ZARISKI'S MULTIPLICITY QUESTION AND ALIGNED SINGULARITIES

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Abstract. We answer positively Zariski's multiplicity question for special classes of non-isolated singularities.

Let $f\colon (\mathbb{C}^n \times \mathbb{C}, \{0\} \times \mathbb{C}) \to (\mathbb{C}, 0)$, $(z_1, \ldots, z_n, t) \mapsto f(z_1, \ldots, z_n, t) = f_t(z_1, \ldots, z_n)$, with $n \geq 3$, be a germ (at the origin) of holomorphic function such that, for all t near 0, the germ f_t is reduced. Let ν_{f_t} be the *multiplicity* of f_t at 0, that is, the number of points of intersection, near 0, of $V_{f_t} := f_t^{-1}(0)$ with a generic (complex) line in $\mathbb{C}^n$ passing arbitrarily close to 0 but not through 0. As we are assuming that f_t is reduced, ν_{f_t} is also the *order* of f_t at 0, that is, the lowest degree in the power series expansion of f_t at 0. Let μ_{f_t} be the Milnor number of f_t at 0. One says that $(f_t)_t$ is *topologically constant* (respectively *μ–constant*, *equimultiple*) if, for all t near 0, there is a germ of homeomorphism $\varphi_t\colon (\mathbb{C}^n, 0) \to (\mathbb{C}^n, 0)$ such that $\varphi(V_{f_t}) = V_{f_0}$ (respectively $\mu_{f_t} = \mu_{f_0}$, $\nu_{f_t} = \nu_{f_0}$). In the special case where $(f_t)_t$ is a family of *isolated* singularities (i.e., when, for all t near 0, f_t has an isolated critical point at 0), if $n \neq 3$, then the topological constancy is equivalent to the μ–constancy (cf. Lê [7], Teissier [16] and Lê–Ramanujam [8]).

In [21], Zariski asked the following question: *if $(f_t)_t$ is topologically constant, then is it equimultiple?* More than thirty years later, the question is, in general, still unsettled (even for isolated hypersurface singularities). The answer is, nevertheless, known to be *yes* in several special cases: for example, for families of plane curve singularities (Zariski [22]), families of convenient Newton nondegenerate (isolated) singularities (Abderrahmane [1] and Saia–Tomazalla [15]), families of semiquasihomogeneous or quasihomogeneous isolated singularities[1] (Greuel [4] and O'Shea [13]), families of isolated singularities of the form $f_t(z) = a(z)+\theta(t)\, b(z)$, where $a, b\colon (\mathbb{C}^n, 0) \to (\mathbb{C}, 0)$ and $\theta\colon (\mathbb{C}, 0) \to (\mathbb{C}, 0)$, $\theta \not\equiv 0$, are germs of holomorphic functions (Greuel [4] and Trotman [19, 20]). For a detailed and more complete list, see the recent author's survey article [3].

In this note, we concentrate our attention on families $f = (f_t)_t$ of the following form:

$$f_t(z_1, \ldots, z_n) = g_t(z_1, \ldots, z_{n-1}) + z_n^2\, h_t(z_1, \ldots, z_n),$$

where $g\colon (\mathbb{C}^{n-1} \times \mathbb{C}, \{0\} \times \mathbb{C}) \to (\mathbb{C}, 0)$, $(z_1, \ldots, z_{n-1}, t) \mapsto g(z_1, \ldots, z_{n-1}, t) = g_t(z_1, \ldots, z_{n-1})$, and $h\colon (\mathbb{C}^n \times \mathbb{C}, \{0\} \times \mathbb{C}) \to (\mathbb{C}, 0)$, $(z_1, \ldots, z_n, t) \mapsto h(z_1, \ldots, z_n, t) = h_t(z_1, \ldots, z_n)$, are germs of holomorphic functions such that, for all t near 0, the germ g_t (and f_t) is reduced.

In [5], Greuel–Pfister already considered families of this type and they proved the following result.

