Symmetries for Borcherds products and Hirzeburch-Zagier divisors

Bernhard Heim and Atsushi Murase

Abstract. We show that the "multiplicative Hecke symmetry" holds for Borcherds products on the Hilbert modular group over a real quadratic field. We also study the action of Hecke operators on Borcherds products and Hirzebruch-Zagier divisors.

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1 Introduction

1.1

Let K be a totally real number field with integer ring $\mathcal{O}_K$. Denote by $S_k(\Gamma_K)$ the space of holomorphic cusp forms of weight k on $\Gamma_K = \mathrm{SL}_2(\mathcal{O}_K)$. When $K/\mathbb{Q}$ is a cyclic extension of prime degree, H. Saito [Sa1] introduced a subspace $\mathcal{S}_k(\Gamma_K)$ of $S_k(\Gamma_K)$ consisting of cusp forms satisfying certain *Hecke symmetries*, and studied the traces of Hecke operators acting on $\mathcal{S}_k(\Gamma_K)$. As a direct consequence of his trace formula, he gave a relation between $\mathcal{S}_k(\Gamma_K)$ and the base changes of elliptic modular forms (see also [Sa2]).

Consider the case where K is a real quadratic field. Then the base changes are Doi-Naganuma lifts or Naganuma lifts (see [DN] and [N]). In particular, Naganuma lifts can be constructed as theta lifts of holomorphic elliptic cusp forms on $\Gamma_0(d)$ of Neben type, where d is the discriminant of K (see [Z], [O1], [O2] and [RS]).

On the other hand, Harvey and Moore [HaMo] and Borcherds [Bo2] constructed the *Borcherds products* on $O(2, m)$, first introduced in [Bo1], as the exponential of regularized theta lifts of weakly holomorphic elliptic modular forms. It is natural to ask whether Borcherds products satisfy certain symmetries.

The object of this paper is to show that Hilbert modular forms over a real quadratic field obtained as Borcherds products satisfy a multiplicative analogue of Saito's Hecke symmetries. In this paper, we give two different proofs of the main results.

The first proof is an analytic one. Borcherds products are obtained, essentially, as the exponentials of regularized integrals of weakly holomorphic modular forms against the Siegel theta series ([HaMo], [Bo2]; see also [BB]). We prove Saito's Hecke symmetries for the Siegel theta series, which seems to be of independent interest. This immediately implies the desired results for the Borcherds products.

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The second proof is an arithmetic one. Recall that the divisor of a Borcherds product is a linear combination of Hirzebruch-Zagier divisors with integral coefficients. We study the action of Hecke operators on Hirzebruch-Zagier divisors, from which follows the multiplicative symmetries for Borcherds products. This proof gives us more precise informations about the action of multiplicative Hecke operators on Borcherds products.

Saito's Hecke symmetries are closely related to the *Hecke duality* introduced and studied in the first named author's paper [H] in the Siegel modular case.

Note that the Hilbert modular forms can be seen as automorphic forms on $O(2,2)$ of $\mathbb{Q}$-rank one. In the forthcoming paper, we will study symmetries for Borcherds products on $O(2,m)$ ($m \geq 2$) of $\mathbb{Q}$-rank two.

The paper is organized as follows. The main results are stated in Section 2. We first recall the definition of Hilbert modular forms and Saito's Hecke symmetries, which we call the *additive Hecke duality* in this paper. We next intorduce the notion of the *multiplicative Hecke duality* for Hilbert modular forms. After recalling the definition of Borcherds products, we state the main results of the paper. The first one (Theorem 2.2) is the multiplicative Hecke duality of Borcherds products. The second one (Theorem 2.4) is concerned with the action of multiplicative Hecke operators on Borcherds products. In Section 3, we recall the definition of Siegel theta series in the Hilbert modular case. In Section 4, we prove the additive Hecke duality of the Siegel theta series (Theorem 4.1), which directly implies Theorem 2.2. This result might be of independent interest. In Section 5, we study the action of Hecke operators on Hirzebruch-Zagier divisors. Parts of the results have been already known (see [Ge]). By using this results, we show that the action of multiplicative Hecke operators is compatible with that of the usual Hecke operators on the "input data" of Borcherds products. This seems to be closely related to one of the open problems stated in [Bo2] (Problem 16.5). It should be noted that a similar compatibility has been studied by Guerzhoy [Gue] in the case of Borcherds products on $O(2,1)$ and by Gritsenko and Nikulin [GN] in some special Borcherds products on $O(2,3)$. In Section 6, we study numerically several examples of Hilbert modular forms in the case of $K = \mathbb{Q}(\sqrt{5})$. Since Theorem 2.2 gives a necessary condition for a Hilbert modular form to be a Borcherds product, we are able to show that several Hilbert modular forms of weight 10 are not Borcherds products.

1.2 Notation

For $z \in \mathbb{C}$, put $\mathbf{e}[z] = \exp(2\pi i z)$. The upper half plane is denote by $\mathfrak{H} = \{\tau \in \mathbb{C} \mid \mathrm{Im}(\tau) > 0\}$. Let δ_{ij} be the Kronecker's delta. We put $\mathbb{C}^1 = \{z \in \mathbb{C} \mid z\overline{z} = 1\}$. For a condition C, we put

$$\delta(C) = \begin{cases} 1 & \text{if } C \text{ holds}, \\ 0 & \text{otherwise.} \end{cases}$$

2 Main results

2.1 Hilbert modular forms

Let K be a real quadratic field of discriminant d. Throughout the paper, we fix an embedding of K into $\mathbb{R}$. Let $z \mapsto z'$ be the nontrivial automorphism of $K/\mathbb{Q}$. For $z \in K$, put $\mathrm{Tr}(z) = z + z'$ and $\mathrm{N}(z) = zz'$. Let $\mathcal{O}_K$ be the integer ring of K and $\mathfrak{d}_K = \sqrt{d}\mathcal{O}_K$ the different of K. Recall that $\mathfrak{d}_K^{-1} = \{z \in K \mid \mathrm{Tr}(zw) \in \mathbb{Z} \text{ for any } w \in \mathcal{O}_K\}$. For an element z of K, we write $z \succ 0$ if z is totally positive (that is, $z > 0$ and $z' > 0$).

Define the action of $GL_2^+(K) = \{g \in \mathrm{GL}_2(K) \mid \det g \succ 0\}$ on the product $\mathfrak{H}^2$ of two copies of $\mathfrak{H}$ by

$$g \cdot (z_1, z_2) = \left(\frac{az_1 + b}{cz_1 + d}, \frac{a'z_2 + b'}{c'z_2 + d'} \right) \qquad \left(g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL_2^+(K), (z_1, z_2) \in \mathfrak{H}^2 \right).$$

Let k be an integer. For a function F on $\mathfrak{H}^2$ and $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2^+(K)$, define

$$F|_k g(z) = j(g, z)^{-k} F(g \cdot z) \qquad (z = (z_1, z_2) \in \mathfrak{H}^2),$$

where $j(g, z) = (cz_1 + d)(c'z_2 + d')$. Let $\Gamma_K = \mathrm{SL}_2(\mathcal{O}_K) \subset GL_2^+(K)$ and χ a character of Γ_K of finite order. Denote by $C_k(\Gamma_K, \chi)$ (respectively $A_k(\Gamma_K, \chi)$) the space of smooth (respectively meromorphic) functions F on $\mathfrak{H}^2$ satisfying $F|_k\gamma = \chi(\gamma)F$ for any $\gamma \in \Gamma_K$. We also let $M_k(\Gamma_K, \chi) = \{F \in A_k(\Gamma_K, \chi) \mid F \text{ is holomorphic on } \mathfrak{H}^2\}$ and $S_k(\Gamma_K, \chi) = \{F \in M_k(\Gamma_K, \chi) \mid F \text{ is cupidal}\}$. If χ is trivial, we often write $C_k(\Gamma_K), A_k(\Gamma_K), M_k(\Gamma_K)$ and $S_k(\Gamma_K)$ for $C_k(\Gamma_K, \chi), A_k(\Gamma_K, \chi), M_k(\Gamma_K, \chi)$ and $S_k(\Gamma_K, \chi)$ respectively.

2.2 Hecke duality

Suppose that the class number of K in the narrow sense is equal to one. Then the discriminat d of K is a prime number with $d \equiv 1 \pmod 4$. Let ϵ_0 be the fundamental unit of K with $\epsilon_0 > 1$. Note that $\mathrm{N}(\epsilon_0) = -1$.

Let $\mathfrak{p}$ be a prime ideal of K and fix a generator π of $\mathfrak{p}$ with $\pi \succ 0$. For $F \in C_k(\Gamma)$, define the Hecke operator by

$$(F|_k T(\mathfrak{p}))(z_1, z_2) = F(\pi z_1, \pi' z_2) + \mathrm{N}(\mathfrak{p})^{-k} \sum_{a \in \mathcal{O}_K/\pi\mathcal{O}_K} F\left(\frac{z_1 + a}{\pi}, \frac{z_2 + a'}{\pi'} \right).$$

Then $F|_k T(\mathfrak{p}) \in C_k(\Gamma_K)$. We also define the multiplicative Hecke operator on $A_k(\Gamma_K, \chi)$ by

$$(F|\mathcal{T}(\mathfrak{p}))(z_1, z_2) = F(\pi z_1, \pi' z_2) \times \prod_{a \in \mathcal{O}_K/\pi\mathcal{O}_K} F\left(\frac{z_1 + a}{\pi}, \frac{z_2 + a'}{\pi'} \right).$$

It is easy to see that $F|\mathcal{T}(\mathfrak{p}) \in C_{(\mathrm{N}(\mathfrak{p})+1)k}(\Gamma_K, \chi')$ with some character χ' of Γ_K.

We say that $F \in C_k(\Gamma_K)$ satisfies the *additive Hecke duality* if

$$F|_k T(\mathfrak{p}) = F|_k T(\mathfrak{p}')$$

holds for any prime ideal $\mathfrak{p}$ of K. We also say that $F \in A_k(\Gamma_K, \chi)$ satisfies the *multiplicative Hecke duality* if

$$F|\mathcal{T}(\mathfrak{p}) = c(\mathfrak{p}, F) \cdot F|\mathcal{T}(\mathfrak{p}')$$

holds for any prime ideal $\mathfrak{p}$ of K with a constant $c(\mathfrak{p}, F) \in \mathbb{C}^1$ depending only on $\mathfrak{p}$ and F.

