FOCAL SETS, MOMENT MAPS AND INDUCED HOPF BUNDLES IN THE QUATERNIONIC PROJECTIVE SPACE

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Abstract. We describe a diagram containing the zero sets of the moment maps associated to the diagonal $U(1)$ and $Sp(1)$ actions on the quaternionic projective space $\mathbb{H}P^n$. These sets are related both to focal sets of submanifolds and to Sasakian-Einstein structures on induced Hopf bundles. As an application, we construct a complex structure on the Stiefel manifolds $V_2(\mathbb{C}^{n+1})$ and $\widetilde{V}_4(\mathbb{R}^{n+1})$, the one on the former manifold being non compatible with its known hypercomplex structure.

1. Introduction

A motivation for the present work has been to understand some aspects of the Riemannian geometry of the focal set $Foc_{\mathbb{H}P^n}\mathbb{C}P^n$, i.e. of the set of points in the quaternionic projective space $\mathbb{H}P^n$ that are critical values of the normal exponential map with respect to the totally geodesic submanifold $\mathbb{C}P^n \subset \mathbb{H}P^n$. Our starting point has been the Habilitationsschrift by J. Berndt [7], [8], where it is proven, in particular, that $Foc_{\mathbb{H}P^n}\mathbb{C}P^n$ fibers in circles over $Gr_2(\mathbb{C}^{n+1})$, the Grassmannian of 2-planes in $\mathbb{C}^{n+1}$. In the simplest cases $n = 1, 2$ these focal points distribute on spheres respectively S^1, S^5 and $n = 3$ seems thus to be the first significant case. Indeed, since $Foc_{\mathbb{H}P^3}\mathbb{C}P^3$ is a circle bundle over the Klein quadric $Gr_2(\mathbb{C}^4) \cong \mathbb{C}Q^4 \subset \mathbb{C}P^5$, both the classical projective geometry of $\mathbb{C}Q^4$ (described for example in [36], pp. 26-35) and its two quaternion Kähler structures [26] can be related to the total space $Foc_{\mathbb{H}P^3}\mathbb{C}P^3$. More generally, the twofold Kähler-Einstein and quaternion Kähler properties of the Grassmannian $Gr_2(\mathbb{C}^{n+1})$ suggested us a link between the focal set $Foc_{\mathbb{H}P^n}\mathbb{C}P^n$ and the Sasakian-Einstein geometry of the induced Hopf bundles considered in our previous work [30].

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This paper begins by giving an explicit identification of the focal set $Foc_{\mathbb{H}P^n}\mathbb{C}P^n$ with the zero level set of the moment map μ associated to the diagonal $U(1)$ - action on $\mathbb{H}P^n$. This is the simplest isometric action giving examples of reductions in quaternion Kähler geometry, and it is well known that the reduced manifold is precisely the quaternion Kähler Wolf space $Gr_2(\mathbb{C}^{n+1})$ [19], [20]. According to the notation in use, we write $\mathbb{H}P^n///U(1) = Gr_2(\mathbb{C}^{n+1})$ to indicate this reduction procedure, that has been proven to be the unique one with respect to isometric circle actions and positive quaternion Kähler manifolds [3], [4]. Next, the diagonal $Sp(1)$-action on $\mathbb{H}P^n$ produces as reduction the quaternion Kähler manifold $\mathbb{H}P^n///Sp(1) = \widetilde{Gr}_4(\mathbb{R}^{n+1})$, the Grassmannian of oriented 4-planes in $\mathbb{R}^{n+1}$. The zero set of the corresponding moment map $\nu : \mathbb{H}P^n \to \mathbb{H}$ can be also related to focal sets in $\mathbb{H}P^n$. (For these identifications cf. theorems 3.1, 3.2).

On the other hand, the zero level sets $\mu^{-1}(0)$ and $\nu^{-1}(0)$ of these moment maps μ and ν can be identified with the total spaces of some induced Hopf S^1-bundles. This enables to define on them respectively a Sasakian-Einstein and a 3-Sasakian metric, and hence to write a diagram where these zero sets $\mu^{-1}(0)$ and $\nu^{-1}(0)$ are base spaces of S^3-bundles, now induced by the Hopf fibration $S^{4n+3} \to \mathbb{H}P^n$. The corresponding total spaces are the Stiefel manifolds respectively $V_2(\mathbb{C}^{n+1})$ and $\widetilde{V}_4(\mathbb{R}^{n+1})$. The 3-Sasakian structure of the fibers S^3 and the Sasakian structure of the base spaces $\mu^{-1}(0)$ and $\nu^{-1}(0)$ allow to construct a complex structure on $V_2(\mathbb{C}^{n+1})$ and on $\widetilde{V}_4(\mathbb{R}^{n+1})$. On the former Stiefel manifold this complex structure is non-compatible with its standard hypercomplex structure and we believe with any of the hypercomplex structures that can be constructed via the 3-Sasakian reduction: [2], [14], [23]. We don't know previous references concerning these complex structures, whose construction is however very much in the spirit of Calabi-Eckmann manifolds, exibiting the Stiefel manifolds as fibrations in complex hermitian products of 3-spheres over quaternion Kähler Wolf spaces.

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2. Preliminaries

We collect here some definitions and basic facts that will be used throughout the paper.

Let (M, g) be a Riemannian manifold and let N be an isometrically immersed submanifold. The critical values of the restriction $exp_{T^\perp N}$ of the exponential map of M to the normal bundle of N are called *focal points of* N and the set of all focal points, here denoted with $Foc_M N$, is called *the focal set of* N (cf. for example [17] p. 23, [18], p. 227, [27] p. 283). Although the focal set of a submanifold may not be a submanifold (for smooth plane curves and for regular surfaces in $\mathbb{R}^3$ the focal sets are respectively the evolutes and the surfaces of the centers, and both of them can have singular points, cf. [18], pp. 237-238 and p. 232), so happens for several examples, including the ones considered in the present paper. Indeed, it seems that not many focal sets of Riemannian submanifolds have been explicitely determined, but we can mention totally geodesic spheres in spheres [31], and hypersurfaces in space forms [15], [16].

The focal set we are mainly interested in is $Foc_{\mathbb{H}P^n}\mathbb{C}P^n$ with respect to the totally geodesic $\mathbb{C}P^n$ in the quaternionic projective space $\mathbb{H}P^n$. This focal set appears in J. Berndt's work [7], [8], there denoted with Q^n, and studied in connection with both the complex Kähler and the quaternion Kähler structure of $Gr_2(\mathbb{C}^{n+1})$, the Grassmannian of complex two-planes in $\mathbb{C}^{n+1}$.

Two geometric structures which appear naturally in our context are the Sasakian and the 3-Sasakian ones. We briefly recall the definitions, sending the reader to the survey [11] for further informations.

