition 3.3 (iii) is $V_{0,2\alpha^2-2}$; as far as we can judge, this is Feigin's result [F]. The restriction of (5.9.2) to $V_L'^0$ is the resolution appearing, perhaps implicitly, in [F]. It is indeed a resolution due to Proposition 4.9 (iii). If α is not generic, then at most the 1st cohomology arise and its structure is provided by Proposition 4.9 (i,ii). The size of the 0th cohomology group also jumps at special values of α as Proposition 3.3 (i-ii) shows.

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