2.3 Borcherds products

Denote by χ_d the quadratic Dirichlet character corresponding to $K/\mathbb{Q}$. Let $\mathcal{W}_k(d, \chi_d)$ be the space of weakly holomorphic modular forms of weight k for $\Gamma_0(d)$ of character χ_d (cf. [BB] §3). Let

$$f(\tau) = \sum_{n \gg -\infty} c_f(n)\mathbf{e}[n\tau]$$

be the Fourier expansion of $f \in \mathcal{W}_k(d, \chi_d)$ at ∞. We put

$$\widetilde{c}_f(n) = \begin{cases} c_f(n), & \text{if } d \nmid n, \\ 2c_f(n), & \text{if } d|n. \end{cases}$$

Let $\mathcal{W}_k^+(d, \chi_d)$ be the subspace of those $f \in \mathcal{W}_k(d, \chi_d)$ such that $c_f(n) = 0$ whenever $\chi_d(n) = -1$.

Let $f \in \mathcal{W}_k^+(d, \chi_d)$. We call a connected component of

$$\mathfrak{H}^2 \setminus \bigcup_{m>0, c_f(-m)\neq 0} S(m)$$

a *Weyl chamber* associated with f, where

$$S(m) = \bigcup_{\lambda \in \mathfrak{d}_K^{-1}, -\lambda\lambda' = m/d} \left\{ z = (z_1, z_2) \in \mathfrak{H}^2 \mid \lambda \mathrm{Im}(z_1) + \lambda' \mathrm{Im}(z_2) = 0 \right\}.$$

Let W be a Weyl chamber. For $\lambda \in K$, we write $(\lambda, W) > 0$ if $\lambda \mathrm{Im}(z_1) + \lambda' \mathrm{Im}(z_2) > 0$ for any $z = (z_1, z_2) \in W$. Define the *Weyl vector* corresponding to f and W by

$$\rho_{f,W} = \sum_{m>0} \widetilde{c}_f(-m) \sum_{\lambda \in R(m,W)} \frac{\lambda}{\epsilon_0^2 - 1} \in K,$$

where $R(m, W)$ is the set of $\lambda \in \mathfrak{d}_K^{-1}$ such that $-\lambda\lambda' = m/d, (\lambda, W) > 0$ and $(-\epsilon_0^{-2}\lambda, W) > 0$ (see [Br3], 3.2 and 3.3).

The theorem of Borcherds is stated as follows ([Bo2] Theorem 13.3; see also [BB] Theorem 9 and [Br3] Theorem 3.44).

THEOREM 2.1 (Borcherds). *Let $f \in \mathcal{W}_0^+(d, \chi_d)$ and assume that $\widetilde{c}_f(n) \in \mathbb{Z}$ for all $n < 0$. Then there exists a meromorphic Hilbert modular form $\Psi(z) = \Psi(z, f)$ for $\Gamma = \mathrm{SL}_2(\mathcal{O}_K)$ (with some multiplier of finite order) satisfying the following conditions.*

(i) The weight of Ψ *is equal to* $c_f(0)$*.*

(ii) The divisor of Ψ *is given by*

$$\sum_{m>0} \widetilde{c}_f(-m) HZ(m),$$

where

$$HZ(m) = \sum_{(a,b,\lambda)\in\mathbb{Z}^2\times\mathfrak{d}_K^{-1},\, ab-\lambda\lambda'=m/d} \left\{(z_1,z_2)\in\mathfrak{H}^2 \mid az_1z_2+\lambda z_1+\lambda' z_2+b=0\right\}$$

is the Hirzebruch-Zagier divisor of discriminant m *(cf. [HZ]).*

(iii) Let W *be a Weyl chamber associated to* f *and put* $N = \mathrm{Min}\{n \mid c_f(n)\neq 0\}$*. The function* Ψ *has the Borcherds product expansion*

$$\Psi(z,f) = \mathbf{e}\left[\rho_{f,W}z_1+\rho'_{f,W}z_2\right] \prod_{\nu\in\mathfrak{d}_K^{-1},\,(\nu,W)>0} \left(1-\mathbf{e}\left[\nu z_1+\nu' z_2\right]\right)^{\widetilde{c}_f(d\nu\nu')},$$

which converges uniformly for all $z=(z_1,z_2)\in W$ *with* $\mathrm{Im}(z_1)\mathrm{Im}(z_2) > |N|/d$ *outside the poles.*

2.4 Multiplicative Hecke duality for Borcherds products

One of the main results of the paper is stated as follows.

Theorem 2.2. *Let* f *and* Ψ *be as in Theorem 2.1. Then* Ψ *satisfies the multiplicative Hecke duality.*

Remark 2.3. Note that, to prove Theorem 2.2, it is sufficient to show that

$$\Psi|\mathcal{T}(\mathfrak{p}) = c(\mathfrak{p},f)\Psi|\mathcal{T}(\mathfrak{p}')$$

holds for any prime ideal $\mathfrak{p}$ dividing a prime p split in $K/\mathbb{Q}$ with $c(\mathfrak{p},f)\in\mathbb{C}^1$.

2.5 Hecke actions on Hirzeburch-Zagier divisors

For $(a,b,\lambda)\in\mathbb{Z}^2\times\mathfrak{d}_K^{-1}$, let

$$D(a,b,\lambda) = \left\{(z_1,z_2)\in\mathfrak{H}^2 \mid az_1z_2+\lambda z_1+\lambda' z_2+b=0\right\}$$

be a divisor in $\mathfrak{H}^2$. Recall that

$$HZ(m) = \sum_{\xi\in X_m} D(\xi) \qquad (m\in\mathbb{Z}_{>0}),$$

where $X_m = \left\{(a,b,\lambda)\in\mathbb{Z}^2\times\mathfrak{d}_K^{-1} \mid ab-\lambda\lambda' = m/d\right\}$. If m is not an integer, we put $X_m=\emptyset$ and $HZ(m)=0$.

For a prime divisor D of $\mathfrak{H}^2$ given by the equation $\varphi(z) = 0$ and $g \in \mathrm{GL}_2^+(K)$, put $D * g = \{z \in \mathfrak{H}^2 \mid \varphi(g \cdot z) = 0\}$. By linearlity, we define the action of $\mathrm{GL}_2^+(K)$ on the divisors of $\mathfrak{H}^2$. It is easily seen that $HZ(m)$ is Γ_K-invariant.

Let p be a prime with $p \nmid d$ and $\mathfrak{p}$ a prime ideal of K dividing p. We fix a totally positive generator π of $\mathfrak{p}$. For a Γ_K-invariant divisor D of $\mathfrak{H}^2$, put

$$D * T(\mathfrak{p}) = D * \begin{pmatrix} \pi & 0 \\ 0 & 1 \end{pmatrix} + \sum_{\mu \in \mathcal{O}_K/\mathfrak{p}} D * \begin{pmatrix} 1 & \mu \\ 0 & \pi \end{pmatrix}$$

Then the following result holds.

Theorem 2.4. *For a positive integer m, we have*

$$\begin{aligned} &HZ(m) * T(\mathfrak{p}) \\ &= \begin{cases} HZ(pm) + p \cdot HZ(p^{-1}m) & \text{if } p \text{ splits in } K/\mathbb{Q}, \\ HZ(p^2m) + p^2 \cdot HZ(p^{-2}m) & \text{if } p \text{ is inert in } K/\mathbb{Q} \text{ and } p|m, \\ HZ(p^2m) + p \cdot HZ(m) & \text{if } p \text{ is inert in } K/\mathbb{Q} \text{ and } p \nmid m. \end{cases} \end{aligned}$$

Remark 2.5. This fact in the case where $p \nmid m$ is already known (see [Ge]).

2.6 Hecke actions on Borcherds products

For a positive integer n, define the Hecke operator $T(n)$ on $\mathcal{W}_0(d, \chi_d)$ by

$$T(n)f(z) = \sum_{0<a'|n,\, aa'=n} \sum_{b=0}^{a'-1} \chi_d(a) f\left(\frac{az+b}{a'}\right)$$

for $f \in \mathcal{W}_0(d, \chi_d)$ (see [Mi], page 142). It is easily seen that, for a prime p with $p \nmid d$, $T(p)$ (respectively $T(p^2)$) leaves $\mathcal{W}_0^+(d, \chi_d)$ invariant when $\chi_d(p) = 1$ (respectively $\chi_d(p) = -1$).

Theorem 2.6. *Let p be a prime with $p \nmid d$ and $\mathfrak{p}$ a prime ideal of K dividing p. Let $f \in \mathcal{W}_0^+(d, \chi_d)$ and put*

$$f' = \begin{cases} T(p)f & \text{if } \chi_d(p) = 1, \\ T(p^2)f + pf & \text{if } \chi_d(p) = -1. \end{cases}$$

Denote by Ψ and Ψ' the Borcherds products associated with f and f' respectively. Then we have $\Psi|\mathcal{T}(\mathfrak{p}) = \gamma(\mathfrak{p}, f)\Psi'$ with $\gamma(\mathfrak{p}, f) \in \mathbb{C}^1$.

Remark 2.7. Theorem 2.2 is a direct consequence of this theorem.

Remark 2.8. For a compatibility of Borcherds products with respect to Hecke operators, see [Gue] in the $O(2, 1)$-case and [GN] for some special cases in the $O(2, 3)$-case.