A *Sasakian* manifold is a $(2n+1)$-dimensional Riemannian manifold (S, g) equipped with a unitary Killing vector field ξ such that the field of endomorphisms $\varphi := \nabla\xi$ satisfy the differential equation

$$\nabla\varphi = Id \otimes \eta - g \otimes \xi,$$

where ∇ is the Levi-Civita connection of g and η is the dual 1-form of ξ. The datum of a Sasakian structure on the manifold S is easily seen to be equivalent to the requirement that the cône metric $dr^2 + r^2 g$, on $\mathbb{R}_+ \times S$ have holonomy contained in $U(n+1)$. Note that η is a contact form on S, hence ξ is its Reeb field. In the simplest example, the canonical sphere S^{2n+1}, the Killing vector field is $\xi = -JU$, J being the standard complex structure of $\mathbb{C}^{n+1}$ and U the unit outward normal to the sphere.

More generally, look at the induced Hopf S^1-bundle $\pi : V \to M$, over a smooth submanifold of $\mathbb{C}P^N$. Its total space V carries a Sasakian structure (V, g, ξ) induced from the one of (S^{2N+1}, can). If the Fubini-Study metric of $\mathbb{C}P^N$ induces an Einstein metric h with Einstein constant α on M, it can be seen (cf. [30], lemma 1) that the Ricci tensor of the metric g has the form

$$Ric(g) = \lambda g + \mu \eta \otimes \eta$$

with $\lambda = \alpha - 2$, $\mu = \dim V + 1 - \alpha$. This is known as the η-*Einstein* property in Sasakian geometry and, following S. Tanno [35], an η-Einstein Sasakian metric can be deformed to a Sasakian-Einstein one by setting

$$g' = Ag + A(A-1)\eta \otimes \eta, \tag{2.1}$$

$\xi' = A^{-1}\xi$, with $A = \frac{\lambda+2}{\dim V+1}$. Since π is a Riemannian submersion with respect to the metrics h and g, to preserve the Riemannian submersion with g', the scaled metric $h' = Ah$ has to be considered on the basis M.

As Sasakian geometry may be viewed as the odd-dimensional counterpart of Kähler geometry, the odd-dimensional correspondent of hyperKähler manifolds are the 3-Sasakian ones (cf. [11]). Precisely, a $(4n+3)$-dimensional Riemannian manifold (S, g) is 3-Sasakian if it is endowed with three mutually orthogonal unit Killing vector fields ξ_1, ξ_2, ξ_3, each one defining a Sasakian structure and satisfying the conditions

$$[\xi_1, \xi_2] = 2\xi_3, [\xi_2, \xi_3] = 2\xi_1, [\xi_3, \xi_1] = 2\xi_2.$$

As above, an equivalent definition requires that the cône metric $dr^2 + r^2 g$ on $\mathbb{R}_+ \times S$ be hyperkähler, i. e. its holonomy group be contained in $Sp(n+1)$. 3-Sasakian manifolds are necessarily Einstein with positive scalar curvature and their Einstein constant is $4n+2$. Given a positive quaternion Kähler manifold P, one constructs its Kähler-Einstein twistor space Z_P and then an S^1 bundle over it whose Chern class is, up to torsion, the one of an induced Hopf bundle. The total space S thus obtained is an $SO(3)$-principal bundle over P with 3-Sasakian structure. Moreover, all three fibrations involved are Riemannian submersions.

We recall now two basic moment maps of quaternion Kähler geometry. Let $[h_0 : h_1 : \ldots : h_n]$ be the homogeneous coordinates on $\mathbb{H}P^n$ and, for each $a = 0, 1, \ldots, n$, we shall write the complex and real components of h_a as follows:

$$h_a = z_a + w_a j = \alpha_a + \beta_a i + \gamma_a j + \delta_a k \tag{2.2}$$

where $z_a = \alpha_a + \beta_a i$, and $w_a = \gamma_a + \delta_a i$. The first moment map, induced by the diagonal action of $U(1)$ on $\mathbb{H}P^n$, is, at the hyperKähler level of the bundle $\mathbb{H}^{n+1}\backslash\{0\} \to \mathbb{H}P^n$, given by:

$$\mu : \mathbb{H}^{n+1}\backslash\{0\} \to \mathbb{H}, \quad \mu = \sum_{a=0}^{n} \overline{h}_a i h_a. \tag{2.3}$$

On $\mathbb{H}P^n$, one has to look at the corresponding moment map μ_{qK} as a section of S^2H, the rank 3 vector bundle of the compatible almost complex structures. In fact, by using the metric, μ_{qK} appears as a 2-form solution of

$$d\mu_{qK} = i_X\Omega,$$

where Ω is the Kähler 4-form and X the Killing vector field generating the $U(1)$-action [34], [1]. However, since the zero level sets $\mu^{-1}(0) \subset \mathbb{H}^{n+1}$ and $\mu^{-1}_{qK}(0) \subset \mathbb{H}P^n$ correspond each other in the bundle projection $\mathbb{H}^{n+1}\backslash\{0\} \to \mathbb{H}P^n$, we shall refer to the definition $\mu = \sum_{a=0}^n \overline{h}_a i h_a$, whose zero set makes sense also when the h_a are the homogeneous coordinates of $\mathbb{H}P^n$ [11]. The reduced manifold $\mu^{-1}(0)/U(1) = \mu^{-1}_{qK}(0)/U(1)$ turns out to be the quaternion Kähler Wolf space $\frac{SU(n+1)}{S(U(n-1)\times U(2))} \cong Gr_2(\mathbb{C}^{n+1})$ [19], [20].

The second moment map to be considered, generated by the action of $Sp(1)$, is (again at the hyperKähler level):

$$\nu : \mathbb{H}^{n+1}\backslash\{0\} \to \mathbb{H}^3, \quad \nu = (\sum_{a=0}^{n} \overline{h}_a i h_a, \sum_{a=0}^{n} \overline{h}_a j h_a, \sum_{a=0}^{n} \overline{h}_a k h_a), \tag{2.4}$$

and its corresponding quaternion Kähler moment map ν_{qK} can be viewed as a triple of 2-forms associated to the frame of Killing vector fields defining the $Sp(1)$-action. The corresponding reduced manifold $\nu^{-1}(0)/Sp(1) = \nu^{-1}_{qK}(0)/Sp(1)$ is now the Wolf space $\frac{SO(n+1)}{SO(n-3)\times SO(4)} \cong \widetilde{Gr}_4(\mathbb{R}^{n+1})$, the Grassmannian of oriented four-planes of $\mathbb{R}^{n+1}$.

