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Abstract

In the paper, a new method constructing of asymptotic solutions to differential equations on manifolds with singularities is presented. This method allows not only to widen essentially the space of asymptotics but also to obtain explicit formulas for asymptotic expansions, in particular, in the case when in a neighbourhood of a singular point there exist strata of different dimensions.

Introduction

In this paper, we present a new method of constructing asymptotic expansions for solutions to differential equations on manifolds with singularities near singular points of these manifolds.

The asymptotics of solutions to equations on non-smooth manifolds near their singular points is a natural object of investigation in this area and at present a certain procedure of obtaining such expansions is developed. To be brief, this procedure is based on the residue theory in the dual space with respect to the

Mellin transform. Then it is natural that the considered function should be at most meromorphic, that is, univalued analytic in the whole plane except for a discrete set of poles.

The corresponding class of asymptotic expansion is now known in the literature as the class of (discrete[1]) conormal asymptotics (see, for example, [1] – [6]) and the modern asymptotic theory on manifolds with singularities deals, in fact, namely with the class of conormal asymptotics. Unfortunately, the class of discrete conormal asymptotics possesses one essential disadvantage. It is not sufficiently wide. A lot of important asymptotic expansions do not have a form of conormal asymptotics (see below). Moreover, this class is not closed in the sense that a conormal asymptotics in the right-hand part of an equation can lead to an asymptotics of solution of more general nature (see the example in Subsection 1.1). The reason of this phenomenon is, for example, in the fact that a function of such kind (the solution) can have *ramification* in the dual (with respect to the Mellin transform) space. In general, such kind of problems ought to be solved (and, in some cases, are solved) with the help of more general class of asymptotics, namely, the asymptotics having the form of application of an *analytic functional* (a hyperfunction with a compact support) to the function r^s (see [4], [5]). And, though such a problem can be solved in certain situations with the help of analytic functionals, the framework of this theory are too narrow for the investigation of the problem in general case, for example, in the situation when the considered function has ramification in a neighbourhood of the infinity. This gives rise to the more general notion of continuous asymptotics with infinite (non-compact) carriers of asymptotics up to infinity, cf. [5] More explicit analysis shows that in the discrete case the asymptotics of the solution can be represented (at least formally) as a sum of series

$$r^{S_1}\sum_{k=0}^{\infty} a_k^{(1)} \ln^{-k} r + r^{S_2}\sum_{k=0}^{\infty} a_k^{(2)} \ln^{-k} r + \dots, \tag{1}$$

where S_j and a_k^j are smooth complex-valued functions and, therefore, cannot be represented via an analytic functional.

It is important to note that the above expression (1) is a sum of *divergent* power series in $\ln r$ and the serious problem is even to give a sense to the above expression. Actually, if, for example $\mathrm{Re} S_1 < \mathrm{Re} S_2$, then each term in the second sum is less in order then each term of the first one. How can one encounter all terms of asymptotics (1)? The answer to this question one can easily obtain in the case when the first series converges. Then, extracting this series from the

[1]We do not consider here the case of the so-called continuous asymptotics, that is, the situation when the singularities of the corresponding function in the dual space are not discrete.

function u we obtain a function for which the second term is a main one, and we therefore can 'see' it. Unfortunately, all the above series are as a rule divergent and, moreover, at some points the effect of 'changing a leadership' can take place. It means that if the argument of the phase functions S_1 and S_2 changes, then the recessive (in our case, second) term of the asymptotic expansion can become the dominant one and the first term can become, on the opposite, a recessive one. Thus, for investigating of asymptotics of the form (1) one must first to work out an appropriate *procedure of summation* of a divergent series and, second, to know how to deal with the effects of the type of 'changing a leadership'.

Such kind of a theory, going back to the classical works by L. Euler, E. Borel, Stieltjes, and G. H. Hardy, got in present its new birth[2] in the remarkable works of J. Ecalle [7], J.-P. Ramis [8] [9], J. Martinet and J.-P. Ramis [10] B. Malgrange [11] – [13], and others. The resurgent functions theory had its further development and application in a set of mathematical and physical papers (see, for example [14] – [21] and others. We remark here that in the above cited papers only the *one-dimensional* theory of resurgent functions was worked out. *Multidimensional* theory of resurgent functions was recently introduced in [22], [23] [24]. This theory can be, in particular, applied also to the construction of a new asymptotic (resurgent) theory on manifolds with singularities.

One of the basic points of our theory is that it is based not on the residue theory but on *the new integral representation* [22], [23], [25], [24], together with the corresponding mathematical apparatus – resurgent analysis – allows one to obtain asymptotics of solutions to equations on manifolds with singularities in the situation of the endlessly-continuable microfunctions (= resurgent functions), that is, practically for any right-hand part of the equation (in the framework of discrete asymptotics).

The first application of this theory to problems on manifolds with singularities was done by the authors in papers [26], [27]. There resurgent asymptotics were constructed for the simplest case when the singularity set of the considered manifold is a smooth manifold. The cases of conical points and smooth edges are included in such a situation. In the present paper we consider the case when the set of singularities can be in turn a manifold with singularities.

Finally, it is worth mentioning that the ideas and methods of resurgent analysis worked out in this and preceding papers can be applied to a very wide class of problems in the considered field, such as degenerate equations, equations with singular coefficients, Sobolev problems and so on.

Shortly about the contents of the paper.

The main aim of the first section is to show how resurgent functions appear in

[2]And, as usual in such a case, the new name – 'the theory of resurgent functions'.

the theory of differential equations on manifolds with singularities. Besides, here we wanted also to show the effects concerning with solvability and uniqueness theorems on manifolds with more than one singular point. To do this we consider the simplest example of such a manifold – the example of a spindel. It occurs, and it will be used in the formulation of the general theory that the condition of unique solvability for such manifolds has essentially non-local character in the sence that for unique solvability the weights of corresponding weighted spaces at each singular point must be related to one another.

In the second section, we recall briefly the theory of resurgent functions of several variables introduced in papers [22], [23], [25], [24] adapted to the functions of the power growth (in one dimensional case such adaptation is presented in [26], [27]).

In the third section, the general theory of constructing resurgent solutions to differential equations is developed.

Finally, in fourth section, we consider the rather representative example of three-faced angle. On this example we illustrate the general method of constructing asymptotic expansions as well as show that for manifolds of such kind the problem can be reduced to an algebraic one.

Short exposition of the results of this paper see [28].

1 Example

1.1 Uniqueness and solvability

Here, we consider an equation on the manifold X which is a result of rotation of the circle arc around its chord (that is, on the surface of a spindle). We denote by r the coordinate along the arc, $r \in [0,1]$ and by φ the angle coordinate corresponding to the rotation. Consider the equation on X of the form

$$\widehat{H}u \stackrel{\text{def}}{=} \left\{ \left[r\,(r-1)\,\frac{\partial}{\partial r}\right]^2 + c^2\frac{\partial^2}{\partial \varphi^2} \right\} u(r,\varphi) = f(r,\varphi) \tag{2}$$

where $c > 0$ is some real parameter. We shall investigate solutions of this problem in the weighted Sobolev spaces $H^s_{\alpha_0,\alpha_1}(M)$ with the norm

$$\|u\|^2_{s,\alpha_0,\alpha_1} = \int_0^1\int_0^{2\pi} r^{-2\alpha_0}\,(1-r)^{-2\alpha_1} \left| \left(1 - \left(r\frac{\partial}{\partial r}\right)^2 - \frac{\partial^2}{\partial\varphi^2}\right)^{\frac{s}{2}} u(r,\varphi)\right|^2 d\varphi \frac{dr}{r(1-r)},$$

and we shall try to choose the weights α_0 and α_1 in such a way that the operator

$$\widehat{H} \;:\; H^s_{\alpha_0,\alpha_1}(M) \to H^{s-2}_{\alpha_0,\alpha_1}(M) \tag{3}$$

is an isomorphisms of these spaces.

Using expansions

$$
\begin{aligned}
u(r,\varphi) &= \sum_{k=-\infty}^{\infty} e^{ik\varphi} u_k(r), \\
f(r,\varphi) &= \sum_{k=-\infty}^{\infty} e^{ik\varphi} f_k(r).
\end{aligned}
$$

of the solution and the right-hand part into the Fourier series, we reduce our equation to the equations for the Fourier coefficients $u_k(r)$ of the solution $u(r,\varphi)$:

$$
\left\{ \left[r\,(r-1)\frac{\partial}{\partial r}\right]^2 - k^2 c^2 \right\} u_k(r) = f_k(r) \tag{4}
$$

for all integer values of $k \in \mathbf{Z}$. Evidently, the solvability of equation (2) is equivalent to the solvability of equations (4) in the corresponding functional spaces (the exact choice of these spaces will be done below).

The fundamental system of solutions to the corresponding homogeneous equation is

$$
u_k^{(1)} = \left(\frac{1-r}{r}\right)^{kc}, \; u_k^{(2)} = \left(\frac{1-r}{r}\right)^{-kc}.
$$

Obviously, if we are solving the initial equation (2) in spaces (3), then we must solve the equation (4) for

$$
u_k \in H^s_{\alpha_0,\alpha_1}(0,1)\,,\; f_k \in H^{s-2}_{\alpha_0,\alpha_1}(0,1)\,, \tag{5}
$$

where the spaces $H^s_{\alpha_0,\alpha_1}(0,1)$ are defined in the obvious way:

$$
|||u_k|||^2_{s,\alpha_0,\alpha_1} = \int_0^1 r^{-2\alpha_0}(1-r)^{-2\alpha_1} \left| \left(1 - \left(r\frac{\partial}{\partial r}\right)^2 + k^2\right)^{\frac{s}{2}} u_k(r,\varphi)\right|^2 \frac{dr}{r(1-r)}.
$$

As it was already mentioned, we search for such values of α_0, α_1 that the operator (3) is an isomorphism. In particular, this means that all equations (4) must have the unique solutions in spaces (5).

It is not hard to see that the solutions $u_k^{(1)}$, $u_k^{(1)}$ belong to the space $H^s_{\alpha_0,\alpha_1}(0,1)$ iff

$$
\alpha_0 < kc \quad \text{and} \quad \alpha_1 < -kc.
$$

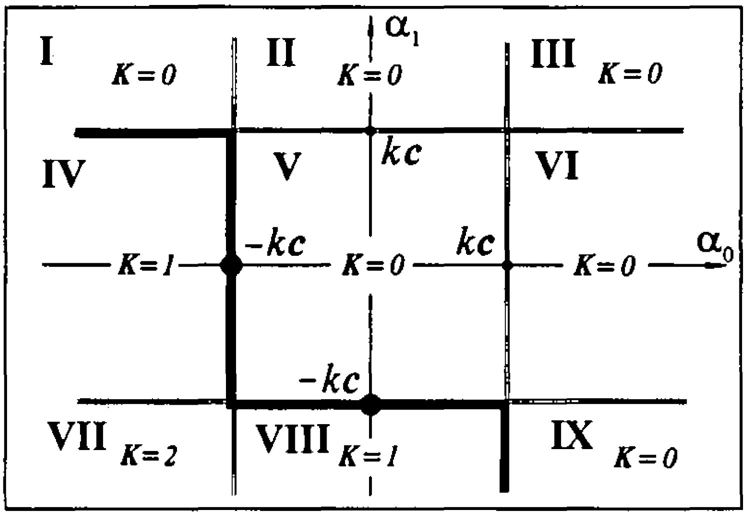

Figure 1: Ker$\widehat{H}_k$ ($k \neq 0$) as a function of α_0, α_1. Thick line is a boundary of the monomorphism region

Thus, the dimension of the kernel of the operator corresponding to equation (4)

$$\widehat{H}_k \; : \; H^s_{\alpha_0,\alpha_1}(0,1) \rightarrow H^{s-2}_{\alpha_0,\alpha_1}(0,1)$$

for different values of α_0, α_1 are such as it is shown on Figure 1 (on this Figure $K = \dim \operatorname{Ker} \widehat{H}_k$).

The dimension of the cokernel of operator (3) can be easily obtained by means of duality. The result for operator (3) is shown on Figure 2. The final result of investigation of the initial equation is shown on Figure 3 where the regions in plane (α_0, α_1) where the considered operator is an isomorphism are shown.

