Mellin and Green pseudo-differential operators associated with non-compact edges

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Abstract

We introduce an algebra of pseudo–differential operators on the wedge $\mathbb{R}^q \times X^\wedge$, with $X^\wedge = \mathbb{R}_+ \times X$ for some closed compact manifold X, and give a notion of ellipticity which is equivalent to the existence of a parametrix.

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Introduction

The present investigation provides techniques to establish algebras of pseudo-differential operators on piecewise smooth manifolds that have non-compact boundaries or edges. In this geometric setting it is essential to control the operators along with their parameter-dependent symbols near these non-compact ends (for more classic situations concerning this task see, e.g., CORDES [3], EGOROV, SCHULZE [5], and SCHROHE [13]). The need for pseudo-differential calculi on non-compact spaces arises naturally even in the theory of compact manifolds. For example, the Lopatinskij–Šapiro condition arises in the study of differential boundary value problems on bounded subsets of a Euclidean space and makes it necessary to deal with (pseudo-)differential operators on the half-space $\mathbb{R}_+$.

One fundamental idea behind the approach of SCHULZE [17], [19] is that adequate operator algebras on manifolds with 'higher' singularities should be reached via iteration of a given calculus on a 'simpler' manifold. Since an edge can locally be viewed as a product of a cone and a Euclidean space this, in particular, means that the analysis of pseudo-differential operators on a manifold with edges will have the structure of a calculus with symbols taking values in the cone algebra.

Boundary value problems comprise an important special case; here the boundary plays the role of an edge and the inner normal the role of the model cone. The local theory of pseudo-differential boundary problems for symbols without the transmission property in terms of an algebra is due to REMPEL, SCHULZE [12] and SCHULZE [19]. This yields a generalization of BOUTET DE MONVEL's [1] algebra with the transmission property. At the same it completes VIŠIK and ESKIN'S [20], [21] pseudo-differential boundary value problems to an algebra in which the asymptotic data are controlled in detail.

A main point of the general edge calculus is a precise description of the asymptotics of solutions. This is reached by establishing a concept of elliptic regularity, which is obtained by a parametrix construction, and requires that the elements of the algebra act between spaces with asymptotics in a specific way. (Note that the transmission property can also be interpreted in this context, since it preserves the Taylor asymptotics, i.e., smoothness up to the boundary). The asymptotics of functions can be characterized by their image under the Mellin transform, yielding meromorphic functions in the complex plane. This behaviour is reflected in the symbolic structure of the operators, i.e., the underlying symbols themselves are required to extend to meromorphic functions. A motivation for arranging the calculus in this specific manner are general functional analytical results conserning (the inversion of) meromorphic operator functions, cf., e.g., GRAMSCH, KABALLO [8].

The strategy to handle non-compact configurations is in some sense analogous to that used to deal with global pseudo-differential operators on $\mathbb{R}^q$: one requires the symbols to have a specific growth in the covariables as well as in the variables. It turns out that the natural Sobolev spaces for such global symbols are weighted variants of the usual Sobolev spaces on $\mathbb{R}^q$. In this paper we modify this approach originated by CORDES, PARENTI, and SHUBIN, to the case of global (abstract) operator-valued symbols. The discussion of the corresponding weighted (abstract) edge Sobolev spaces including the continuity properties of pseudo-differential operators between them requires extensive additional material, and thus will be given elsewhere.

The present paper also develops an algebra of smoothing Mellin and Green operators M and G that extend SCHULZE's theory for manifolds with compact edges to the non-compact situation.

As stated above, the analysis of these operators is the most typical part for understanding the asymptotics of solutions to elliptic problems. Since the symbols of operators in our algebra are families of (smoothing) cone operators, which themselves have symbols that are meromrphic functions, we can associate to each operator a family of meromorphic functions parametrized by the edge variable. A major difference to the calculus for compact edges is that we impose additional conditions on these families, allowing a more subtle control of the spatial dependence of the Laurent coefficients.