3. Statement of results

Theorem 3.1. (i) *The focal set $Foc_{\mathbb{H}P^n}\mathbb{C}P^n$ coincides with $\mu^{-1}(0)$, and it is isometric to the total space of the induced Hopf S^1-bundle through the Plücker embedding $Gr_2(\mathbb{C}^{n+1}) \hookrightarrow \mathbb{C}P^N$.*

(ii) *The metric g_1, induced on $\mu^{-1}(0)$ by the Plücker embedding, allows to define on $\mu^{-1}(0)$ a Sasakian Einstein metric g.*

(iii) *The Stiefel manifold $V_2(\mathbb{C}^{n+1})$ of orthonormal 2-frames in $\mathbb{C}^{n+1}$, diffeomorphic to the total space of the induced Hopf S^3-bundle through* *the embedding $\mu^{-1}(0) \subset \mathbb{H}P^n$, admis an (integrable) complex structure J, non compatible with its standard hypercomplex structure.*

We have a similar statement regarding the moment map ν induced by the action of $Sp(1)$ on $\mathbb{H}P^n$. We need to consider the following mutually congruent, totally geodesic embeddings of $\mathbb{C}P^n$ in $\mathbb{H}P^n$.

$$
\begin{aligned}
\mathbb{C}P^n_i =&\{h \in \mathbb{H}P^n \ ; \ \gamma_a = \delta_a = 0, a = 1, ..., n\},\\
\mathbb{C}P^n_j =&\{h \in \mathbb{H}P^n \ ; \ \beta_a = \delta_a = 0, a = 1, ..., n\},\\
\mathbb{C}P^n_k =&\{h \in \mathbb{H}P^n \ ; \ \beta_a = \gamma_a = 0, a = 1, ..., n\},
\end{aligned}
\tag{3.1}
$$

and of course $\mathbb{C}P^n_i$ is the standard $\mathbb{C}P^n$ appearing in theorem 3.1.

Theorem 3.2. (i) *The zero level set $\nu^{-1}(0)$ coincides with the intersection $M = Foc_{\mathbb{H}P^n}\mathbb{C}P^n_i \cap Foc_{\mathbb{H}P^n}\mathbb{C}P^n_j \cap Foc_{\mathbb{H}P^n}\mathbb{C}P^n_k$ and it is isometric to the total space of the induced Hopf S^1-bundle over the Fano manifold $Z_{\widetilde{Gr}_4(\mathbb{R}^{n+1})}$, through the embeddings $Z_{\widetilde{Gr}_4(\mathbb{R}^{n+1})} \hookrightarrow Gr_2(\mathbb{C}^{n+1}) \hookrightarrow \mathbb{C}P^N$, the first of which is defined by looking at $Z_{\widetilde{Gr}_4(\mathbb{R}^{n+1})}$ as the space of totally isotropic two-planes in $\mathbb{C}^{n+1}$. This isometry allows the construction of a Sasakian Einstein metric on $\nu^{-1}(0)$, and consequently its identification with the homogeneous 3-Sasakian manifold $SO(n+1)/(SO(n-3) \times Sp(1))$.*

(ii) *The Stiefel manifold $\widetilde{V}_4(\mathbb{R}^{n+1})$ admits an (integrable) complex structure, projecting to the complex structure of $Z_{\widetilde{Gr}_4(\mathbb{R}^{n+1})}$.*

Here $Z_{\widetilde{Gr}_4(\mathbb{R}^{n+1})}$ is the twistor space of the quaternion Kähler Wolf space $\widetilde{Gr}_4(\mathbb{R}^{n+1})$, given by the $Sp(1)$ reduction on $\mathbb{H}P^n$. $Z_{\widetilde{Gr}_4(\mathbb{R}^{n+1})}$ is known to be a complex submanifold of the Kähler-Einstein Grassmannian $Gr_2(\mathbb{C}^{n+1})$, cf. for example [24] p. 14 or [22] p. 702.

Statements 3.1, 3.2 give, in particular, some fibrations that can be collected in a larger diagram as follows. Here $V_k(\mathbb{C}^{n+1})$, $\widetilde{V}_k(\mathbb{R}^{n+1})$ denote the Stiefel manifolds of k-frames in $\mathbb{C}^{n+1}$ and of oriented k-frames in $\mathbb{R}^{n+1}$, respectively.

Corollary 3.1. *There is a commutative diagram*

$$\begin{tikzcd}
\widetilde{V}_4(\mathbb{R}^{n+1}) \arrow[r, hook] \arrow[d, "S^3"] & V_2(\mathbb{C}^{n+1}) \arrow[r, hook] \arrow[d, "S^3"] & S^{4n+3} \arrow[d, "S^3"] & \\
\nu^{-1}(0) \arrow[r, hook] \arrow[d, "S^1"] & \mu^{-1}(0) \arrow[r, hook] \arrow[d, "S^1"] & \mathbb{H}P^n & S^{2N-1} \arrow[dl] \\
Z_{\widetilde{Gr}_4(\mathbb{R}^{n+1})} \arrow[r, hook] \arrow[d, "S^2"] & Gr_2(\mathbb{C}^{n+1}) \arrow[r, hook] & \mathbb{C}P^N & \\
\widetilde{Gr}_4(\mathbb{R}^{n+1}) & & &
\end{tikzcd}$$

of principal S^1 and S^3-bundles and Riemannian submersions. The zero level sets $\mu^{-1}(0)$ and $\nu^{-1}(0)$ are thus total spaces of induced Hopf S^1-bundles and as such are minimal submanifolds in the sphere S^{2N-1}.

4. The proofs

Proof of theorem 3.1 (i) and (ii). The first observation is that:

$Foc_{\mathbb{H}P^1}\mathbb{C}P^1$ is the level set of the moment map associated to the standard $U(1)$-action on $\mathbb{H}P^1$.

In the homogeneous coordinates $[h_0, h_1]$ of $\mathbb{H}P^1$, and by using notations from formula (2.1), $\mathbb{C}P^1$ is given by:

$$w_0 = 0, w_1 = 0 \text{ or } \gamma_0 = \delta_0 = \gamma_1 = \delta_1 = 0. \tag{4.1}$$

The pair $\mathbb{C}P^1 \subset \mathbb{H}P^1$ can be identified with $S^2 \subset S^4$, and the focal set of a totally geodesic sphere S^p in S^n is the unit S^{n-p-1} in the $(n-p)$-dimensional orthogonal complement of the $\mathbb{R}^{p+1}$ containing S^p ([31], p. 286). This identification can be made explicit using the coordinate $h = \alpha + \beta i + \gamma j + \delta k = h_0 h_1^{-1}$ in the affine line $h_1 \neq 0$, whose corresponding real coordinates are:

$$\alpha = \frac{\alpha_0\alpha_1 + \beta_0\beta_1 + \gamma_0\gamma_1 + \delta_0\delta_1}{r^2}, \ \beta = \frac{\alpha_1\beta_0 - \alpha_0\beta_1 - \gamma_0\delta_1 + \gamma_1\delta_0}{r^2},$$
$$\gamma = \frac{\alpha_1\gamma_0 - \alpha_0\gamma_1 - \delta_0\beta_1 + \beta_0\delta_1}{r^2}, \ \delta = \frac{\alpha_1\delta_0 - \alpha_0\delta_1 - \gamma_1\beta_0 + \gamma_0\delta_1}{r^2}.$$

where $r^2 = \alpha_1^2 + \beta_1^2 + \gamma_1^2 + \delta_1^2$. If $(x_1, ..., x_5)$ are the coordinates on $\mathbb{R}^5$, the (inverse) stereographic projection $\mathbb{R}^4 \to S^4$ reads:

$$(\alpha, \beta, \gamma, \delta) \mapsto (x_1, ..., x_5) = \left(\frac{2\alpha}{r^2+1}, \frac{2\beta}{r^2+1}, \frac{2\gamma}{r^2+1}, \frac{2\delta}{r^2+1}, \frac{r^2-1}{r^2+1}\right). \tag{4.2}$$