3 Siegel theta series

3.1 The Grassmannian

We define an involution τ of $M_2(K)$ by

$$\tau \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}.$$

Define a $\mathbb{Q}$-vector space V by

$$V = \left\{ X \in M_2(K) \mid {}^tX = X' \right\} = \left\{ \begin{pmatrix} a & \nu \\ \nu' & b \end{pmatrix} \mid a, b \in \mathbb{Q}, \nu \in K \right\}$$

on which a quadratic form Q is given by

$$Q(X) = -2 \det X = -2ab + 2 \,\mathrm{N}(\nu) \qquad \left(X = \begin{pmatrix} a & \nu \\ \nu' & b \end{pmatrix} \in V \right).$$

Since $Q(X) = -\operatorname{tr}(X\tau(X))$, we have

$$Q(X, Y) := \frac{1}{2}\{Q(X+Y) - Q(X) - Q(Y)\} = -\operatorname{tr}(X\tau(Y)) \qquad (X, Y \in V).$$

Let

$$L = V \cap M_2(\mathcal{O}_K) = \left\{ \begin{pmatrix} a & \nu \\ \nu' & b \end{pmatrix} \mid a, b \in \mathbb{Z}, \nu \in \mathcal{O}_K \right\}$$

be a lattice of V. The dual lattice of L with respect to Q is

$$L^* = \left\{ \begin{pmatrix} a & \nu \\ \nu' & b \end{pmatrix} \mid a, b \in \mathbb{Z}, \nu \in \mathfrak{d}_K^{-1} \right\}.$$

Define an action of $G = \mathrm{SL}_2(K)$ on V by

$$g \cdot X = gX\,{}^tg' \qquad (g \in G, X \in V).$$

Then $\Gamma_K \cdot L = L$. We identify $V_\mathbb{R} = V \otimes_\mathbb{Q} \mathbb{R}$ and $V_\mathbb{C} = V \otimes_\mathbb{Q} \mathbb{C}$ with $M_2(\mathbb{R})$ and $M_2(\mathbb{C})$, on which $G_\mathbb{R} = \mathrm{SL}_2(\mathbb{R}) \times \mathrm{SL}_2(\mathbb{R})$ acts by $(g_1, g_2) \cdot X = g_1 X^t g_2$. Let $Gr(V_\mathbb{R})$ be the Grassmannian of $V_\mathbb{R}$. By definition, $Gr(V_\mathbb{R})$ is the set of two dimensional subspaces v of $V_\mathbb{R}$ on which Q is positive definite. Then $G_\mathbb{R}$ acts on $Gr(V_\mathbb{R})$ in a natural manner and the action is transitive. For $\Lambda \in V_\mathbb{R}$ and $v \in Gr(V_\mathbb{R})$, let Λ_v and $\Lambda_{v^\perp}$ be the projections of Λ on v and $v^\perp$ respectively, where $v^\perp$ denotes the orthogonal complement of v with respect to Q. We have $Q(\Lambda) = Q(\Lambda_v) + Q(\Lambda_{v^\perp})$.

Set

$$\begin{aligned} \mathcal{V} &= \{[Z] \in \mathbb{P}(V_\mathbb{C}) \mid Q(Z) = 0, Q(Z, \overline{Z}) > 0\} \\ &= \{[Z] \in \mathbb{P}(M_2(\mathbb{C})) \mid \det Z = 0, -\operatorname{tr}(Z\overline{\tau(Z)}) > 0\}, \end{aligned}$$

where $\overline{Z}$ is the complex conjugate of $Z \in V_{\mathbb{C}}$. Define an action of $G_{\mathbb{R}} = \mathrm{SL}_2(\mathbb{R}) \times \mathrm{SL}_2(\mathbb{R})$ on $\mathcal{V}$ by

$$(g_1, g_2) \cdot [Z] = [g_1 Z^t g_2] \qquad (g_1, g_2 \in \mathrm{SL}_2(\mathbb{R}), [Z] \in \mathcal{V}).$$

It is well-known that $\mathcal{V}$ is a disjoint union of two connected components $\mathcal{V}^+$ and $\mathcal{V}^-$, and $G_{\mathbb{R}}$ acts on $\mathcal{V}^{\pm}$ transitively. Henceforth we choose $\mathcal{V}^+$ so that $\begin{pmatrix} -1 & i \\ i & 1 \end{pmatrix} \in \mathcal{V}^+$. Note that, for $Z = X + iY \in M_2(\mathbb{C})$ $(X, Y \in M_2(\mathbb{R}))$, $[Z] \in \mathcal{V}$ if and only if $Q(X) = Q(Y) > 0$ and $Q(X, Y) = 0$. We put $\eta(Z) = Q(X) = Q(Y)$.

Lemma 3.1. *(i) The mapping $Z = X + iY \mapsto v_Z = \mathbb{R}X + \mathbb{R}Y$ gives rise to a bijection between $\mathcal{V}^+$ and $Gr(V_{\mathbb{R}})$.*

(ii) For $\Lambda \in V_{\mathbb{R}}$ and $Z \in \mathcal{V}^+$, we have

$$\Lambda_{v_Z} = \eta(Z)^{-1} \{Q(\Lambda, X)X + Q(\Lambda, Y)Y\}$$

and

$$Q(\Lambda_{v_Z}) = \frac{|Q(\Lambda, Z)|^2}{\eta(Z)}.$$

For $z = (z_1, z_2) \in \mathbb{C}^2$, put

$$M(z) = \begin{pmatrix} z_1 z_2 & z_1 \\ z_2 & 1 \end{pmatrix}.$$

It is easily seen that $\mathcal{V}^+ = \{[M(z)] \mid z = (z_1, z_2) \in \mathfrak{H}^2\}$ and that $g \cdot M(z) = j(g, z)M(g \cdot z)$ for $g = (g_1, g_2) \in \mathrm{SL}_2(\mathbb{R}) \times \mathrm{SL}_2(\mathbb{R})$ and $z = (z_1, z_2) \in \mathfrak{H}^2$, where $j(g, z) = j(g_1, z_1)j(g_2, z_2)$ and $g \cdot z = ((a_i z_i + b_i)/(c_i z_i + d_i))_{i=1,2}$. We also have $\eta(M(z)) = 2\mathrm{Im}(z_1)\mathrm{Im}(z_2)$.

3.2 Siegel theta series

For $\alpha \in L^*/L, \tau \in \mathfrak{H}$ and $v \in Gr(V_{\mathbb{R}})$, we define

$$\Theta_\alpha(\tau, v) = \sum_{\Lambda \in \alpha + L} \mathbf{e}\left[\frac{\tau}{2} Q(\Lambda_v) + \frac{\overline{\tau}}{2} Q(\Lambda_{v^\perp})\right]. \tag{3.1}$$

Since $\gamma \cdot \Lambda \equiv \Lambda \pmod{L}$ for $\Lambda \in L^*$ and $\gamma \in \Gamma_K$, $v \mapsto \Theta_\alpha(\tau, v)$ is Γ_K-invariant. For $z \in \mathfrak{H}^2$, let v_z be the element of $Gr(V_{\mathbb{R}})$ corresponding to $[M(z)] \in \mathcal{V}^+$ by Lemma 3.1. By abuse of notation, we write $\Theta_\alpha(\tau, z)$ for $\Theta_\alpha(\tau, v_z)$. Henceforth we often identify $\mathfrak{d}_K^{-1}/\mathcal{O}_K$ with L^*/L via $\alpha \mapsto \begin{pmatrix} 0 & \alpha \\ \alpha' & 0 \end{pmatrix}$. The following is a straightforward consequence of Lemma 3.1.

Lemma 3.2. *We have*

$$\Theta_\alpha(\tau, (z_1, z_2)) = \sum_{m,n \in \mathbb{Z}, \lambda \in \alpha + \mathcal{O}_K} \mathbf{e}\left[\frac{i\,\mathrm{Im}(\tau)}{2y_1 y_2} \left|m z_1 z_2 + \lambda z_1 + \lambda' z_2 + n\right|^2 + \overline{\tau}(-mn + \lambda\lambda')\right],$$

where $y_i = \mathrm{Im}(z_i)$.

LEMMA 3.3. *(i) We have* $\Theta_\alpha(\tau, \gamma\cdot z) = \Theta_\alpha(\tau, z)$ *for* $\gamma \in \Gamma_K$.

(ii) We have $\Theta_\alpha = \Theta_{-\alpha} = \Theta_{\alpha'}$.

(iii) We have $\Theta_\alpha(\tau, (z_2, z_1)) = \Theta_\alpha(\tau, (z_1, z_2))$.

PROOF. We have already proved the first assertion. The second one follows from Lemma 3.1 and the fact that $\alpha - \alpha' \in \mathcal{O}_K$ for $\alpha \in \mathfrak{d}_K^{-1}$. By Lemma 3.2, we have $\Theta_\alpha(\tau, (z_2, z_1)) = \Theta_{\alpha'}(\tau, (z_1, z_2)) = \Theta_\alpha(\tau, (z_1, z_2))$. $\square$

For $\alpha \in \mathfrak{d}_K^{-1}/\mathcal{O}_K, z = (z_1, z_2) \in \mathfrak{H}^2, m, n \in \mathbb{Z}$, we put

$$\begin{aligned}\vartheta_\alpha(\tau, z; m, n) = \sum_{\lambda\in\alpha+\mathcal{O}_K} \mathbf{e}\Big[&\frac{i\mathrm{Im}(\tau)}{2y_1y_2}\left\{m(x_1y_2 + x_2y_1) + \lambda y_1 + \lambda' y_2\right\}^2 \\ &+ (m\overline{\tau} + n)\left(mx_1x_2 + \lambda x_1 + \lambda' x_2\right) + \overline{\tau}\lambda\lambda'\Big],\end{aligned}$$

where $x_i = \mathrm{Re}(z_i), y_i = \mathrm{Im}(z_i)$ $(i = 1, 2)$. In the next section, we need the following result due to Borcherds (see [Bo2]).

PROPOSITION 3.4. *We have*

$$\Theta_\alpha(\tau, z) = \sqrt{\frac{y_1y_2}{\mathrm{Im}(\tau)}} \sum_{m,n\in\mathbb{Z}} \mathbf{e}\left[\frac{iy_1y_2}{2\mathrm{Im}(\tau)}|m\tau + n|^2\right] \vartheta_\alpha(\tau, z; m, n).$$

For completeness, we give a sketch of the proof of the proposition.