1.2 Asymptotics of solution

Now we turn our attention to the investigation of the asymptotics of solutions to equation (2) provided that the pair (α_0, α_1) is chosen in such a way that operator (3) is an isomorphism. We recall that the usual asymptotic expansions which can be obtained for solutions to equations of the type (2) are conormal asymptotics

$$u(r,\varphi) = \sum_j r^{S_j} \sum_{k=0}^{m_j} a_{jk} \ln^k r \qquad (6)$$

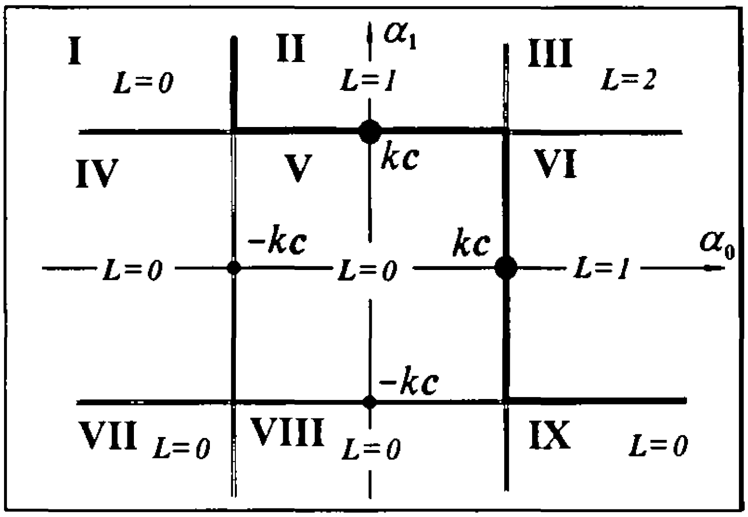

Figure 2: Coker$\widehat{H}_k$ ($k \neq 0$) as a function of α_0, α_1. Thick line is a boundary of the epimorphism region

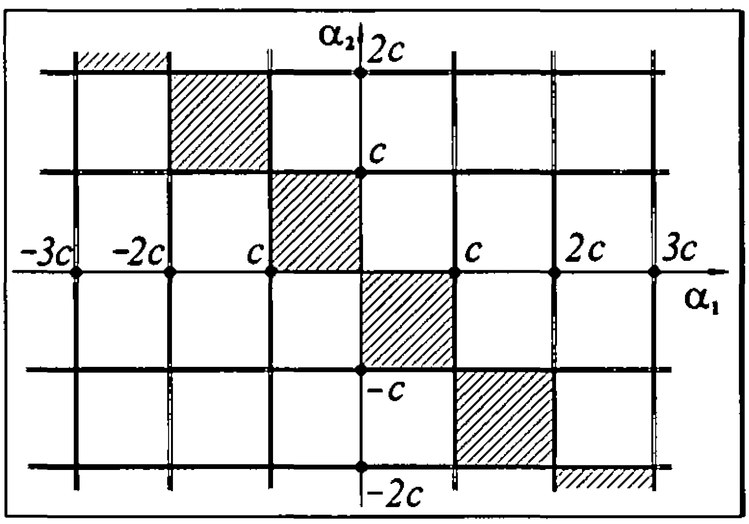

Figure 3: Isomorphism regions of $\widehat{H}$

where r is a local coordinate corresponding to the distance from the singular point, the outer sum is taken over the finite set of indices j; m_j are nonnegative integers and the parameters S_j, a_{jk}, and m_j can in general depend on the angle variable φ (see [4], [5]). However, such class of asymptotics is too narrow for obtaining asymptotics of solutions to equation (2) even if right-hand parts of this equation belongs to this class. The suitable class of asymptotic expansion is given by the notion of resurgent function and we shall try to explain the appearence of this class and its main features on this example.

To do this, we consider equation (2) with the right-hand part

$$f(r,\varphi) = \left(\frac{1-r}{r}\right)^{S(\varphi)} a(\varphi) \tag{7}$$

(where $S(\varphi)$ and $a(\varphi)$ are 2π-periodic complex-valued analytic functions of the variable φ) which has obviously the asymptotic expansion of the type (6) at the both conical points of the manifold M.

To construct a solution to equation (2) with the right-hand part (7) in more or less explicit form, we represent the function (7) with the help of the Cauchy formula as

$$f(r,\varphi) = \frac{1}{2\pi i}\int\limits_{\gamma} \left(\frac{1-r}{r}\right)^{-s} \frac{a(\varphi)}{s-S(\varphi)}\, ds \tag{8}$$

where γ is a contour surrounding the point $s = S(\varphi)$ clockwise. Now we search for the solution to equation (2) in the form

$$u(r,\varphi) = \frac{1}{2\pi i}\int\limits_{\gamma'} \left(\frac{1-r}{r}\right)^{-s} U(s,\varphi)\, ds. \tag{9}$$

The cycle γ' included into the latter formula can differ from that included in the former one. Here $U(s,\varphi)$ is an unknown (in general, *ramifying*) function which is analytic with respect to s. We shall perform the exact choice of the contour later.

Substituting (8) and (9) into (2), we obtain the following equation for the function $U(s,\varphi)$:

$$\left(c^2\frac{\partial^2}{\partial\varphi^2} + s^2\right) U(s,\varphi) = \frac{a(\varphi)}{s-S(\varphi)}.$$

This equation can be solved with the help of the Green function. As a result, we obtain a particular solution of the form

$$U(s,\varphi) = \frac{c}{2s\sin\frac{\pi s}{c}}\int\limits_{\varphi-2\pi}^{\varphi} \cos\frac{s}{c}(\varphi-\theta-\pi)\,\frac{a(\theta)}{s-S(\theta)}\, d\theta. \tag{10}$$

So, one can see that the function $U(s,\varphi)$ is a regular analytic function outside the union of the set

$$\{s = kc\ ,\ k \in \mathbf{Z}\}$$

and the set of values of the function $S(\theta)$ for real values of θ. The latter set forms a closed curve (possibly, with singularities) in a complex plane $\mathbf{C}_s$. However, this function can be continued up to an analytic function in the whole plane $\mathbf{C}_s$ except for a discreet set depending on φ. To do this, we shall treat the integral on the right in (10) as the integral over the segment $[\varphi - 2\pi, \varphi]$ in the *complex* plane $\mathbf{C}_\theta$. The integrand in (10) has singularities at points of the pre image $S^{-1}(s)$ of s under the mapping $\theta \to S(\theta)$. Thus, the integral on the right in (10) is singular in the following three cases:

1) if one of points of $S^{-1}(s)$ coincides with one of the endpoints $\theta = \varphi - 2\pi$ or $\theta = \varphi$ of the integration contour,

2) if two or more points of $S^{-1}(s)$ coincide with one another,

3) if at least one of points of $S^{-1}(s)$ tends to infinity.

First case take place when $s = S(\varphi)$ and the second is realized if $s = S(\varphi^*)$ for some stationary point φ^* of the function S.

These considerations lead us to the following important observation. If we do not suppose some special features (of the type of an analytic continuability) of functions $a(\theta)$ and $S(\theta)$ involved into formula (10), then we can claim only that the function $U(s,\varphi)$ is analytic outside the set of values of the function $S(\theta)$, that is, outside some closed curve in the plane $\mathbf{C}_s$. Thus, the solution is given by the formula (9) where the contour γ' surrounds the mentioned curve. The right-hand part of (9) is none more than the application of an analytic functional (determined by $U(s,\varphi)$) to the function $\left(\frac{1-r}{r}\right)^{-s}$.

However, if we want to obtain the more precise information about the asymptotics of the solution, we must investigate the analytic continuation of the function $U(s,\varphi)$ inside the set bounded by the curve $\{s = S(\theta)\}$. Such a continuation can be obtained with the help of (10) if we suppose that the functions $a(\theta)$ and $S(\theta)$ can be continued up to entire functions to the whole plane $\mathbf{C}_\theta$ (or, at least, as analytic functions with singularities on a discrete set on their Riemannian surfaces)[3]. Then, using the methods of the complex analysis, one can investigate the continuation of $U(s,\theta)$ to the whole plane $\mathbf{C}_s$ and describe the singularities of this continuation.

We remark also that, in the above considerations, we have used essentially the fact that the fundamental solution of the considered equation can be analytically

[3]For general equations, we must require also that the coefficients of the equation can be extended up to entire functions, for the equation in question this requirement is fulfilled automatically.

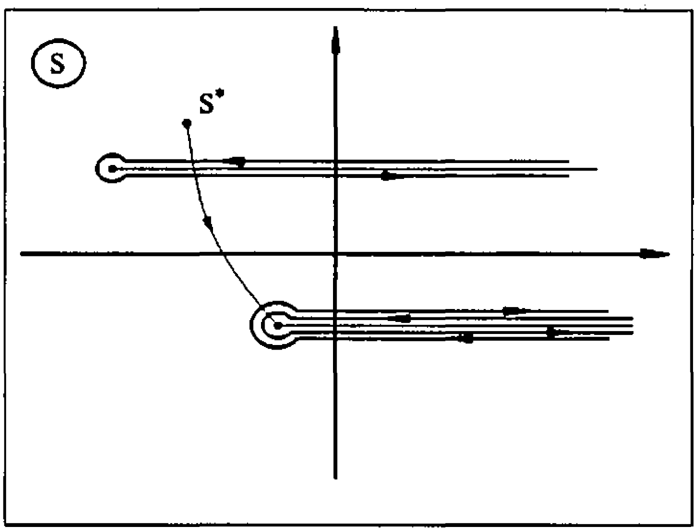

Figure 4: Stokes phenomenon

continued to complex values of its arguments φ, θ.

Let us now investigate the character of singularity of the function $U(s,\varphi)$ given by (10) at the point $s = S(\varphi)$ (this is the only 'moving' point of singularity of $U(s,\varphi)$, that is, such a point whose position on the plane $\mathbf{C}_s$ depends on φ). To do this, we note that when the point s moves along a small loop surrounding $S(\varphi)$, the corresponding points of the pre image moves along a loop surrounding endpoints $\theta = \varphi - 2\pi$ or $\theta = \varphi$ of the integration contour, and thus extracts from this contour two additional contours surrounding these points of the pre image (see Figure 4). Therefore, we see that, in general, the function $U(s,\varphi)$ has at the point $s = S(\varphi)$ the singularity of the logarythmic type (in particular, this point is a point of ramification).

More detailed analysis shows that the asymptotic expansion of the function $U(s,\varphi)$ near the point $s = S(\varphi)$ has the following form

$$U(s,\varphi) = \sum_{j=1}^{\infty} \frac{(s - S(\varphi))^j}{j!} b_j(\varphi) \ln (s - S(\varphi)) \qquad (11)$$

where $b_j(\varphi)$ are some analytic functions in φ (this expansion is written down for values of φ such that $S'(\varphi) \neq 0$).

Now, when the function $U(s,\varphi)$ is found and investigated, let us use the obtained result for constructing the solution (9) and investigating its asymptotics.

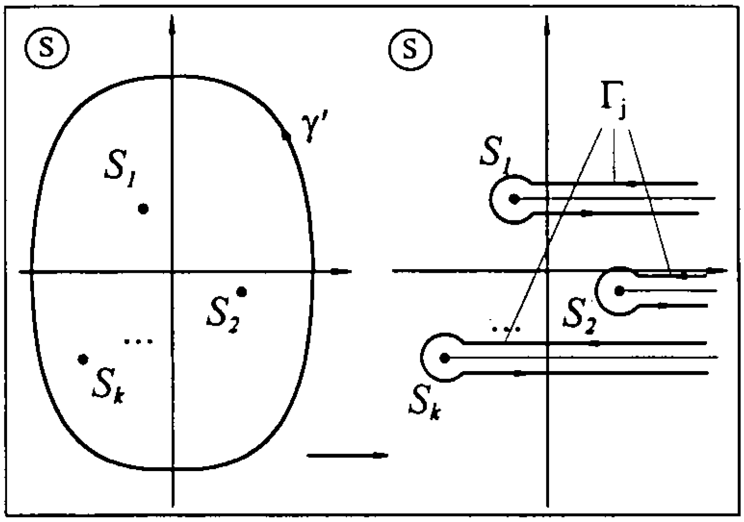

Figure 5: Decomposition of the contour

Since the function U has the singularity of the logarithmic type, one cannot choose the contour γ in this formula to be the same as in (8). However, one can choose the contour γ to surround all the image of the segment $[\varphi - 2\pi, \varphi]$ thus obtaining the desired solution.