Thus, if $x_3 = x_4 = 0$ are the equations of a totally geodesic S^2 in S^4, the focal set of S^2 in S^4 is the S^1 of equations $x_1 = x_2 = x_5 = 0$, corresponding to:

$$\alpha = \beta = 0, \ \gamma^2 + \delta^2 = 1. \tag{4.3}$$

On the other hand, the moment map μ on $\mathbb{H}P^1$ can be written in the non-homogeneous coordinate $h = h_0 h_1^{-1}$ as $\mu = \overline{h} i h + i$. Its level set $\mu^{-1}(0)$ is thus:

$$\alpha_1^2 + \beta_1^2 - \gamma_1^2 - \delta_1^2 + 1 = 0, \ \alpha\delta - \beta\gamma = 0, \ \alpha\gamma + \beta\gamma = 0, \tag{4.4}$$

and the systems (4.3) and (4.4) are equivalent.

We now treat the case $n > 1$. From the definition of the focal set, we only need to look at geodesics normal to $\mathbb{C}P^n$. These are normal to all the complex projective lines $\mathbb{C}L^1 \subset \mathbb{C}P^n$, thus their focal points with respect to $\mathbb{C}P^n$ are also focal points for all the lines $\mathbb{C}L^1$ contained in $\mathbb{C}P^n$. Any such line $\mathbb{C}L^1$ in $\mathbb{H}P^n$ belongs to a unique quaternionic projective line $\mathbb{H}L^1$ and the latter is totally geodesic in $\mathbb{H}P^n$. It follows that any geodesic that is normal to a $\mathbb{C}L^1$ and tangent to the corresponding $\mathbb{H}L^1$ at a given point, remains tangent to $\mathbb{H}L^1$ for its entire length. Hence, if we show that through any point x of $\mathbb{C}P^n$ and for any $v \in T_x^\perp \mathbb{C}P^n$ there exists a projective quaternionic line $\mathbb{H}L^1$ containing v, we may deduce:

$Foc_{\mathbb{H}P^n} \mathbb{C}P^n$ is the union of all the focal sets $Foc_{\mathbb{H}L^1} \mathbb{C}L^1$.

This can be seen through the diagram:

$$\begin{array}{ccc} \mathbb{C}^{n+1} \backslash \{0\} & \hookrightarrow & \mathbb{H}^{n+1} \backslash \{0\} \\ \downarrow & & \downarrow \\ \mathbb{C}P^n & \hookrightarrow & \mathbb{H}P^n. \end{array}$$

by looking at the complex planes $L_2^{\mathbb{C}}$ in $\mathbb{C}^{n+1}$ containig the fibre $\mathbb{C}^*$ and the corresponding hypercomplex 2-planes $L_2^{\mathbb{H}}$ in $\mathbb{H}^{n+1}$ containig the fibre $\mathbb{H}^*$. Observe that for any vector $\overrightarrow{v} \in \mathbb{H}^{n+1}$, normal to the standard embedded $\mathbb{C}^{n+1}$, there exist a $L_2^{\mathbb{C}}$ and a $L_2^{\mathbb{H}}$ with $\mathrm{span}_{\mathbb{R}}\{L_2^{\mathbb{C}}, \overrightarrow{v}\} \subset L_2^{\mathbb{H}}$:

indeed, if $\mathbb{H}^{n+1} = \text{span}_{\mathbb{H}}\{\overrightarrow{e_0}, ..., \overrightarrow{e_n}\}$ and $\mathbb{C}^{n+1} = \text{span}_{\mathbb{C}}\{\overrightarrow{e_0}, ..., \overrightarrow{e_n}\}$, we have:

$$\overrightarrow{v} \perp \mathbb{C}^{n+1} \text{ if and only if } \overrightarrow{v} = \sum_{a=0}^{n} \lambda_a j \overrightarrow{e_a} + \mu_a k \overrightarrow{e_a}.$$

Hence, if

$$\overrightarrow{w} = -j\overrightarrow{v} = \sum_{a=0}^{n} \lambda_a \overrightarrow{e_a} - \mu_a i \overrightarrow{e_a},$$

we obtain

$$L_2^{\mathbb{C}} = \text{span}_{\mathbb{C}}\{\overrightarrow{e_0}, \overrightarrow{w}\}$$

and

$$L_2^{\mathbb{H}} = \text{span}_{\mathbb{R}}\{\overrightarrow{e_0}, i\overrightarrow{e_0}, j\overrightarrow{e_0}, k\overrightarrow{e_0}, \overrightarrow{w}, i\overrightarrow{w}, j\overrightarrow{w}, k\overrightarrow{w}\}$$

satisfy the condition.

The identification of $Foc_{\mathbb{H}P^n}\mathbb{C}P^n$ with $\mu^{-1}(0)$ is then completed by the following:

The subset $\mu^{-1}(0) \subset \mathbb{H}P^n$ is the union of the zero sets of the moment maps μ_1 associated to the standard $U(1)$-action on all the projective quaternionic lines $\mathbb{H}L^1 \subset \mathbb{H}P^n$.

Let $\{p_0, p_1, ..., p_n\}$ be the canonical frame of $\mathbb{H}P^n$ with unit point u and let $[h_0 : ... : h_n]$ be the homogeneous coordinates in this frame. Accordingly, the moment map reads $\mu(h) = \sum_{a=0}^{n} \overline{h}_a i h_a$. Fix a $q \in \mu^{-1}(0)$, *i.e.* $\sum_{a=0}^{n} \overline{q}_a i q_a = 0$ and note that q cannot be real; however, we may suppose $q_0 \neq 0$.

Let $\mathbb{H}L^1 \cong \mathbb{H}P^1$ be the quaternionic projective line through q and p_1. To compare the intersection $\mu^{-1}(0) \cap \mathbb{H}L^1$ with the zero level set of the moment map in $\mathbb{H}L^1 \cong \mathbb{H}P^1$, we change the frame in $\mathbb{H}P^n$ as follows. We want new homogeneous coordinates $[k_0 : k_1 : ... : k_n]$ such that $q = [1 : j : 0 : ... : 0]$ and the coordinates of p_1,..., p_n remain unchanged. With the new unit point $v = [q_0 : 1 : -1 : -1 : ... : -1]$, the matrix of the change of coordinates is

$$A = \begin{pmatrix} q_0 & 0 & 0 & \dots & 0 \\ q_1 - j & 1 & 0 & \dots & 0 \\ q_2 & 0 & -1 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ q_n & 0 & 0 & \dots & -1 \end{pmatrix}$$

Thus, ${}^t[h_0,...,h_n] = A^t[k_0,...,k_n]$ and the moment map is

$$\mu = \sum_{a=0}^{n} \overline{h_a} i h_a = \overline{k_0}(\sum_{a=0}^{n} \overline{q_a} i q_a) k_0 - \overline{k_0} jij k_0 - \sum_{b=1}^{n} \overline{k_b} i k_b = - \sum_{b=0}^{n} \overline{k_b} i k_b.$$

It follows that $\mu^{-1}(0)$ is invariant under these projective changes of coordinates. As the coordinates on $\mathbb{H}L^1$ are $[k_0 : k_1]$, its moment map is $\mu_1 = -\overline{k_0} i k_0 - \overline{k_1} i k_1$ with the same level set $\mu_1^{-1}(0)$ described for $\mathbb{H}P^1$. This proves the inclusion $\mu^{-1}(0)|_{\mathbb{H}L^1} \subset \mu_1$. As the converse inclusion is clear, for any projective line $\mathbb{H}L^1$ in $\mathbb{H}P^n$, the proof of the identification $Foc_{\mathbb{H}P^n}\mathbb{C}P^n = \mu^{-1}(0)$ is complete.

