On the Kodaira dimension of the moduli space of $K3$ surfaces II

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Introduction

Let $\mathcal{K}_{2n}$ be the coarse moduli space of $K3$ surfaces with a primitive polarization of degree $2n$. It follows from Torelli theorem for $K3$ surfaces that $\mathcal{K}_{2n}$ is described as an arithmetic quotient $\mathcal{D}_{2n}/\Gamma_{2n}$ where $\mathcal{D}_{2n}$ is a 19-dimensional bounded symmetric domain of type IV and Γ_{2n} is an arithmetic subgroup of $O(2,19)_{\mathbf{Q}}$. Recently Borcherds [2] constructed a remarkable automorphic form Φ of weight 12 on a 26-dimensional bounded symmetric domain of type IV. By dividing Φ by a product of linear functions vanishing on $\mathcal{D}_{2n}$ and by restricting it to $\mathcal{D}_{2n}$, we have an automorphic form Φ_{2n} on $\mathcal{D}_{2n}$ with respect to Γ_{2n} (see Borcherds et al. [3]). The purpose of this note is to show:

THEOREM. *Assume that* $n = 42, 43, 51, 53, 55, 57, 59, 61, 66, 67, 69, 74, 83, 85, 105, 119$ *or* 133. *Then* Φ_{2n} *is a cusp form of weight* 19 *on* $\mathcal{D}_{2n}$ *with respect to* Γ_{2n}.

An automorphic form of weight 19 on $\mathcal{D}_{2n}$ gives a section of the canonical line bundle of the smooth locus of $\mathcal{D}_{2n}/\Gamma_{2n}$. Moreover it is known that a cusp form of weight 19 gives a global section of the canonical line bundle of a smooth model of a compactification of $\mathcal{D}_{2n}/\Gamma_{2n}$ (Bauermann [1]). Thus we have:

COROLLARY 1. *Assume that* n *is the same as above. Then the Kodaira dimension of* $\mathcal{K}_{2n} = \mathcal{D}_{2n}/\Gamma_{2n}$ *is non negative.*

If $m = n{\cdot}l^2$ for some natural number l, then $\Gamma_{2n} \supset \Gamma_{2m}$ and we have a dominant map from $\mathcal{D}_{2m}/\Gamma_{2m}$ to $\mathcal{D}_{2n}/\Gamma_{2n}$ (O'Grady [12], Kondō [6], Lemma 3.2). Hence we have:

COROLLARY 2. *Assume that* n *is the same as above and* $m = n{\cdot}l^2$ *for some* l. *Then the Kodaira dimension of* $\mathcal{K}_{2m}$ *is non negative.*

We remark that if $1 \leq n \leq 11, 17$ or 19, then $\mathcal{K}_{2n}$ is unirational and in particular its Kodaira dimension is $-\infty$. For $n \leq 4$, this is classical and for other n this was shown by Mukai [7], [8], [9]. On the other hand, in the paper [6], the author proved that if $n = p^2$ and p is a sufficiently large prime number, then $\mathcal{K}_{2n}$ is of general type, i.e. the Kodaira dimension

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of $\mathcal{K}_{2n}$ is equal to $dim\ \mathcal{K}_{2n}$ $(= 19)$. However the author did not have an effective estimate of p. Also Gritsenko [5] proved that the Kodaira dimension of the double cover $\mathcal{D}_{2n}/\bar{\Gamma}_{2n}$ of $\mathcal{D}_{2n}/\Gamma_{2n}$ is non-negative if n is not perfect square and positive if n is square free and $n > 3$ by using the lifting of Jacobi forms, where $\bar{\Gamma}_{2n} = \Gamma_{2n} \cap SO(2,19)_{\mathbf{Q}}$.