LEMMA 3.5. *We have*

$$\begin{aligned}\Theta_\alpha(\tau, (z_1, z_2)) = \sqrt{\frac{y_1y_2}{\mathrm{Im}(\tau)}} \sum_{m,n\in\mathbb{Z},\lambda\in\alpha+\mathcal{O}_K} \mathbf{e}\Bigg[&\frac{i\mathrm{Im}(\tau)}{2y_1y_2}\left|mz_1z_2 + \lambda z_1 + \lambda' z_2\right|^2 + \overline{\tau}\lambda\lambda' \\ &+ \frac{iy_1y_2}{2\mathrm{Im}(\tau)}\left\{m\overline{\tau} + n - \frac{i\mathrm{Im}(\tau)}{y_1y_2}\left(m(x_1x_2 - y_1y_2) + \lambda x_1 + \lambda' x_2\right)\right\}^2\Bigg]\end{aligned}$$

PROOF. We have

$$\begin{aligned}&\Theta_\alpha(\tau, z) \\ &= \sum_{m,n\in\mathbb{Z},\lambda\in\alpha+\mathcal{O}_K} \mathbf{e}\left[\frac{i\mathrm{Im}(\tau)}{2y_1y_2}\left|n + \lambda z_1 + \lambda' z_2 + mz_1z_2\right|^2 + \overline{\tau}(-mn + \lambda\lambda')\right] \\ &= \sum_{m\in\mathbb{Z},\lambda\in\alpha+\mathcal{O}_K} \mathbf{e}\left[\frac{i\mathrm{Im}(\tau)}{2y_1y_2}\left|\lambda z_1 + \lambda' z_2 + mz_1z_2\right|^2 + \overline{\tau}\lambda\lambda'\right] \sum_{n\in\mathbb{Z}} f(n),\end{aligned}$$

where

$$\begin{aligned}&f(u) = \mathbf{e}[iAu^2 + Bu], \\ &\quad A = \frac{\mathrm{Im}(\tau)}{2y_1y_2},\ B = -m\overline{\tau} + \frac{i\mathrm{Im}(\tau)}{y_1y_2}\left(\lambda x_1 + \lambda' x_2 + m(x_1x_2 - y_1y_2)\right).\end{aligned}$$

Observe that the Fourier transform of f is equal to $\sqrt{2A}^{-1}\mathbf{e}\left[\frac{i}{4A}(u+B)^2\right]$. Then Poisson summation formula implies

$$\Theta_\alpha(\tau,z)=\sqrt{\frac{y_1y_2}{\mathrm{Im}(\tau)}}\sum_{m,n\in\mathbb{Z},\lambda\in\alpha+\mathcal{O}_K}\mathbf{e}\left[\frac{i\mathrm{Im}(\tau)}{2y_1y_2}\left|mz_1z_2+\lambda z_1+\lambda' z_2\right|^2+\overline{\tau}\lambda\lambda'\right.$$
$$\left.+\frac{y_1y_2}{2\mathrm{Im}(\tau)}\left\{n-m\overline{\tau}+\frac{i\mathrm{Im}(\tau)}{y_1y_2}\left(m(x_1x_2-y_1y_2)+\lambda x_1+\lambda' x_2\right)\right\}^2\right].$$

Changing (m,λ) into $(-m,-\lambda)$, we get the lemma (note that $\Theta_{-\alpha}=\Theta_\alpha$). $\square$

Proposition 3.4 is derived from Lemma 3.5 by a tedious but straightforward calculation.

3.3 Regularized theta integrals and Borcherds products

For $f\in\mathcal{W}_0^+(d,\chi_d)$, put

$$\Phi(z,f,s)=\lim_{t\to\infty}\int_{\mathcal{F}_t}\langle f(\tau),\overline{\Theta_0(\tau,z)}\rangle v^{-s}\frac{dudv}{v^2}, \tag{3.2}$$

where $\mathcal{F}_t=\{\tau=u+iv\in\mathfrak{H}\mid |\tau|\geq 1, |u|\leq\frac{1}{2}, |v|\leq t\}$ is the truncated fundamental domain and $\langle,\rangle$ is the pairing given in [Br3] Proposition 3.32. The limit (3.2) exists for $\mathrm{Re}(s)\gg 0$ and is continued to a meromorphic function of s on $\mathbb{C}$. The regularized theta integral $\Phi(z,f)$ is defined to be the constant term in the Laurent expansion of $\Phi(z,f,s)$ at $s=0$. Then the Borcherds product $\Psi(z,f)$ satisfies

$$\Phi(z,f)=-4\log\left|\Psi(z,f)(\mathrm{Im}(z_1)\mathrm{Im}(z_2))^{c_f(0)/2}\right|-2c_f(0)\left(\log(2\pi)+\Gamma'(1)\right) \tag{3.3}$$

in the domain of convergence of the infinite product for $\Psi(z,f)$ in Theorem 2.1 (iii).

4 Hecke duality for Siegel theta series

4.1

From now on, we assume that the class number of K in the narrow sense is equal to one, and fix a $\mathbb{Z}$-basis $\{1,\omega\}$ of $\mathcal{O}_K$. Put $s=\mathrm{Tr}(\omega)$ and $t=\mathrm{N}(\omega)$. Let $p=\mathfrak{p}\mathfrak{p}'$ be a prime split in $K/\mathbb{Q}$. Take and fix a totally positive element π of $\mathcal{O}_K$ such that $\mathfrak{p}=\pi\mathcal{O}_K$. Then $p=\pi\pi'$. In this section, we prove the following result:

Theorem 4.1. *For any $\alpha\in\mathfrak{d}_K^{-1}/\mathcal{O}_K$, we have*

$$\Theta_\alpha(*,z)|T(\mathfrak{p})=\Theta_\alpha(*,z)|T(\mathfrak{p}').$$

Remark 4.2. In view of (3.3), this theorem implies Theorem 2.2.

4.2

The following result follows from Proposition 3.4.

LEMMA 4.3. *We have*

$$\Theta_\alpha(\tau,z)|T(\mathfrak{p}) = \sqrt{\frac{py_1y_2}{\mathrm{Im}(\tau)}}\left\{\sum_{m,n\in\mathbb{Z}} \mathbf{e}\left[p\,\frac{iy_1y_2}{2\mathrm{Im}(\tau)}|m\tau+n|^2\right] I_+^{\uparrow}(\alpha,\tau,z;m,n)\right.$$
$$\left. +p^{-1}\sum_{m,n\in\mathbb{Z}} \mathbf{e}\left[p^{-1}\,\frac{iy_1y_2}{2\mathrm{Im}(\tau)}|m\tau+n|^2\right] I_-^{\uparrow}(\alpha,\tau,z;m,n)\right\}$$

and

$$\Theta_\alpha(\tau,z)|T(\mathfrak{p}') = \sqrt{\frac{py_1y_2}{\mathrm{Im}(\tau)}}\left\{\sum_{m,n\in\mathbb{Z}} \mathbf{e}\left[p\,\frac{iy_1y_2}{2\mathrm{Im}(\tau)}|m\tau+n|^2\right] I_+^{\downarrow}(\alpha,\tau,z;m,n)\right.$$
$$\left. +p^{-1}\sum_{m,n\in\mathbb{Z}} \mathbf{e}\left[p^{-1}\,\frac{iy_1y_2}{2\mathrm{Im}(\tau)}|m\tau+n|^2\right] I_-^{\downarrow}(\alpha,\tau,z;m,n)\right\},$$

where

$$\begin{aligned}
I_+^{\uparrow}(\alpha,\tau,z;m,n) &= \vartheta_\alpha(\tau,(\pi z_1,\pi' z_2);m,n),\\
I_+^{\downarrow}(\alpha,\tau,z;m,n) &= \vartheta_\alpha(\tau,(\pi' z_1,\pi z_2);m,n),\\
I_-^{\uparrow}(\alpha,\tau,z;m,n) &= \sum_{c=0}^{p-1}\vartheta_\alpha\left(\tau,\left(\frac{z_1+c}{\pi},\frac{z_2+c}{\pi'}\right);m,n\right),\\
I_-^{\downarrow}(\alpha,\tau,z;m,n) &= \sum_{c=0}^{p-1}\vartheta_\alpha\left(\tau,\left(\frac{z_1+c}{\pi'},\frac{z_2+c}{\pi}\right);m,n\right).
\end{aligned}$$

The proof of Theorem 4.1 is now reduced to that of the following:

THEOREM 4.4. *(i) If $p|m$ and $p|n$, we have*

$$\begin{aligned}
I_-^{\uparrow}(\alpha,\tau,z;m,n) &= p\,I_+^{\downarrow}(\alpha,\tau,z;p^{-1}m,p^{-1}n),\\
I_-^{\downarrow}(\alpha,\tau,z;m,n) &= p\,I_+^{\uparrow}(\alpha,\tau,z;p^{-1}m,p^{-1}n).
\end{aligned}$$

(ii) If $p\nmid m$ or $p\nmid n$, we have

$$I_-^{\uparrow}(\alpha,\tau,z;m,n) = I_-^{\downarrow}(\alpha,\tau,z;m,n).$$

4.3

In what follows, we fix $\alpha\in\mathfrak{d}_K^{-1}/\mathcal{O}_K,\tau\in\mathfrak{H}$ and $z=(z_1,z_2)\in\mathfrak{H}^2$, and put $x_i=\mathrm{Re}(z_i), y_i=\mathrm{Im}(z_i)$. We write $I_\pm^{\updownarrow}(m,n)$ for $I_\pm^{\updownarrow}(\alpha,\tau,z;m,n)$ to simplify the notation. Let $\pi=a+b\omega$. Then $a^2+sab+tb^2=p$ and $p\not| b$. To prove Theorem 4.4, we need the following results.

Lemma 4.5. *We have*

$$I_+^{\uparrow}(m,n)=\sum_{\lambda\in\alpha+\mathcal{O}_K}\mathbf{e}\Big[p^{-1}\frac{i\mathrm{Im}(\tau)}{2y_1y_2}\left\{\pi\lambda y_1+\pi'\lambda' y_2+mp(x_1y_2+x_2y_1)\right\}^2 + (m\overline{\tau}+n)\left(\lambda\pi x_1+\lambda'\pi' x_2+mpx_1x_2\right)+\overline{\tau}\lambda\lambda'\Big], \tag{4.1}$$

$$I_+^{\downarrow}(m,n)=\sum_{\lambda\in\alpha+\mathcal{O}_K}\mathbf{e}\Big[p^{-1}\frac{i\mathrm{Im}(\tau)}{2y_1y_2}\left\{\pi'\lambda y_1+\pi\lambda' y_2+mp(x_1y_2+x_2y_1)\right\}^2 + (m\overline{\tau}+n)\left(\lambda\pi' x_1+\lambda'\pi x_2+mpx_1x_2\right)+\overline{\tau}\lambda\lambda'\Big], \tag{4.2}$$

$$\begin{aligned} I_-^{\uparrow}(m,n) &= \sum_{\lambda\in\alpha+\mathcal{O}_K}\sum_{c=0}^{p-1}\mathbf{e}\Big[p\frac{i\mathrm{Im}(\tau)}{2y_1y_2}\left\{\left(\frac{\lambda}{\pi}+\frac{mc}{p}\right)y_1+\left(\frac{\lambda'}{\pi'}+\frac{mc}{p}\right)y_2+\frac{m}{p}(x_1y_2+x_2y_1)\right\}^2 \\ &+(m\overline{\tau}+n)\left(\left(\frac{\lambda}{\pi}+\frac{mc}{p}\right)x_1+\left(\frac{\lambda'}{\pi'}+\frac{mc}{p}\right)x_2+\frac{m}{p}x_1x_2+\frac{\lambda}{\pi}c+\frac{\lambda'}{\pi'}c+\frac{m}{p}c^2\right)+\overline{\tau}\lambda\lambda'\Big], \end{aligned} \tag{4.3}$$

$$\begin{aligned} I_-^{\downarrow}(m,n) &= \sum_{\lambda\in\alpha+\mathcal{O}_K}\sum_{c=0}^{p-1}\mathbf{e}\Big[p\frac{i\mathrm{Im}(\tau)}{2y_1y_2}\left\{\left(\frac{\lambda}{\pi'}+\frac{mc}{p}\right)y_1+\left(\frac{\lambda'}{\pi}+\frac{mc}{p}\right)y_2+\frac{m}{p}(x_1y_2+x_2y_1)\right\}^2 \\ &+(m\overline{\tau}+n)\left(\left(\frac{\lambda}{\pi'}+\frac{mc}{p}\right)x_1+\left(\frac{\lambda'}{\pi}+\frac{mc}{p}\right)x_2+\frac{m}{p}x_1x_2+\frac{\lambda}{\pi'}c+\frac{\lambda'}{\pi}c+\frac{m}{p}c^2\right)+\overline{\tau}\lambda\lambda'\Big]. \end{aligned} \tag{4.4}$$