Next, we prove that $Foc_{\mathbb{H}P^n}\mathbb{C}P^n$ is isometric to the total space of the induced Hopf bundle over the Grassmannian $Gr_2(\mathbb{C}^{n+1})$. In fact, since $Foc_{\mathbb{H}P^n}\mathbb{C}P^n$ is simply connected ([7], p. 17), a diffeomorphism with the induced Hopf bundle is a consequence of the following observation:

Let $\pi : P \to Gr_2(\mathbb{C}^{n+1})$ be a principal circle bundle with simply connected P. Then M is diffeomorphic with the total space of the induced Hopf bundle of $S^{2N-1} \to \mathbb{C}P^N$, $N = \binom{n+1}{2} - 1$, through the Plücker embedding $Gr_2(\mathbb{C}^{n+1}) \hookrightarrow \mathbb{C}P^N$.

In fact, the principal S^1 -bundles over the base B are classified by their Chern class in $H^2(B,\mathbb{Z})$. Since $H^2(Gr_2(\mathbb{C}^{n+1}),\mathbb{Z}) \cong \mathbb{Z}$ is generated by the class of the Kähler form, one can denote by P_a the S^1-bundle over $Gr_2(\mathbb{C}^{n+1})$ associated to $a \in \mathbb{Z}$. Observe that P_1 (resp. P_{-1}) is the circle bundle associated to the canonical line bundle $O_{Gr_2(\mathbb{C}^{n+1})}(1)$ (resp. $O_{Gr_2(\mathbb{C}^{n+1})}(-1)$, dual of the former). But dual complex line bundles are diffeomorphic as real vector bundles. Hence P_1 is diffeomorphic with P_{-1}.

Let us now show that the if P_a is simply connected, then $a = \pm 1$. From $\pi_1(P) = 0$, we have $H^1(P,\mathbb{Z}) = 0$ and $H^2(P,\mathbb{Z})$ torsion free. The Gysin sequence of π:

$$\begin{aligned} 0 \to H^1(P,\mathbb{Z}) \to H^0(Gr_2(\mathbb{C}^{n+1}),\mathbb{Z}) &\to H^2(Gr_2(\mathbb{C}^{n+1}),\mathbb{Z}) \\ &\to H^2(P,\mathbb{Z}) \to 0 \end{aligned}$$

reduces thus to:

$$0 \longrightarrow \mathbb{Z} \xrightarrow{\cup c_1} \mathbb{Z} \longrightarrow H^2(P,\mathbb{Z}) \longrightarrow 0,$$

where c_1 is the Chern class of π. Hence, $c_1 \neq \pm 1$ implies $H^2(P,\mathbb{Z}) \cong \mathbb{Z}_n$ for some $n \geq 2$. Consequently, c_1 is, up to sign, the Chern class of the induced Hopf bundle.

As regards the identification of the metric g_1 inherited by $Foc_{\mathbb{H}P^n}(\mathbb{C}P^n)$ from (S^{2N-1}, can) as total space of the induced Hopf bundle and the metric g induced on $Foc_{\mathbb{H}P^n}(\mathbb{C}P^n)$ from $\mathbb{H}P^n$, we note that both $(Foc_{\mathbb{H}P^n}(\mathbb{C}P^n), g_1)$ and $(Foc_{\mathbb{H}P^n}(\mathbb{C}P^n), g)$ are Riemannian submersions with geodesic fibres S^1 over $Gr_2(\mathbb{C}^{n+1})$ (cf. [30] for g_1 and [7] for g). Thus theorem 9.59 in [6], p. 249, can be used to conclude that $g = g_1$. This completes the proof of (i) and (ii) in theorem 3.1.

Remark 4.1. An alternative way of recognizing that $Foc_{\mathbb{H}P^n}\mathbb{C}P^n = \mu^{-1}(0)$ is to look at the standard isometric action of $SU(n+1)$ on $\mathbb{H}P^n$ and to see that the two subset are obtained as homogeneous spaces of $SU(n+1)$ with the same isotropy groups. The homogeneous space is in fact $\frac{SU(n+1)}{SU(2)\times SU(n-1)}$, showed in [7], p. 17 (and there denoted by Q^n) to be a singular orbit of the action of $SU(n+1)$. A similar description of $\mu^{-1}(0)$ as homogeneous space is in [1], p. 65, in relation with the problem of studying local compatible complex structures in $\mathbb{H}P^n$. Cfr. also [33], p. 171, where $\frac{SU(n+1)}{SU(2)\times SU(n-1)}$ is called the "Grassmannian of *oriented* two-planes" of $\mathbb{C}^{n+1}$.

Remark 4.2. We proved in [30] that the metric g_1 is Sasakian and η-Einstein, and this is in accordance with formulas following prop. 9 in [7]. As for the extrinsic geometry of $Foc_{\mathbb{H}P^n}(\mathbb{C}P^n)$, J. Berndt proves (cor. 1 in [7]) that the immersion in $\mathbb{H}P^n$ is minimal. It is interesting to observe that the same holds for the immersion $(Foc_{\mathbb{H}P^n}\mathbb{C}P^n, g_1)$ in (S^{2N-1}, can), cf. [30].

Proof of Theorem 3.2 (i). The identification of $\nu^{-1}(0)$ with the intersection M is an immediate consequence of the definition of ν and of the first statement of theorem 3.1. To see that $\nu^{-1}(0)$ is isometric to the above mentioned induced Hopf bundle, we notice that the last observation in the proof of theorem 3.1 (i) and (ii) still holds for principal circle bundles with simply connected P over any complex algebraic projective submanifold B of $\mathbb{C}P^n$ with $H^2(B,\mathbb{Z}) \cong \mathbb{Z}$. This is the case for our Z, hence it remains to prove that $\nu^{-1}(0)$ is simply connected. To this purpose we note that $\nu^{-1}(0)$ can be looked at as homogeneous space $\frac{SO(n+1)}{SO(n-3)\times Sp(1)}$ through the transitive action of $SO(n+1)$ on $\nu^{-1}(0)$. This last action comes, in fact, from the natural action of $SO(n+1)$ on $\mathbb{H}P^n$ and J. Berndt's observation that $SU(n+1)$ acts transitively on $Foc_{\mathbb{H}P^n}(\mathbb{C}P^n)$. His argument can be triplicated to produce transitive actions of groups $SU_i(n+1)$, $SU_j(n+1)$, $SU_k(n+1)$ (the first one is the standard $SU(n+1)$ in $Sp(n+1)$, the other two are similarly defined by interchanging the rôles of the unit quaternions i, j, k) on the zero level sets $\mu_i^{-1}(0)$, $\mu_j^{-1}(0)$, $\mu_k^{-1}(0)$. The isotropy

