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On spherical nilpotent orbits and beyond

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Introduction

Let $\mathfrak g$ be a semisimple Lie algebra and G its adjoint group. The ground field $\Bbbk$ is algebraically closed and of characteristic zero. We continue investigations started in [Pa94], which are primarily concerned with the *complexity* of nilpotent G-orbits (conjugacy classes) in $\mathfrak g$. Let $\mathcal N \subset \mathfrak g$ be the nilpotent cone and $\{0\} \ne \mathcal O \subset \mathcal N$ an orbit. It was proved in [Pa94] that $\mathcal O$ is spherical (i.e., of complexity 0) if and only if $\mathrm{ht}\,(\mathcal O) \le 3$. The number $\mathrm{ht}\,(\mathcal O)$, the height of $\mathcal O$, can be defined as $\max\{n \in \mathbb N \mid (\mathrm{ad}\, e)^n \ne 0,\ e \in \mathcal O\}$. In this article we give yet another characterization of spherical nilpotent orbits in terms of minimal Levi subalgebras intersecting them, see (3.2). As an outcome, a kind of the canonical form for such orbits is proposed (3.4):

an orbit $\mathcal O \subset \mathcal N$ is spherical if and only if it contains a representative which is a sum of root vectors corresponding to orthogonal simple roots.

Along the way, in sect. 2 and 3, we prove several auxiliary results about the height and the type of $\mathcal O$. Minimal non-spherical orbits in simple Lie algebras are described in sect. 4. These are of complexity 1 for SL_N and of complexity 2 for all other simple groups. In section 5, we study the complexity of nilpotent orbits for Vinberg's θ-groups. Recall that associated with a finite order automorphism θ of $\mathfrak g$, one has the periodic grading $\mathfrak g = \bigoplus_{j \in \mathbb Z_m} \mathfrak g_j$ and the connected reductive group G_0 acting on $\mathfrak g_1$. In this situation we are interested in the complexity of G_0-orbits in $\mathcal N \cap \mathfrak g_1$. The main results are:

- a monotonicity result for the complexity of Ge and G_0e ($e \in \mathcal N \cap \mathfrak g_1$), see (5.1) ;
- a formula for the complexity of G_0e in terms of a bi-grading of $\mathfrak g$, see (5.4);
- in case θ is of order 2, an almost complete description of spherical orbits is found, see (5.7).

However, the situation for θ-groups is not so simple, as it could be. It may happen that irreducible components of $Ge \cap \mathfrak g_1$, which are G_0-orbits, have different complexity, see (5.10). Finally, section 6 is a collection of observations and questions related to spherical nilpotent orbits.

As usual, algebraic groups are denoted by capital Roman letters, and their Lie algebras by the corresponding small Gothik letters. For $x \in \mathfrak g$, we write Gx in place of $(\mathrm{ad}\, G)x$.

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1 Recollections on nilpotent orbits and the complexity

Let $\mathfrak{g}$ be a semisimple Lie algebra with a fixed triangular decomposition $\mathfrak{g}=\mathfrak{u}_-\oplus\mathfrak{t}\oplus\mathfrak{u}_+$, Δ the corresponding root system, and $\Pi=\{\alpha_1,\dots,\alpha_p\}$ the set of simple roots. Let $\mathcal{N}\subset\mathfrak{g}$ be the nilpotent cone. By the Morozov-Jacobson theorem, any nonzero element $e\in\mathcal{N}$ can be included in an $\mathfrak{sl}_2$-triple $\{e,h,f\}$ (i.e., $[e,f]=h,\ [h,e]=2e,\ [h,f]=-2f$). The semisimple element h defines the $\mathbb{Z}$-grading in $\mathfrak{g}$:

$$\mathfrak{g}=\bigoplus_{i\in\mathbb{Z}}\mathfrak{g}(i)\ ,$$

where $\mathfrak{g}(i)=\{\,x\in\mathfrak{g}\mid [h,x]=ix\,\}$. It is well known that all $\mathfrak{sl}_2$-triples containing e are G_e-conjugate. Therefore the structure of this $\mathbb{Z}$-grading does not depend on a particular choice of h.

Following E.B. Dynkin, we shall say that h is a *characteristic* of e. The orbit Gh contains a unique element h_+ such that $h_+\in\mathfrak{t}$ and $\alpha(h_+)\ge 0$ for all $\alpha\in\Pi$. The Dynkin diagram of $\mathfrak{g}$ equipped with the numerical labels $\alpha_i(h_+)$, $\alpha_i\in\Pi$ at the corresponding nodes is called the *weighted Dynkin diagram* of e. After Dynkin and Kostant, it is known (see e.g. [SpSt]) that

(a) $\alpha_i(h_+)\in\{0,1,2\}$;

(b) elements $e,e'\in\mathcal{N}$ are G-conjugate *if and only if* their characteristics h and h' are G-conjugate *if and only if* their weighted Dynkin diagrams coincide.

We shall need the following standard results on the structure of the stabilizer $G_e\subset G$ and the centralizer $\mathfrak{g}_e\subset\mathfrak{g}$ (see [SpSt, ch. III]).

1.1 Proposition. (i) *The Lie algebra $\mathfrak{g}_e$ (resp. $\mathfrak{g}_f$) is positively (resp. negatively) graded; $\mathfrak{g}_e=\bigoplus_{i\ge 0}\mathfrak{g}_e(i)$, where $\mathfrak{g}_e(i)=\mathfrak{g}_e\cap\mathfrak{g}(i)$, and likewise for $\mathfrak{g}_f$;*
(ii) *Let $L:=G(0)$ be the connected subgroup in G corresponding to $\mathfrak{g}(0)$ and $K:=L_e$*[1]*. Then $K=G_e\cap G_f$ and it is a maximal reductive subgroup in both G_e and G_f;*
(iii) *For any i, there are K-stable decompositions:*

$$\mathfrak{g}(i)=\mathfrak{g}_e(i)\oplus[f,\mathfrak{g}(i+2)],\qquad \mathfrak{g}(i)=\mathfrak{g}_f(i)\oplus[e,\mathfrak{g}(i-2)].$$

In particular, $\mathrm{ad}\,f:\mathfrak{g}(i)\to\mathfrak{g}(i-2)$ is injective when $i\ge 1$ and surjective when $i\le 1$;
(iv) $(\mathrm{ad}\,f)^i:\mathfrak{g}(i)\to\mathfrak{g}(-i)$ *is one-to-one.* □

The notation related to the $\mathbb{Z}$-grading associated with a nilpotent orbit will be fixed throughout the paper.

1.2 Proposition. *For i even (resp. odd), $\mathfrak{g}(i)$ is an orthogonal (resp. symplectic) K-module. In particular, $\dim\mathfrak{g}(i)$ is even for i odd.*

[1] K may not be connected.

Proof. For $i \geq 0$, consider the bilinear form Ψ_i on $\mathfrak{g}(i)$ given by $(x, y) \mapsto \langle (\operatorname{ad} f)^i x, y \rangle$, where $\langle \cdot, \cdot \rangle$ is a G-invariant inner product on $\mathfrak{g}$. By (1.1)(ii),(iv), Ψ_i is nondegenerate and K-invariant. It follows from G-invariance of $\langle \cdot, \cdot \rangle$ that Ψ_i is symmetric for i even and alternate for i odd. $\square$

Recall that e or Ge is called

- *even* whenever $\mathfrak{g}(i) = 0$ for i odd or, equivalently, if all $\alpha_i(h_+) \in \{0, 2\}$;
- *distinguished*, if $\mathfrak{g}_e$ contains no semisimple elements, i.e., K is finite.

Let R be a reductive group and X an irreducible R-variety. Then X is called (R)-*spherical* if B_R, a Borel subgroup of R, has an open orbit in X. The *complexity* of X relative to R, which is denoted by $c_R(X)$, is equal to the minimal codimension of B_R-orbits in X. Clearly, $c_R(X) = c_{R^o}(X)$, where R^o stands for the identity component of R.

2 The height of a nilpotent orbit

Definition. The integer $\max\{i \mid \mathfrak{g}(i) \neq 0\}$ is called the *height* of e or the orbit $\mathcal{O} = Ge$ and is denoted by $\operatorname{ht}(e)$ or $\operatorname{ht}(\mathcal{O})$.