Theorem 0.1 (Greuel–Pfister [5, Proposition 3.2])**.** *Let $f = (f_t)_t$ with $f_t(z_1, \ldots, z_n) = g_t(z_1, \ldots, z_{n-1}) + z_n^2\, h_t(z_1, \ldots, z_n)$ as above. Suppose that, for all t near 0, the germ f_t has an isolated critical point at 0 and the germ g_0 is semiquasihomegeneous (or the germ f_t has an isolated critical at 0 and $n = 3$). If $(f_t)_t$ is topologically constant, then $(g_t)_t$ is equimultiple. In*

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[1]In this case, it suffices to assume the semiquasihomogeneity or quasihomogeneity only for the germ f_0.

particular, if, moreover, for all t near 0, the multiplicity at 0 of g_t is less than or equal to the order at 0 of the (nonreduced) germ $(z_1, \ldots, z_n) \mapsto z_n^2\, h_t(z_1, \ldots, z_n)$, then $(f_t)_t$ is equimultiple.

We extend here Greuel–Pfister's result (concerning *isolated* singularities) to a special class of higher dimensional singularities. We also prove similar results in the case where g_t, all small t, is convenient Newton nondegenerate or of the form $a(z') + \theta(t)\, b(z')$, where $z' = (z_1, \ldots, z_{n-1})$.

Theorem 0.2. *Let $f = (f_t)_t$ with $f_t(z_1, \ldots, z_n) = g_t(z_1, \ldots, z_{n-1}) + z_n^2\, h_t(z_1, \ldots, z_n)$ as above. Assume that, for all t near 0, the germ f_t has an s–dimensional aligned singularity at 0. Also suppose that $(f_t)_t$ is topologically constant. Let $(t_k)_k$ be an infinite sequence of points in $\mathbb{C}$ tending to 0. Assume that the coordinates $z = (z_1, \ldots, z_n)$, or some circular permutation of them, form an aligning set of coordinates at 0 for f_0 and for f_{t_k}, for all $k \in \mathbb{N}$. Finally suppose that at least one of the following four conditions is satisfied:*

1. *for all t near 0, the germ g_t is convenient and has a nondegenerate Newton principal part with respect to the coordinates $z' = (z_1, \ldots, z_{n-1})$;*
2. *for all t near 0, the germ g_t is of the form $g_t(z') = a(z') + \theta(t)\, b(z')$, where $a, b \colon (\mathbb{C}^{n-1}, 0) \to (\mathbb{C}, 0)$ and $\theta \colon (\mathbb{C}, 0) \to (\mathbb{C}, 0)$, $\theta \not\equiv 0$, are germs of holomorphic functions;*
3. *g_0 is the germ of a semiquasihomogeneous polynomial with respect to z';*
4. *$n = 3$.*

Then $(g_t)_t$ is equimultiple. In particular, if, moreover, for all t near 0, the multiplicity at 0 of the germ g_t is less than or equal to the order at 0 of the (nonreduced) germ $(z_1, \ldots, z_n) \mapsto z_n^2\, h_t(z_1, \ldots, z_n)$, then $(f_t)_t$ is equimultiple.

For the definition of *aligned* singularities and *aligning* sets of coordinates, see Massey [9]. For the basic material about Newton polyhedra, we refer to Kouchnirenko [6] and Oka [11, 12].

Aligned singularities were introduced by Massey in [9]. They generalize isolated singularities (obtained for $s = 0$) and smooth one–dimensional singularities (in particular line singularities). Regarding this class of singularities, Massey proved the following *reduction* theorem.

Theorem 0.3 (Massey [9, Theorem 7.9])**.** *The following are equivalent:*

1. *for all $n \geq 4$, the answer to Zariski's multiplicity question is positive for families $(f_t)_t$ of reduced analytic hypersurfaces with isolated singularities;*
2. *for all $n \geq 4$, there exists an integer s such that the answer to Zariski's multiplicity question is positive for families $(f_t)_t$ of reduced analytic hypersurfaces with s–dimensional aligned singularities (i.e., for all t near 0, f_t has an s–dimensional aligned singularity at 0);*
3. *for all $n \geq 4$, for all integer s, the answer to Zariski's multiplicity question is positive for families $(f_t)_t$ of reduced analytic hypersurfaces with s–dimensional aligned singularities.*

The proof of Theorem 0.2 is a combination of Massey's proof of Theorem 0.3 and Greuel–Pfister's proof of Theorem 0.1, together combined with the results of Zariski [22], Abderrahmane [1], Saia–Tomazella [15], Greuel [4], O'Shea [13] and Trotman [19, 20]. Note, nevertheless, that Theorem 0.2 is not an immediate consequence of Theorems 0.3 and 0.1 (cf. Remark 0.4).