subgroup of the action on $\nu^{-1}(0)$ at the point $[1 : i : j : k : 0... : 0]$ is then $Sp(1) \times SO(n-3)$. The homotopy sequence associated to the homogeneous manifold $\frac{SO(n+1)}{Sp(1)\times SO(n-3)}$ thus shows that $\nu^{-1}(0)$ is simply connected, so that it is diffeomorphic with the total space of the induced Hopf bundle, now over the Fano manifold $Z_{\widetilde{Gr}_4(\mathbb{R}^{n+1})}$.
As an induced S^1-Hopf bundle, $\nu^{-1}(0)$ has a Sasakian η - Einstein structure. This can be deformed to a Sasakian-Einstein metric, which is still a Riemannian submersion after rescaling the standard Kähler-Einstein metric of $Z_{\widetilde{Gr}_4(\mathbb{R}^{n+1})}$ [30]. On the other hand, the composition of the fiberings

$$\nu^{-1}(0) \xrightarrow{S^1} Z_{\widetilde{Gr}_4(\mathbb{R}^{n+1})} \xrightarrow{S^2} \widetilde{Gr}_4(\mathbb{R}^{n+1})$$

is an $SO(3)$-bundle which endows $\nu^{-1}(0)$ with a 3-Sasakian structure *via* the inversion theorem 4.6 of [10].

Proof of Theorem 3.1 (iii) and Theorem 3.2 (ii). In both cases we use the same idea: we construct a complex structure on the total space of a Riemannian submersion with fibre S^3 and with a Sasakian structure on its base. As both Stiefel manifolds under discussion are induced Hopf S^3-bundles (as easily recognized by looking at them as homogeneous spaces), we may endow them naturally with metrics making these S^3-bundles Riemannian submersions. By naturality, these are the usual homogeneous metrics on the considered Stiefel manifolds. Now the proof follows from:

Proposition 4.1. *Let $\pi : P \to B$ be a totally geodesic Riemannian submersion with fibre S^3. Suppose the base B has a Sasakian structure $(\varphi, \xi, \eta, g^B)$. Then P admits a complex structure with respect to which g^P is Hermitian, but not Kählerian.*

Proof. For any $Y \in \mathcal{X}(B)$ we let X^* be its horizontal lift on P. Let ξ_1, ξ_2, ξ_3 be the unit Killing vector fields which give the usual 3-Sasakian structure of S^3 and η_1, η_2, η_3 their dual w.r.t. the canonical metric of S^3. ξ_i may be viewed as vector fields on P. Let $\hat{\eta}_i$ be their dual forms w.r.t. the metric g^P. Restricted to any fibre, $\hat{\eta}_i$ coincide with η_i. The usual splitting of TP into vertical and horizontal parts may be refined to

$$TP = \mathrm{span}\{\xi_1, \xi_2, \xi_3\} \oplus \mathrm{span}\{\xi^*\} \oplus \mathcal{H}'$$

where $\mathcal{H}'$ represents the horizontal vector fields orthogonal to ξ^*.
Define an almost complex structure J on P as:

- $J\xi_1 = \xi_2, \quad J\xi_2 = -\xi_1.$
- $J\xi_3 = \xi^*, \quad J\xi^* = -\xi_3$

- $JX^* = (\phi X)^*$ for any $X \in \mathcal{X}(B)$ orthogonal to ξ.

To show J is integrable, we compute its Nijenhuis tensor field:

$[J, J](A, B) = [A, B] + J[JA, B] + J[A, JB] - [JA, JB], \quad A, B \in \mathcal{X}(P).$

We shall need the following

Lemma 4.1. $[\xi_i, Y^*] = 0$ *for any* $Y \in \mathcal{X}(B)$ *and* $i = 1, 2, 3.$

Proof. It is enough to show that $[\xi_i, Y^*]$ is both horizontal and vertical. Indeed, on one hand:

$$d\hat{\eta}_i(\xi_j, Y^*) = g(\varphi_i \xi_j, Y^*) = g(\epsilon_{ijk}\xi_k, Y^*) = 0$$

and also

$$d\hat{\eta}_i(\xi_j, Y^*) = -\hat{\eta}_i([\xi_i, Y^*]).$$

Hence $[\eta_i, Y^*]$ is horizontal. And on the other hand $[\xi_i, Y^*]$ is vertical (as the bracket of any vertical field with a projectable one, cf. [6] p.240). The conclusion follows. □

We analyse separately the different possible positions of A, B, noting that, due to the tensorial character of $[J, J]$, when dealing with horizontal vector fields it is enough to work with basic ones.

1) Clearly $[J, J](\xi_1, \xi_2) = 0$.

2) $[J, J](\xi_a, \xi_3) = 0$ for $a = 1, 2$ because (computation for $a = 1$, similar for $a = 2$):

$$\begin{aligned} [J, J](\xi_1, \xi_3) &= [\xi_1, \xi_3] + J[J\xi_1, \xi_3] + J[\xi_1, J\xi_3] - [J\xi_1, J\xi_3] = \\ &= -2\xi_2 + J[\xi_2, \xi_3] + J[\xi_1, \xi^*] - [\xi_2, \xi^*] = \\ &= -2\xi_2 + J(2\xi_1) = \\ &= 0 \end{aligned}$$

where in the middle equality the last two brackets are zero according to the lemma.

3) $[J, J](\xi_a, \xi^*) = 0$ because (computation for $a = 1$, similar for $a = 2$):

$$\begin{aligned} [J, J](\xi_1, \xi^*) &= [\xi_1, \xi^*] + J[J\xi_1, \xi^*] + J[\xi_1, J\xi^*] - [J\xi_1, J\xi^*] = \\ &= J[\xi_2, \xi^*] + J[\xi_1, -\xi_3] - [\xi_2, -\xi_3] = \\ &= J(2\xi_2) + 2\xi_1 = 0 \end{aligned}$$

(note that some of the brackets are zero by the lemma).

4) $[J, J](\xi_a, \xi^*) = 0$ because :

$$\begin{aligned} [J, J](\xi_3, \xi^*) &= [\xi_3, \xi^*] + J[J\xi_3, \xi^*] + J[\xi_3, J\xi^*] - [J\xi_3, J\xi^*] = \\ &= [\xi_3, \xi^*] + J[\xi_2, \xi^*] + J[\xi^*, \xi^*] + J[\xi_3, -\xi_3] - [\xi^*, -\xi_3] = 0, \end{aligned}$$

using again the lemma to cancel the first, second and third terms.

For the last four cases, $X \in \mathcal{X}(B)$ will be orthogonal to ξ.