Since $e \in \mathfrak{g}(2)$, we have $\operatorname{ht}(e) \geq 2$ for any $e \in \mathcal{N} \setminus \{0\}$. Let $\Lambda \in \Delta_+$ be the highest root, $\Lambda = \sum_{i=1}^{p} n_i \alpha_i$. Clearly, we then have

(2.1)
$$\operatorname{ht}(e) = \Lambda(h_+) = \sum_{i=1}^{p} \alpha_i(h_+) n_i.$$

An immediate consequence of (1.1) is an intrinsic characterization of the height:

(2.2)
$$\operatorname{ht}(e) = \max\{\, n \in \mathbb{N} \mid (\operatorname{ad} e)^n \neq 0 \,\}.$$

For the classical Lie algebras $\mathfrak{sl}(V)$, $\mathfrak{sp}(V)$, and $\mathfrak{so}(V)$, it is sometimes more convenient to describe nilpotent orbits by the sizes of blocks in the Jordan normal form, i.e., in terms of partitions $(a_1, \dots, a_t)$, where $a_1 \geq a_2 \geq \dots \geq a_t$ and $\sum_{i=1}^{t} a_i = \dim V$. As is well known, this correspondence is one-to-one in case of $\mathfrak{sl}(V)$. For $\mathfrak{so}(V)$ and $\mathfrak{sp}(V)$, there is a correspondence between the nilpotent orbits and partitions satisfying a special condition. That is, in the symplectic (resp. orthogonal) case, one considers the partitions whose *odd* (resp. *even*) parts occur pairwise. This correspondence turns out to be a bijection, the only exception being that for $\mathfrak{so}(V)$ with $\dim V \equiv 0 \pmod 4$, each partition whose all parts are even ("a very even partition") arises from the two $SO(V)$-orbits, see [CM93, 5.1]. Since these two $SO(V)$-orbits form a single $O(V)$-orbit, these have the same height and complexity. In the sequel, we shall identify "classical" nilpotent orbits with corresponding partitions, keeping in mind this exception.

Let us give simple formulas for the height of nilpotent orbits in the classical Lie algebras.

2.3 Theorem. *Let $\mathcal{O} = (a_1, \dots, a_t)$ be a nilpotent orbit in a classical Lie algebra $\mathfrak{g}$ $(a_1 \ge a_2 \ge \dots \ge a_t)$.*

1. *If $\mathfrak{g} = \mathfrak{sl}(V)$ or $\mathfrak{sp}(V)$, then $\mathrm{ht}\,(\mathcal{O}) = 2(a_1 - 1)$;*
2. *If $\mathfrak{g} = \mathfrak{so}(V)$, then $\mathrm{ht}\,(\mathcal{O}) = \begin{cases} a_1 + a_2 - 2, & \text{if} \quad a_2 \ge a_1 - 1 \\ 2a_1 - 4, & \text{if} \quad a_2 \le a_1 - 2 \,. \end{cases}$*

In particular, either $\mathrm{ht}\,(\mathcal{O})$ is even or $\mathrm{ht}\,(\mathcal{O}) \equiv 3 \pmod 4$.

Proof. Let $\mathfrak{a} \subset \mathfrak{g}$ be a simple 3-dimensional subalgebra containing $e \in \mathcal{O}$. Denote by $R(n)$ a simple $\mathfrak{a}$-module of dimension $n+1$. Considering $\mathfrak{g}$ as the $\mathfrak{a}$-module, say $\mathfrak{g} = \bigoplus_i R(n_i)$, one sees that $\mathrm{ht}\,(e) = \max\{n_i\}$. On the other hand, $V = \bigoplus_{i=1}^t R(a_i - 1)$ as $\mathfrak{a}$-module. The relationship between V and the adjoint representation is well known:

$$\mathfrak{g} = \begin{cases} V \otimes V^* \ominus \mathbb{1} & \text{for} \quad \mathfrak{sl}(V) \\ S^2 V & \text{for} \quad \mathfrak{sp}(V) \\ \wedge^2 V & \text{for} \quad \mathfrak{so}(V) \,. \end{cases}$$

Combining these relations with the Clebsch–Gordan formula $R(n) \otimes R(m) = R(n + m) + R(n-1) \otimes R(m-1)$ and with the decomposition of $S^2 R(n_i)$ and $\wedge^2 R(n_i)$, one easily detects the biggest $\mathfrak{a}$-submodule in $\mathfrak{g}$. Whence the formulas for the height. In the orthogonal case, the constraint on pairity must be satisfied. That is, the equality $a_2 = a_1 - 1$ is only possible, if a_1 is odd. □

Remark. The above relationship between the adjoint and the simplest representations of classical algebras has been used in [El85] for obtaining a quick classification of distinguished nilpotent elements.

The orbits with odd height, in all simple Lie algebras, are not numerous and my feeling is that these ought to have some interesting properties. The following is a simple observation for them:

2.4 Proposition. *Suppose $\mathrm{ht}\,(e)$ is odd. Then the weighted Dynkin diagram of e contains no 2's.*

Proof. By (2.3), such elements e do not exist in $\mathfrak{sl}(V)$ and $\mathfrak{sp}(V)$. For $\mathfrak{so}(V)$, we must then have $a_2 = a_1 - 1$. Then a formula for the weighted Dynkin diagram (see [SpSt, 2.32] or [CM93, 5.3]) shows that $\alpha_i(h_+) \in \{0, 1\}$. In the exceptional cases, one can consult Elashvili's tables for the weighted Dynkin diagrams [El75] (see also [CM93, ch. 8]). □

The $\mathbb{Z}$-gradings associated with $\mathfrak{sl}_2$-triples form only a small part among all possible $\mathbb{Z}$-gradings. Many interesting features of the former were described in [Ka80]. The following assertion concerns the same subject.

2.5 Proposition. *Suppose that $\mathfrak{g}(2n) = 0$ for the $\mathbb{Z}$-grading associated with a nilpotent element e. Then $\mathrm{ht}\,(e) \le 2n - 1$.*

Proof. It suffices to prove that $\mathfrak{g}(2n+1) = 0$. Suppose not. Then $\mathfrak{g}(2n-1) \neq 0$ as well. To reach $\mathfrak{g}(2n+1)$ from $\mathfrak{g}(2n-1)$, we must have in $\mathfrak{g}(2)$ a root vector e_{α_i} for some $\alpha_i \in \Pi$. Thus, the weighted Dynkin diagram must contain a label "2". Then (2.4) says that $\mathrm{ht}\,(e)$ is even, which is not the case, if $\mathfrak{g}(2n+1) \neq 0$. $\square$

A relationship between the complexity and the height of nilpotent orbits is given by the following theorem proved in [Pa94]:

2.6 Theorem. *A nilpotent orbit Ge is spherical if and only if* $\mathrm{ht}\,(e) \le 3$. $\square$

3 The type of a nilpotent orbit and sphericity

In this section we characterize spherical nilpotent orbits in terms of minimal Levi subalgebras intersecting them. For any $e \in \mathcal{N}$, there exists a unique, up to conjugation, minimal Levi subalgebra $\mathfrak{z}$ intersecting Ge and, moreover, the orbit $Ze \subset \mathfrak{z}' := [\mathfrak{z}, \mathfrak{z}]$ is distinguished. This fact is usually attributed to Bala and Carter [BC76]. Not everybody has observed that a much more general assertion, in the context of graded Lie algebras, has independently been proved by E.B. Vinberg (see [Vi75] and [Vi79]). Construction itself is quite simple. Let $\mathfrak{h}$ be a Cartan subalgebra in $\mathfrak{k} = \mathfrak{g}_e(0)$. Then the centralizer $\mathfrak{z} := \mathfrak{z}_{\mathfrak{g}}(\mathfrak{h})$ is the desired Levi subalgebra. Put $\mathfrak{q} := \mathfrak{z}_{\mathfrak{g}}(\mathfrak{h})'$. Obviously, the elements e, h, and f lie in $\mathfrak{z}$ and hence in $\mathfrak{q}$. We also have $\mathfrak{z} = \bigoplus_{i \in \mathbb{Z}} \mathfrak{z}(i)$ and $\mathfrak{q} = \bigoplus_{i \in \mathbb{Z}} \mathfrak{q}(i)$.