Theorem 0.2 answers positively Zariski's multiplicity question for special classes of high–dimensional singularities without any assumption on the topological constancy, that is, without any assumption on the homeomorphisms φ_t. We recall that, under some additional hypotheses on the φ_t's, positive answers to Zariski's question for high–dimensional singularities already exist. For example, it is known that the multiplicity is an embedded C^1 invariant (cf. Ephraim [2] and Trotman [17, 18, 20]) and an embedded 'right–left bilipschitz' invariant (cf. Risler–Trotman [14]).

Let's give an example where Theorem 0.2 applies. Set $g_t(z_1, z_2) = z_1^2 + z_2^2 + (1-t)\, z_1^3$ and $h_t(z_1, z_2, z_3) = t\, z_2^2$, so that $f_t(z_1, z_2, z_3) = z_1^2 + z_2^2 + (1-t)\, z_1^3 + z_3^2\, t\, z_2^2$. For all t sufficiently close to 0, the singular locus of f_t is just the z_3–axis (so, f_t has an 1–dimensional aligned singularity at 0). The coordinates (z_1, z_2, z_3) are not aligning, but one checks easily that (z_3, z_1, z_2) are aligning for f_t, all t. Since the singular locus of $f\colon (z_1, z_2, z_3, t) \mapsto z_1^2 + z_2^2 + (1-t)\, z_1^3 + z_3^2\, t\, z_2^2$ is nothing but the plane in $\mathbb{C}^4$ defined by $z_1 = z_2 = 0$ and the Milnor number of $f_{t,z_3}\colon (z_1, z_2) \mapsto z_1^2 + z_2^2 + (1-t)\, z_1^3 + z_3^2\, t\, z_2^2$ is independent of t and z_3 (in fact, $\mu_{f_{t,z_3}} = 1$ for all t, z_3 near 0), it follows from Massey [10, Proposition p. 47] that $(f_t)_t$ is topologically constant. Hence Theorem 0.2 (4) applies. Since g_t is convenient and Newton nondegenerate with respect to (z_1, z_2) and semiquasihomogeneous with respect to (z_1, z_2), this example also shows that the special classes of high–dimensional singularities that we consider in the cases (1) and (3) (and, obviously, (2) too) are not empty.