Now let us consider the asymptotics of the obtained solution near, say, the point $r = 0$. To construct this asymptotics, we replace the contour γ' included into (9) by the sum of contours Γ_j of special type which is homological to the initial contour γ:

$$u(r,\varphi) = \sum_j u_j(r,\varphi) = \sum_j \frac{1}{2\pi i} \int\limits_{\Gamma_j} \left(\frac{1-r}{r}\right)^{-s} U(s,\varphi)\, ds. \tag{12}$$

Each of the Γ_j's will be a contour surrounding in positive direction exactly one point of singularity of the function $U(s,\varphi)$ and going to infinity along the positive direction of the real axis in the s-plane (see Figure 5). Due to the fact that we consider the solution in a neighbourhood of the point $r = 0$, the function will decrease exponentially along these contours and all the integrals over Γ_j's will converge. From the theory of asymptotic expansions of integrals of the Laplace-Borel type (see, for example [26], [27]) it follows that each of the integrals

$$u_j(r,\varphi) = \frac{1}{2\pi i} \int\limits_{\Gamma_j} \left(\frac{1-r}{r}\right)^{-s} U(s,\varphi)\, ds$$

included into (12) has the asymptotic expansion of the form

$$u_j(r,\varphi) \simeq r^{S_j(\varphi)} \sum_{k=0}^{\infty} b_{jk}(\varphi) \ln^{-k} r \tag{13}$$

as $r \to 0$ if the function $U(s,\varphi)$ has the asymptotic expansion of the type (11) near the origin point $s = S_j(\varphi)$ of the contour Γ_j. Can one say, that the asymptotics of the whole function $u(r,\varphi)$ is simply the sum of asymptotics of the type (13)? Unfortunately, this is not so, since the asymptotic representation, say, of the type

$$u(r,\varphi) \simeq r^{S_1(\varphi)} \sum_{k=0}^{\infty} b_{1k}(\varphi) \ln^{-k} r + r^{S_2(\varphi)} \sum_{k=0}^{\infty} b_{2k}(\varphi) \ln^{-k} r, \tag{14}$$

as have been already mentioned in the Introduction, is absolutely unclear if (for some fixed value of φ) $\operatorname{Re} S_1(\varphi) > \operatorname{Re} S_2(\varphi)$ and the series in the first term on the right in the latter formula diverges. In the Introduction we had presented also some reasons concerning the verification of expansion (14) as an asymptotics. Let us consider here these arguments in more detail. The matter is that before writing down terms of asymptotic expansion with power type $r^{S_2(\varphi)}$ one must to take into account all the terms with power type $r^{S_1(\varphi)}$. In fact, *before writing down the second term on the right in* (14), *one must resummate the first term*, that is, to replace it with some function for which this term will be an asymptotic expansion. Certainly, the coefficients of the second term (we call it *a recessive term* of expansion (14)) *will depend on the resummation method* used in the first term (which is called *a dominant term* of the expansion). The adequate resummation method is given, for example, by Borel resummation (see, for example, [7], [8]). In our case, this resummation procedure is determined by integral (12). Now, if the value of φ changes, then, in general, the effect of 'change of the leadership' can occur. This means that in different regions in φ we obtain different asymptotics (the corresponding asymptotics have simply different dominant terms). The asymptotics in this case changes by jump. Such phenomenon in the theory of divergent parametric power series is known as *the Stokes phenomenon.* The nature of this phenomenon is as follows. We have already said that the values of coefficients $b_{2k}(\varphi)$ of the recessive term depend on the choice of the resummation method used for the dominant one. However, even if we use one and the same resummation method based on integral representations of the type (12), in the case when the considered asymptotic expansion involves an additional parameter (such as the parameter φ in expansion (14)), the coefficients in the recessive term can have jumps for some values of the parameter. These jumps take place exactly for those values of the parameter for which some point

s^* of singularity of the integrand in (12) moves across one of the contours Γ_j of integration included into this formula. As it is shown on Figure 4, this point will extract from the contour Γ_j some additional contour Γ'_j originated from the point s^*. Certainly, such a situation can lead not only to jumps in coefficients of recessive terms, but also to appearance of new ones or to cancelling out the old ones.

This is a program of investigation of asymptotic expansions in the class of resurgent functions.

2 Resurgent functions of power type (multidimensional theory)

In the papers [26], [27], we introduced the notion of the resurgent function of power type of one variable r. Here we need the generalization of this notion to the function of power type in *several variables*. Essentially, this notion is a 'composition' of the notion of resurgent functions of several variables introduced in [22], [23] with the logarithmic change of variables; that is why we present here only the main definitions and statements of the main theorems. The reader can find the details in the above cited papers.

We remark that, as it can be seen from the example above, we need to construct the theory of resurgent functions depending on a parameter. However, the presence of the parameter is not essential, at least at the first stage of the theory, and in the beginning of this Section we postpone the investigation of dependence on a parameter in order to simplify the notation.

By $\mathbf{R}^k_+$, we denote the direct product of k copies of the half-plane $\mathbf{R}_+$ with coordinates $(r_1, \ldots, r_k)$. The tuple $(r_1, \ldots, r_k)$ we denote also by r. We introduce the variables $\rho = (\rho_1, \ldots, \rho_k)$ which are related to the variables $(r_1, \ldots, r_k)$ with the help of logarithmic map

$$\rho_j = \ln r_j, \quad j = 1, \ldots, k.$$

We shall widen the range of the variables ρ up to the space $\mathbf{C}^k$.

Let now $\mathbf{C}_s$ be a complex plane of one complex variable s and let $s = S(\rho)$ be a homogeneous analytic function in ρ of order 1 which is, in general, ramifying. We denote by $\mathcal{M}^{\text{cont}}_S$ the space of endlessly-continuable microfunctions $F(s, \rho)$ at the point $s = S(\rho)$ which are homogeneous of order -1 in (s, ρ). We recall that a microfunction $F(s)$ at the point $s = S$ is an element of the quotient space

$$\mathcal{M}_S = \mathcal{A}_S \,/\, \mathcal{O}_S$$

where $\mathcal{A}_S$ is a space of germs of (in general, ramifying) analytic functions in the deleted neighbourhood of the point $s = S(\rho)$ and $\mathcal{O}_S$ is a space of germs of functions holomorphic in a (full) neighbourhood of this point.

Definition 2.1 The microfunction $F(s,\rho)$ is called to be *endlessly continuable* (in the variable s) if for any positive constant L there exists a discrete set $\Sigma_L \subset \mathbf{C}_s$ such that some representative $F^*(s,\rho)$ (which does not depend on L) of the class $F(s,\rho)$ can be continued along any path in $\mathbf{C}_s$ originated from the neighbourhood of $S(\rho)$ with the length less than L.

Let Σ be an analytic set in $\mathbf{C}_s \times \mathbf{C}^k_\rho$ such that the intersection Σ_ρ between Σ and $\{\rho = \text{const}\}$ contains only a finite number of points in each left half-plane $\{s : \operatorname{Re} s < A\}$ for any value of A. Then *a resurgent function with the support* Σ is an element of the direct product

$$\prod_{S_j \in \Sigma} \mathcal{M}^{\text{cont}}_{S_j}$$

where $S_j = S_j(\rho)$ are different values of the ramifying function S which describes the set Σ:

$$\Sigma = \{(s,\rho) : \ s = S(\rho)\}.$$

The space of resurgent functions (with different supports) we denote by $\widetilde{\mathcal{R}}\left(\mathbf{R}^k_+\right)$.

Now let $\mathcal{P}\left(\mathbf{R}^k_+\right)$ be a space of germs at the origin of functions on $\mathbf{R}^k_+$ of power type. We recall ([26], [27]) that the function $f(r)$ has the power type at the origin if it satisfies the inequality

$$|f(r)| \le Cr^\alpha \tag{15}$$

for a positive constant C and a real multiindex $\alpha = (\alpha_1, \dots, \alpha_k)$ in a neighbourhood of the origin (here $r^\alpha = r_1^{\alpha_1} r_2^{\alpha_2} \dots r_k^{\alpha_k}$). We also denote by $\mathcal{P}_\alpha\left(\mathbf{R}^k_+\right)$ the subspace of $\mathcal{P}\left(\mathbf{R}^k_+\right)$ consisting of functions which satisfy inequality (15) with the given value of α. Finally, we pose

$$\mathcal{P}_{-\infty}\left(\mathbf{R}^k_+\right) = \bigcap_{\alpha \in \mathbf{R}^k} \mathcal{P}_\alpha\left(\mathbf{R}^k_+\right).$$

The elements of $\mathcal{P}_{-\infty}\left(\mathbf{R}^k_+\right)$ will be referred below as *rapidly decreasing functions* (of power type).

On each space $\mathcal{M}^{\text{cont}}_S$ we define the mapping

$$\ell : \mathcal{M}^{\text{cont}}_S \to \mathcal{P}\left(\mathbf{R}^k_+\right) / \mathcal{P}_{-\infty}\left(\mathbf{R}^k_+\right)$$

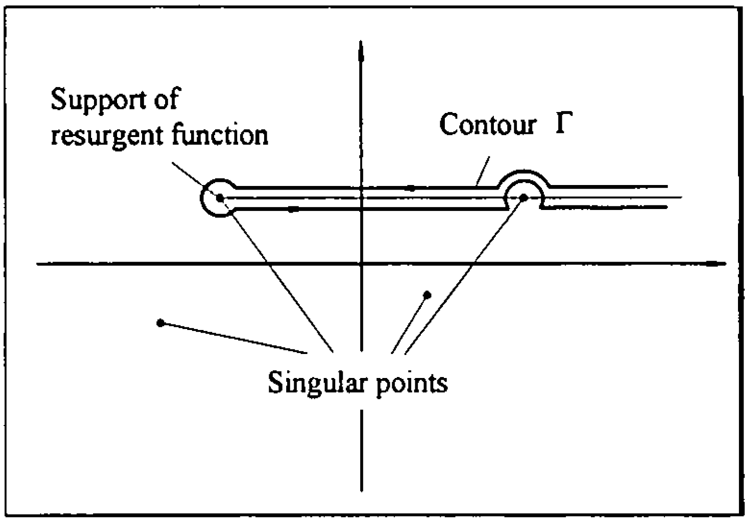

Figure 6: Shape of the contour

given by the formula

$$\ell(F) \stackrel{\text{def}}{=} \frac{1}{2\pi i} \int_h e^{-s} F(s,\rho)\, ds. \tag{16}$$

The contour representing the homology class h involved in the latter integral is shown on Figure 6. Certainly, the representation (16) is valid locally, that is, in some conical ($= R_+$-invariant) neighbourhood of the fixed point ρ_0 of the space $\mathbf{C}^k_\rho$. The globalization of this representation as well as the globalization of the notion of resurgent function (see below) can be done with the help of the so-called *transition homomorphism* which we discuss in the end of this subsection.

We do not present here the investigation of convergence of the integral (16). We remark only that this integral can be defined as the element of the quotient space

$$\mathcal{P}\left(\mathbf{R}^k_+\right) / \mathcal{P}_{-\infty}\left(\mathbf{R}^k_+\right)$$

without any growth conditions on the function $F(s,\rho)$ in the integrand on the right in (16).Thus, this integral is defined modulo rapidly decreasing functions of r. By linearity, one can extend the operator ℓ defined by (16) up to the operator

$$\ell : \widetilde{\mathcal{R}}\left(\mathbf{R}^k_+\right) \to \mathcal{P}\left(\mathbf{R}^k_+\right) / \mathcal{P}_{-\infty}\left(\mathbf{R}^k_+\right).$$

Definition 2.2 The image $\ell\left(\widetilde{\mathcal{R}}\left(\mathbf{R}_+^k\right)\right)$ will be called *a space of resurgent functions of power type on* $\mathbf{R}_+^k$.

We denote this space by $\mathcal{R}\left(\mathbf{R}_+^k\right)$.

Remark 2.1 We note that each resurgent function can be represented as a sum of integrals of the type (16) with one and the same hyperfunction in the integrand. This hyperfunction[4] will also be denoted by $F(s,\rho)$.