For operators of the form $1 + M + G$ (with $M + G$ having order zero) we establish a notion of ellipticity that incorporates the corresponding concept for compact edges as well as new features responsible for controlling the behaviour of the symbols near infinity. The ellipticity of a given operator turns out to be equivalent to the existence of a parametrix, which again looks like $1 + M + G$.

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1 Basic structures

1.1 Cone Sobolev spaces with asymptotics

This section is devoted to the definition of all spaces needed in this paper and the description of their basic properties. For detailed proofs and further information we refer to [17], [19], [5].

In general $\omega(t)$, $\tilde{\omega}(t)$ and $\omega_j(t)$, $\tilde{\omega}_j(t)$ $(j \in \mathbb{N}_0)$ always will denote real valued functions in $C_0^\infty(\overline{\mathbb{R}}_+)$, which are identically 1 in a neighbourhood of $t = 0$.

Let $\mathcal{M} : C_0^\infty(\mathbb{R}_+) \to \mathcal{A}(\mathbb{C})$, where the latter space is that of all entire functions, denote the Mellin transform given by

$$\mathcal{M}u(z) = \int_0^\infty t^{z-1} u(t)\, dt.$$

The (left–)inverse of this mapping is obtained by

$$\mathcal{M}^{-1}g(t) = \frac{1}{2\pi i} \int_{\Gamma_\beta} t^{-z} g(z)\, dz.$$

Here $\Gamma_\beta = \{z \in \mathbb{C};\ \mathrm{Re}\, z = \beta\}$ for all real β. For $s, \gamma \in \mathbb{R}$ let $\mathcal{H}^{s,\gamma}(\mathbb{R}_+)$ be the completion of $C_0^\infty(\mathbb{R}_+)$ with respect to the norm

$$\|u\|_{s,\gamma}^2 = \int_{\Gamma_{1/2-\gamma}} (1 + |z|^2)^s |\mathcal{M}u(z)|^2\, |dz|. \tag{1.1}$$

Then the Mellin transform extends by continuity to linear operators

$$\mathcal{M}_\gamma : \mathcal{H}^{s,\gamma}(\mathbb{R}_+) \to \langle z \rangle^{-s} L^2(\Gamma_{1/2-\gamma}).$$

There is a canonical inner product on $\mathcal{H}^{s,\gamma}(\mathbb{R}_+)$, and $\mathcal{H}^{s,\gamma}(\mathbb{R}_+)$ is a subset of $H^s_{loc}(\mathbb{R}_+)$, the space of distributions that locally belong to the usual Sobolev space of smoothness s on $\mathbb{R}_+$.

A natural class of operators acting between those spaces are the so called Mellin pseudo-differential operators that are defined by

$$\mathrm{op}_M^\gamma(h)u(t) = \{\mathcal{M}_{\gamma,z\to t}^{-1}(h(z)\mathcal{M}_{\gamma,t'\to z}u)\}(t) \tag{1.2}$$

for appropriate functions h on $\Gamma_{1/2-\gamma}$ (for details see Section 2.1).
For a Fréchet space E, which is a left module over an algebra A, we set

$$[a]E = \overline{\{ae;\ e \in E\}}, \qquad a \in A,$$

where the closure is taken in the topology of E. Then define spaces

$$\mathcal{K}^{s,\gamma}(\mathbb{R}_+) = [\omega]\mathcal{H}^{s,\gamma}(\mathbb{R}_+) + [1-\omega]H^s(\mathbb{R}_+)$$

equipped with the topology of a non direct sum of Hilbert spaces, which thus are Hilbert spaces themselves. This construction is independent of the special choice of ω. Especially it holds $\mathcal{K}^{0,0}(\mathbb{R}_+) = L^2(\mathbb{R}_+)$.
For $\gamma \in \mathbb{R}$ and an interval $\Theta =]\vartheta, 0]$, $\vartheta < 0$, we call Q a discrete asymptotic type with respect to (γ, Θ), and write $Q \in \mathrm{As}(\gamma, \Theta)$, if

$$Q = \{(q_j, m_j) \in \mathbb{C} \times \mathbb{N}_0;\ 1/2 - \gamma + \vartheta < \mathrm{Re}\, q_j < 1/2 - \gamma,\ j = 0, \ldots, N\},$$