Now, let's sketch the proof of Theorem 0.2. We start as in [9, Proof of Theorem 7.9]. Let $\zeta = (\zeta_1, \ldots, \zeta_n)$ be a circular permutation of the coordinates $z = (z_1, \ldots, z_n)$. We use the notation $\zeta_p := z_n$ for the 'special' coordinate z_n. Suppose that ζ is aligning for f_0 and for f_{t_k} at 0, all k. Then, since $(f_t)_t$ is a topologically constant family of aligned singularities, the Lê numbers (cf. [9, Definition 1.11]) $\lambda^i_{f_0,\zeta}$ $(0 \leq i \leq n-1)$ of f_0 at 0 with respect to ζ are equal to the Lê numbers $\lambda^i_{f_{t_k},\zeta}$ of f_{t_k} at 0 with respect to ζ, for all k large enough (cf. [9, Corollary 7.8]). Hence, by an inductive application of the Massey's generalized Iomdine–Lê formula (cf. [9, Theorem 4.5 and Corollary 4.6]), for all integers $j_1, \ldots, j_s$ such that $0 \ll j_1 \ll j_2 \ll \ldots \ll j_s$, the germs $f_0 + \zeta_1^{j_1} + \ldots + \zeta_s^{j_s}$ and $f_{t_k} + \zeta_1^{j_1} + \ldots + \zeta_s^{j_s}$ have an isolated singularity at 0 and the same Milnor number at 0, provided k is large enough[2]. In particular, by the upper semicontinuity of the Milnor number, this implies that, for all t sufficiently close to 0, the germ $f_t + \zeta_1^{j_1} + \ldots + \zeta_s^{j_s}$ has an isolated singularity at 0 and the same Milnor number, at 0, as $f_0 + \zeta_1^{j_1} + \ldots + \zeta_s^{j_s}$. In other words, the family $(f_t + \zeta_1^{j_1} + \ldots + \zeta_s^{j_s})_t$ is a μ–constant family of isolated singularities. This implies, in particular, that $g_t + \zeta_1^{j_1} + \ldots \zeta_s^{j_s}$, where, if $1 \leq p \leq s$, the term $\zeta_p^{j_p}$ is omitted, has an isolated singularity at 0[3] for all small t. Hence, as in [5, Proof of Proposition 3.2], by applying [5, Lemma 3.1] to the family $(f_t + \zeta_1^{j_1} + \ldots + \zeta_s^{j_s})_t$ with the hyperplane in $\mathbb{C}^n$ defined by $\zeta_p = 0$, one gets that $(g_t + \zeta_1^{j_1} + \ldots + \zeta_s^{j_s})_t$, where again, if $1 \leq p \leq s$, the term $\zeta_p^{j_p}$ is omitted, is also a μ–constant family of isolated singularities. Now, according to the case (1) or (3) that we consider, it follows from our hypotheses that, if the j_i's are chosen sufficiently large, then for all t sufficiently close to 0, the germ $g_t + \zeta_1^{j_1} + \ldots + \zeta_s^{j_s}$ ($\zeta_p^{j_p}$ omitted) is convenient and has a nondegenerate Newton principal part with respect to the coordinates $\tilde{\zeta}'$ (case (1)) or $g_0 + \zeta_1^{j_1} + \ldots + \zeta_s^{j_s}$ ($\zeta_p^{j_p}$ omitted) is the germ of a semiquasihomogeneous polynomial with respect to $\tilde{\zeta}'$ (case (3)). Since the j_i's can be chosen arbitrarily large, Theorem 0.2 then follows from the results of Abderrahmane [1] and Saia–Tomazella [15] (case (1)), Greuel [4] and Trotman [19, 20] (case (2)), Greuel [4] and O'Shea [13] (case (3)), and Zariski [22] (case (4)).

Remark 0.4. If one replaces the word *semiquasihomogeneous* by *quasihomogeneous* in Theorem 0.2 Part (3), the argument above does *not* work. Indeed, in this case, $g_0 + \zeta_1^{j_1} + \ldots + \zeta_s^{j_s}$ ($\zeta_p^{j_p}$ is omitted) is neither quasihomogeneous with an isolated singularity nor semiquasihomogeneous, so that we cannot apply the result of Greuel [4] and O'Shea [13] (we recall that

[2] According to [9], since we are using the coordinates $(\zeta_1, \ldots, \zeta_n)$ for the germ f_t, we use the coordinates $\tilde{\zeta} = (\zeta_{s+1}, \zeta_{s+2}, \ldots, \zeta_n, \zeta_1, \ldots, \zeta_s)$ for the germ $f_t + \zeta_1^{j_1} + \ldots + \zeta_s^{j_s}$.

[3] For the germ g_t, we use the coordinates $\zeta' = (\zeta_1, \ldots, \zeta_n)$, where ζ_p is omitted. For the germ $g_t + \zeta_1^{j_1} + \ldots + \zeta_s^{j_s}$, where, if $1 \leq p \leq s$, the term $\zeta_p^{j_p}$ is omitted, we use the coordinates $\tilde{\zeta}' = (\zeta_{s+1}, \zeta_{s+2}, \ldots, \zeta_n, \zeta_1, \ldots, \zeta_s)$, where ζ_p is omitted.