Remark 2.2 In the case of *one* variable r (that is, for $k=1$) the above definition of a space of resurgent functions coincides with that given in [26], [27]. Actually, in this case one has

$$\ell(F) = \frac{1}{2\pi i}\int\limits_h e^{-s}F(s,\rho)\,ds \tag{17}$$

where the function $F(s,\rho)$ is a homogeneous function of the two complex parameters (s,ρ). Therefore one has

$$F(s,\rho) = \rho^{-1}F\left(\frac{s}{\rho},1\right).$$

Substituting the latter formula into (17) and performing the variable change $s/\rho = -\tilde{s}$ one reduces the definition (17) of the mapping ℓ to the form

$$\ell(F) = \frac{1}{2\pi i}\int\limits_h e^{\rho\tilde{s}}F(\tilde{s},1)\,d\tilde{s}$$

which, after the variable change $\rho = \ln r$ coincides with the definition

$$\ell(F) = \frac{1}{2\pi i}\int\limits_h r^s F(s)\,ds$$

given in the above cited paper.

The following two theorems describe the main features of the introduced notions.

Theorem 2.1 *The space of resurgent functions of power type on* $\mathbf{R}_+^k$ *is an algebra with respect to the usual multiplication. The operator* ℓ *determines an algebra homomorphism*

$$\ell\,:\,\widetilde{\mathcal{R}}\left(\mathbf{R}_+^k\right) \to \mathcal{R}\left(\mathbf{R}_+^k\right)$$

where $\widetilde{\mathcal{R}}\left(\mathbf{R}_+^k\right)$ *is considered as an algebra with respect to convolution of microfunctions in* s.

[4]The definition of hyperfunctions the reader can find, for example, in [20], [29]

Theorem 2.2 *The formulas*

$$r_j \frac{\partial}{\partial r_j} \ell(F(s,\rho)) = \ell\left(\left(\frac{\partial}{\partial s}\right)^{-1} \frac{\partial}{\partial \rho_j} F(s,\rho)\right), \ j = 1, \dots, k$$

and

$$r^\alpha \ell(F(s,\rho)) = \ell\left(\widehat{T}_{\alpha\rho} F(s,\rho)\right)$$

are valid. Here $\widehat{T}_{\alpha\rho}$ *is a shift to* $\alpha\rho = \alpha_1\rho_1 + \ldots + \alpha_k\rho_k$ *in the s-plane:*

$$T_{\alpha\rho}(F(s,\rho)) = F(s - \alpha\rho, \rho).$$

Let us mention here one more notion in the resurgent analysis. This is a notion of a resurgent function with simple singularities.

We recall ([23]) that the resurgent function $f(r)$ is called *a resurgent function with simple singularities* iff the corresponding function $F(s,\rho)$ has the asymptotic expansion of the form

$$F(s,\rho) \simeq \frac{a_0(\rho)}{s - S(\rho)} + \ln(s - S(\rho)) \sum_{k=0}^{\infty} \frac{(s - S(\rho))^k}{k!} a_{k+1}(\rho) \tag{18}$$

near each singular point $s = S(\rho)$ (we restrict ourselves to the case when the 'polar part' of $F(s,\rho)$ is of order 1 but one can also consider poles of an arbitrary order). Here $a_j(\rho)$ are homogeneous functions in ρ of order $-j$. In this case function (18) is a homogeneous microfunction of order -1.

If f is a resurgent function with simple singularities and its support consists (for given value of ρ) of a single point, then it has the asymptotical expansion of the form[5]

$$f(r) = e^{S(\ln r)} \sum_{k=0}^{\infty} a_k(\ln r)$$

where $\ln r = (\ln r_1, \dots, \ln r_k)$. We remark that the series on the right in the latter formula is, in general, divergent (unless the series on the right in (18) converges in the whole plane $\mathbf{C}_s$). So, as it is shown in the above example, one needs to use a resummation procedure for series (18). Such resummation procedure is given by the operator ℓ.

Let us now turn our attention to the investigation of the case when the considered resurgent functions depend on some *additional parameters* $x = (x^1, \dots, x^n)$.

[5]Certainly, this kind of asymptotic expansion is valid outside the set of ramification of the function $S(\rho)$. The investigation of the asymptotic behavior of a resurgent function near ramification points of S (the so-called *focal points*) is given in the paper [23].

Then the set of singularities of the function $F(s,\rho,x)$ computed for each given value x of the parameter, will depend on this value. Thus, one can imagine that the points of singularity of the function F are moving over the plane $\mathbf{C}_s$ while the values of x (as well as the values of ρ) are changed. As above, we denote by $\Sigma_{\rho,x}$ the intersection between the set Σ of singularities of the function F and the set $\{\rho = \text{const}, x = \text{const}\}$. Then the following objects can be considered.

1) The set $\mathcal{F}$ of ramification of the analytic function $S(\rho,x)$ describing the singularity set of the function F. We shall call this set *the set of focal points* corresponding to the considered resurgent function.

2) *The Stokes surface* corresponding to the considered resurgent function. This surface is a set in the space $C^{n+k}_{\rho,x}$ consisting of points such that at least two points of $\Sigma_{\rho,x}$ have coinciding imaginary parts.

One can see that if the point (ρ,x) intersects the Stokes surface, then the support of the resurgent function can have a jump (the reason for such changing the reader can see from Figure 4). This observation gives rise to the homomorphism

$$\mathcal{S}_{(\rho,x)} : \bigoplus_{S_j\in\Sigma} \mathcal{M}^{\text{cont}}_{S_j} \to \bigoplus_{S_j\in\Sigma} \mathcal{M}^{\text{cont}}_{S_j}$$

which is called a *transition homomorphism* corresponding to the given point (ρ,x) of the Stokes surface. This homomorphism allows one to compare the *formal monodromy* of a resurgent function (that is, the monodromy of its pre image $F(s,\rho,x)$ and its *real monodromy*, that is, the monodromy of the function

$$f(\rho,x) = \ell(F(s,\rho,x))$$

while moving around the set of focal points.

We remark that the investigation of the behavior of a resurgent function near its focal points is a distinct task which can be done with the help of the Maslov's canonical operator theory [30], [31]. We shall not stand on this point; the interested reader can find these considerations in [23].

3 Construction of resurgent solutions (general theory)

3.1 Statement of the problem

Now, we proceed with the construction of the general theory of constructing resurgent solutions to partial differential equations on a manifold M with singularities near its vertex V. We suppose that the manifold M has near V the following structure(see Figures 7 and 8)

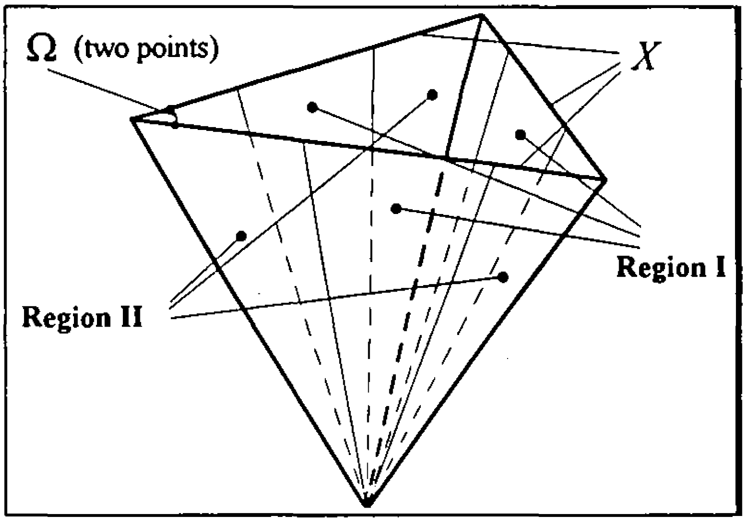

Figure 7: Manifold M, $\dim M = 2$

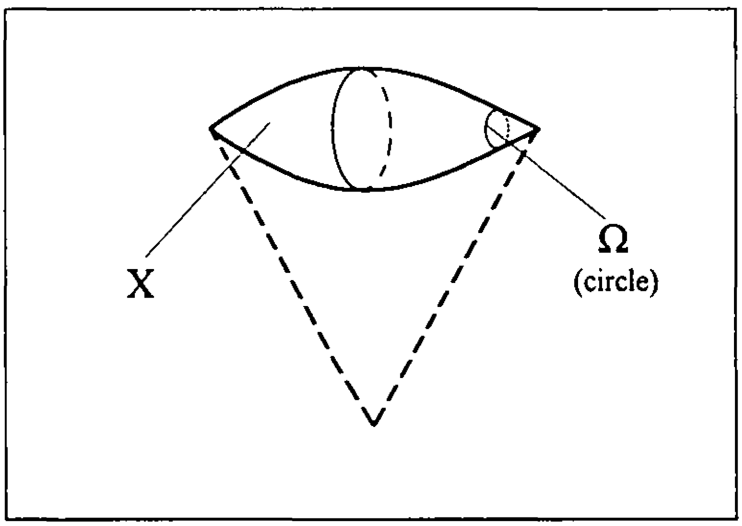

Figure 8: Manifold M, $\dim M = 3$

$$\{[0,1] \times X\} \,/\, \{\{0\} \times X\}\,. \tag{19}$$

For simplicity, we shall consider below the case when the manifold X is a manifold with smooth edges; the general case can be investigated in the similar manner. Thus, we suppose that the manifold X includes edges $Y_1, \dots Y_N$ which are smooth manifolds, and the structure of X near each edge Y_j is described as follows:

$$\{\{[0,1] \times \Omega\} \,/\, \{\{0\} \times \Omega\}\} \times Y_j \tag{20}$$

with some smooth compact manifold Ω.

Let us introduce now special coordinate systems on M which will be used below. We denote by t a point of the segment $[0,1]$ involved in (19). Later on, the coordinate transversal to any edge Y_j of X will be denoted by r. The local coordinate systems on the manifolds Y_j and Ω will be denoted by $y = (y_1, \dots, y_k)$ and $\omega = (\omega_1, \dots, \omega_l)$ correspondingly. The local coordinates on X outside its edges we denote by $x = (x_1, \dots, x_n)$.

Consider a partial differential operator on the manifold M near the vertex V. This operator has the form

$$\widehat{H} = t^{-m} \sum_{j=0}^{m} \widehat{a}_j(t) \left(t\frac{\partial}{\partial t}\right)^j \tag{21}$$

where $\widehat{a}_j(t)$, $j = 0, \dots, m$ are partial differential operators on the manifold X which is, in term, a manifold with singularities. This means, in particular, that these operators have the form

$$\widehat{a}_j(t) = \sum_{|\alpha| \leq m-j} a_{j\alpha}(x,t) \left(\frac{\partial}{\partial x}\right)^\alpha$$

near points of X which are far away from edges and

$$\widehat{a}_j(t) = r^{-(m-j)} \sum_{l=0}^{m-j} \widehat{b}_{jl}(r,t) \left(r\frac{\partial}{\partial r}\right)^l \tag{22}$$

where $\widehat{b}_{jl}(r,t)$ are partial differential operators with smooth coefficients of order $m - l - j$ on the manifold Ω.

We shall search for the resurgent solutions of the equation

$$\widehat{H} u = f \tag{23}$$

provided that the right-hand part f of this equation is a resurgent function. This means that

$$u(t) = \ell\left[U(s,\tau)\right] = \frac{1}{2\pi i}\int\limits_{\Gamma} e^{-s} U(s,\tau)\,ds, \tag{24}$$

the function $U(s,\tau)$ is an endlessly-continuable function in the variable s with values in some weighted Sobolev space $H^s_\alpha(X)$ (α being a real-valued multiindex) which is homogeneous of order -1 in (s,τ). To give an exact description of the mentioned functional space we introduce the following notions.

First of all, one can suppose without loss of generality that the manifolds Ω involved in the representation (20) of the manifold X near each its edge Y_j are connected.

Let us introduce the *weight* function χ which has the representation

$$\chi = \chi(r,\omega) = r^{-2\alpha_j}\chi_j(r,\omega)$$

near each edge Y_j of the manifold X with some smooth function $\chi_j(r,\omega)$ on $[0,1]\times\Omega$ which does not vanish at $r=0$:

$$\chi_j(0,\omega) \neq 0 \text{ for any } \omega \in \Omega.$$

Then the Sobolev space $H^s_\alpha(X)$, $\alpha = (\alpha_1,\dots,\alpha_k)$ is defined with the help of the norm

$$\|u\|_{s,\alpha} = \int\limits_X \chi\left|(1-\Delta)^{\frac{s}{2}} u\right|^2 dV$$

where Δ is a Beltrami-Laplace operator on X with respect to some metrics, dV is a volume measure with respect to the same metrics.