with some $N \in \mathbb{N}_0$. $Q = \emptyset$ is called empty asymptotic type and will from now on be denoted by $\mathcal{O}$. The projection of Q to the complex plane is written as

$$\pi_{\mathbb{C}} Q = \{q_j;\ j = 0, \ldots, N\}.$$

With such Q we associate finite dimensional vector spaces

$$\tilde{\mathcal{E}}_Q(\mathbb{R}_+) = \Big\{t \mapsto \sum_{j=0}^{N}\sum_{k=0}^{m_j} \xi_{jk} t^{-q_j} \log^k t;\ \xi_{jk} \in \mathbb{C}\Big\}, \qquad [\phi]\tilde{\mathcal{E}}_Q = \{\phi f;\ f \in \tilde{\mathcal{E}}_Q\},$$

for some function ϕ on $\mathbb{R}_+$. If we give

$$\mathcal{K}_\Theta^{s,\gamma}(\mathbb{R}_+) = \cap_{\varepsilon>0}\mathcal{K}^{s,\gamma-\vartheta-\varepsilon}(\mathbb{R}_+)$$

the topology of a projective limit we obtain that

$$\mathcal{K}_Q^{s,\gamma}(\mathbb{R}_+) = \mathcal{K}_\Theta^{s,\gamma}(\mathbb{R}_+) + [\omega]\tilde{\mathcal{E}}_Q(\mathbb{R}_+) = \mathcal{K}_\Theta^{s,\gamma}(\mathbb{R}_+) + \mathcal{E}_Q(\mathbb{R}_+)$$

is a direct sum of Fréchet spaces. If, as in the latter definition, a space constructed with help of $[\omega]\tilde{\mathcal{E}}_Q(\mathbb{R}_+)$ is independent of the choice of ω, we will replace $[\omega]\tilde{\mathcal{E}}_Q(\mathbb{R}_+)$ simply by $\mathcal{E}_Q(\mathbb{R}_+)$. Again as a projective limit we define

$$\mathcal{K}_Q^{\infty,\gamma}(\mathbb{R}_+) = \cap_{s\in\mathbb{R}}\mathcal{K}_Q^{s,\gamma}(\mathbb{R}_+),$$

and finally

$$\mathcal{S}_Q^\gamma(\mathbb{R}_+) = [\omega]\mathcal{K}_Q^{\infty,\gamma}(\mathbb{R}_+) + [1-\omega]\mathcal{S}(\mathbb{R}_+),$$

with arbitrary choice of ω and $\mathcal{S}(\mathbb{R}_+)$ being the space of all restrictions of rapidly decreasing functions to $\mathbb{R}_+$. This vector space can be written as a projective limit of Hilbert spaces, $\mathcal{S}_Q^\gamma(\mathbb{R}_+) = \cap_{k\in\mathbb{N}} E_Q^k(\mathbb{R}_+)$, with

$$E_Q^k(\mathbb{R}_+) = [\omega]\{\mathcal{K}^{k,\gamma-\vartheta-c_k}(\mathbb{R}_+) + \mathcal{E}_Q(\mathbb{R}_+)\} + [1-\omega]\,\langle t\rangle^{-k} H^k(\mathbb{R}_+), \quad k \in \mathbb{N}, \tag{1.3}$$

where $c_k = c_Q/k$, and c_Q is chosen in a way that $\operatorname{Re} q_j > 1/2 - \gamma + \vartheta + c_Q$ for all j. These spaces coincide for various choices of ω.

For a discrete subset D of an open set $U \subset \mathbb{C}$, the function χ is called a D-excision function (with respect to U), if

i) χ is smooth, and $0 \le \chi \le 1$,

ii) there exist open bounded sets U_1, U_2 with $D \subset U_1 \subset \overline{U}_2 \subset U$, such that $\chi \equiv 0$ on U_1, and $\chi \equiv 1$ on $U \setminus U_2$.