a quasihomogeneous polynomial is *not* semiquasihomogeneous if it has a nonisolated critical point at 0). By contrast, one can replace *semiquasihomogeneous* by *quasihomogeneous* in Theorem 0.1. Indeed, the hypothesis for the f_t's of having an isolated critical point at 0 automatically implies a similar property for the g_t's and, consequently, if g_0 is quasihomogeneous, then it is necessarily semiquasihomogeneous too. This shows that Theorem 0.2 is not an immediate consequence of Theorems 0.3 and 0.1. Note that one can replace *semiquasihomogeneous* by *quasihomogeneous with an isolated singularity* in Theorem 0.2 Part (3)

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References

1. Ould. M. Abderrahmane, *On the deformation with constant Milnor number and Newton polyhedron*, Preprint (Saitama University, 2004).
2. R. Ephraim, *C^1 preservation of multiplicity*, Duke Math. J. **43** (1976) 797–803.
3. C. Eyral, *Zariski's multiplicity question – A survey*, Preprint (Max-Planck Institut für Mathematik, 2005).
4. G.-M. Greuel, *Constant Milnor number implies constant multiplicity for quasihomogeneous singularities*, Manuscripta Math. **56** (1986) 159–166.
5. G.-M. Greuel and G. Pfister, *Advances and improvements in the theory of standard bases and syzygies*, Arch. Math. **66** (1996) 163–176.
6. A.G. Kouchnirenko, *Polyèdres de Newton et nombres de Milnor*, Invent. Math. **32** (1976) 1–32.
7. Lê D.T., *Topologie des singularités des hypersurfaces complexes*, Atérisque **7/8** (Singularités à Cargèse) (1973) 171–182.
8. Lê D.T. and C.P. Ramanujam, *The invariance of Milnor number implies the invariance of the topological type*, Amer. J. Math. **98** (1976) 67–78.
9. D. Massey, *Lê cycles and hypersurface singularities*, Lecture Notes Math. **1615** (Springer–Verlag, Berlin, 1995).
10. D. Massey, *The Lê–Ramanujam problem for hypersurfaces with one-dimensional singular sets*, Math. Ann. **282** (1988) 33–49.
11. M. Oka, *On the bifurcation of the multiplicity and topology of the Newton boundary*, J. Math. Soc. Japan **31** (1979) 435–450.
12. M. Oka, *Non-degenerate complete intersection singularity*, Actualités Mathématiques (Hermann, Paris, 1997).
13. D. O'Shea, *Topologically trivial deformations of isolated quasihomogeneous hypersurface singularities are equimultiple*, Proc. Amer. Math. Soc. **101** (1987) 260–262.
14. J.-J. Risler and D. Trotman, *Bilipschitz invariance of the multiplicity*, Bull. London Math. Soc. **29** (1997) 200–204.
15. M.J. Saia and J.N. Tomazella, *Deformations with constant Milnor number and multiplicity of complex hypersurfaces*, Glasg. Math. J. **46** (2004) 121–130.
16. B. Teissier, *Cycles évanescents, sections planes et conditions de Whitney*, Astérique **7/8** (Singularités à Cargèse) (1973) 285–362.
17. D. Trotman, *Multiplicity is a C^1 invariant*, Preprint (University Paris 11 (Orsay), 1977).
18. D. Trotman, *Multiplicity as a C^1 invariant*, Real analytic and algebraic singularities (Nagoya / Sapporo / Hachioji, 1996), Pitman Res. Notes Math. **381**, Longman, Harlow (1998) 215–221 (based on the Orsay Preprint [17]).
19. D. Trotman, *Partial results on the topological invariance of the multiplicity of a complex hypersurface*, Séminaire A'Campo–MacPherson (University Paris 7, 1977).
20. D. Trotman, *Equisingularité et conditions de Whitney*, Thesis (thèse d'état), Orsay, 1980.
21. O. Zariski, *Open questions in the theory of singularities*, Bull. Amer. Math. Soc. **77** (1971) 481–491.
22. O. Zariski, *On the topology of algebroid singularities*, Amer. J. Math. **54** (1932) 453–465.

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