The space of resurgent functions with values in the Sobolev space $H^s_\alpha(X)$ we shall denote by

$$\mathcal{R}^s_\alpha(M) = \mathcal{R}\left([0,1], H^s_\alpha(X)\right).$$

Let us introduce the operator family

$$\widehat{H}(z) \stackrel{\text{def}}{=} \sum_{j=0}^{m} \widehat{a}_j(0) z^j$$

parameterized by points of the complex plane $\mathbf{C}_z$.

We suppose the following condition to be fulfilled.

Condition 3.1 There exists a 'weight' $\alpha = (\alpha_1,\dots,\alpha_k)$ such that the operator family

$$\widehat{H}(z) : H^s_\alpha(X) \to H^{s-m}_\alpha(X)$$

is *invertible* in the space scale $H^s_\alpha(X)$ for any values of z in the complex plane except for some discrete set Σ_H. The resolving operator $\widehat{R}(z)$ is an (in general *ramifying*) operator function of the parameter z outside Σ_H. We suppose also that the resolving operator can be written down in the form

$$\left(\widehat{R}(z)f\right)(x) = \int K\left(x,x',z\right)f\left(x'\right)dx', \tag{25}$$

where $K(x,x',z)$ (the fundamental solution to equation $\widehat{H}(z)u = 0$) can be analytically continued[6] in x, x' up to (ramifying) analytic function in the complexification of the manifold X.

Remark 3.1 As it can be seen from Section 1, Condition 3.1 is of global and terminal character in the sense that it cannot be localized further at singular points of the manifold M. Actually, as it is shown in the Example (see Section 1) for a family satisfying Condition 3.1 the weights $\alpha_1,\dots,\alpha_k$, in general, cannot be chosen in arbitrary way

From the other hand, in any concrete case the singularities of the resolvent operator can be investigated (see, in particular, Section 4), and in the situation of the manifold with conical singularities this Condition is valid for an elliptic operator (see, for example, [2]).

In what follows we investigate the solvability of equation (23) in spaces of resurgent functions.

3.2 The solvability theorem

Here, we present the reduction of the considered problem to a family of equations with the complex parameter z.

First of all, we rewrite equation (23) in the form

$$\sum_{j=0}^{m}\widehat{a}_j(t)\left(t\frac{\partial}{\partial t}\right)^j u = f_1 \stackrel{\text{def}}{=} t^m f$$

or, expanding the coefficients of the latter equation into the Taylor series,

$$\sum_{j=0}^{m}\sum_{k=0}^{\infty}\widehat{a}_{jk}t^k\left(t\frac{\partial}{\partial t}\right)^j u = f_1. \tag{26}$$

[6]Investigation of the analytic continuation of fundamental solution can be performed with the help of analytic continuation of solutions to integral equations, see [32].

Here the operators $\widehat{a}_{jk}$ on the manifold X has the representation of the form (22) with coefficients independent of t.

Now let us substitute resurgent representation (24) to equation (26). Using Theorem 2.2 above, we obtain the following equation for functions $U(s,\tau)$:

$$\sum_{j=0}^{m}\sum_{k=0}^{\infty}\widehat{a}_{jk}\widehat{T}_{k\tau}\left(\left(\frac{\partial}{\partial s}\right)^{-1}\frac{\partial}{\partial \tau}\right)^{j}U=F \tag{27}$$

where the function $F=F(s,\tau)$ is an endlessly-continuable function corresponding to the function $f_1(t)$ via the representation (24).

Similar to the papers [26], [27], one can see that the solution U to equation (27) can be found in the form of the series

$$U(s,\tau)=\sum_{k=0}^{\infty}U^{(k)}(s,\tau) \tag{28}$$

where $U^{(k)}(s,\tau)$ are solutions to the following recurrent system of equations:

$$\begin{gathered}\sum_{j=0}^{m}\widehat{a}_{j0}\left(\left(\frac{\partial}{\partial s}\right)^{-1}\frac{\partial}{\partial \tau}\right)^{j}U^{(0)}=F,\\ \sum_{j=0}^{m}\widehat{a}_{j0}\left(\left(\frac{\partial}{\partial s}\right)^{-1}\frac{\partial}{\partial \tau}\right)^{j}U^{(k)}=-\sum_{j=0}^{m}\sum_{k'+k''=k}\widehat{a}_{jk'}\widehat{T}_{k'\tau}\left(\left(\frac{\partial}{\partial s}\right)^{-1}\frac{\partial}{\partial \tau}\right)^{j}U^{(k'')},\\ k\geq 1.\end{gathered} \tag{29}$$

It is evidently sufficient to investigate the solvability of the first equation in (29) since the operator on the left in all equations of this system is the same. To prove the solvability of this equation, we pass to $\partial/\partial s$-transform [33], [34][7]. We obtain the equation

$$\sum_{j=0}^{m}\widehat{a}_{j0}z^{j}\widetilde{U}^{(0)}(s,z)=\widetilde{F}(s,z) \tag{30}$$

where $\widetilde{U}^{(0)}(s,z)$ and $\widetilde{F}(s,z)$ are $\partial/\partial s$-transforms of the functions $U^{(0)}(s,\tau)$ and $F(s,\tau)$.

The solvability of this equation follows now directly from Condition 3.1. We shall present a more detailed analysis of the support of the resurgent function u. Namely, the singularity set of solutions to equation (30) can be investigated with the help of formula (25) (for example, see computations in the next

[7]This transform is an analogue of the real Fourier-Maslov $\partial/\partial\tau$-transform [30] in the situation of ramifying analytic functions.

subsection). Then the support $\operatorname{supp} u^0$ of the corresponding resurgent function $u^0 = \left(F^{\partial/\partial_s}\right)^{-1} \tilde{U}^0$ can be computed with the help of the Thom theorem. Now the support of the solution to equation (23) consists of the set of lattices with the step τ whose origins are situated either at points of $\operatorname{supp} u^0$ (see Condition 3.1) or at points of support of the right-hand part f of the equation.

One can investigate the resurgent solutions to equation (23) from the different points of view.

First, if we are intended to obtain the asymptotics of the solution only for real values of t, then the variable τ, which is related to the variable t by

$$t = e^{\tau} \tag{31}$$

in (24) is real with sufficiently large in module negative values (we recall that we construct the resurgent solutions near the vertex V of the manifold M, that is, for t sufficiently close to zero).

Second, one can investigate *the analytic continuation* of the solution to equation (23) to the complex domain (that is, to the complex values of t sufficiently close to zero in module). In this case, due to the relation (31) between t and τ, the variable τ must belong to some left half-plane in the complex plane $\mathbf{C}_\tau$:

$$\operatorname{Re} \tau < A$$

for some positive real A.

In both cases the value of $\operatorname{Re} \tau$ increases along each above mentioned lattice involved into the support $\operatorname{supp} u$ of the solution u and, hence, series (28) converge in the sense of the resurgent function theory (see [23]). Thus, for both cases the following statement is valid.

Theorem 3.1 *If Condition* 3.1 *is valid, then equation* (23) *is solvable in the spaces* $\mathcal{R}^s_\alpha(M)$.

3.3 Investigation of singularity set of the solution

Let us investigate in more detaile the singularity set of solutions to equation (27), which gives us an information about the asymptotic behavior of solutions to the initial equation.

To simplify our considerations, we shall suppose that the two following conditions are fulfilled.

i). The support $\operatorname{supp} f$ of the right-hand part of equation (23) consists of a single point $s_0(\tau) = \sigma_0 \tau$ for each fixed value of τ.

ii). The resolving operator $\widehat{R}(q)$ has only polar singularities at points of the set Σ_H (see Condition 3.1).

We remark that the first of these two conditions is not restrictive at all since every resurgent function can be represented as a sum of resurgent functions each supported in a single point. The second condition is, of course, more restrictive, but, it is satisfied for elliptic differential operators.

Now let us proceed with the analysis of the resurgent solution to equation (23). To begin with, we shall investigate the singularity set of the solution $U^{(0)}(s,\tau)$ to the first of equations (29).. First of all, we note that the solution to the equation (30) is given by the formula

$$\widetilde{U}^{(0)}(s,z)=\widehat{R}(z)\widetilde{F}(s,z).$$

Hence, the solution to the first equation in (29) has the form

$$U^{(0)}(s,\tau)=F^{\partial/\partial s}_{z\to\tau}\left\{\widehat{R}(z)\widetilde{F}(s,z)\right\}$$

where $F^{\partial/\partial s}_{z\to\tau}$ is the inverse $\partial/\partial s$-transform

$$F^{\partial/\partial s}_{z\to\tau}\widetilde{U}(s,z)=\left(\frac{i}{2\pi}\right)^{1/2}\frac{\partial}{\partial s}\int\limits_{h(s,\tau)}\widetilde{U}(s+z\tau,z)\,dz$$

and $h(s,\tau)$ is some special relative homology class (see [33], [34]):

$$h(s,\tau)\in H_1\left(\mathbf{C}_z\setminus\Sigma_H,\Sigma_{\widetilde{F}}\right).$$

Since

$$\widetilde{U}(s,z)=F^{\partial/\partial s}_{\tau\to z}U(s,\tau)=\int\limits_{h(s,\tau)}U(s-z\tau,\tau)\,d\tau,$$

we finally obtain the expression for the solution to the first equation in (29):

$$U^{(0)}(s,\tau)=\left(\frac{i}{2\pi}\right)^{1/2}\frac{\partial}{\partial s}\int\limits_{H(s,\tau)}\widehat{R}(z)F(s+z\left(\tau-\tau'\right),\tau')\,d\tau'\wedge dz \qquad (32)$$

with some relative homology class $H(s,\tau)\in H_2\left(C^2_{z,\tau}\setminus\Sigma_H,\Sigma_F\right)$. Now the singularity set of the function $U^{(0)}(s,\tau)$ can be computed with the help of the Thom theorem [35].

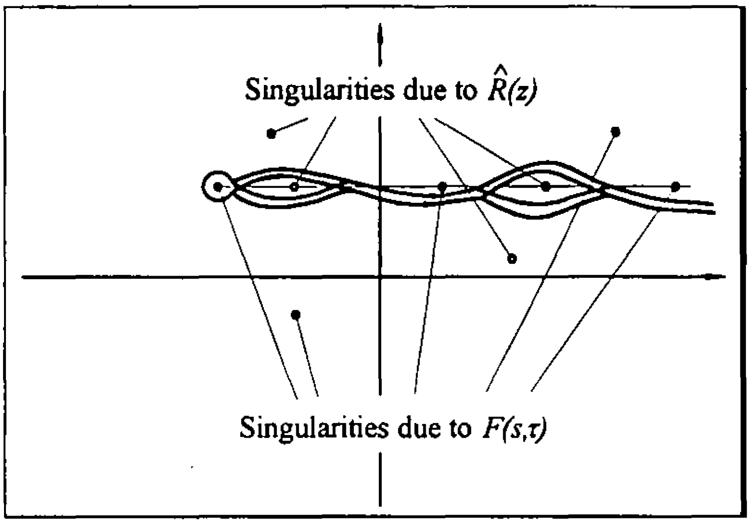

Figure 9: Admissible contours

Namely, the integrand in (32) have singularities exactly on the union of the set Σ_H and the set Σ_F of singular points of the function $F(s,\tau)$. These singularities are described by the formulas

$$z = z_j,\ j = 1, 2, \ldots \text{ and } s + z(\tau - \tau') = \sigma_0\tau'.$$

Now the standard considerations using the Thom theorem show that the singularities of the function $U^{(0)}(s,\tau)$ lye at the points

$$s = \sigma_0\tau \quad \text{and} \quad s = z_j\tau,\ j = 0, 1, \ldots.$$

To construct the first term $u^{(0)}(t)$ of the expansion of the solution $u(t)$ corresponding to expansion (28), we must use the resurgent representation of the form (24) with the function $U^{(0)}(s,\tau)$ in the integrand along the contour which is homological to that of the representation for the function $f(t)$ in the space $C \setminus \Sigma_f$. However, there remains a degree of freedom in the choice of this contour outside the singular set $\Sigma_f \cup \Sigma_H$ (see Figure 9). It is not so hard to see that functions obtained by integration along different admissible contours differ from one another by solutions to the corresponding homogeneous equation, which, due to condition **ii)** above, has the form of the conormal asymptotics

$$\sum_j t^{z_j} \sum_{k=0}^{m_j} c_{kj} \ln^k t \tag{33}$$

where the inner sum is taken over such j that all values of z_j involved in the latter equation lye in some fixed right half-plane of the complex plane $\mathbf{C}_q$ and c_{kj} are functions on the manifold X (which certainly may have singularities at singular points of this manifold). The examination of solutions to the homogeneous equations the reader can find in [4], [5].