Proof. The lemma is immediate from the definitions. □

Lemma 4.6. *Let $\lambda\in\mathfrak{d}_K^{-1}$. Then $p|\operatorname{Tr}(\pi'\lambda)$ (respectively $p|\operatorname{Tr}(\pi\lambda)$) if and only if $\lambda\in\pi\mathfrak{d}_K^{-1}$ (respectively $\lambda\in\pi'\mathfrak{d}_K^{-1}$).*

Proof. Let $\lambda=x+y\theta$. Suppose that $p|\operatorname{Tr}(\pi'\lambda)$. Then

$$\pi'\lambda=\frac{1}{\sqrt{d}}\left\{(a+sb)x+tby+(-bx+ay)\omega\right\}$$

and hence $p|(-bx+ay)$. Since

$$b\left\{(a+sb)x+tby\right\}=(a+sb)(bx-ay)+py\equiv 0 \pmod p$$

and $p\nmid b$, we have $p|((a+sb)x+tby)$. This implies $\pi'\lambda\in p\mathfrak{d}_K^{-1}$ and hence $\lambda\in\pi\mathfrak{d}_K^{-1}$. The other assertion is similarly proved. □

Lemma 4.7. *There exists an element β of $\mathfrak{d}_K^{-1}$ such that $\alpha-\pi\beta,\alpha-\pi'\beta\in\mathcal{O}_K$. Furthermore we have $(\alpha+\mathcal{O}_K)\cap\pi\mathfrak{d}_K^{-1}=\pi(\beta+\mathcal{O}_K)$ and $(\alpha+\mathcal{O}_K)\cap\pi'\mathfrak{d}_K^{-1}=\pi'(\beta+\mathcal{O}_K)$.*

PROOF. Since $\mathfrak{d}_K + \pi\mathcal{O}_K = \mathcal{O}_K$, we have $\mathcal{O}_K + \pi\mathfrak{d}_K^{-1} = \mathfrak{d}_K^{-1}$. Thus there exists a $\beta \in \mathfrak{d}_K^{-1}$ such that $\alpha - \pi\beta \in \mathcal{O}_K$. Since $\pi - \pi' \in \mathfrak{d}_K$, we have $\alpha - \pi'\beta = \alpha - \pi\beta + (\pi - \pi')\beta \in \mathcal{O}_K$. It is easily seen that $(\alpha + \mathcal{O}_K) \cap \pi\mathfrak{d}_K^{-1} \supset \pi(\beta + \mathcal{O}_K)$. Let $\lambda \in (\alpha + \mathcal{O}_K) \cap \pi\mathfrak{d}_K^{-1}$. Then $\lambda - \pi\beta = \lambda - \alpha + (\alpha - \pi\beta \in \mathcal{O}_K)$ and hence $\pi^{-1}(\lambda - \pi\beta) \in \pi^{-1}\mathcal{O}_K \cap \mathfrak{d}_K^{-1} = \mathcal{O}_K$. This implies that $\lambda \in \pi(\beta + \mathcal{O}_K)$ and hence we have proved $(\alpha + \mathcal{O}_K) \cap \pi\mathfrak{d}_K^{-1} = \pi(\beta + \mathcal{O}_K)$. The proof of the remaining assertion is similar. $\square$

4.4 The case where $p|m$

In this subsection, we assume that $p|m$ and prove Theorem 4.4 in this case. We put $m' = p^{-1}m$.

LEMMA 4.8. *We have*

$$I_-^{\uparrow}(m,n) = \sum_{\lambda \in \alpha + \mathcal{O}_K} \mathbf{e}\Big[p\,\frac{i\mathrm{Im}(\tau)}{2y_1y_2}\left\{\frac{\lambda}{\pi}y_1 + \frac{\lambda'}{\pi'}y_2 + \frac{m}{p}(x_1y_2 + x_2y_1)\right\}^2 + (m\overline{\tau} + n)\left(\frac{\lambda}{\pi}x_1 + \frac{\lambda'}{\pi'}x_2 + \frac{m}{p}x_1x_2\right) + \overline{\tau}\lambda\lambda'\Big]\sum_{c=0}^{p-1}\mathbf{e}\left[\frac{cn}{p}\,\mathrm{Tr}(\pi'\lambda)\right],$$

$$I_-^{\downarrow}(m,n) = \sum_{\lambda \in \alpha + \mathcal{O}_K} \mathbf{e}\Big[p\,\frac{i\mathrm{Im}(\tau)}{2y_1y_2}\left\{\frac{\lambda}{\pi'}y_1 + \frac{\lambda'}{\pi}y_2 + \frac{m}{p}(x_1y_2 + x_2y_1)\right\}^2 + (m\overline{\tau} + n)\left(\frac{\lambda}{\pi'}x_1 + \frac{\lambda'}{\pi}x_2 + \frac{m}{p}x_1x_2\right) + \overline{\tau}\lambda\lambda'\Big]\sum_{c=0}^{p-1}\mathbf{e}\left[\frac{cn}{p}\,\mathrm{Tr}(\pi\lambda)\right].$$

PROOF. The first assertion is proved by changing λ into $\lambda - \pi m'c$ in (4.3). The remaining one is similarly proved. $\square$

First suppose that $p|n$ and put $n' = p^{-1}n$. By Lemma 4.8, we obtain

$$\begin{aligned}
&I_-^{\uparrow}(m,n)\\
&= p\sum_{\lambda \in \alpha + \mathcal{O}_K}\mathbf{e}\Big[p^{-1}\,\frac{i\mathrm{Im}(\tau)}{2y_1y_2}\left\{\lambda(\pi'y_1) + \lambda'(\pi y_2) + m'\left((\pi'x_1)(\pi y_2) + (\pi x_2)(\pi' y_1)\right)\right\}^2\\
&\qquad + (m'\overline{\tau} + n')\left(\lambda(\pi'x_1) + \lambda'(\pi x_2) + m'(\pi'x_1)(\pi x_2)\right) + \overline{\tau}\lambda\lambda'\Big]\\
&= p\,I_+^{\downarrow}(m,n),
\end{aligned}$$

which proves the first assertion of Theorem 4.4 (i). The second one is similarly proved.

Next suppose that $p \nmid n$. Let β be an element of $\mathfrak{d}_K^{-1}$ satisfying $\alpha - \pi\beta, \alpha - \pi'\beta \in \mathcal{O}_K$ (cf. Lemma 4.7). Lemma 4.6 and Lemma 4.7 imply that, for $\lambda \in \alpha + \mathcal{O}_K$,

$$\sum_{c=0}^{p-1}\mathbf{e}\left[\frac{cn}{p}\,\mathrm{Tr}(\pi'\lambda)\right] = \begin{cases} p & \text{if } \lambda \in \pi(\beta + \mathcal{O}_K),\\ 0 & \text{otherwise.}\end{cases}$$

Then, by Lemma 4.8, we obtain

$$(4.5) \qquad I_-^{\uparrow}(m,n) = \sum_{\mu \in \beta + \mathcal{O}_K} \mathbf{e}\Big[p \frac{i\mathrm{Im}(\tau)}{2y_1y_2}\left\{\mu y_1 + \mu' y_2 + \frac{m}{p}(x_1y_2 + x_2y_1)\right\}^2 + (m\overline{\tau} + n)\left(\mu x_1 + \mu' x_2 + \frac{m}{p}x_1x_2\right) + p\overline{\tau}\mu\mu'\Big].$$

A similar argument shows that $I_-^{\downarrow}(m,n)$ is equal to the right-hand side of (4.5), which proves Theorem 4.4 (ii) in the case where $p|m$ and $p \nmid n$.

4.5 The case where $p \nmid m$

In this subsection, we assume that $p \nmid m$ and prove the remaining part of Theorem 4.4 (ii). Let β be as in Lemma 4.7. The following is easily verified.