By $\mathcal{A}_Q^{s,\gamma}$, we denominate the space of all functions f that are meromorphic in the strip $\{1/2 - \gamma + \vartheta < \operatorname{Re} z < 1/2 - \gamma\}$ with poles in $q_j \in \pi_\mathbb{C} Q$ of order less or equal to $m_j + 1$, and satisfy

i) $\lim_{\delta\to 0+} \langle 1/2 - \gamma - \delta + i\varrho\rangle^s f(1/2 - \gamma - \delta + i\varrho)$ exists in $L^2(\mathbb{R}_\varrho)$,

ii) $\sup\{\|\chi_Q f\|_{s,\beta};\ 1/2 - \gamma + \vartheta + \varepsilon \le \beta < 1/2 - \gamma\} < \infty \quad \forall \varepsilon > 0$.

Here we have used the notation

$$\|g\|_{s,\beta}^2 = \frac{1}{2\pi} \int_{\Gamma_\beta} (1 + |z|^2)^s |g(z)|^2\, |dz|, \tag{1.4}$$

and χ_Q is an arbitrary $\pi_\mathbb{C} Q$-excision function. The topology of $\mathcal{A}(\mathbb{C}\setminus\pi_\mathbb{C} Q)$ and the semi-norms from ii) induce a Fréchet topology on $\mathcal{A}_Q^{s,\gamma}$.

Similarly, $\overline{\mathcal{A}}_Q^{s,\gamma}$ is defined as the space of all functions in $\mathcal{A}_Q^{s,\gamma}$ satisfying in addition

i') $\lim_{\delta\to 0+} \langle 1/2 - \gamma + \vartheta + \delta + i\varrho\rangle^s f(1/2 - \gamma + \vartheta + \delta + i\varrho)$ exists in $L^2(\mathbb{R}_\varrho)$,

ii') $\sup\{\|\chi_Q f\|_{s,\beta};\ 1/2 - \gamma + \vartheta < \beta < 1/2 - \gamma\} < \infty$.

Now, in an obvious manner we get a Fréchet topology, again. To simplify the notation from now on we will omit writing "$\mathbb{R}_+$" in various spaces, e.g., $\mathcal{H}^{s,\gamma}$ instead of $\mathcal{H}^{s,\gamma}(\mathbb{R}_+)$.

1.2 Global pseudo–differential operators

Here we give a summary of the calculus for global pseudo–differential operators with operator valued symbols. A detailed and comprehensive approach to this subject can be found in [4].

1.1 Definition. Let E be a Banach space. A set $\{\kappa_\lambda;\ \lambda > 0\} \subset \mathcal{L}(E)$ of isomorphisms is called a *(strongly continuous) group action* on E if

i) $\kappa_\lambda \kappa_\varrho = \kappa_{\lambda\varrho} \quad \forall \lambda, \varrho > 0$ and $\kappa_1 = \mathrm{id}_E$.

ii) For each $e \in E$ the function $\lambda \mapsto \kappa_\lambda e : \mathbb{R}_+ \to E$ is continuous.

1.2 Example. For each $\lambda > 0$ define mappings $\kappa_\lambda : \mathcal{D}'(\mathbb{R}_+) \to \mathcal{D}'(\mathbb{R}_+)$ by

$$\langle \kappa_\lambda u, \phi \rangle = \langle u, \lambda^{-1/2}\phi(\lambda^{-1}t) \rangle, \quad \phi \in C_0^\infty(\mathbb{R}_+).$$

Note that if $u \in L^1_{loc}(\mathbb{R}_+)$, then $(\kappa_\lambda u)(t) = \lambda^{1/2}u(\lambda t)$. On the spaces $\mathcal{H}^{s,\gamma}$, $\mathcal{K}^{s,\gamma}$, E^k_Q, and $\tilde{\mathcal{E}}_Q$, defined in Section 1.1, we now have group actions arising from the restriction of κ_λ to the corresponding spaces. From now on these group actions will be fixed when dealing with those spaces.