3.4 Asymptotics of solutions near the vertex

Our next goal is to investigate the obtained solutions near edges emanated from the vertex V of the manifold M. The matter is that our above considerations give the appropriate asymptotics of solutions when the point approaches the vertex in the direction which does not coincide with some edge emanated from V, that is, when $t \to 0$ and the corresponding point on X do not tend to some singular point of this manifold (the region I on Figure 7). Actually, in this case the coefficients of resurgent representation (24) are smooth and the asymptotic behavior of the solution is determined only by the variable t. Quite another situation takes place when the point approaches the vertex along the region II (Figure 7) since in this case the corresponding point on X tends to some singular point of this manifold and, hence, the coefficients of the resurgent representation are in turn singular. Certainly, one can use for this region the results of the paper [26] which show that (under the assumption that the right-hand part of equation (23) is a resurgent function not only in t but also in r) the solution to equations (27) is a resurgent function in r near each edge of the manifold X. These considerations show that the obtained solution is a resurgent function in t with values in the space of resurgent functions in r.

However, such a description of solutions to equation (23) is rather implicit and there arises a problem of the more explicit description of the asymptotics of the solution in region II. The following theorem is the first step in solving this problem.

Theorem 3.2 *Any resurgent function of the variable τ with values in the space of resurgent functions of the variable $\rho = \ln r$ is a resurgent function with respect to (τ, ρ).*

Proof. First of all, we remark that it is sufficient to prove this Theorem for regular resurgent functions, that is, for resurgent functions u such that the corresponding function U has only integrable singularities. Actually, to reduce the problem to an integrable case one can apply to the considered functions the operator $(\partial/\partial s)^{-N}$ with sufficiently large N.

Thus, let us consider the resurgent function in τ:

$$u(\tau) = \int_{\gamma} e^{-s} U(s,\tau)\, ds \tag{34}$$

where γ is a ray directed along the positive part of the real axis and emanated from the point $s^*(\tau)$ (the support of the resurgent function $u(\tau)$). Suppose that the function $U(s,\tau)$ is an infinitely continuable in s function homogeneous of order -1 in (s,τ) with values in space of resurgent functions in ρ:

$$U(s,\tau) = U(s,\tau,\rho) = \int_{\gamma_1} e^{-s_1} U_1(s,\tau,s_1,\rho)\, ds_1 \tag{35}$$

where $U_1(s,\tau,s_1,\rho)$ is an endlessly-continuable in s function which is homogeneous in (s_1,ρ) (as well as in (s,τ), of course). Here the contour γ_1 is a ray emanated from a point $s_1^*(\rho)$ along the direction of the positive part of the real axis. Substituting (35) into (34) we obtain a representation for the function $u(\tau)$ of the form

$$u(\tau) = u(\tau,\rho) = \int_{\gamma} e^{-s} \left[\int_{\gamma_1} e^{-s_1} U_1(s,\tau,s_1,\rho)\, ds_1 \right] ds.$$

The latter expression can be rewritten as follows:

$$\begin{aligned} u(\tau,\rho) &= \int_0^\infty d\xi \int_0^\infty \exp\left[-\left(s^*(\tau) + s_1^*(\rho) + \xi + \eta\right)\right] \\ &\times \; U_1(s^*(\tau)+\xi, \tau, s_1^*(\rho)+\eta, \rho)\, d\eta. \end{aligned}$$

Rewriting the latter integral as a multiple integral over the positive quarter $\mathbf{R}^2_+$ of the plane $R^2_{\xi\eta}$ and introducing the variable change

$$\xi = \xi, \quad \varsigma = \xi + \eta$$

we obtain the formula

$$u(\tau,\rho) = \int_0^\infty \exp\left[-\left(s^*(\tau) + s_1^*(\rho) + \varsigma\right)\right] U_2(\varsigma,\tau,\rho)\, d\varsigma$$

where

$$U_2(\varsigma,\tau,\rho) = \int_0^\varsigma U_1\left(s^*(\tau)+\xi, \tau, s_1^*(\rho)+\varsigma-\xi, \rho\right)\, d\xi.$$

The two last formulas can be rewritten in the form

$$u(\tau,\rho)=\int\limits_{\gamma_2} e^{-s}U_*(s,\tau,\rho)\,ds \tag{36}$$

where the function $U_*(s,\tau,\rho)$ is given by

$$U_*(s,\tau,\rho)=\int\limits_{\gamma_3} U_1(s',\tau,s-s',\rho)\,ds' \tag{37}$$

and the contour γ_3 is the segment with the endpoint $s^*(\tau)$ and $s-s_1^*(\rho)$. It is easy to check that the function $U_*(s,\tau,\rho)$ given by (37) is a homogeneous function of order -1 with respect to (s,τ,ρ) which is infinitely continuable in the variable s for any fixed values of (τ,ρ). The proof is complete.

Since we know now that the solution to equation (24) is a resurgent function near edges (that is, in the region II on Figure 7, then the second step in constructing asymptotics for this solution is to substitute representation (36) directly into the initial equation[8] (after the change of variables $t=e^\tau$, $r=e^\rho$). To do this we rewrite equation (24) in the form

$$r^{-m}t^{-m}H\left(r,t,r\frac{\partial}{\partial r},rt\frac{\partial}{\partial t}\right)u=f. \tag{38}$$

After the mentioned variable change the equation becomes

$$H\left(e^\rho,e^\tau,\frac{\partial}{\partial\rho},e^\rho\frac{\partial}{\partial\tau}\right)u(\rho,\tau)=f_1(\rho,\tau)$$

(we had also multiplied equation (38) by $r^{-m}t^{-m}$). Similar to the above considerations, substituting (36) into the latter equation and expanding its coefficients into the Taylor series with respect to the first two arguments, we obtain

$$\sum_{j,k=0}^{\infty}\widehat{T}_{j\rho+k\tau}H_{jk}\left(\left(\frac{\partial}{\partial s}\right)^{-1}\frac{\partial}{\partial\rho},\left(\frac{\partial}{\partial s}\right)^{-1}\frac{\partial}{\partial\tau}\right)U(s,\rho,\tau)=F(s,\rho,\tau). \tag{39}$$

Here, as above, $\widehat{T}_{j\rho+k\tau}$ is a shift operator in the plane $\mathbf{C}_s$ to the value $j\rho+k\tau$. The operators

$$\widehat{H}_{jk}=H_{jk}\left(\left(\frac{\partial}{\partial s}\right)^{-1}\frac{\partial}{\partial\rho},\left(\frac{\partial}{\partial s}\right)^{-1}\frac{\partial}{\partial\tau}\right)$$

[8] We present here rather brief description of the procedure of constructing resurgent solutions to the considered equation. The reader can find the detailed presentation of this construction (for somewhat different situation) in [26], [27].

have the operator-valued coefficients (which are differential operators on the manifold Ω) and can be found from the relation

$$H\left(e^{\rho}, e^{\tau}, \frac{\partial}{\partial\rho}, e^{\rho}\frac{\partial}{\partial\tau}\right) = \sum_{j,k=0}^{\infty} e^{j\rho+k\tau} H_{jk}\left(\left(\frac{\partial}{\partial s}\right)^{-1}\frac{\partial}{\partial\rho}, \left(\frac{\partial}{\partial s}\right)^{-1}\frac{\partial}{\partial\tau}\right). \quad (40)$$

We remark that, due to the presence of the factor e^{ρ} before the derivative $\partial/\partial\tau$, this derivative will not be included into the principal part $\widehat{H}_{00}$ of expansion (40). Hence, equation (39) can be rewritten in the form

$$H_{00}\left(\left(\frac{\partial}{\partial s}\right)^{-1}\frac{\partial}{\partial\rho}\right) U = F - \sum_{j,k=0}^{\infty} \widehat{T}_{j\rho+k\tau} H_{jk}\left(\left(\frac{\partial}{\partial s}\right)^{-1}\frac{\partial}{\partial\rho}, \left(\frac{\partial}{\partial s}\right)^{-1}\frac{\partial}{\partial\tau}\right) U$$

and the sum on the right in the latter expression does not contain the term with $j = k = 0$.

As above, the solution to the latter equation can be constructed with the help of a recurrent procedure in the form

$$U(s,\rho,\tau) = \sum_{j,k=0}^{\infty} U^{(jk)}(s,\rho,\tau),$$

where the functions $U^{(jk)}(s,\rho,\tau)$ satisfy the following recurrent system:

$$\begin{aligned} &H_{00}\left(\left(\frac{\partial}{\partial s}\right)^{-1}\frac{\partial}{\partial\rho}\right) U^{(00)} = F, \\ &H_{00}\left(\left(\frac{\partial}{\partial s}\right)^{-1}\frac{\partial}{\partial\rho}\right) U^{(jk)} = \\ &-\sum \widehat{T}_{j'\rho+k'\tau} H_{j'k'}\left(\left(\frac{\partial}{\partial s}\right)^{-1}\frac{\partial}{\partial\rho}, \left(\frac{\partial}{\partial s}\right)^{-1}\frac{\partial}{\partial\tau}\right) U^{(j''k'')}, \end{aligned} \quad (41)$$

where the sum on the left in the latter relation is taken over all nonnegative integers j', k', j'', k'' such that $j' + j'' = j$ and $k' + k'' = k$ except for the term with $j' = k' = 0$. Thus, again it is sufficient to solve the first of the above equations because this system is a triangle one with one and the same operator on the diagonal. As usual, we shall apply the $\partial/\partial s$-transform to this equation thus reducing it to the following family of equations

$$\widehat{H}_{00}(z)\widetilde{U}(s,z,\tau) = \widetilde{F}(s,z,\tau), \quad (42)$$

where $\widetilde{U}$ and $\widetilde{F}$ are the $\partial/\partial s$-transforms of the functions U and F with respect to the variable ρ,

$$\widetilde{U}(s,z,\tau) = F^{\partial/\partial s}_{\rho\to z} U(s,\rho,\tau),$$

$$\widetilde{F}(s,z,\tau) = F^{\partial/\partial s}_{\rho\to z} F(s,\rho,\tau),$$

z is a dual variable (one can notice that the variable τ is simply a parameter in the latter equation). One should remember that the operator $\widehat{H}_{00}(z)$ included into (42) is a differential operator of order m on the smooth manifold Ω:

$$\widehat{H}_{00}(z) = \sum_{|\alpha|\le m} a_\alpha(\omega,z)\left(\frac{\partial}{\partial\omega}\right)^\alpha$$

with smooth coefficients with polynomial dependence of z. This operator can be expressed as

$$\widehat{H}_{00}(z) = \widehat{H}(0,0,z,0) \tag{43}$$

where $\widehat{H}$ is an (operator-valued) Hamiltonian included in the representation (38) of the considered operator.

It is natural that the following condition must be fulfilled.

Condition 3.2 The operator family (43) depending on the parameter z is analytically invertible, that is, there exists an operator $\widehat{R}_{00}(z)$ (which depends on z analytically in the whole plane C_z except for a discrete set $\Sigma = \{z_1, z_2, \ldots\}$) such that

$$\widehat{H}_{00}(z)\widehat{R}_{00}(z) = \widehat{R}_{00}(z)\widehat{H}_{00}(z) = \widehat{1}$$

where $\widehat{1}$ is the identity operator. The operator $\widehat{R}_{00}(z)$ is an integral operator with a kernel admitting an analytic continuation to $\Omega_{\mathbf{C}} \times \Omega_{\mathbf{C}}$, where $\Omega_{\mathbf{C}}$ is a complexification of the manifold Ω (see formula (25) above).