3.1 Lemma. *1.* $\mathfrak{q}(i) = \mathfrak{z}(i)$ *for* $i \neq 0$*;*
2. $\mathfrak{q}(i) = 0$ *whenever* i *is odd.*

Proof. 1. Since $\mathfrak{z}$ contains h, the centre of $\mathfrak{z}$ is contained in $\mathfrak{g}(0) \cap \mathfrak{z} = \mathfrak{z}(0)$.
2. Obviously, $\mathfrak{q}(0)_e = \mathfrak{q}_e(0) = 0$, i.e., e is distinguished in $\mathfrak{q}$. By a result of Bala-Carter [BC76] and Vinberg [Vi79], any distinguished nilpotent element is even[2]. $\square$

The Cartan label of the semisimple subalgebra $\mathfrak{q} \subset \mathfrak{g}$ is called the *type* of Ge. Indication of the type forms a part of the notation for the nilpotent orbits in the exceptional Lie algebras used in [BC76][3]. In case of two root lengths, if a simple component of $\mathfrak{q}$ involves only short roots, then one places tilde over its Cartan label. The type does not determine the orbit uniquely. For instance, if e is distinguished in $\mathfrak{g}$, then $\mathfrak{h} = \{0\}$ and $\mathfrak{q} = \mathfrak{g}$. In order to distinguish different distinguished orbits, the Cartan label is accompanied by additional symbols, but we do not need this in our exposition. See [BC76] or [CM93, 8.4] for more details.

[2]Both proofs were case-by-case. An elegant *a priori* proof was found by Jantzen, see [Ka80, Note added in proof]

[3]Actually, this is a truncation of the data that were already used by E.B. Dynkin in [Dy52].

The notion of type applies to the classical Lie algebras as well and it is worth to write down explicit formulas for the type of a nilpotent orbit in this case. The partitions corresponding to the distinguished orbits has been pointed out in [Vi75], but the general formulas, though being known to experts, seem to be not in print.

Let $\mathcal{O} = (a_1, \dots, a_t)$ be a 'classical' nilpotent orbit.

For $SL(V)$: By the theory of Jordan normal form, we have $\mathfrak{q} = \mathbf{A}_{a_1-1} + \ldots + \mathbf{A}_{a_t-1}$.

For $Sp(V)$: If all the a_i's are distinct and even, then e is distinguished and $\mathfrak{q} = \mathbf{C}_n$, where $n = (\dim V)/2$. In general, each pair of equal parts (equal Jordan blocks) $a_i = a_{i+1}$ gives rise to a summand $\tilde{\mathbf{A}}_{a_i-1}$. After deleting all equal pairs, we obtain a partition with distinct *even* parts. This little partition determines the last summand in $\mathfrak{q}$, a smaller symplectic algebra. Because there are two root lengths, one has to distinguish between $\mathbf{A}_1$ and $\tilde{\mathbf{A}}_1$. The answer is that $\mathbf{A}_1$ occurs if and only if one obtains at the very end the partition (2). That is, formally $\mathbf{C}_1 = \mathbf{A}_1$.

For $SO(V)$: The procedure is similar. Each pair of equal Jordan blocks gives rise to a summand $\mathbf{A}_{a_i-1}$ in $\mathfrak{q}$. After deleting all equal pairs, we obtain a partition with distinct *odd* parts. This partition determines the last summand in $\mathfrak{q}$, a smaller orthogonal algebra. We have again to distinguish between $\mathbf{A}_1$ and $\tilde{\mathbf{A}}_1$. The answer is that $\tilde{\mathbf{A}}_1$ occurs if and only if one obtains at the very end the partition (3). That is, formally $\mathbf{B}_1 = \tilde{\mathbf{A}}_1$.

Modulo the description of distinguished orbits, the proof immediately amounts to the claim that the orbit(s) corresponding to the partition (n, n) is of type $\mathbf{A}_{n-1}$ in $\mathfrak{so}_{2n}$ and of type $\tilde{\mathbf{A}}_{n-1}$ in $\mathfrak{sp}_{2n}$.

Examples. 1. $\mathcal{O} = (4, 4, 4, 3, 3, 1, 1) \in \mathfrak{sp}_{20}$. Then $\mathfrak{q} = \tilde{\mathbf{A}}_3 + \tilde{\mathbf{A}}_2 + \mathbf{C}_2$.
2. $\mathcal{O} = (3, 3, 3, 2, 2, 2, 2) \in \mathfrak{so}_{17}$. Then $\mathfrak{q} = \mathbf{A}_2 + 2\mathbf{A}_1 + \tilde{\mathbf{A}}_1$.

3.2 Theorem. *Let $e \in \mathcal{N} \setminus \{0\}$. The following conditions are equivalent:*

1. $\mathrm{ht}\,(e) \le 3$;
2. $\mathfrak{g}(4) = 0$;
3. *Ge is of type $r\mathbf{A}_1 + l\tilde{\mathbf{A}}_1$;*
4. *There exist pairwise orthogonal simple roots $\beta_1, \dots, \beta_t$ such that Ge contains an element of the form $\sum_{i=1}^t e_i$, where $e_i \in \mathfrak{g}_{\beta_i} \setminus \{0\}$. (Then $t = r + l$ and there are r long and l short roots among the β_i's.)*

Proof. 1⇒2 – Obvious.
2⇒3. It follows from the definition of $\mathfrak{q}$ and (3.1) that $\mathfrak{q} = \mathfrak{q}(-2) \oplus \mathfrak{q}(0) \oplus \mathfrak{q}(2)$, $e \in \mathfrak{q}(2)$, and $h \in \mathfrak{q}(0)$. Since $\mathfrak{q}_e(0) = \{0\}$, we have $\dim \mathfrak{q}(0) = \dim \mathfrak{q}(2)$. On the other hand, $\mathfrak{q}(2)$ is a spherical $Q(0)$-module by [Pa94, 3.2]. Hence $Q(0)$ is a torus. Clearly, a semisimple Lie algebra having such a grading is just a sum of several 3-dimensional simple algebras.
3⇔4. This follows at once from the definition of the type.
4⇒1. Consider the Levi subalgebra $\mathfrak{f}$ corresponding to the roots $\beta_1, \dots, \beta_t$ and the corresponding connected subgroup $F \subset G$. We may assume that $e = \sum_{i=1}^t e_i$ and $\mathfrak{a} =$

$\langle e,h,f\rangle$ is embedded diagonally in $\mathfrak{f}'\simeq(\mathfrak{sl}_2)^t$. Take the unique F-stable decomposition $\mathfrak{g}=\mathfrak{f}\oplus\mathfrak{m}$ and consider an arbitrary irreducible F-submodule $V\subset\mathfrak{m}$. I claim that F has finitely many orbits in V. Indeed, as F is a Levi subgroup of G, there is a $\mathbb{Z}^{p-t}$-grading of $\mathfrak{g}$ whose 'zero'-part is $\mathfrak{f}$. (Recall that $p=\operatorname{rk}\mathfrak{g}$.) The F-module V is contained in some homogeneous subspace of this polygrading (it will actually be equal to some homogeneous subspace, but we do not need this fact). Now, finiteness follows by famous Vinberg's lemma, see [Vi76, Lemma in §2]. Let s be the number of simple factors of F acting non-trivially on V. Then $V\simeq R(d_1)\otimes\ldots\otimes R(d_s)$, where all $d_i\ge 1$. Since F has a dense orbit in V, we have $\dim((SL_2)^s\times\Bbbk^*)\ge\dim V$, i.e. $3s+1\ge(d_1+1)\ldots(d_s+1)\ge 2^s$. This inequality has exactly three solutions:

$$\begin{gathered} s=3,\quad d_1=d_2=d_3=1;\\ s=2,\quad d_1\le 2, d_2=1;\\ s=1,\quad d_1\le 3.\end{gathered}$$

In each case we have $\sum_i d_i\le 3$. Together with the Clebsch–Gordan formula, this shows that the biggest irreducible $\mathfrak{a}$-module that can occur in V is $R(3)$. Since $\mathfrak{f}|_{\mathfrak{a}}\simeq tR(2)+(p-t)R(0)$, the same is true for the whole Lie algebra $\mathfrak{g}$. But, this means precisely that $\operatorname{ht}(e)\le 3$. □

Equivalence of conditions (1) and (2) is a particular case of Prop. 2.5. But the last proof does not appeal to case-by-case considerations as in (2.4).

3.3 Corollary (of the proof). *For any $\mu\in\Delta$, let $\{\alpha_{i_1},\ldots,\alpha_{i_s}\}\subset\Pi$ be a set of pairwise orthogonal roots such that $\langle\alpha_{i_j},\mu\rangle\ne 0$, $j=1,\ldots,s$. Then $s\le 3$ (even $s\le 2$, if an addition μ is long and one of the α_{i_j}'s is short).*

Proof. Replacing μ by a suitable root $\mu'=\mu+\sum_j n_j\alpha_{i_j}$, one can achieve that $\langle\alpha_{i_j},\mu'\rangle>0$ for all j. Then μ' is the highest weight of an irreducible $(SL_2)^s$-submodule V of $\mathfrak{g}$. Since all copies of SL_2 act non-trivially on V, the proof of 4⇒1 shows that $s\le 3$ (even $s\le 2$, if $d_1\ge 2$). □

Remarks. 1. In case $\mu\in\Pi$, we obtain the well-known assertion that the number of other simple roots that are not orthogonal to μ is at most 3. That is, we have given an invariant-theoretic proof of it.