For the following considerations, we fix pairs $(E_j, \{\kappa_{j,\lambda}\})$, $j = 0, 1, 2$, of Banach spaces with corresponding group actions. Furthermore, we choose a smooth and strictly positive function

$$\eta \mapsto [\eta] : \mathbb{R}^q \to \mathbb{R}_+ \quad \text{with} \quad [\eta] = |\eta| \text{ for } |\eta| \geq c$$

for a certain constant $c > 0$ and set for abbreviation

$$\kappa(\eta) = \kappa_{[\eta]}.$$

1.3 Definition. For $\nu, m \in \mathbb{R}$ let

$$S^{\nu,m}(\mathbb{R}^q \times \mathbb{R}^q; E_0, E_1)$$

denote the space of all functions $a \in C^\infty(\mathbb{R}^q \times \mathbb{R}^q, \mathcal{L}(E_0, E_1))$ satisfying

$$p_{\alpha\beta}(a) = \sup_{y,\eta \in \mathbb{R}^q} \left\{ \|\kappa_1^{-1}(\eta)\partial_\eta^\alpha \partial_y^\beta a(y,\eta)\kappa_0(\eta)\|_{E_0,E_1} [\eta]^{|\alpha|-\nu}[y]^{|\beta|-m} \right\} < \infty$$

for all multiindices $\alpha, \beta \in \mathbb{N}_0^q$. The system of semi-norms $p_{\alpha\beta}(\cdot)$ induces a Fréchet topology on $S^{\nu,m}(\mathbb{R}^q \times \mathbb{R}^q; E_0, E_1)$ and the definition is independent of the concrete choice of the function $[\cdot]$. As usual, set

$$S^{-\infty,m}(\mathbb{R}^q \times \mathbb{R}^q; E_0, E_1) = \cap_{\nu \in \mathbf{R}} S^{\nu,m}(\mathbb{R}^q \times \mathbb{R}^q; E_0, E_1)$$

equipped with the topology of a projective limit. Analogously we have spaces $S^{\nu,-\infty}(\mathbb{R}^q \times \mathbb{R}^q; E_0, E_1)$ and $S^{-\infty,-\infty}(\mathbb{R}^q \times \mathbb{R}^q; E_0, E_1)$. Furthermore, for a function a we write

$$a \in S^{\nu,m}(\mathbb{R}^q \times \mathbb{R}^q; E_0, E_1) \text{ for large } |(y,\eta)|,$$

if there exists a $\phi \in C^\infty(\mathbb{R}^{2q})$ with $\phi \equiv 0$ in some neighborhood of 0 and $\phi \equiv 1$ near infinity such that $\phi a \in S^{\nu,m}(\mathbb{R}^q \times \mathbb{R}^q; E_0, E_1)$.

It is easily seen that

$$S^{\nu,m}(\mathbb{R}^q \times \mathbb{R}^q; E_1, E_2) \cdot S^{\nu',m'}(\mathbb{R}^q \times \mathbb{R}^q; E_0, E_1) \subset S^{\nu+\nu',m+m'}(\mathbb{R}^q \times \mathbb{R}^q; E_0, E_2),$$

and in case of $E_1 \hookrightarrow E_2$ and $\kappa_{2,\lambda} = \kappa_{1,\lambda}$ on E_1 it holds

$$S^{\nu,m}(\mathbb{R}^q \times \mathbb{R}^q; E_0, E_1) \hookrightarrow S^{\nu,m}(\mathbb{R}^q \times \mathbb{R}^q; E_0, E_2). \tag{1.5}$$

As in the scalar case we can associate to a given symbol $a \in S^{\nu,m}(\mathbb{R}^q \times \mathbb{R}^q; E_0, E_1)$ a continuous operator

$$\mathrm{op}(a) : \mathcal{S}(\mathbb{R}^q, E_0) \to \mathcal{S}(\mathbb{R}^q, E_1),$$

where $\mathcal{S}(\mathbb{R}^q, E)$ is the Schwartz space of rapidly decreasing functions taking values in a Fréchet space E, by

$$\mathrm{op}(a)u(y) = \{\mathcal{F}^{-1}_{\eta\to y}(a(y,\eta)\mathcal{F}_{y'\to\eta}u)\}(y) = \iint e^{i(y-y')\eta} a(y,\eta)u(y')\, dy' đ\eta.$$