Due to Condition 3.2 equation (42) is solvable and its solution is given by

$$\widetilde{U}(s,z,\tau) = \widehat{R}_{00}(z)\widetilde{F}(s,z,\tau).$$

Now the solution to equation (42) can be written down in the form

$$U(s,\rho,\tau) = F^{\partial/\partial s}_{z\to\rho}\left\{\widehat{R}_{00}(z)\widetilde{F}(s,z,\tau)\right\} = \left\{F^{\partial/\partial s}_{z\to\rho}\widehat{R}_{00}(z)F^{\partial/\partial s}_{z\to\rho}\right\}F(s,\rho,\tau).$$

Similar to formula (32) above, we can rewrite the latter relation in the integral form

$$U = \widehat{\mathcal{R}}[F] = \left(\frac{i}{2\pi}\right)^{1/2}\frac{\partial}{\partial s}\int\limits_{H(s,\tau)} \widehat{R}_{00}(z)F(s+z(\rho-\rho'),\rho',\tau)\,d\rho'\wedge dz. \tag{44}$$

Formula (44) is the main tool for investigation of solutions to system (41). Namely, this solution is given by

$$\begin{aligned} U^{(00)} &= \widehat{\mathcal{R}}[F], \\ U^{(jk)} &= -\widehat{\mathcal{R}}\left[\sum \widehat{T}_{j'\rho+k'\tau} H_{j'k'}\left(\left(\frac{\partial}{\partial s}\right)^{-1}\frac{\partial}{\partial \rho}, \left(\frac{\partial}{\partial s}\right)^{-1}\frac{\partial}{\partial \tau}\right) U^{(j''k'')}\right], \\ & \quad j \geq 0 \,\text{or}\, k \geq 0. \end{aligned}$$

Now the investigation of singularities of functions $U^{(jk)}$ can be performed exactly in the same manner as it was done for functions $U^{(k)}(s,\tau)$ in the end of the previous Section. The result of this investigation can be formulated as follows.

Theorem 3.3 *The set of singularities of function* (44) *is a union of the set of singularities of the function* $F(s,\rho,\tau)$ *and the characteristic conoid of the latter set with respect to the operator* $\widehat{R}(z)$.

The investigation of the asymptotical expansions of solutions in the region I (see Figure 7) admits no further detalization since in this zone the coefficients in the resurgent representation (33) are smooth functions. We shall show that the resurgent representation for solutions to equation (23):

$$u(\tau,\rho) = \int\limits_{\Gamma} e^{-s} U(s,\tau,\rho)\, ds \tag{45}$$

can be reduced to the one-dimensional representation (24) when $r = e^{\rho} \geq \delta > 0$. Actually, let us consider representation (45) for finite values of ρ (that is, $|\rho| \leq C$ for some positive constant C). Expanding the function $U(s,\tau,\rho)$ in powers of ρ (this can be done for large values of τ due to the homogeneity of the function $U(s,\tau,\rho)$), we come to the relation

$$u(\tau,\rho) = \sum_{j=0}^{\infty} \frac{\rho^j}{j!} \int\limits_{\Gamma} e^{-s} \frac{\partial^j U}{\partial \rho^j}(s,\tau,0)\, ds. \tag{46}$$

The functions $\partial^j U/\partial\rho^j$ under the integral sign on the right of the latter formula are homogeneous of order $-1-j$ (not of the order -1 as it is required in the definition (24) of the one-dimensional resurgent representation). To improve this, we shall integrate by parts under the integral sign in (46) j times. As a result, we obtain a representation

$$u(\tau,\rho) = \int\limits_{\Gamma} e^{-s} \sum_{j=0}^{\infty} \frac{\rho^j}{j!} \left(\left(\frac{\partial}{\partial s}\right)^{-1} \frac{\partial}{\partial \rho}\right)^j U(s,\tau,0)\, ds.$$

Substituting $\tau = \ln t$, $\rho = \ln r$ in this representation, we have the following one-dimensional resurgent representation for the function u

$$u(t) = \int_{\Gamma} e^{-s} \sum_{j=0}^{\infty} \frac{(\ln r)^j}{j!} \left(\left(\frac{\partial}{\partial s} \right)^{-1} \frac{\partial}{\partial \rho} \right)^j U(s, \ln t, 0)\, ds$$

(the variable t on the left in the latter formula is omitted since we consider here the function u as a function of t with values in a function space of functions depending on r). We remark that the obtained relation is considered in the domain where $r \geq \delta > 0$, and, hence, the function $\ln r$ is a regular one.

3.5 The case of resurgent functions with simple singularities

In this subsection, we briefly describe the explicit forms of asymptotic expansions of resurgent solutions provided that the right-hand part of equation (23) is a *resurgent function with simple singularities.* We recall that the function $u(t, r)$ is called to be a resurgent function with simple singularities if and only if the corresponding (due to representation (45)) function $U(s, \tau, \rho)$ has the form[9]

$$U(s, \tau, \rho) = \frac{G_1(s, \tau, \rho)}{(s - S(\tau, \rho))^k} + G_2(s, \tau, \rho) \ln (s - S(\tau, \rho)) \tag{47}$$

with holomorphic functions $G_1(s, \tau, \rho)$ and $G_2(s, \tau, \rho)$ near each singular point of $U(s, \tau, \rho)$. Here $S(\tau, \rho)$ is an (in general, ramifying) analytic function such that the singularity set of the function $U(s, \tau, \rho)$ is given by

$$s - S(\tau, \rho) = 0.$$

We recall that the ramification points of the function $S(\tau, \rho)$ are called *focal points* of the corresponding resurgent function. Certainly, representation (47) do not work in a neighbourhood of focal points and must be replaced there by some other representation. This representation, using singular charts of the corresponding Lagrangian manifold, was presented in the paper [23] by the authors and we shall not discuss it here.

Remark 3.2 We claim that *equation (23) is solvable in classes of resurgent functions with simple singularities.* In other words, if the right-hand part of this equation is a resurgent function with simple singularities, then the equation has a solution in the same class.

[9] We remind that all the functions involved in the formula below depend on local coordinates (ω, y) on $\Omega \times Y_j$. This dependence is omitted here for brevity.

Let us turn our mind to the non singular case (47). We remark that in this representation the function $S(\tau,\rho)$ must be a homogeneous function of order 1 (in this case the function $U(s,\tau,\rho)$ determined by (47) will be homogeneous *in the sense of hyperfunctions* provided that the functions $G_1(s,\tau,\rho)$ and $G_2(s,\tau,\rho)$ are homogeneous of the corresponding orders).

Now we are able to write down the form of asymptotic expansions of the solution to equation (23) in different regions. First of all we shall investigate the asymptotic solutions in the region II. As it is shown in the above cited papers, the asymptotics corresponding to expansion (47) has the form[10]

$$u(t,r) = \sum_i e^{-S_i(\ln t, \ln r)} \sum_{j=j_0}^{-\infty} a_j(\ln t, \ln r) \tag{48}$$

where the inner sum is taken over all points of the support of the resurgent function u, $S_i(\tau,\rho)$ are the corresponding branches of the (ramifying) function $S(\tau,\rho)$ (which is homogeneous of order 1), and the coefficients $a_j(\tau,\rho)$ are functions homogeneous of order $-j$ in (τ,ρ) which are determined via the Taylor coefficients of functions $G_1(s,\tau,\rho)$ and $G_2(s,\tau,\rho)$ at the point $s = S_i(\tau,\rho)$. We remark that, as it follows from the considerations of this Subsection, the support of the constructed solution consists of a number of double lattices originated from points of the support of the right-hand part of equation (23) (and possibly from some of the points of singularity originated by the operator itself) with steps $\rho = \ln r$ and $\tau = \ln t$. We remark also that the series over j included into the right-hand side of relation (48) are, as a rule, divergent and the resurgent representation of the form (45) gives, in particular, the unified resummation procedure for these divergent series. The problem of computing supports of the constructed solution is solved in the region II (that is, near edges) similar to the general case. Relation (48) gives us a general form of the asymptotic expansion of the solution near edges of the manifold M. The computation of the exact values of the functions $a_j(\tau,\rho)$ can be carried out by the asymptotic expansion (by smoothness) of integrals of the type (45) with the function $U(s,\tau)$ given by (44).

Let us detalize the obtained asymptotic expansion. First of all, we note that asymptotics (48) contains two kind of terms. They are asymptotic expansions corresponding to the points of the support coinciding with the support of the right-hand part. The form of these terms cannot be detalized further; the actions S_i for these terms coincide with that for the right-hand part of the equation. The second kind of terms are all other terms. To establish the form of these terms in more detail, we remark that, due to system (41), the phase function of these

[10]We remark that functions $S(\tau,\rho)$ and $a_j(\tau,\rho)$ depend also on the local coordinates ω and y on manifolds Ω and Y_j (see the beginning of this Section).

terms must satisfy the corresponding Hamilton-Jacobi equation

$$\frac{\partial S}{\partial \rho} = z_j,$$

where z_j are points of singularity of the operator $\widehat{R}_{00}(z)$ (see Condition 3.2 above). Hence, we have

$$S(\tau, \rho, \omega, y) = z_j \rho + S_1(\tau, \omega, y)$$

for some function $S_1(\tau, \omega, y)$ which is homogeneous of order 1 in τ. Then it is evident that

$$S(\tau, \rho, \omega, y) = z_j \rho + \tau Z(\omega, y)$$

with some smooth function $Z(\omega, y)$. Thus, those terms included into the asymptotics of the solution whose supports do not coincide with the support of the right-hand part of the considered equation have the form

$$\begin{aligned} & e^{-z_j \ln r - Z(\omega,y) \ln t} \sum_{j=j_0}^{-\infty} a_j(\ln t, \ln r, \omega, y) \\ &= r^{-z_j} t^{-Z(\omega,y)} \sum_{j=j_0}^{-\infty} a_j(\ln t, \ln r, \omega, y). \end{aligned} \tag{49}$$

Let us proceed now with the investigation of the asymptotics in the region I (that is, for $t \to 0$ and $r \geq \delta > 0$). Similar to the previous case, resurgent function of one variable with simple singularities is a function given by (24) such that for the corresponding function $U(s, \tau, r, \omega, y)$ the expansion

$$U(s, \tau, r, \omega, y) = \frac{G_{1i}(s, \tau, r, \omega, y)}{(s - S_i(\tau, r, \omega, y))^k} + G_{2i}(s, \tau, r, \omega, y) \ln (s - S_i(\tau, r, \omega, y))$$

is valid near each its singular point

$$s = S_i(\tau) = S_i(\tau, r, \omega, y).$$

In this case, since S_i are homogeneous functions of order 1 in one-dimensional variable τ, they have the form

$$S_i(\tau, r, \omega, y) = \sigma_i(r, \omega, y)\tau \tag{50}$$

for some complex-valued function σ_i, the constructed asymptotic expansion in the region I (that is, when a point approaches the vertex along the direction not close to the edge) has the form

$$u(t) = \sum_i t^{-\sigma_i(r,\omega,y)} \sum_{j=m}^{-\infty} (\ln t)^j \, a_j(r, \omega, y) \tag{51}$$

where $a_j(r,\omega,y)$ are some functions on the non singular part of X determined by the Taylor expansion of the functions $G_{1i}(s,\tau,r,\omega,y)$ and $G_{2i}(s,\tau,r,\omega,y)$ at the point (50); we remark that the coefficients of these expansions, being homogeneous functions of one-dimensional variable τ are simply proportional to the corresponding powers of this variable. The computation of the exact values of the coefficients a_j in (49) and (51) can be carried out by the asymptotic expansion (by smoothness) of integrals of the type (24) with the corresponding function U.

One can easily see that the forms (49) and (51) are in a good agreement with each other on their mutual domain of definition (we emphasize that since the constructed solution is a resurgent function in t with values in the space of resurgent functions in r, the function $\sigma_i(r,\omega,y)$ must have the asymptotic expansion $\sigma_i(r,\omega,y) = z_j \ln r + \dots$ with some constants z_j as $r \to 0$.