2. If we drop the assumption that the α_{i_j}'s are pairwise orthogonal, then it is easy to give an example with $s=5$ (at least).

Combining (2.6) and (3.2), we obtain a kind of normal form for spherical nilpotent orbits:

3.4 Theorem. *Suppose $e\in\mathcal{N}$. Then the orbit Ge is spherical if and only if it contains an element of the form $\sum_{i=1}^t e_i$, where $e_i\in\mathfrak{g}_{\beta_i}$ and $\beta_1,\ldots,\beta_t$ are pairwise orthogonal simple roots.* □

Examples. 1. The spherical nilpotent orbits in $\mathfrak{sp}_{2p}$ are $\mathcal{O}_i=(2^i,1^{2p-2i})$[4] $(0\le i\le p)$. Then the type of $\mathcal{O}_i$ is $\begin{cases} l\tilde{\mathbf{A}}_1, & \text{if } i=2l,\\ l\tilde{\mathbf{A}}_1+\mathbf{A}_1, & \text{if } i=2l+1.\end{cases}$
In particular for $i=p=2l+1$, the respective set of simple roots is $\alpha_1,\alpha_3,\dots,\alpha_{2l+1}$. (The last of them is long.)

2. The type of spherical orbit $(3,2^2,1^l)$ in $\mathfrak{g}=\mathfrak{so}(V)$ is $\begin{cases} 3\mathbf{A}_1, & \text{if } l \text{ is odd },\\ \mathbf{A}_1+\tilde{\mathbf{A}}_1, & \text{if } l \text{ is even }.\end{cases}$

4 Orbits of small complexity

Recall, with some variations, a formula for the complexity of Ge. Keep the notation of sect. 1. Let S be a *stabilizer in general position* (= s.g.p.) for the K-action on $\mathfrak{g}(2)$. We shall use the notation $S=\text{s.g.p.}(K,\mathfrak{g}(2))$ as a shorthand. The reader is referred to [VP89, §7] for the basic facts on s.g.p. As $\mathfrak{g}(2)$ is an orthogonal K-module, a result of D. Luna asserts that S is reductive. Set $\mathfrak{g}(\ge j)=\oplus_{i\ge j}\mathfrak{g}(i)$.

4.1 Theorem. *1.* $c_G(Ge)=c_L(\mathfrak{g}(2))+c_S(\mathfrak{g}(\ge 3))$*;*
2. $c_G(Ge)=c_L(\mathfrak{g}(\ge 2))$.

Proof. 1. Since $Le\simeq L/K$ is the dense orbit in $\mathfrak{g}(2)$, we have $c_L(\mathfrak{g}(2))=c_L(L/K)$. Hence (1) is nothing but the first formula in [Pa94, 2.3].

2. By definition, $c_L(\mathfrak{g}(\ge 2))=\dim\mathfrak{g}(\ge 2)-\dim B_L+\dim(B_L)_*$, where B_L is a Borel subgroup of L and $(B_L)_*:=\text{s.g.p.}(B_L,\mathfrak{g}(2))$. To compute $(B_L)_*$, we use the decomposition $\mathfrak{g}(\ge 2)=\mathfrak{g}(2)\oplus\mathfrak{g}(\ge 3)$. Namely, if $B_2:=\text{s.g.p.}(B_L,\mathfrak{g}(2))$, then $(B_L)_*=\text{s.g.p.}(B_2,\mathfrak{g}(\ge 3))$. Whence

$$c_L(\mathfrak{g}(\ge 2))=c_L(\mathfrak{g}(2))+\min_{v\in\mathfrak{g}(\ge 3)}\operatorname{codim}_{\mathfrak{g}(\ge 3)}B_2v\ .$$

I claim that $(B_2)^o=B_S$[5]. Granting this, we see that the last term in the above formula is just $c_S(\mathfrak{g}(\ge 3))$, as required. It remains to demonstrate the claim. By [Pa87] (see also [Pa90, sect. 1]), $(B_2)^o$ is equal to a Borel subgroup of an s.g.p.$(L,\mathfrak{g}(2)\oplus\mathfrak{g}(2)^*)$. To compute the latter, we use the same trick as for $(B_L)_*$:

$$\text{s.g.p.}(L,\mathfrak{g}(2)\oplus\mathfrak{g}(2)^*)=\text{s.g.p.}(K,\mathfrak{g}(2)^*)=\text{s.g.p.}(K,\mathfrak{g}(2))=S\ .$$

The first equality follows because L/K is the dense orbit in $\mathfrak{g}(2)$. $\square$

Remark. The meaning of these formulas is that computing of the complexity of Ge is reduced to that for a representation space. In case of representations, there is an explicit algorithm for doing this [Pa87]. Actually, given a representation $R\to GL(V)$ of

[4] As usual in the theory of partitions, $a^j:=a,\dots,a$ (j times).

[5] Because s.g.p.'s are defined up to conjugacy in the ambient group, this means that one obtains the equality under an appropriate choice of all the groups involved.

a reductive group R, the algorithm says how to find s.g.p.$(R, V\oplus V^*) =: R_*$. The group R_* is reductive and has some other nice properties. Then $c_R(V) = \dim V - \dim B_R + \dim B_{R_*}$.

4.2 Proposition. *If* $\dim\mathfrak g(4)\ge 2$ *or* $\mathrm{ht}\,(e)\ge 5$, *then* $c_G(Ge)\ge 2$.

Proof. Actually, it will be shown that $c_S(\mathfrak g(\ge 3))\ge 2$.
Let us detect two algebraically independent B_S-invariant rational functions on $\mathfrak g(\ge 3)$. It follows from (3.2) that in both cases $\mathfrak g(4)\ne 0$. Since $\mathrm{ad}\, f:\mathfrak g(4)\to\mathfrak g(2)$ is injective, there is a K-module W_1 such that $\mathfrak g(2)\simeq\mathfrak g(4)\oplus W_1$. Because $S=\text{s.g.p.}(K,\mathfrak g(2))$, we have $K(\mathfrak g(2)^S)$ is dense in $\mathfrak g(2)$. It follows that $\mathfrak g(4)^S\ne 0$. Write $\mathfrak g(4)=\mathfrak g(4)^S\oplus W_2$, where W_2 is a K-module. Since $\mathfrak g(4)$ is an orthogonal K-module (1.2), the same holds for W_2. Hence in case $\dim\mathfrak g(4)\ge 2$ we have at least two algebraically independent functions in $\Bbbk[\mathfrak g(4)]^S$. If $\dim\mathfrak g(4)=1$, then we examine the following possible cases:

(a) Let $\mathfrak g(6)\ne 0$. Then repeating the previous argument with $(\mathrm{ad}\, f)^2$ in place of $\mathrm{ad}\, f$, we obtain $\mathfrak g(6)^S\ne 0$.

(b) Let $\mathfrak g(5)\ne 0$. Then $\mathfrak g(5)$ and $[f,\mathfrak g(5)]\subset\mathfrak g(3)$ are two different isomorphic S-submodules in $\mathfrak g(\ge 3)$. Obviously, this produces a non-constant B_S-invariant rational function on $\mathfrak g(\ge 3)$. □

Remark. Similar arguments prove that if $\mathrm{ht}\,(e)\ge 2n{+}1$ $(n\ge 2)$, then $c_G(Ge)\ge n$. But I think there ought to exist a quadratic polynomial $n\mapsto\phi(n)$ such that $c_G(Ge)\ge\phi(n)$.