§1. Preliminaries

A *lattice* L is a free $\mathbf{Z}$-module of finite rank endowed with an integral symmetric bilinear form $\langle\ ,\ \rangle$. If L_1 and L_2 are lattices, then $L_1 \oplus L_2$ denotes the orthogonal direct sum of L_1 and L_2. An isomorphism of lattices preserving the bilinear forms is called an *isometry.* A sublattice S of L is called *primitive* if L/S is torsion free.

A lattice L is *even* if $\langle x, x\rangle$ is even for each $x \in L$. A lattice L is *non-degenerate* if the discriminant $d(L)$ of its bilinear form is non zero, and *unimodular* if $d(L) = \pm 1$. If L is a non-degenerate lattice, the *signature* of L is a pair (t_+, t_-) where $t_\pm$ denotes the multiplicity of the eigenvalues ± 1 for the quadratic form on $L \otimes \mathbf{R}$.

Let L be a non-degenerate even lattice. The bilinear form of L determines a canonical embedding $L \subset L^* = \mathrm{Hom}(L, \mathbf{Z})$. The factor group L^*/L, which is denoted by A_L, is an abelian group of order $|d(L)|$. We extend the bilinear form on L to the one on L^*, taking value in $\mathbf{Q}$, and define

$$q_L : A_L \to \mathbf{Q}/2\mathbf{Z},\ \ q_L(x + L)\ =\ \langle x, x\rangle +\ 2\mathbf{Z}\ \ (x \in L^*).$$

We call q_L the *discriminant quadratic form* of L.

We denote by H the hyperbolic lattice defined by $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ which is an even unimodular lattice of signature (1,1), and by A_m, D_n or E_l an even negative definite lattice associated with the Dynkin diagram of type A_m, D_n or E_l ($m \geq 1$, $n \geq 4$, $l =$ 6,7,8). A *root* of a lattice L is a vector r in L with $\langle r, r\rangle = -2$. For an even negative definite lattice L, we denote by $R(L)$ the sublattice of L generated by all roots in L which is called the *root sublattice* of L and isometric to a direct sum of some A_m, D_n, E_l.

Let X be a $K3$ surface. Put $L = H^2(X, \mathbf{Z})$. Then L admits a canonical structure of a lattice induced from the cup product. It is an even unimodular lattice with signature (3,19) and hence isometric to $H \oplus H \oplus H \oplus E_8 \oplus E_8$ (e.g. Nikulin [11], Theorem 1.1.1). Let h be a primitive vector of L with $\langle h, h\rangle = 2n$. Then the orthogonal complement of h in L is isometric to $L_{2n} = H \oplus H \oplus E_8 \oplus E_8 \oplus < -2n >$. Put $\Omega_{2n} = \{[\omega] \in \mathbf{P}(L_{2n} \otimes \mathbf{C}) : \langle \omega, \omega\rangle = 0, \langle \omega, \bar{\omega}\rangle > 0\}$. Then Ω_{2n} consists of two connected components. We denote by $\mathcal{D}_{2n}$ either one connected component, which is a bounded symmetric domian of type IV and of dimension 19. Let Γ_{2n} be the group of isometries of L which fix h and preserve the component $\mathcal{D}_{2n}$. Then Γ_{2n} acts on $\mathcal{D}_{2n}$ properly discontinuously, and hence by Cartan's theorem $\mathcal{D}_{2n}/\Gamma_{2n}$ has a canonical structure of normal analytic space. We call $\mathcal{D}_{2n}$ the *period space* of $K3$ surfaces with a primitive polarization of degree $2n$. It follows from Torelli theorem for $K3$ surfaces that $\mathcal{D}_{2n}/\Gamma_{2n}$ is a coarse moduli space of $K3$ surface with a primitive polarization of degree $2n$.