LEMMA 4.9. *For a function φ on K, we have*

$$\sum_{\lambda \in \alpha + \mathcal{O}_K} \sum_{c=0}^{p-1} \varphi\left(\frac{\lambda}{\pi} + \frac{mc}{p}\right) = \sum_{\mu \in p\beta + \mathcal{O}_K} \varphi\left(\frac{\mu}{p}\right).$$

LEMMA 4.10. *For $\mu \in \mathfrak{d}_K^{-1}$, there uniquely exist $c(\pi,\mu) \in \mathbb{Z}$ with $0 \leq c(\pi,\mu) \leq p-1$ and $z(\pi,\mu) \in \mathfrak{d}_K^{-1}$ such that*

$$\mu = \pi' z(\pi,\mu) + mc(\pi,\mu).$$

PROOF. Since $\mathrm{Tr}(\pi)^2 = (2a+sb)^2 = 4p + db^2$, $\mathrm{Tr}(\pi)$ is not divisible by p. It follows that there exists an integer c satisfying $0 \leq c \leq p-1$ and $p|\mathrm{Tr}(\pi(\mu - mc))$. Then, by Lemma 4.6, we have $\mu - mc \in \pi'\mathfrak{d}_K^{-1}$, which proves the existence of $c(\pi,\mu)$ and $z(\pi,\mu)$. The uniqueness is clear. □

LEMMA 4.11. *For $\mu \in \mathfrak{d}_K^{-1}$, we have*

$$c(\pi,\mu)\,\mathrm{Tr}(\mu) - mc(\pi,\mu)^2 \equiv c(\pi',\mu)\,\mathrm{Tr}(\mu) - mc(\pi',\mu)^2 \pmod{p}.$$

PROOF. Put $c_1 = c(\pi,\mu)$ and $c_2 = c(\pi',\mu)$. We have $m\,\mathrm{Tr}(\pi)c_1 \equiv \mathrm{Tr}(\pi\mu) \pmod{p}$ and $m\,\mathrm{Tr}(\pi)c_2 \equiv \mathrm{Tr}(\pi'\mu) \pmod{p}$. Then

$$\begin{aligned}
&\mathrm{Tr}(\pi)\left\{(c_1\,\mathrm{Tr}(\mu) - mc_1^2) - (c_2\,\mathrm{Tr}(\mu) - mc_2^2)\right\} \\
&= (c_1 - c_2)\left\{\mathrm{Tr}(\pi)\,\mathrm{Tr}(\mu) - m\,\mathrm{Tr}(\pi)c_1 - m\,\mathrm{Tr}(\pi)c_2\right\} \\
&\equiv (c_1 - c_2)\left\{\mathrm{Tr}(\pi)\,\mathrm{Tr}(\mu) - \mathrm{Tr}(\pi\mu) - \mathrm{Tr}(\pi'\mu)\right\} \pmod{p} \\
&= 0,
\end{aligned}$$

which proves the lemma since $p \not| \mathrm{Tr}(\pi)$. □

We are now ready to prove Theorem 4.4 (ii). By Lemmas 4.5, 4.9, 4.10, we have

$$I_-^{\uparrow}(m,n) = \sum_{\mu \in p\beta + \mathcal{O}_K} f_{\tau,z,m,n}(\mu)\, \mathbf{e}\left[\frac{n}{p}\left\{c(\pi,\mu)\operatorname{Tr}(\mu) - mc(\pi,\mu)^2\right\}\right],$$
$$I_-^{\downarrow}(m,n) = \sum_{\mu \in p\beta + \mathcal{O}_K} f_{\tau,z,m,n}(\mu)\, \mathbf{e}\left[\frac{n}{p}\left\{c(\pi',\mu)\operatorname{Tr}(\mu) - mc(\pi',\mu)^2\right\}\right],$$

where

$$f_{\tau,z,m,n}(\mu) = \mathbf{e}\Big[p^{-1}\frac{i\mathrm{Im}(\tau)}{2y_1y_2}\left(\mu y_1 \mu' y_2 + m(x_1y_2 + x_2y_1)\right) \\ + p^{-1}(m\overline{\tau} + n)\left(\mu x_1 + \mu' x_2 + m x_1 x_2\right) + p^{-1}\overline{\tau}\mu\mu'\Big].$$

Then Lemma 4.11 implies that $I_-^{\uparrow}(m,n) = I_-^{\downarrow}(m,n)$, which completes the proof of Theorem 4.4.

5 Hecke action on Hirzebruch-Zagier divisors and Borcherds products

5.1

We first show that Theorem 2.4 implies Theorem 2.6. Let $f \in \mathcal{W}_0^+(d,\chi_d)$. Hereafter we make a convention that $\widetilde{c}_f(n) = 0$ if n is not an integer.

Assume that $p = \mathfrak{p}\mathfrak{p}'$ is a prime split in $K/\mathbb{Q}$. Recall that

$$f'(z) = T(p)f(z) = f(pz) + \sum_{a=0}^{p-1} f\left(\frac{z+a}{p}\right).$$

Since $\widetilde{c}_{f'}(n) = p\,\widetilde{c}_f(p\,n) + \widetilde{c}_f(p^{-1}n)$, the divisor of Ψ' is given by

$$\sum_{m>0} \widetilde{c}_{f'}(-m)HZ(m) = \sum_{m>0}\left\{p\widetilde{c}_f(-pm) + \widetilde{c}_f(-p^{-1}m)\right\}HZ(m) \\ = \sum_{m>0}\widetilde{c}_f(-m)\left\{HZ(pm) + p\,HZ(p^{-1}m)\right\}.$$

On the other hand, by Theorem 2.4, the divisor of $\Psi|\mathcal{T}(\mathfrak{p})$ is given by

$$\sum_{m>0}\widetilde{c}_f(-m)HZ(m) * T(\mathfrak{p}) \\ = \sum_{m>0}\widetilde{c}_f(-m)\left\{HZ(pm) + p\cdot HZ(p^{-1}m)\right\}.$$

Thus the divisor of $\Psi|\mathcal{T}(\mathfrak{p})$ coincides with that of Ψ', which implies $\Psi|\mathcal{T}(\mathfrak{p}) = \gamma(\mathfrak{p},f)\Psi'$ with $\gamma(\mathfrak{p},f) \in \mathbb{C}^\times$. In view of the infinite product expansions of $\Psi|\mathcal{T}(\mathfrak{p})$ and Ψ' (see Theorem 2.1 (iii)), we see that $|\gamma(\mathfrak{p},f)| = 1$.

The proof of Theorem 2.6 in the inert case is similar and we omit it.

5.2

We need several preparations to show Theorem 2.4. For $l \in \mathbb{Z}$ and $\xi = (a, b, \lambda) \in X_m$, put $l \star \xi = (la, lb, l\lambda) \in X_{l^2 m}$. Note that $D(l \star \xi) = D(\xi)$.

Set

$$q = \pi\pi' = \begin{cases} p & \text{if } p \text{ splits in } K/\mathbb{Q}, \\ p^2 & \text{if } p \text{ is inert in } K/\mathbb{Q}. \end{cases}$$

We define mappings $f, g_\mu \colon X_m \to X_{qm}$ $(\mu \in \mathcal{O}_K)$ by

$$\begin{aligned} f(\xi) &= (qa, b, \pi\lambda), \\ g_\mu(\xi) &= (a, qb + \mathrm{Tr}(\pi'\lambda\mu) + a\mu\mu', \pi'\lambda + a\mu') \end{aligned}$$

for $\xi = (a, b, \lambda)$. Note that f and g_μ are injective and that $g_{\mu_1}(X_m) = g_{\mu_2}(X_m)$ if $\mu_1 - \mu_2 \in p\mathcal{O}_K$.

Lemma 5.1. *We have*

$$HZ(m) * T(\mathfrak{p}) = \sum_{\xi \in X_m} f(\xi) + \sum_{\mu \in \mathcal{O}_K/p\mathcal{O}_K} \sum_{\xi \in X_m} g_\mu(\xi).$$

Proof. This follows from

$$D(\xi) * \begin{pmatrix} \pi & 0 \\ 0 & 1 \end{pmatrix} = D(f(\xi)),\ D(\xi) * \begin{pmatrix} 1 & \mu \\ 0 & \pi \end{pmatrix} = D(g_\mu(\xi)).$$

□

For $\Xi \in X_{qm}$, set

$$N(\Xi) = \#\{\mu \in \mathcal{O}_K/\mathfrak{p} \mid \Xi \in g_\mu(X_m)\} + \delta(\Xi \in f(X_m)).$$

To show Theorem 2.4, it suffices to prove

Proposition 5.2. *Let $\Xi \in X_{qm}$.*

(i) If p splits in $K/\mathbb{Q}$, we have

$$N(\Xi) = \begin{cases} p+1 & \text{if } \Xi \in p \star X_{p^{-1}m}, \\ 1 & \text{if } \Xi \in X_{pm} \setminus p \star X_{p^{-1}m}. \end{cases}$$

(ii) If p is inert in $K/\mathbb{Q}$, we have

$$N(\Xi) = \begin{cases} p^2+1 & \text{if } \Xi \in p^2 \star X_{p^{-2}m}, \\ p+1 & \text{if } p \nmid m \text{ and } \Xi \in p \star X_m \setminus p^2 \star X_{p^{-2}m}, \\ 1 & \text{if } p | m \text{ and } \Xi \in p \star X_m \setminus p^2 \star X_{p^{-2}m}, \\ 1 & \text{if } \Xi \in X_{p^2m} \setminus p \star X_m. \end{cases}$$

The following elementary lemma is useful in later discussions.

Lemma 5.3. *Let* $\Xi = (A, B, \Lambda) \in X_{qm}$.

(i) We have $\Xi \in f(X_m)$ *if and only if* $q|A$ *and* $\Lambda \in \pi\mathfrak{d}_K^{-1}$.

(ii) Let $\mu \in \mathcal{O}_K$. *We have* $\Xi \in g_\mu(X_m)$ *if and only if*

$$\Lambda - A\mu' \in \pi'\mathfrak{d}_K^{-1} \quad \textit{and} \quad B \equiv \mathrm{Tr}(\Lambda\mu) - A\mu\mu' \pmod q.$$

5.3 The proof of Proposition 5.2 (i)

Throughout this subsection, we assume that p splits in $K/\mathbb{Q}$. We need several results on the arithmetic of K.