By (2.6) and (4.2), the nilpotent orbits of complexity 1 are contained among those having height 4. The latter exist in all simple Lie algebras. However, routine computations lead to the following conclusion:

4.3 Theorem. *Nilpotent orbits of complexity 1 exist only for* $G=SL_n$, $n\ge 3$. *Moreover, such an orbit is unique. The corresponding weighted Dynkin diagram is* 2–0-…-0–2 *and the partition is* $(3,1^{n-3})$. *For all other simple groups, the minimal non-spherical orbits are of complexity 2.*

Proof. For the classical groups, classification of the minimal non-spherical (= m.n.s.) orbits follows from (2.3), (2.6), and an explicit description, due to Gerstenhaber and Hesselink, of the closure ordering, see [CM93, 6.2]. For the exceptional groupes, one uses (3.2)(3) and the Hasse diagrams for the closure ordering, see [Spal, IV.2].
1. For SL_n, there is a unique m.n.s. orbit $\mathcal O=(3,1^{n-3})$. It is even and of height 4. For $n=3$, $\mathcal O$ is the regular nilpotent orbit and the assertion follows by counting dimensions. Let $n>3$. Then $L=SL_{n-2}\times(\Bbbk^*)^2$ and the L-modules $\mathfrak g(2)$ and $\mathfrak g(4)$ are as follows: $\mathfrak g(2)=R(\varphi_1)\otimes\varepsilon+R(\varphi_1)^*\otimes\mu$, $\mathfrak g(4)=R(0)\otimes\varepsilon\mu$. Here $R(\lambda)$ stands for the irreducible representation of the semisimple part with highest weight λ, while ε and μ are basic characters of the central torus; $\dim\mathfrak g(2)=2n-4$ and $\dim\mathfrak g(4)=1$. Obviously, $\mathfrak g(2)$ is a spherical L-module. It is not hard to compute that $K\simeq SL_{n-3}\times\Bbbk^*$, $S\simeq SL_{n-4}\times\Bbbk^*$, and S acts trivially on $\mathfrak g(4)$. Applying then (4.1), we get $c_G(\mathcal O)=1$.

The minimal nilpotent orbit lying 'over' $\mathcal{O}$ is

$\mathcal{O}' = \begin{cases} (4), & \text{if } n=4 \\ (3,2,1^{n-5}), & \text{if } n\ge 5 \end{cases}$. Then it already turns out that $c_G(\mathcal{O}')=2$.

2. There are two m.n.s. orbits for Sp_{2n} $(n\ge 3)$ and $\mathbf{F}_4$, and one orbit in the other cases. More precisely, all groups of rank ≥ 3 have a unique orbit of type $\mathbf{A}_2$ (of type $\tilde{\mathbf{A}}_2$ for Sp_{2n}), which is m.n.s. Besides, $\mathbf{F}_4$ has the m.n.s. orbit of type $\tilde{\mathbf{A}}_2$, and Sp_{2n} $(n\ge 2)$ has the m.n.s. orbit of type $\mathbf{C}_4$, which corresponds to the partition $(4,1^{2n-4})$. Finally, the m.n.s. orbit for $\mathbf{G}_2$ is 10-dimensional (and distinguished). All these orbits are of complexity 2.

For instance, consider the orbit of type $\tilde{\mathbf{A}}_2$ for $G=\mathbf{F}_4$. Here $L=Spin_7\times\Bbbk^*$, $\mathfrak{g}(2)=R(\varphi_3)\otimes\varepsilon$, and $\mathfrak{g}(4)=R(\varphi_1)\otimes\varepsilon^2$; $\dim\mathfrak{g}(2)=8$, $\dim\mathfrak{g}(4)=7$. This information is easily being extracted from the weighted Dynkin diagram given in the Table 1 in sect. 5. Then $K\simeq\mathbf{G}_2$ and $c_L(L/K)=0$. Next, $\mathfrak{g}(2)$ affords the sum of the simplest (7-dimensional) and the trivial 1-dimensional representation of $\mathbf{G}_2$. Therefore $S\simeq SL_3$. It is not hard to compute that $\mathfrak{g}(4)|_{SL_3}=R(\varphi_1')+R(\varphi_2')+R(0)$. Whence $c_S(\mathfrak{g}(4))=2$. $\square$

This result confirms a claim in [Vi86, n. 9] concerning orbits of complexity 1 in the universe that "it appears there should be few of them".

5 The complexity of nilpotent orbits of θ-groups

Let θ be an automorphism of $\mathfrak{g}$, of finite order m. Fix a primitive m-th root of unity ζ. Consider the periodic grading

$$\mathfrak{g}=\bigoplus_{j\in\mathbb{Z}_m}\mathfrak{g}_j\ ,$$

where $\mathfrak{g}_j$ is the θ-eigenspace of $\mathfrak{g}$ corresponding to ζ^j. Following Vinberg, we shall say that the connected reductive group G_0 acting linearly on $\mathfrak{g}_1$ is a *θ-group*. The main references on θ-groups are [Vi75], [Vi76], [Vi79]. One of the basic results is that $\overline{G_0e}\ni 0$ $(e\in\mathfrak{g}_1)$ if and only if $e\in\mathcal{N}$. Such G_0-orbits are called nilpotent, too. Our aim is to study the complexity of them. Throughout this section, it is assumed that $e\in\mathcal{N}\cap\mathfrak{g}_1$. The first result is:

5.1 Theorem. $c_G(Ge)\ge c_{G_0}(G_0e)$.

To demonstrate the theorem, we need the following variation on Vinberg's themes:

5.2 Lemma. *Let $G\to GL(V)$ be a representation of a reductive group, $H\subset G$ a reductive subgroup, and $W\subset V$ an H-stable subspace. Choose Borel subgroups in H and G such that $B_H\subset B_G$. Suppose that*

$(*)\qquad\qquad \mathfrak{b}_Hv=\mathfrak{b}_Gv\cap W\qquad$ *for all $v\in W$.*

For any G-stable locally-closed subvariety $X \subset V$ and each irreducible component Y of $X \cap W$, we then have

$$c_G(X) \geq c_H(Y) \ .$$

Proof. By [Vi86], the complexity $c_G(X)$ is equal to the *modality* of B_G-action on X, i.e., to $\max\limits_{X' \subset X} \operatorname{trdeg} k(X')^{B_G}$, where X' runs through the irreducible B_G-stable subvarieties of X. On the other side, it follows from $(*)$ and [Vi76, §2] that $B_G v \cap W$ is a union of finitely many B_H-orbits for all $v \in W$, each B_H-orbit being an irreducible component of $B_G v \cap W$. Therefore, if $Y' \subset Y$ is B_H-stable and irreducible, then $\operatorname{trdeg} k(Y')^{B_H} = \operatorname{trdeg} k(\overline{B_G{\cdot}Y'})^{B_G}$. □

Proof of (5.1). The lemma applies to $V = \mathfrak{g}$, $H = G_0$, and $W = \mathfrak{g}_1$. The condition $(*)$ follows from presence of periodic grading. As $(*)$ holds also for $\mathfrak{g}_0$ and $\mathfrak{g}$ in place of $\mathfrak{b}_H$ and $\mathfrak{b}_G$, Vinberg's lemma [Vi76, §2] implies that each irreducible component of $Ge \cap \mathfrak{g}_1$ is a G_0-orbit. In particular, one of the components is $G_0 e$. □

It follows that, given a spherical orbit Ge, each irreducible component of $Ge \cap \mathfrak{g}_1$ is a spherical G_0-orbit, too. But a naïve hope for the converse fails to be true. For, all simple Lie algebras have a periodic grading such that G_0 is a torus. Then all G_0-orbits in $\mathfrak{g}_1$ are spherical, while this is not always the case for the G-orbits in $G\mathfrak{g}_1$. To develop a technique for dealing with the complexity of G_0-orbits and, in particular, for classifying the spherical ones, we need some preparations.

By a modification of the Morozov-Jacobson theorem, one may assume that $\{e, h, f\}$ is *adapted* to θ, i.e., $h \in \mathfrak{g}_0$ and $f \in \mathfrak{g}_{-1}$. Then $\mathfrak{g}$ gains a $\mathbb{Z} \times \mathbb{Z}_m$-grading

$$\mathfrak{g} = \bigoplus_{i \in \mathbb{Z}} \bigoplus_{j \in \mathbb{Z}_m} \mathfrak{g}(i)_j \ .$$

We have $e \in \mathfrak{g}(2)_1$, $h \in \mathfrak{g}(0)_0$, and $f \in \mathfrak{g}(-2)_{-1}$. By [Vi79, Th. 1], all adapted triples containing e are $(G_0)_e$-conjugate. Therefore the structure of this bi-grading does not depend on choice of an adapted triple. One may say that $G_0 e$ determines a refinement $\mathfrak{g}(i) = \bigoplus_{j \in \mathbb{Z}_m} \mathfrak{g}(i)_j$ of the $\mathbb{Z}$-grading associated with Ge. However, it is worth noting that another irreducible component of $Ge \cap \mathfrak{g}_1$ may and usually does determine another refinement of the same $\mathbb{Z}$-grading. We defer discussing of this and related problems until (5.9).