A *boundary component* of $\mathcal{D}_{2n}$ is a maximal connected complex analytic subset in $\bar{\mathcal{D}}_{2n} \setminus \mathcal{D}_{2n}$ where $\bar{\mathcal{D}}_{2n}$ is the topological closure of $\mathcal{D}_{2n}$ in $\{[\omega] \in \mathbf{P}(L_{2n} \otimes \mathbf{C}) : \langle \omega, \omega \rangle = 0\}$. A boundary component is called *rational* if its stabilizer subgroup of $O(L_{2n} \otimes \mathbf{R})$ is defined over $\mathbf{Q}$. It is known that the set of rational boundary components of $\mathcal{D}_{2n}$ bijectively corresponds to the set of all primitive totally isotropic sublattices of L_{2n} (Scattone [13], (2.1.7)). If E is a totally isotropic sublattice, then the corresponding rational boundary component is defined by $\mathbf{P}(E \otimes \mathbf{C}) \cap \bar{\mathcal{D}}_{2n}$. Since the signature of L_{2n} is (2,19), the dimension of a rational boundary component is either 0 or 1.

For simplicity we now assume that n is square-free. Let F be a rational boundary component and E the corresponding primitive totally isotropic sublattice. If $dim\ F = 0$, then F is unique modulo Γ_{2n} and if $dim\ F = 1$, then we have an orthogonal decomposition

$$L_{2n} = H \oplus H \oplus K$$

where $H \oplus H$ contains E and $K \simeq E^{\perp}/E$ (Scattone [13], Theorem 4.0.1, Lemma 5.2.1). Let $\{e_1, ..., e_{21}\}$ be a base of L_{2n} such that

$$(\langle e_i, e_j \rangle)_{1 \leq i,j \leq 21} = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & Q & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix}$$

where $\{e_1, e_2\}$ is a base of E and Q is the intersection matrix of K. In case $dimF = 0$, F is unique modulo Γ_{2n} and hence we may assume that E is generated by e_1. Denote by q_0 (resp. q_1) the symmetric bilinear form associated with the matrix $(\langle e_i, e_j \rangle)_{2 \leq i,j \leq 20}$ (resp. Q). Also denote by L_0 the sublattice $e_1^{\perp}/e_1$ of L_{2n}. Let $z = \Sigma z_i e_i$ be a homogeneous coordinate of $\mathbf{P}(L_{2n} \otimes \mathbf{C})$. Then we have the following unbounded realization of $\mathcal{D}_{2n}$.

(1.1) In case $dimF = 0$:

$$\mathcal{D}_{2n} \simeq \{z = (z_2, ..., z_{20}) \in \mathbf{C}^{19} : (Im(z_i)) \in C(F)\}$$

where $C(F) = \{y = (y_2, ..., y_{20}) \in \mathbf{R}^{19} : q_0(y, y) > 0, y_2 > 0\}$.

(1.2) In case $dim\ F = 1$:

$$\mathcal{D}_{2n} \simeq \{(z, w, \tau) = (z_2, z_3, ..., z_{19}, z_{20}) \in H^+ \times \mathbf{C}^{17} \times H^+ : Im(z) - Re(h_\tau(w, w)) > 0\}$$

where $h_\tau(w, w') = \{-q_1(w, \bar{w}') + q_1(w, w')\}/4Im(\tau)$ (see Kondo [6], §2).

Let f be an automorphic form on $\mathcal{D}_{2n}$ with respect to Γ_{2n}. Then f has an expansion with respect to (1.1) which is called the *Fourier expansion* of f at F:

$$f = \sum_{\rho \in \bar{C}(F) \cap L_0^*} c_\rho \cdot exp(2\pi\sqrt{-1} \cdot q_0(\rho, z))$$

where $\bar{C}(F)$ is the closure of $C(F)$. Also with respect to (1.2), f has an expansion which is called the *Fourier-Jacobi expansion* of f at F:

$$f = \sum_{m \geq 0} \theta_m(\tau, w) \cdot exp(2\pi\sqrt{-1}mz).$$

We call f a *cusp form* if for any rational boundary components the initial term of the above expansion vanishes, i.e. $c_0 \equiv 0$ if $dim\ F = 0$ and $\theta_0(\tau, w) \equiv 0$ if $dim\ F = 0$. Note that $\theta_0(\tau, w)$ does not depend on w. For more details we refer the reader to Borcherds [2], Gritsenko [5], Kondo [6].