Lemma 5.4. *For* $B \in \mathbb{Z}$ *and* $\Lambda \in (\mathfrak{d}_K^{-1} \setminus \pi\mathfrak{d}_K^{-1}) \cap \pi'\mathfrak{d}_K^{-1}$, *we have*

$$\#\{\mu \in \mathcal{O}_K/\pi\mathcal{O}_K \mid \mathrm{Tr}(\Lambda\mu) \equiv B \pmod p\} = 1.$$

Proof. Put $\varphi(\mu) = \mathrm{Tr}(\Lambda\mu) \pmod p$. Then φ gives rise to a homomorphism of $\mathcal{O}_K/\pi\mathcal{O}_K$ to $\mathbb{Z}/p\mathbb{Z}$. To prove the lemma, it is sufficient to show that φ is injective. Suppose that $\mathrm{Tr}(\Lambda\mu) \equiv 0 \pmod p$. Put $\lambda = \Lambda/\pi'$. Then $\lambda \in \mathfrak{d}_K^{-1} \setminus \pi\mathfrak{d}_K^{-1}$ and $p|\mathrm{Tr}(\pi'\lambda\mu)$. By Lemma 4.6, we have $\lambda\mu \in \pi\mathfrak{d}_K^{-1}$ and hence $\mu \in \pi\mathcal{O}_K$, which shows the injectivity of φ. □

Lemma 5.5. *For* $\Lambda \in \mathfrak{d}_K^{-1}$ *and* $A \in \mathbb{Z}$ *with* $p \nmid A$, *we have*

$$\#\left\{\mu \in \mathcal{O}_K/\pi\mathcal{O}_K \mid \Lambda - A\mu' \in \pi'\mathfrak{d}_K^{-1}\right\} = 1.$$

Proof. Since $A\pi' \in (\mathfrak{d}_K^{-1} \setminus \pi\mathfrak{d}_K^{-1}) \cap \pi'\mathfrak{d}_K^{-1}$, there exists a $\mu \in \mathcal{O}_K$ such that $\mathrm{Tr}(A\pi'\mu) \equiv \mathrm{Tr}(\pi\Lambda) \pmod p$ by Lemma 5.4. Since $p|\mathrm{Tr}(\pi(\Lambda - A\mu'))$, we have $\Lambda - A\mu' \in \pi'\mathfrak{d}_K^{-1}$ by Lemma 4.6. To prove the uniqueness, suppose that $\Lambda - A\mu_1', \Lambda - A\mu_2' \in \pi'\mathfrak{d}_K^{-1}$. Then $\mu_1 - \mu_2 \in \pi\mathfrak{d}_K^{-1} \cap \mathcal{O}_K = \pi\mathcal{O}_K$, which completes the proof of the lemma. □

We are now ready to show Proposition 5.2 (i). Let $\Xi = (A, B, \Lambda) \in X_{pm}$.

First suppose that $\Xi \in p \star X_{p^{-1}m}$. Then, by Lemma 5.3, we have $\Xi \in f(X_m)$ and $\Xi \in g_\mu(X_m)$ for any $\mu \in \mathcal{O}_K$, which implies $N(\Xi) = p + 1$.

We next consider the case where $\Xi \in X_{pm} \setminus p \star X_{p^{-1}m}$. Note that, if $p|A$, either $\Lambda \in \pi\mathfrak{d}_K^{-1}$ or $\Lambda \in \pi'\mathfrak{d}_K^{-1}$ holds, since $AB - \Lambda\Lambda' = pm/d$.

Suppose that $p|A$ and $\Lambda \in \pi\mathfrak{d}_K^{-1}$. By Lemma 5.3, we have $\Xi \in f(X_m)$. If $\Xi \in g_\mu(X_m)$ for some $\mu \in \mathcal{O}_K$, by Lemma 5.3, we have $\Lambda \in \pi'\mathfrak{d}_K^{-1}$ and hence $\Lambda \in p\mathfrak{d}_K^{-1}$. This implies that $p|B$, which contradicts to the assumption $\Xi \notin p \star X_{p^{-1}m}$. Thus we have proved $N(\Xi) = 1$.

Next suppose that $p|A$ and $\Lambda \in (\mathfrak{d}_K^{-1} \setminus \pi\mathfrak{d}_K^{-1}) \cap \pi'\mathfrak{d}_K^{-1}$. Then $\Xi \notin f(X_m)$. Take a $\mu \in \mathcal{O}_K$ with $\Lambda - A\mu' \in \pi'\mathfrak{d}_K^{-1}$ (see Lemma 5.5). Since

$$dA(B - \mathrm{Tr}(\Lambda\mu) + A\mu\mu') = pm + d\,\mathrm{N}(\Lambda - A\mu') \equiv 0 \pmod p,$$

we have $B \equiv \mathrm{Tr}(\Lambda\mu) - A\mu\mu' \pmod{p}$ and hence $\Xi \in g_\mu(X_m)$ by Lemma 5.3. Using again Lemma 5.5, we see that such a μ modulo $\pi\mathcal{O}_K$ uniquely exists and hence $N(\Xi) = 1$.

Finally suppose that $p \nmid A$. Then $\Xi \notin f(X_m)$. An argument similar to the above one shows that $\Xi \in g_\mu(X_m)$ if and only if $\Lambda - A\mu' \in \pi'\mathfrak{d}_K^{-1}$. Lemma 5.5 now implies that $N(\Xi) = 1$, which completes the proof of Proposition 5.2 (i).

5.4 The proof of Proposition 5.2 (ii)

Throughout this subsection, we assume that p is inert in $K/\mathbb{Q}$. Then $q = p^2$. The following two lemmas are easily verified.

Lemma 5.6. *For $\Lambda \in \mathfrak{d}_K^{-1}$ and $A \in \mathbb{Z}$ with $p \nmid A$, there uniquely exists an element $\mu \in \mathcal{O}_K/p\mathcal{O}_K$ such that $\Lambda - A\mu' \in p\mathfrak{d}_K^{-1}$. We also have $d^2\,\mathrm{N}(\Lambda) \equiv d^2\,\mathrm{N}(\mu) \pmod{p}$.*

Lemma 5.7. *For $c \in \mathbb{Z}$, we have*

$$\#\{\mu \in \mathcal{O}_K/p\mathcal{O}_K \mid \mu\mu' \equiv c \pmod{p}\} = \begin{cases} p+1 & \text{if } p \nmid c, \\ 1 & \text{if } p|c. \end{cases}$$

For $\xi = (a, b, \lambda) \in X_m$, set

$$n(\xi) = \#\{\mu \in \mathcal{O}_K/p\mathcal{O}_K \mid b \equiv \mathrm{Tr}(\lambda\mu) - a\mu\mu' \pmod{p}\}.$$

Lemma 5.8. *Let $\xi = (a, b, \lambda) \in X_m$.*

(i) If $p \nmid m$, we have

$$n(\xi) = \begin{cases} p & \text{if } p|a, \\ p+1 & \text{if } p \nmid a. \end{cases}$$

(ii) If $p|m$, we have

$$n(\xi) = \begin{cases} p^2 & \text{if } \xi \in p \star X_{p^{-2}m}, \\ 0 & \text{if } \xi \in X_m \setminus p \star X_{p^{-2}m} \text{ and } p|a, \\ 1 & \text{if } p \nmid a. \end{cases}$$

Proof. First consider the case $p|a$. If $p|m$, then $\lambda \in p\mathfrak{d}_K^{-1}$ and hence

$$n(\xi) = \begin{cases} p^2 & \text{if } \xi \in p \star X_{p^{-2}m}, \\ 0 & \text{if } \xi \in p \star X_m \setminus X_{p^{-2}m}. \end{cases}$$

Suppose that $p \nmid m$. Then $\lambda \in \mathfrak{d}_K^{-1} \setminus p\mathfrak{d}_K^{-1}$. Let $\lambda = (\alpha + \beta\omega)/\sqrt{d}$ $(\alpha, \beta \in \mathbb{Z}, p \nmid \alpha \text{ or } p \nmid \beta)$. For $\mu = x + y\omega \in \mathcal{O}_K$, we have $\mathrm{Tr}(\lambda\mu) = \beta x + (\alpha + \beta s)y$. Then

$$\begin{aligned} n(\xi) &= \#\{(x, y) \in (\mathbb{Z}/p\mathbb{Z})^2 \mid \beta x + (\alpha + \beta s)y \equiv b \pmod{p}\} \\ &= p. \end{aligned}$$

Next consider the case where $p \nmid a$. Take $a_1, d_1 \in \mathbb{Z}$ and $\lambda_1 \in \mathcal{O}_K$ such that

$$aa_1 \equiv 1 \pmod{p},\ dd_1 \equiv 1 \pmod{p},\ \lambda - \lambda_1 \in p\mathfrak{d}_K^{-1}.$$

Then, for $\mu \in \mathcal{O}_K$, we have

$$\begin{aligned} a_1\left(b - \mathrm{Tr}(\lambda\mu) + a\mu\mu'\right) &\equiv \mathrm{N}(\mu - a_1\lambda_1') + a_1 b - a_1^2\,\mathrm{N}(\lambda_1) \pmod{p} \\ &\equiv \mathrm{N}(\mu - a_1\lambda_1') + a_1^2(ab - \mathrm{N}(\lambda_1)) \pmod{p}. \end{aligned}$$

By Lemma 5.6, we have

$$d^2(ab - \mathrm{N}(\lambda_1)) \equiv d^2(ab - \mathrm{N}(\lambda)) = dm \pmod{p}.$$

It follows that

$$n(\xi) = \#\{\mu \in \mathcal{O}_K/p\mathcal{O}_K \mid \mathrm{N}(\mu - a_1\lambda_1') \equiv -a_1^2 d_1 m \pmod{p}\}.$$

By Lemma 5.6, we obtain

$$n(\xi) = \begin{cases} p+1 & \text{if } p \nmid m, \\ 1 & \text{if } p | m. \end{cases}$$

□

We now prove Proposition 5.2 (ii). Let $\Xi = (A, B, \Lambda) \in X_{p^2m}$. Recall that $\Xi \in f(X_m)$ if and only if $p^2 | A$ and $\Lambda \in p\mathfrak{d}_K^{-1}$, and that $\Xi \in g_\mu(X_m)$ if and only if $\Lambda - A\mu' \in p\mathfrak{d}_K^{-1}$ and $B \equiv \mathrm{Tr}(\Lambda\mu) - A\mu\mu' \pmod{p^2}$ (see Lemma 5.3).

First assume that $\Xi \in p^2 \star X_{p^{-2}m}$. Then $\Xi \in f(X_m)$ and $\Xi \in g_\mu(X_m)$ for any $\mu \in \mathcal{O}_K$, which shows that $N(\Xi) = p^2 + 1$.