Denote by L_0 and K_0 the identity components of the θ-fixed subgroups in L and K respectively. Then $\mathfrak{l}_0 = \mathfrak{g}(0)_0$ and it follows from (1.1) that $[\mathfrak{l}_0, e] = \mathfrak{g}(2)_1$ and $(\mathfrak{l}_0)_e = \mathfrak{k}_0$. Clearly, each $\mathfrak{g}(i)_j$ is a K_0-module and $\langle \mathfrak{g}(i)_j, \mathfrak{g}(i')_{j'} \rangle = 0$ unless $i + i' = 0$ and $j + j' = 0$. An extension of (1.2) to the bi-grading is given by

5.3 Proposition. *For all $i, j \in \mathbb{Z}$ we have*
1. $\mathfrak{g}(2i)_i$ and $\mathfrak{g}(2i)_j \oplus \mathfrak{g}(2i)_{2i-j}$ are orthogonal K_0-modules;
2. $\mathfrak{g}(2i+1)_j$ and $\mathfrak{g}(2i+1)_{2i+1-j}$ are dual K_0-modules.
(The subscripts are being considered as elements of $\mathbb{Z}/m\mathbb{Z}$.)

Proof. Let $i \geq 0$. Since $(\mathrm{ad}\, f)^{2i} : \mathfrak{g}(2i)_i \to \mathfrak{g}(-2i)_{-i}$ is bijective, the restriction of Ψ_i (see (1.2)) to $\mathfrak{g}(2i)_i$ is non-degenerate, and so on ... $\square$

Denote by M an s.g.p. for the K_0-action on $\mathfrak{g}(2)_1$. Again, M is reductive by Luna's result, since $\mathfrak{g}(2)_1$ is orthogonal. Note that there is no relation between M and $S =$ s.g.p.$(K, \mathfrak{g}(2))$.

5.4 Theorem. *1.* $c_{G_0}(G_0 e) = c_{L_0}(\mathfrak{g}(2)_1) + c_M(\mathfrak{g}(\geq 3)_1)$*;*
2. $c_{G_0}(G_0 e) = c_{L_0}(\mathfrak{g}(\geq 2)_1)$.

Proof. 1. As well as Theorem 4.1(1), it will be a consequence of [Pa94, 1.2]. Namely, to write a formula for the complexity of $G_0 e \simeq G_0/(G_0)_e$, we use an embedding of $(G_0)_e$ into some parabolic subgroup in G_0. Set $\mathfrak{p} = \bigoplus_{i\geq 0} \mathfrak{g}(i)$. It is a parabolic subalgebra in $\mathfrak{g}$ and $\mathfrak{l} = \mathfrak{g}(0)$ is a Levi subalgebra in it. Obviously, $\mathfrak{p}_0 = \bigoplus_{i\geq 0} \mathfrak{g}(i)_0$ is parabolic in $\mathfrak{g}_0$ and $\mathfrak{l}_0$ is a Levi subalgebra in it. Denote by A^u the unipotent radical of an algebraic group A. By (1.1), we have $G_e \subset P$ and $(G_e)^u \subset P^u$. Write N for $(G_e)^u \cap G_0$. Then the identity component of $(G_0)_e$ is $K_0 N$ and $N = (K_0 N)^u$. Clearly, $K_0 N \subset P_0$ and $N \subset (P_0)^u$. That is, we have the *right* embedding in terminology of [Pa94]. Since the component group of the stabilizer does not affect the complexity, we may apply [Pa94, 1.2] to conclude

$$c_{G_0}(G_0 e) = c_{G_0}(G_0/K_0 N) = c_{L_0}(L_0/K_0) + c_M((\mathfrak{p}_0)^u/\mathfrak{n}) \ .$$

Since K_0 is the identity component of $(L_0)_e$ and $L_0 e$ is dense in $\mathfrak{g}(2)_1$, we have $c_{L_0}(L_0/K_0) = c_{L_0}(\mathfrak{g}(2)_1)$. It follows from (1.1)(iii) that

$$\mathfrak{p}^u = \mathfrak{g}(\geq 1) = (\mathfrak{g}_e)^u \oplus \mathrm{ad}\, f{\cdot}\mathfrak{g}(\geq 3) \ .$$

Whence

$$(\mathfrak{p}_0)^u = (\mathfrak{p}^u)_0 = (\mathfrak{g}_e)^u_0 \oplus \mathrm{ad}\, f{\cdot}\mathfrak{g}(\geq 3)_1 = \mathfrak{n} \oplus \mathrm{ad}\, f{\cdot}\mathfrak{g}(\geq 3)_1 \ .$$

Thus, $(\mathfrak{p}_0)^u/\mathfrak{n}$ is isomorphic to $\mathfrak{g}(\geq 3)_1$ as K_0- and hence M-module.

2. The proof is analogous that in (4.1)(2). $\square$

Remarks. 1. All the previous results hold without changes if $\mathfrak{g} = \oplus_j \mathfrak{g}_j$ is a $\mathbb{Z}$-grading, i.e., formally $m = \infty$.

2. The paper [Pa94] has dealt not only with the complexity, but also with the *rank* of nilpotent orbits. Although the notions are quite different, the formulas for the both and the proofs turned out to be the same. This also holds in case of θ-groups. For instance, making use of the above embedding $K_0 N \subset P_0$ and Theorem 1.2 in [loc cit.], one proves the formula for the rank of $G_0 e$: $r_{G_0}(G_0 e) = r_{L_0}(\mathfrak{g}(2)_1) + r_M(\mathfrak{g}(\geq 3)_1)$.

Although it is already possible to give some estimates for $c_{G_0}(G_0 e)$, these are isolated and do not allow us to achieve attractive results (yet?). For this reason, we restrict ourselves to the case where θ is *involutory*. That is, from now on $m = 2$ and we intend to describe spherical nilpotent orbits for the isotropy representation of a symmetric variety G/G_0. Now, an adapted $\mathfrak{sl}_2$-triple gives rise to the decomposition of K_0-modules $\mathfrak{g}(i)=\mathfrak{g}(i)_0\oplus\mathfrak{g}(i)_1$ $(i\in\mathbb{Z})$ and as an immediate corollary of (5.3) we have

5.5 Lemma.
(i) *For i odd, $\mathfrak{g}(i)_0$ and $\mathfrak{g}(i)_1$ are dual K_0-modules;*
(ii) *For i even, $\mathfrak{g}(i)_0$ and $\mathfrak{g}(i)_1$ are orthogonal K_0-modules.* □

Our idea is to characterize spherical G_0-orbits in terms of the G-orbits in $\mathfrak{g}$ these generate. In view of (5.1), one has to realize which nonspherical G-orbits may arise in this way. To this end, our main tool is Theorem 5.4.

5.6 Theorem. *Suppose θ is involutory and $e\in\mathfrak{g}_1\cap\mathcal{N}$. If $\mathrm{ht}\,(e)\ge 5$ or $\mathfrak{g}(4)_1\ne 0$, then $c_{G_0}(G_0e)>0$.*

Proof. Actually, we shall prove that $\Bbbk[\mathfrak{g}(\ge 3)_1]^M\ne\Bbbk$, which certainly implies that $c_M(\mathfrak{g}(\ge 3)_1)>0$, cf. (4.2). The situation splits into 3 cases.

(a) Assume that $\mathfrak{g}(5)\ne 0$. By (5.5), $\mathfrak{g}(5)_1$ and $\mathfrak{g}(5)_0$ are dual K_0-modules and hence dual M-modules. Then $(\mathrm{ad}\,f)\mathfrak{g}(5)_0$ and $\mathfrak{g}(5)_1$ are dual M-modules in $\mathfrak{g}(\ge 3)_1$.

(b) Assume that $\mathfrak{g}(4)_1\ne 0$. By (5.5), it is an orthogonal K_0-module and hence M-module. Thus, $\Bbbk[\mathfrak{g}(\ge 3)_1]^M\ne\Bbbk$.