1.4 Theorem. *If* $a \in S^{\nu,m}(\mathbb{R}^q \times \mathbb{R}^q; E_1, E_2)$ *and* $b \in S^{\nu',m'}(\mathbb{R}^q \times \mathbb{R}^q; E_0, E_1)$ *then* $\mathrm{op}(a)\mathrm{op}(b) = \mathrm{op}(a\#b)$*, where for each* $N \in \mathbb{N}$

$$(a\#b)(y,\eta) = \sum_{\alpha<N} \frac{1}{\alpha!}(\partial_\eta^\alpha a)(D_y^\alpha b) \; + \; r_N(y,\eta)$$

with a remainder $r_N \in S^{\nu+\nu'-N, m+m'-N}(\mathbb{R}^q \times \mathbb{R}^q; E_0, E_2)$ *that equals*

$$N \sum_{|\sigma|=N} \int_0^1 \frac{(1-\theta)^{N-1}}{\sigma!} \iint e^{-ix\xi} \partial_\eta^\sigma a(y, \eta + \theta\xi) D_y^\sigma b(y+x, \eta)\, dx đ\xi d\theta.$$

Now we extend the definition of $S^{\nu,m}(\mathbb{R}^q \times \mathbb{R}^q; E_0, E_1)$ to the case of E_1 being a Fréchet space, which can be written as a projective limit

$$E_1 = \cap_{k\in\mathbb{N}} E_1^k$$

with Banach spaces $E_1^1 \hookleftarrow E_1^2 \hookleftarrow \cdots$ such that the group action given on E_1^1 induces the corresponding group action on each E_1^k. Then we set

$$S^{\nu,m}(\mathbb{R}^q \times \mathbb{R}^q; E_0, E_1) = \cap_{k\in\mathbb{N}} S^{\nu,m}(\mathbb{R}^q \times \mathbb{R}^q; E_0, E_1^k) \tag{1.6}$$

equipped with the topology of a projective limit.

1.3 Parameter dependent cut–off operator

We consider the function $\eta \mapsto \omega(t[\eta])$, where the right hand side has to be understood as an operator of multiplication with the function $t \mapsto \omega(t[\eta])$ for each fixed $\eta \in \mathbb{R}^q$. Further denote by $\mathbf{M}_f$ the multiplication with f (in some function space). The here derived Proposition 1.11 will be an important technical tool for later sections.

1.5 Lemma. *Choose* ω_0 *such that* $\omega_0(t)\omega(t[\eta]) = \omega(t[\eta]) \quad \forall \eta \in \mathbb{R}^q,\ \forall t \in \mathbb{R}_+$. *Then we have*

$$\eta \mapsto \omega(t[\eta]) \in C^\infty(\mathbb{R}^q,\ \mathcal{L}(\mathcal{H}^{s,\gamma}, [\omega_0]\mathcal{H}^{s,\gamma})), \qquad \eta \mapsto \omega(t[\eta]) \in C^\infty(\mathbb{R}^q,\ \mathcal{L}(\mathcal{K}^{s,\gamma}, [\omega_0]\mathcal{K}^{s,\gamma})).$$

PROOF: At first it is clear that $\eta \mapsto \omega(t[\eta]) \in C^\infty(\mathbb{R}^q,\ C_0^\infty(\overline{\mathbb{R}}_+))$. In [17], Proposition 7, p. 27 it has be shown the continuity of $\mathbf{M} : C_0^\infty(\overline{\mathbb{R}}_+) \to \mathcal{L}(\mathcal{H}^{s,\gamma}, \mathcal{H}^{s,\gamma})$ and the analogous property with $\mathcal{K}^{s,\gamma}$ instead of $\mathcal{H}^{s,\gamma}$. Now the result follows by construction of ω_0. ∎