§2. Automorphic forms on the period domain of $K3$ surfaces

In the following, as in Borcherds et al. [3], we identify $L_{2n} = H \oplus H \oplus E_8 \oplus E_8 \oplus < -2n >$ with the sublattice $H \oplus H \oplus E_8 \oplus E_8 \oplus \mathbf{Z}v$ of $II_{2,26} = H \oplus H \oplus E_8 \oplus E_8 \oplus E_8$ where v is a primitive vector in E_8 with $\langle v, v \rangle = -2n$. Let U be the orthogonal complement of v in E_8. Let $\mathcal{D}$ be the 26-dimensional bounded symmetric domain of type IV associated with $II_{2,26}$. Under the above identification, $\mathcal{D}_{2n} \subset \mathcal{D}$. In the paper [2], Example 2 of section 10, Borcherds constructed an automorphic form Φ of weight 12 on $\mathcal{D}$ with respect to the group of isometries of $II_{2,26}$ which preserve $\mathcal{D}$. The restriction of Φ on $\mathcal{D}_{2n}$ is identically 0 whenever U contains a root. So first divide Φ by a product of linear functions vanishing on the divisors of each of these roots in U, and then restrict it on $\mathcal{D}_{2n}$. Then we have an automorphic form Φ_{2n} on $\mathcal{D}_{2n}$ with respect to Γ_{2n} which has the following properties (Borcherds et al. [3], also see Borcherds [2], Theorem 13.1 and its proof):

(2.1) Φ_{2n} vanishes on the hyperplanes of $\mathcal{D}_{2n}$ each of which is orthogonal to a vector $r' \in L_{2n}^*$ with $-2 \leq \langle r', r' \rangle < 0$. Here r' is the projection of a root r of $II_{2,26}$.

(2.2) The weight of Φ_{2n} is equal to the weight (=12) of Φ plus half the number of roots of U.

Let $\{e_i\}_{1 \leq i \leq 8}$ be a set of simple roots of E_8 which satisfies the following:

$\langle e_i, e_i \rangle = -2$, $\langle e_1, e_2 \rangle = \langle e_2, e_3 \rangle = \langle e_3, e_4 \rangle = \langle e_3, e_5 \rangle = \langle e_5, e_6 \rangle = \langle e_6, e_7 \rangle = \langle e_7, e_8 \rangle = 1$ and $\langle e_i, e_j \rangle = 0$ for other i, j.

Let R be a root lattice isometric to $A_2 \oplus A_2 \oplus A_1$ or $A_3 \oplus A_1$. Note that the number of roots in R is 14. In case $R = A_2 \oplus A_2 \oplus A_1$, we consider R as one of the following sublattices of E_8:

$(a) < e_1, e_2, e_4, e_6, e_7 >$, $(b) < e_1, e_2, e_5, e_6, e_8 >$, $(c) < e_1, e_2, e_5, e_7, e_8 >$,

$(d) < e_1, e_3, e_4, e_6, e_7 >$, $(e) < e_1, e_3, e_4, e_7, e_8 >$.

In case $R = A_3 \oplus A_1$, we consider R as one of the following sublattices of E_8:

$(f) < e_1, e_2, e_3, e_6 >$, $(g) < e_1, e_2, e_3, e_7 >$, $(h) < e_1, e_2, e_3, e_8 >$,

$(i) < e_2, e_3, e_4, e_7 >$, $(j) < e_2, e_3, e_5, e_7 >$, $(k) < e_2, e_3, e_5, e_8 >$,

$(l) < e_3, e_4, e_5, e_7 >$, $(m) < e_1, e_5, e_6, e_7 >$, $(n) < e_4, e_5, e_6, e_7 >$,

$(o) < e_1, e_6, e_7, e_8 >$, $(p) < e_2, e_6, e_7, e_8 >$, $(q) < e_4, e_6, e_7, e_8 >$.