5) We compute $[J,J](\xi^*, X^*)$. Here we have to observe that ξ and ξ^* (resp. X and X^*) being π-related, one has $[\xi^*, X^*] = [\xi, X]^*$. Then:

$$
\begin{aligned}
[J,J](\xi^*, X^*) &= [\xi, X]^* + J[J\xi^*, X^*] + J[\xi^*, JX^*] - [J\xi^*, X^*] = \\
&= [\xi, X]^* + J[-\xi_3, X^*] + J[\xi^*, (\varphi X)^*] - [-\xi_3, X^*] = \\
&= [\xi, X]^* + J[\xi^*, (\varphi X)^*] = \\
&= ([\xi, X] + \varphi[\xi, \varphi X])^*.
\end{aligned}
$$

where again in the middle equality two terms are zero by the lemma But, on B, we have:

$$
\begin{aligned}
[\xi, X] &= \nabla_\xi X - \nabla_X \xi = \nabla_\xi X - \varphi X; \\
[\xi, \varphi X] &= \nabla_\xi(\varphi X) - \nabla_(\varphi X)\xi = \\
&= (\nabla_\xi \varphi)X + \varphi\nabla_\xi X - \varphi^2 X = \\
&= \eta(X)\xi - g^B(\xi, X)\xi + \varphi\nabla_\xi X + X - \eta(X)\xi = \\
&= X + \varphi\nabla_\xi X - \eta(X)\xi; \\
\varphi[\xi, \varphi X] &= \varphi X - \nabla_\xi X + \eta(\nabla_\xi X)\xi.
\end{aligned}
$$

Finally, by $g^B(\xi, X) = 0$, taking the derivative along ξ, using $\nabla_\xi \xi = \varphi\xi = 0$, we get $\eta(\nabla_\xi X) = g(\xi, \nabla_\xi X) = 0$. Then

$$\varphi[\xi, \varphi X] = \varphi X - \nabla_\xi X,$$

and the proof of this case is complete.

6) $[J,J](\xi_a, X^*) = 0$, $a = 1, 2$ taking into account the Lemma and the fact that $JX^* = (\varphi X)^*$. Next:

7) For $[J,J](\xi_3, X^*)$ we compute:

$$
\begin{aligned}
[J,J](\xi_3, X^*) &= [\xi_3, X^*] + J[\xi^*, X^*] + J[\xi_3, (\varphi X)^*] - [\xi^*, (\varphi X)^*] = \\
&= J[\xi, X]^* - [\xi, (\varphi X)^*] = \text{ as above} \\
&= (\varphi[\xi, X] - X - \varphi\nabla_\xi X + \eta(X)\xi)^* = 0.
\end{aligned}
$$

Finally, for any $X, Y \perp \xi$,

$$[J,J](X^*, Y^*) = [\varphi, \varphi](X, Y)^* = 0.$$

due to the normality property of the Sasakian structure of B.

The compatibility of J with g^P is immediate.

The Kähler form of (P, g^P, J) is

$$\omega = d\pi^*\eta + \pi^*\eta \wedge \eta_3 - d\eta_3,$$

that shows that $d\omega \neq 0$. □

Remark 4.3. A similar computation proves that $L_{\xi^*}J = L_{\xi_3}J = 0$, thus ξ^* and ξ_3 are infinitesimal automorphisms of the constructed complex structure. Moreover, ξ^* and ξ_3 are also Killing fields with respect to g^P.

Note that the complex structure obtained in this way projects onto the complex structure underlying the Kähler structure of $Gr_2(\mathbb{C}^{n+1})$. This latter is not compatible with the quaternion Kähler structure of this Grassmannian. But it is precisely this quaternion Kähler structure which is lifted to a 3-Sasakian structure and then, by means of an appropriate circle bundle, to a hypercomplex structure on $V_2(\mathbb{C}^{n+1})$, cf. [2], [14]. Hence our complex structure is not compatible with the standard hypercomplex one of $V_2(\mathbb{C}^{n+1})$.

This completes the proof of the theorem.

Proof of Corollary 3.1. It remains only to show that the total space of an induced S^1 Hopf bundle is minimal in (S^{2N-1}, can). In fact, more generally, in a commutative diagram

$$\begin{array}{ccc} \overline{N} & \xrightarrow{\bar{i}} & \overline{M} \\ {\scriptstyle \pi_N}\big\downarrow & & \big\downarrow{\scriptstyle \pi_M} \\ N & \xrightarrow{i} & M \end{array}$$

of Riemannian submersions π_N, π_M with totally geodesic fibres, and immersions i (resp. $\bar{i}$), one has that N is minimal in M if and only if $\overline{N}$ is minimal in $\overline{M}$. This follows by a direct computation of the mean curvature vector fields of i and $\bar{i}$ using Gauss' formula of a submanifold and formula (9.25 a) in [6].

5. Further Observations

As mentioned in the introduction, the zero level set $\mu^{-1}(0) \subset \mathbb{H}P^2$ is diffeomorphic to a sphere S^5 and the projection to the reduced manifold can be identified with the Hopf fibration $S^5 \to \mathbb{C}P^2$. Going to the next case $n = 3$, the projection $\mu^{-1}(0) \to Gr_2(\mathbb{C}^4)$, now an induced Hopf fibration, can be described by looking at the families of submanifolds in $Gr_2(\mathbb{C}^4)$ that are either Kähler-Einstein or quaternion Kähler. The Grassmannian $Gr_2(\mathbb{C}^4)$ admits in fact, besides a natural complex Kähler structure, two distinct quaternion Kähler structures in connection with the isomorphism of vector bundles $TGr_2(\mathbb{C}^4) \cong V \otimes V^{\perp}$ and to the (almost) hypercomplex structure on the tautological vector bundle V or on its orthogonal complement $V^{\perp}$ (cf. [26]). The families of submanifolds we want to look at on $Gr_2(\mathbb{C}^4)$ are described as follows (cf. [7], [8], [25], [32]).

There are two families $\mathcal{F}$, $\mathcal{F}'$ of complex projective planes $\mathbf{C}P^2$, $\mathbf{C}P^{2'}$, a family $\mathcal{F}''$ of products $\mathbb{C}P^1 \times \mathbb{C}P^1$, and a family $\mathcal{F}'''$ of spheres S^4; they are given by:

$$\begin{aligned}
\mathcal{F} =& \{\text{planes contained in a 3-space of } \mathbb{C}^4\},\\
\mathcal{F}' =& \{\text{planes through a line of } \mathbb{C}^4\},\\
\mathcal{F}'' =& \{\text{planes that are invariant for a hypercomplex}\\
& \qquad \text{structure } J \text{ of } \mathbb{C}^4\},\\
\mathcal{F}''' =& \{\text{planes given by pairs of lines in two fixed orthogonal}\\
& \qquad \text{planes of } \mathbb{C}^4\}.
\end{aligned}$$

All these 4-dimensional submanifolds of $Gr_2(\mathbb{C}^4)$ have nice intersection properties, some of which can also be formulated in the framework of projective geometry of the lines in the 3-dimensional space $\mathbb{C}P^3$, the context where the Klein quadric $\mathbb{C}Q^4 \subset \mathbb{C}P^5$, isometric to $Gr_2(\mathbb{C}^4)$, was first introduced. Instead of listing these intersection properties on $Gr_2(\mathbb{C}^4)$ (cf. [25], pp. 508-512, for some of them, and the remaining ones can be similarly deduced), we formulate the corresponding properties for the families of 5-dimensional submanifolds obtained at the Sasakian Einstein level as induced Hopf bundles over the members of families $\mathcal{F}$,..., $\mathcal{F}'''$.