(c) If $\mathrm{ht}\,(e)\ge 5$, $\mathfrak{g}(5)=0$, and $\mathfrak{g}(4)\subset\mathfrak{g}_0$, then $0\ne(\mathrm{ad}\,e)\mathfrak{g}(4)=\mathfrak{g}(6)=\mathfrak{g}(6)_1$. Consider $(\mathrm{ad}\,f)^2\mathfrak{g}(6)_1=(\mathrm{ad}\,f)\mathfrak{g}(4)\subset\mathfrak{g}(2)_1$. It is a K_0-submodule. Since $M=$ s.g.p.$(K_0,\mathfrak{g}(2)_1)$, we have $K_0(\mathfrak{g}(2)_1^M)$ is dense in $\mathfrak{g}(2)_1$. Whence $((\mathrm{ad}\,f)^2\mathfrak{g}(6)_1)^M\ne 0$ and, finally, $\mathfrak{g}(6)_1^M\ne 0$. Thus, $\Bbbk[\mathfrak{g}(\ge 3)_1]^M$ contains a linear function. □

From (5.1) and (5.6) it follows that there is a coherent description for all G-orbits except for those of height 4:

(5.7) *If θ is involutory and $\mathcal{O}$ is a nilpotent orbit such that $\mathcal{O}\cap\mathfrak{g}_1\ne\varnothing$, then for* $\mathrm{ht}\,(\mathcal{O})\le 3$ *each irreducible component of $\mathcal{O}\cap\mathfrak{g}_1$ is G_0-spherical, while for* $\mathrm{ht}\,(\mathcal{O})\ge 5$ *none of the irreducible components is G_0-spherical. This holds regardless of θ provided only that $\mathcal{O}\cap\mathfrak{g}_1\ne\varnothing$.*

For the G-orbits of height 4, the answer already depends on relationship between θ and $\mathcal{O}$, i.e., the complexity of the irreducible components of $\mathcal{O}\cap\mathfrak{g}_1$ may or may not be equal to zero, depending on θ. A necessary condition of sphericity is that $\mathfrak{g}(4)\subset\mathfrak{g}_0$, but it is not sufficient. I am going to consider this problem in a forthcoming paper. Note that, given e and θ, it is easy to realize when $Ge\cap\mathfrak{g}_1\ne\varnothing$. By a result of L. Antonyan [An82, Th. 1], we have $Ge\cap\mathfrak{g}_1\ne\varnothing\Longleftrightarrow Gh\cap\mathfrak{g}_1\ne\varnothing$, the latter being immediately seen from the weighted Dynkin diagram of e and the Satake diagram of θ.

5.8 Example. Using (5.4), one can give a recipe for producing triples $(\mathfrak{g},\theta,e)$ such that $e\in\mathfrak{g}_1$, $c_G(Ge)>0$, and $c_{G_0}(G_0e)=0$. Let e be an even nilpotent element with $\mathrm{ht}\,(e)=4$. Then $c_G(Ge)>0$. Define a $\mathbb{Z}_2$-grading of $\mathfrak{g}$ by the formulas $\mathfrak{g}_0=$

$\mathfrak g(-4)\oplus\mathfrak g(0)\oplus\mathfrak g(4)$, $\mathfrak g_1=\mathfrak g(-2)\oplus\mathfrak g(2)$. Then $L_0=L$, $\mathfrak g(2)=\mathfrak g(2)_1$, and $\mathfrak g({\ge}3)_1=0$. Therefore $c_{G_0}(G_0e)=c_L(\mathfrak g(2))$. All such orbits with spherical L-module $\mathfrak g(2)$ are listed in table 1. Content-Length: 16782 X-Lines: 353

Table 1:

$\mathfrak g$	characteristic	type of $\mathcal O$	$\dim\mathfrak g(2)$	$\mathfrak g_0$
$\mathfrak{sl}_n$ $(n\ge 3)$	2–0–...–0–2	$\mathbf A_2$	$2n-4$	$\mathfrak{sl}_{n-2}\oplus\mathfrak{sl}_2\oplus\Bbbk$
$\mathfrak{sl}_6$	0–2–0–2–0	$2\mathbf A_2$	8	$\mathfrak{sl}_4\oplus\mathfrak{sl}_2\oplus\Bbbk$
$\mathfrak{sp}_{2n}$ $(n\ge 3)$	0–2–0-...$\Leftarrow$0	$\mathbf A_2$	$4n-8$	$\mathfrak{sp}_{2n-4}\oplus\mathfrak{sp}_4$
$\mathfrak f_4$	2–0$\Leftarrow$0–0	$\tilde{\mathbf A}_2$	8	$\mathfrak{so}_9$
$\mathfrak e_6$	0 2–0–0–0–2	$2\mathbf A_2$	16	$\mathfrak{so}_{10}\oplus\Bbbk$

5.9 On behaviour of irreducible components. In general, different irreducible components of $\mathcal O\cap\mathfrak g_1$ determine non-isomorphic bi-gradings, e.g. the groups L_0 can be different, and so on. One may address the following questions in this regard:

Is it possible that these components have different complexity relative to G_0?

Is it possible that spherical and non-spherical components occur together?

The answer to the first question is "yes" and we present below an example of triple $(\mathfrak g,\theta,e)$ such that the complexity of irreducible components of $\mathcal O\cap\mathfrak g_1$ takes three values. As for the second question, such situation might only occur when $\mathrm{ht}\,(\mathcal O)=4$. But my calculations support the negative answer. I have the following partial result based on the classification of orbits and explicit calculations:

Suppose that either $\mathfrak g$ is exceptional or $\mathfrak g=\mathfrak{sl}_N$ and θ is inner. If $\mathrm{ht}\,(\mathcal O)=4$ and $\mathcal O\cap\mathfrak g_1$ has a spherical irreducible component, then all the components are spherical.

So, I believe the second answer ought to be "no" and it would be desirable to find a classification-free proof.

5.10 Example. Let $\mathfrak g=\mathfrak{sl}_N$. Denote by θ_n an inner involution of $\mathfrak g$ such that $\mathfrak g_0\simeq\mathfrak{sl}_n\oplus\mathfrak{sl}_{N-n}\oplus\Bbbk$. To emphasize the dependence on n, we shall write $G_0^{(n)}$ and $\mathfrak g_1^{(n)}$. The elements of $\mathfrak g_1^{(n)}$ can be thought of as pairs of counter operators (A,B), where $A\in\mathbf{Hom}\,(V,W)$, $B\in\mathbf{Hom}\,(W,V)$, $\dim V=n$, and $\dim W=N-n$. The orbit classification for pairs of counter operators was first obtained in [DP65]. (From the modern point of view, this is a special case of the quiver theory.) We explain this classification using the language of ab-diagrams introduced in [KP79, sect. 4]. Given $\mathcal O=(a_1,\dots,a_t)$, a corresponding ab-diagram is obtained if one writes a string of consecutive symbols "a" and "b", of length a_i, in place of part "a_i". Two ab-diagrams are proclaimed to be equivalent if these are obtained form each other by reordering ab-strings of equal length. There is a bijection between the irreducible components of $\mathcal O\cap\mathfrak g_1^{(n)}$ and the classes of equivalent ab-diagrams such that the total number of a's is n.

For $N \geq 6$, consider $\mathcal{O} = (3,3,1^{N-6})$. Then $\mathcal{O} \cap \mathfrak{g}_1^{(n)} \neq \varnothing$ if and only if $2 \leq n \leq N-2$. From the point of view of complexity, essentially different irreducible components correspond to the following ab-diagrams:

I : $(bab, bab, \dots)$; II : $(aba, aba, \dots)$; III : $(aba, bab, \dots)$.

The strings of length 1 are uniquely determined by the requirement that the total number of the symbol "a" is as prescribed. Note that case I (resp. II) occurs if and only if $N-n \geq 4$ (resp. $n \geq 4$) and case III occurs if and only if $3 \leq n \leq N-3$. Making use of these ab-diagrams, one can write explicitly an $\mathfrak{sl}_2$-triple adapted to θ_n. An explicit matrix form of the latter allows us to determine the decompositions $\mathfrak{g}(i)_0 \oplus \mathfrak{g}(i)_1$ and then to compute the complexity. The answer is that the complexity relative to $G_0^{(n)}$ of the irreducible components of $\mathcal{O} \cap \mathfrak{g}_1^{(n)}$ is equal to: $n-2$ in case I; $N-n-2$ in case II; 1 in case III. For instance, if $N=9$ and $n=4$ then there are irreducible components of complexity $1, 2$, and 3. Since $c_G(\mathcal{O}) = N-2$, one also obtains a nice illustration to Theorem 5.1.

6 Questions, observations, remarks

6.1 Irreducibility. A direct verification shows that $\mathcal{N}^{sph}$, the union of all nilpotent spherical G-orbits, is irreducible. The explicit expression for the maximal spherical orbit in the classical case is derived from [Pa94, sect. 4]; in the exceptional case, look at the tables in [Spal, IV.2]. It might be interesting to give an *a priori* proof and to describe the ideal of this variety. However, the analogous property does not hold for arbitrary θ-groups. For instance, $(\mathcal{N} \cap \mathfrak{g}_1)^{sph}$ has 2 irreducible components for an involution of SL_2. A more exotic example is an involution of $\mathbf{F}_4$ of maximal rank.