If $R =< e_\alpha, e_\beta, e_\gamma, e_\delta, e_\kappa >$ (resp. $R =< e_\alpha, e_\beta, e_\gamma, e_\delta >$), then we take a vector v as a sum

$$v = e_\lambda^* + e_\mu^* + e_\nu^* \text{ (resp. } v = e_\kappa^* + e_\lambda^* + e_\mu^* + e_\nu^*)$$

where $\{\alpha, \beta, ..., \nu\} = \{1, 2, ..., 8\}$ and e_α^* is the dual of e_α (see Bourbaki [4], Planche VII). Since E_8 is unimodular, $v \in E_8$. A direct calculation shows that the number $-\langle v, v\rangle/2$ is equal to:

(a) 61, (b) 55, (c) 67, (d) 43, (e) 66, (f) 57, (g) 69, (h) 83, (i) 59, (j) 42, (k) 53, (l) 51, (m) 85, (n) 74, (o) 133, (p) 105, (q) 119.

Since $\langle v, v\rangle/2$ is square free, v is primitive in E_8. Obviously any root in U $(= v^\perp$ in $E_8)$ is contained in R. Hence the number of roots of U is 14. Thus we have an automorphic form Φ_{2n} of weight 19 $(= 12 + 14/2)$ by (2.2).

THEOREM. Φ_{2n} *is a cusp form of weight* 19.

Proof. It suffices to see that Φ_{2n} vanishes on the top dimensional (= 1-dim.) rational boundary components of $\mathcal{D}_{2n}$. Since n is square free, the set of primitive totally isotropic sublattices of rank two corresponds to the set of isomorphy classes of even negative definite lattices K such that $L_{2n} \simeq H \oplus H \oplus K$ (Scattone [13], Theorem 5.0.2). Here totally isotropic sublattice is contained in $H \oplus H$. Since $q_K \simeq q_{L_{2n}} \simeq -q_U$, we have an even negative definite unimodular lattice N of rank 24 such that K and U are primitive sublattices of N, $K^\perp \simeq U$ and $U^\perp \simeq K$ (Nikulin [11], Corollary 1.6.2). Thus each K is obtained as the orthogonal complement of a primitive sublattice U of some N. Let $U \subset N$ be a primitive embedding. Since U contains R, N also contains a root. Such lattices N are classified into 23 isomorphism classes which are characterized by its root sublattice $R(N)$ (Niemeier [10]). Since *rank* $R(N) = 24$, there exists a root r of N whose projection r' into K^* has a negative norm : $-2 \leq \langle r', r'\rangle < 0$. Note that the totally isotropic primitive sublattice of rank 2 corresponding to K is contained in the orthogonal complement of r'. It follows from (2.1) that Φ_{2n} vanishes the hyperplane orthogonal to r'. Let

$$\Phi_{2n} = \Sigma\theta_m(\tau, w) \cdot exp(2\pi\sqrt{-1}mz)$$

be the Fourier-Jacobi expansion of Φ_{2n} at this boundary component. Since

$$\theta_0(\tau, w) = \lim_{Im(z)\to\infty} \Phi_{2n}$$

and $\theta_0(\tau, w)$ does not depend on w, the above implies that $\theta_0(\tau, w) \equiv 0$. *q.e.d.*

Remark. There are another embeddings of R into E_8. In the above we take embeddings such that the number $\langle v, v\rangle/2$ is square free. Also we can take v as a linear combination of $e_\lambda^*, e_\mu^*, e_\nu^*$ (resp. $e_\kappa^*, e_\lambda^*, e_\mu^*, e_\nu^*$) with positive coefficients. Then Φ_{2n} is a cusp form of weight 19 whenever $\langle v, v\rangle/2$ is square free and Corollaries 1, 2 hold for these n.

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