1.6 Lemma. *For all $s, \varrho, \gamma \in \mathbb{R}$ the mapping*

$$\phi \mapsto \mathbf{M}_\phi : C_0^\infty(\mathbb{R}_+) \to \mathcal{L}(\mathcal{H}^{s,\gamma}, \mathcal{H}^{s,\varrho})$$

is linear and continuous. Also one could replace $\mathcal{H}^{s,\gamma}$ by $\mathcal{K}^{s,\gamma}$ and $\mathcal{H}^{s,\varrho}$ by $\mathcal{K}^{s,\varrho}$. Further for some asymptotic type Q there is continuity of

$$\phi \mapsto \mathbf{M}_\phi : C_0^\infty(\mathbb{R}_+) \to \mathcal{L}(\tilde{\mathcal{E}}_Q, \mathcal{H}^{s,\gamma}).$$

PROOF: Consider the decomposition $\mathbf{M}_\phi = \mathbf{M}_{t^{\gamma-\varrho}\phi(t)}\mathbf{M}_{t^{-\gamma+\varrho}} =: T_2(\phi)T_1$. Now $T_1 \in \mathcal{L}(\mathcal{H}^{s,\gamma}, \mathcal{H}^{s,\varrho})$ and $\phi \mapsto T_2(\phi) : C_0^\infty(\mathbb{R}_+) \to \mathcal{L}(\mathcal{H}^{s,\varrho}, \mathcal{H}^{s,\varrho})$ is continuous (see [17], Proposition 7, p. 27). The second claim follows from elementary norm calculations using (1.1). ∎

1.7 Definition. For $Q \in \mathrm{As}(\gamma, \Theta)$, $\Theta =]\vartheta, 0]$, and $c_k = c_Q/k$ as in (1.3) we define

$$Z_Q^k = \mathcal{H}^{k,\gamma-\vartheta-c_k} \cap \mathcal{H}^{k,\gamma}, \quad k \in \mathbb{N}.$$

1.8 Lemma. *For $Q \in As(\gamma, \Theta)$ it holds*

a) $\mathbf{M}_\omega \in \mathcal{L}(E_Q^k, Z_Q^k + \mathcal{E}_Q)$,

b) $\omega_n - \omega \to 0$ *in* $C_0^\infty(\mathbb{R}_+)$ *for* $n \to \infty$ *implies* $\mathbf{M}_{\omega_n} \to \mathbf{M}_\omega$ *in* $\mathcal{L}(E_Q^k, Z_Q^k + \mathcal{E}_Q)$ *for* $n \to \infty$,

c) $\phi \mapsto \mathbf{M}_\phi : C_0^\infty(\mathbb{R}_+) \to \mathcal{L}(E_Q^k, Z_Q^k)$ *is linear and continuous,*

a') $\mathbf{M}_\omega \in \mathcal{L}(Z_Q^k + \mathcal{E}_Q, E_Q^k)$,

b') $\omega_n - \omega \to 0$ *in* $C_0^\infty(\mathbb{R}_+)$ *for* $n \to \infty$ *implies* $\mathbf{M}_{\omega_n} \to \mathbf{M}_\omega$ *in* $\mathcal{L}(Z_Q^k + \mathcal{E}_Q, E_Q^k)$ *for* $n \to \infty$,

c') $\phi \mapsto \mathbf{M}_\phi : C_0^\infty(\mathbb{R}_+) \to \mathcal{L}(Z_Q^k + \mathcal{E}_Q, E_Q^k)$ *is linear and continuous.*

PROOF: a),b) For abbreviation set $Z^k = Z_Q^k$. Choose a cut–off function $\tilde{\omega}$ such that $\omega\tilde{\omega} = \omega$ and $\omega_n\tilde{\omega} = \omega_n$. Now let $f_n \to 0$ in E_Q^k. Hence we find sequences $(f_n^1) \subset [\tilde{\omega}]\mathcal{K}^{k,\gamma-\vartheta-c_k}$, $(f_n^2) \subset \mathcal{E}_Q$, $(f_n^3) \subset [1-\tilde{\omega}]\,\langle t\rangle^{-k} H^k(\mathbb{R}_+)$, tending to zero in the corresponding spaces and $f_n = f_n^1 + \tilde{\omega} f_n^2 + f_n^3$. Then

$$\|\mathbf{M}_\omega f_n\|_{Z^k+\mathcal{E}_Q} = \|\omega f_n^1\|_{Z^k} + \|\omega f_n^2\|_{\mathcal{E}_Q} \xrightarrow{n\to\infty} 0$$