Making use of the known normality results (due to Broer and Kraft & Procesi), one sees that $\mathcal{N}^{sph}$ is normal for $G = \mathbf{A}_p$, $\mathbf{B}_{2p}$, $\mathbf{C}_p$, $\mathbf{D}_p$, $\mathbf{F}_4$; and not normal for $\mathbf{B}_{2p+1}$ and $\mathbf{G}_2$. It is likely that $\mathcal{N}^{sph}$ is normal for $\mathbf{E}_p$ and there is a unified proof for **A**-**D**-**E**.

6.2 Dimension. By definition, $\dim \mathcal{O} \leq \dim B_G$ for $\mathcal{O} \subset \mathcal{N}^{sph}$. Our aim here is a bit more exact inequality. Without loss of generality, we may assume that $\mathfrak{g}$ is simple. Denote by k_0 (resp. k_1) the number of even (resp. odd) exponents of $\mathfrak{g}$. Then $k_0 + k_1 = \mathrm{rk}\, \mathfrak{g} =: p$.

Proposition. *If* $\mathcal{O} \subset \mathcal{N}^{sph}$, *then* $\dim \mathcal{O} \leq \dim B_G - k_0$.

Proof. This is easily verified case-by-case, but we give a conceptual proof. Let $\mathfrak{g} = \mathfrak{g}_0^{max} \oplus \mathfrak{g}_1^{max}$ be the decomposition corresponding to an involution of maximal rank θ^{max}. This means $\mathfrak{g}_1^{max}$ contains a Cartan subalgebra (= C.s.a.) of $\mathfrak{g}$. By [An82, th. 2], we have $\mathcal{O} \cap \mathfrak{g}_1^{max} \neq \varnothing$. By [KR71, prop. 5], the equality $\dim(\mathcal{O} \cap \mathfrak{g}_1) = \frac{1}{2} \dim \mathcal{O}$ holds for any involution (provided that $\mathcal{O} \cap \mathfrak{g}_1 \neq \varnothing$). As each irreducible component of $\mathcal{O} \cap \mathfrak{g}_1^{max}$

is G_0^{max}-spherical (5.1), we have

$$\frac{1}{2}\dim \mathcal{O} = \dim(\mathcal{O}\cap \mathfrak{g}_1^{max}) \le \dim B_0^{max} \ ,$$

where B_0^{max} is a Borel subgroup of G_0^{max}. Thus, it remains to demonstrate that $\dim B_0^{max} = \frac{1}{2}(\dim B_G - k_0)$. Since $\dim G_0^{max} = \dim B_G - p$, it amounts to proving that $\mathrm{rk}\, G_0^{max} = k_1$, see next lemma. □

Lemma. $\mathrm{rk}\, G_0^{max} = k_1$.

Proof. Let $\mathfrak{h}$ be a θ^{max}-stable C.s.a. of $\mathfrak{g}$ such that $\mathfrak{h}_0$ is a C.s.a. of $\mathfrak{g}_0^{max}$. Since the G-orbit of a characteristic of a regular nilpotent element intersects $\mathfrak{g}_0^{max}$ (consider an adapted $\mathfrak{sl}_2$-triple!), we see that $\mathfrak{h}_0$ contains regular semisimple elements. Set $\sigma := \theta^{max}\mid_{\mathfrak{h}}$. By [Sp74, 6.5], the eigenvalues of σ are ε_i^{-1} $(1\le i\le p)$, where the ε_i's are the "eigenvalues of θ^{max} on the set of basic invariants". The latter means that homogeneous generators $F_1,\dots,F_p$ of the polynomial algebra $\Bbbk[\mathfrak{g}]^G$ can be chosen so that $\theta^{max}{\cdot}F_i = \varepsilon_i F_i$. (Of course, $\varepsilon_i \in \{1,-1\}$.) On the other side, there is a C.s.a. $\mathfrak{h}' \subset \mathfrak{g}_1^{max}$, i.e., $\sigma' := \theta^{max}\mid_{\mathfrak{h}'} = -id$. By [Sp74, 6.4(v)], the eigenvalues of σ' are $\varepsilon_i^{-1}(-1)^{m_i}$ $(1\le i\le p)$, where $m_1,\dots,m_p$ are the exponents of $\mathfrak{g}$. Whence $\varepsilon_i = 1$ if m_i is odd and $\varepsilon_i=-1$ if m_i is even. Finally, $\mathrm{rk}\, G_0^{max} = \dim\mathfrak{h}_0 = \#\{i \mid \varepsilon_i = 1\} = k_0$. □

We have proven that $\dim\mathcal{N}^{sph} \le \dim B_G - k_0$. Actually, the equality holds, but I do not know a unified proof. A promising approach is discussed in the next subsection.

There is another curious coincidence related to another involution. By [An82, Th.3(2)], there is a unique, up to G-conjugation, *inner* involution θ^{int} of $\mathfrak{g}$ such that *all* orbits in $\mathcal{N}$ have non empty intersection with $\mathfrak{g}_1^{int}$. ($\theta^{int} \ne \theta^{max}$ for $\mathbf{A}_p$, $\mathbf{D}_{2p+1}$, and $\mathbf{E}_6$.) One can prove that $\dim\mathfrak{g}_0^{int} - \dim\mathfrak{g}_1^{int} = k_0 - k_1$. Since $\mathrm{rk}\, G_0^{int} = \mathrm{rk}\, G$, this implies $\dim\mathfrak{g}_1^{int} = \dim B_G - k_0$. That is, $\dim\mathcal{N}^{sph} = \dim\mathfrak{g}_1^{int}$.

6.3 A relationship with the index of B_G. The index of an algebraic group (or its Lie algebra) is the minimal codimension of its orbits in the coadjoint representation. The index $\mathrm{ind}\, B_G$ of B_G for all simple groups was computed in [Tr83, §4]. This result can be stated as: $\mathrm{ind}\, B_G = k_0$. It seems however that no explanation of this equality is known. Our observation is:

Proposition. *If $\mathcal{O}\subset\mathcal{N}^{sph}$, then $\dim\mathcal{O} \le \dim B_G - \mathrm{ind}\, B_G$.*

Proof. By the very definition of sphericity, there is $x\in\mathcal{O}$ such that $\mathfrak{g}_x + \mathfrak{b}_G = \mathfrak{g}$. Taking the orthocomplements, we obtain

$$(\heartsuit) \qquad [\mathfrak{g},x]\cap[\mathfrak{b}_G,\mathfrak{b}_G] = \{0\}\ .$$

Let $\overline{x}$ be the image of x in $\mathfrak{g}/[\mathfrak{b}_G,\mathfrak{b}_G]$. The latter is identified with the B_G-module $(\mathfrak{b}_G)^*$. Since $x\in[\mathfrak{g},x]$, we have $\overline{x}\ne 0$. Because $[\mathfrak{g},x]=[\mathfrak{b}_G,x]$, equality ($\heartsuit$) implies that $\dim B_G - \mathrm{ind}\, B_G \ge \dim B_G\overline{x} = \dim B_G x = \dim\mathcal{O}$. □

Given an $\xi\in(\mathfrak{b}_G)^*\simeq\mathfrak{g}/[\mathfrak{b}_G,\mathfrak{b}_G]$, it would be interesting to realize, is there $x\in\mathfrak{g}$ over ξ such that Gx is spherical?

6.4 In search of a uniform proof. Recently, Fan and Stembridge found a relationship in the *simply laced* case between the spherical nilpotent orbits and the "commutative" elements in the Weyl group W. They proved (see [FS97, Th. 3.1(a)]) that, for a natural map $\phi : W \to \mathcal{N}/G$, the set W_c of commutative elements is just the preimage of $\mathcal{N}_4/G$, where $\mathcal{N}_4 = \{e \in \mathcal{N} \mid \mathrm{ht}\,(e) < 4\}$. But their hope [FS97, 3.4(d)] that better understanding of the fibres of ϕ may lead to a uniform proof of (2.6) seems to me groundless. For: (1) the proof of Theorem 3.1(a) in [loc.cit.] also uses classificational results; (2) while the concept of a spherical orbit is quite general, the above relationship breaks in the non simply laced case, even if one replaces "commutative elements" in W by "fully commutative" or "short-braid avoiding" ones. To see this, it suffices to consider the groups Sp_4 and $\mathbf{G}_2$.

Actually, case-by-case (classificational) arguments in proving (2.6) were used only for the orbits of height 3. My opinion is that the right way consists in better understanding of the orbits of height 3, as special case of orbits of odd height.

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