Proposition 5.1. *The 9-dimensional Sasakian Einstein focal set $Foc_{\mathbb{H}P^3}\mathbb{C}P^3 \cong \mu^{-1}(0) \subset \mathbb{H}P^3$ contains the following families of 5-dimensional submanifolds, each of which fibers in circles over a complex or a quaternionic submanifold of $Gr_2(\mathbb{C}^4)$. There are two families $\mathcal{E}$, $\mathcal{E}'$ of Sasakian 5-spheres S^5, S'^5, a family $\mathcal{E}''$ of Sasakian $S^3 \times S^2$ and a family $\mathcal{E}'''$ of products $S^4 \times S^1$. The induced metrics on members of the families $\mathcal{E}$, $\mathcal{E}'$,$\mathcal{E}''$ are Sasakian η-Einstein, and can be modified to Sasakian Einstein metrics through formula (2.1). The intersection properties of these 5-dimentional submanifolds are the following.*

(a) Pairs of 5-spheres in the same family intersect in a circle, and pairs of 5-spheres in different families either do not intersect or intersect in a S^3.

(b) Pairs of submanifolds of type $S^3 \times S^2$ intersect either in a S^3 or in a pair of disjoint circles. A submanifold of type $S^3 \times S^2$ intersects a 5-sphere in a S^3.

(c) Any submanifold of type $S^4 \times S^1$ intersects a 5-sphere in a circle, intersects a $S^3 \times S^2$ in two disjoint circles, and any pair submanifolds of type $S^4 \times S^1$ intersect in two disjoint circles.

This kind of geometry of submanifolds, now described for the level set $\mu^{-1}(0) \subset \mathbb{H}P^3$, can be formulated for all the zero level sets of moment maps appearing in the diagram of corollary 3.1. There is such a level set for any odd dimension. For example, $\nu^{-1}(0) \subset \mathbb{H}P^4$ is diffeomorphic to a sphere S^7, yielding as reduced manifold $\mathbb{H}P^1$ and fibering in circles over its twistor space $\mathbb{C}P^3$, Kähler submanifold of the Grassmannian $Gr_2(\mathbb{C}^5)$. Thus Sasakian (η)-Einstein submanifolds of type S^5 and $S^3 \times S^2$ can be determined in S^7 (cf. [30]), and intersection properties like in prop. 5.1 are obtained.

The 11-dimensional example is the level set $\nu^{-1}(0) \subset \mathbb{H}P^5$, identified in theorem 3.2 with an intersection of three focal sets in $\mathbb{H}P^5$. This 11-dimensional manifold is diffeomorphic to the (unique) 3-Sasakian homogeneous manifold projecting in $SO(3)$ over the Wolf space $\widetilde{Gr}_4(\mathbb{R}^6)$, and this latter manifold is isometric to $Gr_2(\mathbb{C}^4)$. Thus again the geometry of 3-Sasakian and of Sasakian (η)-Einstein submanifolds $\nu^{-1}(0) \subset \mathbb{H}P^4$ is obtained from the same starting point of proposition 5.1.

Besides this kind of geometry of submanifolds, the Sasakian-Einstein level sets $\mu^{-1}(0)$ seem to share with (some of) the 3-Sasakian level sets $\nu^{-1}(0)$ a common expression of their Poincaré polynomials. We have in fact:

Proposition 5.2. *The Poincaré polynomial of $\mu^{-1}(0) \subset \mathbb{H}P^n$ is given by:*

$$Poin_{\mu^{-1}(0)}(t) = \sum_{i=0}^{[\frac{n-1}{2}]} (t^{4i} + t^{4n-3-4i})$$

This is obtained by the Gysin sequence of the S^1-bundle $\mu^{-1}(0) \to Gr_2(\mathbb{C}^{n+1})$, where the connecting homomorphism

$$H^p(Gr_2(\mathbb{C}^{n+1})) \to H^{p+2}(Gr_2(\mathbb{C}^{n+1}))$$

is given by the wedge product with the Kähler form of the Grassmannian. Since this wedge product is injective up to the middle real dimension $p + 2 = 2n - 2$ (cf. [21], lemma 3.1, for the similar 3-Sasakian situation), the Gysin sequence reduces to a series of short exact sequences up to

$$H^{2n-4}(Gr_2(\mathbb{C}^{n+1})) \to H^{2n-2}(Gr_2(\mathbb{C}^{n+1})) \to H^{2n-2}(\mu^{-1}(0)).$$

This allows to compute the Betti numbers of $(\mu^{-1}(0))$ by differences of consecutive even Betti numbers in $Gr_2(\mathbb{C}^{n+1})$. The Poincaré polynomial of $Gr_2(\mathbb{C}^{n+1})$ is well known (for example [9], p.292) and by writing it as:

(for $n+1=2m$)

$$Poin_{Gr_2(\mathbb{C}^{n+1})} = (1+t^2+...+t^{2n-2})(1+t^4+t^{4m-4}),$$

(for $n+1=2m+1$)

$$Poin_{Gr_2(\mathbb{C}^{n+1})} = (1+t^4+...+t^{4m-4})(1+t^2+t^{2n}),$$

the conclusion is easily obtained.

The following table contains the Poincaré polynomial of $\mu^{-1}(0) \subset \mathbb{H}P^n$ for low values of n

$n=3$	$1+t^4+t^5+t^9$
$n=4$	$1+t^4+t^9+t^{13}$
$n=5$	$1+t^4+t^8+t^9+t^{13}+t^{17}$
$n=6$	$1+t^4+t^8+t^{13}+t^{17}+t^{21}$
$n=7$	$1+t^4+t^8+t^{12}+t^{13}+t^{17}+t^{21}+t^{25}$
$n=8$	$1+t^4+t^8+t^{12}+t^{17}+t^{21}+t^{25}+t^{29}$

The Poincaré polynomial of $\mu^{-1}(0)$ can be compared with that of $\nu^{-1}(0)$, computed as for the homogeneous 3-Sasakian manifold $SO(n+1)/(SO(n-3)\times Sp(1))$. The latter has two different expressions, according to whether $n+1$ is even or odd (see [21] or [11], p. 28). For odd values of $n+1=2k+3$ this expression is:

$$Poin_{\nu^{-1}(0)}(t) = \sum_{i=0}^{k-1}(t^{4i}+t^{8k-1-4i}),$$

and taking account of the dimensions, this is the same formula given in proposition 5.2 for $Poin_{\mu^{-1}(0)}(t)$.

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