Next assume that $\Xi \in p \star X_m \setminus p^2 \star X_{p^{-2}m}$. It is easy to see that $\Xi \in f(X_m)$ if and only if $p^2 | A$. Let $\Xi = p \star \xi$ with $\xi = (a, b, \lambda) \in X_m$. Then $\Xi \in g_\mu(X_m)$ if and only if $b \equiv \mathrm{Tr}(\lambda\mu) - a\mu\mu' \pmod{p}$. We thus have $N(\Xi) = n(\xi) + \delta(p^2 | A)$. By Lemma 5.8, we obtain

$$N(\Xi) = \begin{cases} p+1 & \text{if } p \nmid m, \\ 1 & \text{if } p | m. \end{cases}$$

Finally assume that $\Xi \in X_{p^2m} \setminus p \star X_m$. Suppose tha $p^2 | A$. Then we have $\Lambda \in p\mathfrak{d}_K^{-1}$ and $p \nmid B$, which implies $\Xi \in f(X_m)$. For every $\mu \in \mathcal{O}_K$, we have $\mathrm{Tr}(\Lambda\mu) - A\mu\mu' \equiv 0 \pmod{p}$ and hence $\Xi \notin g_\mu(X_m)$. It follows that $N(\Xi) = 1$. Suppose that $A \in p\mathbb{Z} \setminus p^2\mathbb{Z}$. We then have $p | B$ and $\Lambda \in p\mathfrak{d}_K^{-1}$, a contradiction. Suppose that $p \nmid A$. Then $\Xi \notin f(X_m)$. Note that, if $\Lambda - A\mu' \in p\mathfrak{d}_K^{-1}$, we have $dA(B - \mathrm{Tr}(\Lambda\mu) + A\mu\mu') \equiv d\,\mathrm{N}(\Lambda - A\mu') \equiv 0 \pmod{p^2}$ and hence $B \equiv \mathrm{Tr}(\Lambda\mu) - A\mu\mu' \pmod{p^2}$. This and Lemma 5.6 imply that $N(\Xi) = 1$.

6 Examples

In this section, we present several examples of Borcherds products and non Borcherds products in the case $K = \mathbb{Q}(\sqrt{5})$. We then have $d = 5$ and $\epsilon_0 = (1 + \sqrt{5})/2$. The structure of the graded ring $\bigoplus_{k \geq 0} M_k(\Gamma_K)$ has been studied by K. -B. Gundlach [Gun] and R. Müller [M]. We now recall Müller's construction.

We first recall the definition of the theta constants. For $Z \in \mathfrak{H}_2$ (the upper half space of degree 2) and $a, b \in \{0, 1\}^2$, set

$$\vartheta(Z; a, b) = \sum_{l \in \mathbb{Z}^2} \mathbf{e}\left[\frac{1}{2}{}^t(l + a/2)Z(l + a/2) + \frac{1}{2}{}^t lb\right].$$

It is known that $\vartheta(Z; a, b) \not\equiv 0$ if and only if ${}^t ab$ is even. The even pairs are listed as follows:

${}^t a_0 = (0, 0), {}^t b_0 = (0, 0);\ {}^t a_1 = (1, 1), {}^t b_1 = (0, 0);\ {}^t a_2 = (0, 0), {}^t b_2 = (1, 1);\ {}^t a_3 = (1, 1), {}^t b_3 = (1, 1),$
${}^t a_4 = (0, 1), {}^t b_4 = (0, 0);\ {}^t a_5 = (1, 0), {}^t b_5 = (0, 0);\ {}^t a_6 = (0, 0), {}^t b_6 = (0, 1);\ {}^t a_7 = (1, 0), {}^t b_7 = (0, 1),$
${}^t a_8 = (0, 0), {}^t b_8 = (1, 0);\ {}^t a_9 = (0, 1), {}^t b_9 = (1, 0).$

Let $\iota\colon \mathfrak{H}^2 \to \mathfrak{H}_2$ be a modular embedding given by

$$\iota(z_1, z_2) = \frac{1}{\sqrt{5}}\begin{pmatrix} \epsilon_0 z_1 - \epsilon_0' z_2 & z_1 - z_2 \\ z_1 - z_2 & -\epsilon_0' z_1 + \epsilon_0 z_2 \end{pmatrix}.$$

For $a \in \mathbb{Z}_{\geq 0}$ and $i_1, \ldots, i_r \in \{0, 1\}$, we put

$$\theta^a_{i_1 \cdots i_r} = (\theta_{i_1} \cdots \theta_{i_r})^a,$$

where

$$\theta_i(z) = \vartheta(\iota(z); a_i, b_i) \qquad (0 \leq i \leq 9, z \in \mathfrak{H}^2).$$

Let $M_k^{sym}(\Gamma_K) = \{F \in M_k(\Gamma_K) \mid F(z_1, z_2) = F(z_2, z_1)\}$. The graded ring $\bigoplus_{k \geq 0} M_{2k}^{sym}(\Gamma_K)$ is generated by

$$\begin{aligned} g_2 &= 2^{-2}\left(\theta_0^4 + \theta_1^4 + \theta_2^4 - \theta_3^4 + \theta_4^4 + \theta_5^4 + \theta_6^4 - \theta_7^4 + \theta_8^4 - \theta_9^4\right), \\ s_6 &= 2^{-8}\left(\theta^2_{012478} + \theta^2_{012569} + \theta^2_{034568} + \theta^2_{236789}\right), \\ s_{10} &= 2^{-12}\,\theta^2_{0123456789}. \end{aligned}$$

This implies that $M_{10}^{sym}(\Gamma_K)$ is spanned by $s_{10}, t_{10} = g_2^2 s_6$ and $u_{10} = g_2^5$. It is known that the weight of any Borcherds product is divisible by 5, and that s_{10} is a Borcherds product (see [Br3], page 161).

For $F = s_{10}, t_{10}$ and u_{10}, we calculated the values $F(\pi z_1, \pi' z_2), F(\pi' z_1, \pi z_2)$ and the quotient $F|\mathcal{T}(\mathfrak{p})(z_1, z_2) \cdot F|\mathcal{T}(\mathfrak{p}')(z_1, z_2)^{-1}$ by Mathematica, where $z_1 = 2\sqrt{-1}, z_2 = \sqrt{-1}$ and $\mathfrak{p} = (\pi)$ with $\pi = 4 + \sqrt{5}$. The result is as follows:

	$F(\pi z_1, \pi' z_2)$	$F(\pi' z_1, \pi z_2)$	$F\|\mathcal{T}(\mathfrak{p})(z_1, z_2) \cdot F\|\mathcal{T}(\mathfrak{p}')(z_1, z_2)^{-1}$
$F = s_{10}$	$4.27068550613 \cdots \times 10^{-27}$	$4.58279473089948 \cdots \times 10^{-24}$	$1.00000000000 \cdots$
$F = t_{10}$	$3.23264624182 \cdots \times 10^{-13}$	$2.14286904632 \cdots \times 10^{-13}$	$1.59928132099 \cdots$
$F = u_{10}$	$1.00000000019 \cdots$	$1.00000000128 \cdots$	$1661.00964313 \cdots$

In view of Theorem 2.2, this numerical computation shows that neither of t_{10} and u_{10} is a Borcherds product.

References

[Bo1] R. E. Borcherds, *Automorphic forms on $O_{s+2,2}(\mathbb{R})$ and infinite products*, Invent. Math. **120** (1995), 161–213.

[Bo2] R. E. Borcherds, *Automorphic forms with singularities on Grassmannians*, Invent. Math. **132** (1998), 491–562.

[Bo3] R. E. Borcherds, *The Gross-Kohnen-Zagier theorem in higher dimensions*, Duke Math. J. **97** (1999), 219–233.

[Br1] J. H. Bruinier, *Borcherds Products and Chern Classes of Hirzebruch-Zagier Divisors*, Invent. Math. **138** (1999), 51–83.

[Br2] J. H. Bruinier, *Borcherds Products on $O(2,l)$ and Chern Classes of Heegner Divisors*, Lecture Notes in Math. **1780** (2002), Springer Verlag.

[Br3] J. H. Bruinier, *Hilbert Modular Forms and Their Applications*, in The 1-2-3 of Modular Forms (2007), Springer Verlag.

[BB] J. H. Bruinier and M. Bundschuh, *On Borcherds products associated with lattices of prime discriminat*, The Ramanujan J. **7** (2003), 49–61.

[DN] K. Doi and H. Naganuma, *On the functional equation of certain Dirichlet series*, Invent. Math. **9** (1969), 1–14.

[Ge] G. van der Geer, *Hilbert Modular Surfaces*, Springer.

[GN] V. A. Gritsenko and V. V. Nikulin, *Automorphic forms and Lorentzian kac-Moody algebras. Part II*, International J. Math. **9** (1998), 201-275.

[Gue] P. Guerzhoy, *On the Hecke equivariance of the Borcherds isomorphism*, Bull. London Math. Soc. **38** (2006), 93–96.

[Gun] K. -B. Gundlach, *Die Bestimmung der Funktionen zur Hilbertschen Modulgruppe des Zahlkörpers* $\mathbb{Q}(\sqrt{5})$, Math. Ann. **152** (1963), 226–256.

[HaMo] J. A. Harvey and G. Moore, *Algebras, BPS states, and strings*, Nuclear Physics B **463** (1996), 315–368.

[H] B. Heim, *On the Spezialschar of Maass*, preprint.

[HZ] F. Hirzebruch and D. Zagier, *Intersection numbers of curves on Hilbert modular surfaces and modular forms of Nebentypus*, Invent. Math. **36** (1976), 57–113.

[Mi] T. Miyake, *Modular Forms*, Springer.

[M] R. Müller, *Hilbertschen Modulformen und Modulfunktionen zu* $\mathbb{Q}(\sqrt{5})$, Arch. Math. **45** (1985), 239–251.

[N] H. Naganuma, *On the coincidence of two Dirichlet series associated with cusp forms of Hecke's "Neben"-type and Hilbert modular forms over a real quadratic field*, J. Math. Soc. Japan **25** (1973), 547–555.

[O1] T. Oda, *On modular forms associated with indefinite quadratic forms of signature* $(2, n-2)$, Math. Ann. **231** (1977), 97–144.

[O2] T. Oda, *Periods of Hilbert modular forms*, Progress in Math. **19** (1982), Birkhäuser.

[RS] S. Rallis and G. Schiffmann, *On a relation between* SL_2 *cusp forms and cusp forms on tube domains associated to orthogonal groups*, Trans. AMS **263** (1981), 1–58.

[Sa1] H. Saito, *Automorphic Forms and Algebraic Extensions of Number Fields*, Lectures in Math., Department of Math., Kytoto Univ., **8**, Kinokuniya.

[Sa2] H. Saito, *Automorphic Forms and Algebraic Extensions of Number Fields*, Proc. Japan Acad., **51** (1975), 229–233.

[Z] D. Zagier, *Modular forms associated to real quadratic fields*, Invent. Math. **30** (1975), 1–46.

Bernhard Heim
Max-Planck Insitut für Mathematik, Vivatsgasse 7, 53111 Bonn, Germany
e-mail: heim@mpim-bonn.mpg.de

Atsushi Murase
Department of Mathematics, Faculty of Science, Kyoto Sangyo University, Motoyama, Kamigamo, Kita-ku, Kyoto 603-8555, Japan
e-mail: murase@cc.kyoto-su.ac.jp