4 Two-dimensional problem

In this section, we shall show how the singularities of the above introduced operator families can be computed for two-dimensional manifold M. Also we illustrate the method of obtaining resurgent solution on the two-dimensional model. Since in this case the manifold X is one-dimensional, its 'edges' Y_j are single points. Therefore, the operators $\widehat{a}_j(t)$ are in this case ordinary differential operators of Fuchsian type

$$\widehat{a}_j(t) = r^{-(m-j)} \sum_{l=0}^{m-j} b_{jl}(r,t) \left(r\frac{\partial}{\partial r} \right)^l$$

near each singular point Y_j of X. Here r is a coordinate on X near the singular point Y_j.

4.1 Resurgent solutions

In this subsection, we shall briefly describe the procedure of constructing a resurgent solution to equation (23) for the considered particular case.

Similar to Subsection 3.2, equation (23) can be rewritten in the form

$$\sum_{j=0}^{m} \widehat{a}_j(t) \left(t\frac{\partial}{\partial t} \right)^j u = f_1 \stackrel{\text{def}}{=} t^m f. \tag{52}$$

We search for resurgent solutions to equation (52) provided that the right-hand part f_1 of this equation is a resurgent function. Let

$$u(t) = \ell(U)$$

(see formula (16) above and let $U(s,\tau)$ be an endlessly-continuable function of s for any fixed value of τ. Substituting the latter relation to (52) and expanding the coefficients $\widehat{a}_j(t)$ into the Taylor series in t:

$$\widehat{a}_j(t) = \sum_{k=0}^{\infty} t^k \widehat{a}_{jk}$$

(where $\widehat{a}_{jk}$ are differential equations on X of the form

$$\widehat{a}_{jk} = r^{-(m-j)} \sum_{l=0}^{m-j} b_{jkl}(r) \left(r\frac{\partial}{\partial r}\right)^l$$

near each singular point Y_j), we come to the equation

$$\sum_{j=0}^{m} \sum_{k=0}^{\infty} \widehat{a}_{jk}\widehat{T}_{k\tau} \left(\left(\frac{\partial}{\partial s}\right)^{-1} \frac{\partial}{\partial \tau}\right)^j U(s,\tau) = F(s,\tau), \tag{53}$$

where $\widehat{T}_{k\tau}$ is a shift operator in the plane s to the value $k\tau$.

Equation (53) can be solved with the help of the recurrent procedure. Namely, the function $U(s,\tau)$ can be represented in the form

$$U(s,\tau) = \sum_{k=0}^{\infty} U^{(k)}(s,\tau)$$

where the functions $U^{(k)}(s,\tau)$ satisfy the following recurrent system of equations

$$\begin{aligned} &\sum_{j=0}^{m} \widehat{a}_{j0} \left(\left(\frac{\partial}{\partial s}\right)^{-1} \frac{\partial}{\partial \tau}\right)^j U^{(0)} = F, \\ &\sum_{j=0}^{m} \widehat{a}_{j0} \left(\left(\frac{\partial}{\partial s}\right)^{-1} \frac{\partial}{\partial \tau}\right)^j U^{(k)} = \\ &-\sum_{j=0}^{m} \sum_{k'+k''=k} \widehat{T}_{k'\tau}\widehat{a}_{jk} \left(\left(\frac{\partial}{\partial s}\right)^{-1} \frac{\partial}{\partial \tau}\right)^j U^{(k'')}, \quad k = 1,2,\dots . \end{aligned} \tag{54}$$

Now it is clear that one must investigate solvability of the first equation in (54) since all other equations of this system have the same operator in the left-hand side. To do this, we apply the $\partial/\partial s$-transformation to this equation and obtain

$$\left(\sum_{j=0}^{m} \widehat{a}_{j0} z^j \right) \widetilde{U}(s,z) = \widetilde{F}(s,z), \tag{55}$$

where $\widetilde{U}^{(0)}(s,z)$ and $\widetilde{F}(s,z)$ are $\partial/\partial s$-transforms of $U^{(0)}(s,\tau)$ and $F(s,\tau)$ correspondingly. For solving the latter equation we need the following condition

Condition 4.1 The family of operators

$$\widehat{H}(z) = \sum_{j=0}^{m} \widehat{a}_{j0} z^j$$

on the manifold X with conical points is invertible for any complex values of z except for a discreet set $\{z_1, z_{2,\ldots}\}$ in the complex plane $\mathbf{C}_q$. The inverse operator $R(z)$ is a ramifying analytic operator-valued function of z in $\mathbf{C}_z \setminus \{z_1, z_{2,\ldots}\}$.

Thus, to construct the resurgent solution to equation (23) in the considered particular case, one has to investigate the solvability of partial differential equations of the type (55) on one-dimensional manifolds with conical points. The next subsection is aimed at such an investigation.

4.2 Solvability of analytic family of one-dimensional problems

In the beginning of this subsection, we omit the parameter z since now the dependence of z is at the moment unessential for us. So, let us consider the ordinary differential equation

$$\widehat{H}u = f \tag{56}$$

on a one-dimensional manifold X with singular points $Y_1, \ldots, Y_N$. Certainly, we suppose, as above, that the operator $\widehat{H}$ involved into the latter equation has the form

$$\widehat{H} = \sum_{j=0}^{m} a_j(r) \left(r\frac{\partial}{\partial r} \right)^j \tag{57}$$

near each singular point Y_j of X, where r is a local coordinate on X. We suppose that the coefficients $a_j(r)$ are real-analytic functions of r near the origin. As we already know (see Section 1), the solvability of equation (56) strongly depends on the functional spaces in which this equation is considered. To introduce suitable functional spaces, we note that the manifold X can be decomposed into the union of segments l_j, which are glued together at points Y_j, $j = 1, \ldots, N$. It is evident that the equation can be considered on each segment l_j separately.

Now, let us represent some segment l_j as the segment $[0,1]$ by the choice of the local coordinate r. We use, as above, the Sobolev space $H^s_{\alpha_0,\alpha_1}(0,1)$ with the norm

$$
\begin{aligned}
\|u\|^2_{s,\alpha_0,\alpha_1} &= \int\limits_0^{1/2} r^{-2\alpha_0} \left| \left(1 - \left(r\frac{\partial}{\partial r}\right)^2\right)^{\frac{s}{2}} u(r) \right|^2 \frac{dr}{r} \\
&+ \int\limits_{1/2}^{1} (1-r)^{-2\alpha_1} \left| \left(1 - \left((1-r)\frac{\partial}{\partial r}\right)^2\right)^{\frac{s}{2}} u(r) \right|^2 \frac{dr}{1-r}.
\end{aligned}
$$

Let us now consider the operator (57) as operator in spaces

$$\widehat{H} \,:\, H^s_{\alpha_0,\alpha_1}(0,1) \rightarrow H^{s-m}_{\alpha_0,\alpha_1}(0,1). \tag{58}$$

To begin with, we shall investigate the kernel of this operator. As it is known from the theory of Fuchsian equations (see, for example, [20]; to be short, we consider the generic position), the equation

$$\widehat{H}u = 0 \tag{59}$$

has in the vicinity of the point 0 the fundamental system of solutions of the form

$$u_j^{(0)}(r) = r^{\lambda_j} v_j^{(0)}(r), \quad j = 1, \dots, m \tag{60}$$

where $v_j(r)$ are analytic functions near the origin. Similar, near the point 1 we have another fundamental system of solutions

$$u_k^{(1)}(r) = (1-r)^{\mu_k}\, v_k^{(1)}(r), \quad k = 1, \dots, m. \tag{61}$$

The numbers $\lambda_1, \dots, \lambda_m$ can be determined as the roots of the algebraic equation

$$\sum_{j=0}^{m} a_j(0)\lambda^j = 0,$$

and the similar equation can be written down for the numbers $\mu_1, \dots, \mu_m$.

Due to the existence theorems for ordinary differential equations, both the systems (60) and (61) can be continued up to systems of solutions determined on the whole segment $[0,1]$. Then it is evident that there exists a (constant) invertible matrix $\|A_{jk}\|$ such that

$$u_j^{(0)}(r) = \sum_{k=0}^{m} A_{jk} u_k^{(1)}(r). \tag{62}$$

The matrix $\|A_{jk}\|$ will be referred below as *a transition matrix.* Let us try to construct the element of the kernel of operator (58). First of all, we note that any solution to homogeneous equation (59) has the form

$$u(r) = \sum_{j=0}^{m} C_j u_j^{(0)}(r) \tag{63}$$

with arbitrary constants C_j, $j = 1, \ldots, m$. However, not all the constants C_j may not vanish if we want to construct a solution which belongs to the space $H^s_{\alpha_0,\alpha_1}(0,1)$. To describe the requirements which must be fulfilled for solution (63) to belong the space $H^s_{\alpha_0,\alpha_1}(0,1)$ at the left endpoint of the segment $[0,1]$, we suppose that[11]

$$\mathrm{Re}\lambda_1 < \mathrm{Re}\lambda_2 < \ldots < \mathrm{Re}\lambda_{m-l} < \alpha_0 < \mathrm{Re}\lambda_{m-l+1} < \mathrm{Re}\lambda_m \tag{64}$$

for some value of l (the cases $l = 0$ and $l = m + 1$ are not excluded and must be understood in the natural way), so that l is a number of λ_j's which have their real parts more than α_0. Then one can see that for the solution to belong to the required functional space, it is necessary that

$$C_j = 0,\ j = 1, \ldots, m - l$$

in (63), that is, that

$$u(r) = \sum_{j=m-l+1}^{m} C_j u_j^{(0)}(r). \tag{65}$$

Now we consider the behavior of solution (65) near the right endpoint 1 of the segment $[0,1]$. To do this, we use a transition matrix $\|A_{jk}\|$:

$$u(r) = \sum_{j=m-l+1}^{m} C_j \sum_{k=0}^{m} A_{jk} u_k^{(1)}(r) = \sum_{k=0}^{m} \left(\sum_{j=m-l+1}^{m} C_j A_{jk} \right) u_k^{(1)}(r).$$

Similar to (64), let us suppose that

$$\mathrm{Re}\mu_1 < \mathrm{Re}\mu_2 < \ldots < \mathrm{Re}\mu_n < \alpha_1 < \mathrm{Re}\mu_{n+1} < \mathrm{Re}\mu_m.$$

Thus, for the element $u(r)$ of the kernel of operator (58) we obtain

$$\sum_{j=m-l+1}^{m} A_{jk} C_j = 0,\ k = 1, \ldots, n. \tag{66}$$

We had come to the following result

[11] For simplicity, we suppose that the numbers $\mathrm{Re}\lambda_j$ are different for different values of j.

Proposition 4.1 *Let us denote by*

$$\widehat{a}_{\alpha_0\alpha_1} \; : \; \mathbf{C}^l \to \mathbf{C}^n \tag{67}$$

the finite-dimensional operator with matrix

$$\|A_{jk}\|_{k=1,\ldots,n}^{j=m-l+1,\ldots,m} .$$

Then, if the numbers α_0, α_1 do not coincide with numbers Reλ_i *and* Reμ_j *correspondingly (this case we shall name a non resonance one), then the dimension of kernels of operators* (57) *and* (67) *coincide with each other.*

From the dual considerations one can easily obtain that the cokernel of the operator (57) coincides with the cokernel of the finite-dimensional operator (67).

Let us return now to the investigation of the case of equation (55) when the operator analytically depends on a complex parameter z. In this case the operator $\widehat{a}_{\alpha_0\alpha_1}$ and, hence, the entries A_{jk} of the matrix $\|A_{jk}\|$ will analytically depend on the parameter z:

$$A_{jk} = A_{jk}(z).$$

Thus, we see that the following result is valid.

Proposition 4.2 *Condition* 4.1 *is valid if and only if:*

i) *Matrix A_{jk} is quadratic, that is $l = n$.*

ii) *The determinant*

$$\det \|A_{jk}(z)\|_{k=1,\ldots,n}^{j=m-l+1,\ldots,m} \tag{68}$$

is not identically zero in z.

Thus, the set $\{z_1, z_2, \ldots\}$ mentioned in Condition 4.1 is simply the set of zeros of analytic function (68). Certainly, all the above conditions must be valid for any segment l_j, $j = 1, \ldots, J$. In particular, the set of singularities of the inverse operator $R(z)$ (see Condition 4.1) is the union of zero sets of determinants (68) taken over all segments involved in the manifold X. Therefore, in the two-dimensional case we have reduced the problem of finding singular points of the analytic family of operators to an *algebraic equation*.

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