(note that $Z^k + \mathcal{E}_Q$ is a direct sum). This shows a). For b) let $f \in E_Q^k$ and $f = f^1 + \tilde{\omega} f^2 + f^3$ with a partition analogous to that of f_n above, and write $Z^k + \mathcal{E}_Q = Z^k + [\omega_0]\tilde{\mathcal{E}}_Q$. Then

$$\begin{aligned} \|(\mathbf{M}_{\omega_n} - \mathbf{M}_\omega)f\|_{Z^k+\mathcal{E}_Q} &= \|(\omega_n - \omega)f^1 - (\omega_n - \omega)f^2\|_{Z^k} \\ &\leq \|\mathbf{M}_{\omega_n-\omega}\|_{\mathcal{K}^{k,\gamma-\vartheta-c_k},Z^k}\|f^1\|_{\mathcal{K}^{k,\gamma-\vartheta-c_k}} + \|\mathbf{M}_{(\omega_n-\omega)\omega_0}\|_{\tilde{\mathcal{E}}_Q,Z^k}\|f^2\|_{\tilde{\mathcal{E}}_Q} \end{aligned}$$

Thus we can conclude that

$$\|\mathbf{M}_{\omega_n} - \mathbf{M}_\omega\|_{E_Q^k, Z^k+\mathcal{E}_Q} \leq \max\left\{\|\mathbf{M}_{\omega_n-\omega}\|_{\mathcal{K}^{k,\gamma-\vartheta-c_k},Z^k}, \|\mathbf{M}_{(\omega_n-\omega)\omega_0}\|_{\tilde{\mathcal{E}}_Q,Z^k}\right\} \xrightarrow{n\to\infty} 0,$$

where the convergence is consequence of Lemma 1.6. The remaining parts of this Lemma can be proved in a similar way. ∎

1.9 Corollary. *For $Q \in As(\gamma, \Theta)$ we obtain*

$$\eta \mapsto \omega(t[\eta]) \in C^\infty(\mathbb{R}^q,\, \mathcal{L}(E^k_Q, Z^k + \mathcal{E}_Q)), \qquad \eta \mapsto \omega(t[\eta]) \in C^\infty(\mathbb{R}^q,\, \mathcal{L}(Z^k + \mathcal{E}_Q, E^k_Q)).$$

Another consequence of Lemma 1.5 and 1.8 is

1.10 Corollary. *For $Q, \mathcal{O} \in As(\gamma, \Theta)$ and each $k \in \mathbb{N}$ we have*

$$\eta \mapsto \omega(t[\eta]) \in C^\infty(\mathbb{R}^q,\, \mathcal{L}(\tilde{\mathcal{E}}_Q, E^k_Q)), \qquad \eta \mapsto \omega(t[\eta]) \in C^\infty(\mathbb{R}^q,\, \mathcal{L}(\mathcal{H}^{k,\gamma-\vartheta}, E^k_{\mathcal{O}})).$$

The mapping $I : \mathcal{D}'(\mathbb{R}_+) \to \mathcal{D}'(\mathbb{R}_+)$ defined by $\langle Iu, \phi\rangle = \langle u, t^{-1}\phi(t^{-1})\rangle$, $\phi \in C_0^\infty(\mathbb{R}_+)$, induces isomorphisms $\mathcal{H}^{s,\gamma} \to \mathcal{H}^{s,-\gamma}$. Then Lemma 1.5 implies that $\eta \mapsto 1 - \omega(t[\eta]) \in C^\infty(\mathbb{R}^q,\, \mathcal{L}(\mathcal{H}^{s,\gamma},\, \mathcal{H}^{s,\gamma+\varepsilon}))$ $\forall \varepsilon \geq 0$. If we now keep in mind that

$$\kappa_\lambda^{-1}\omega(t[\lambda\eta])\kappa_\lambda = \omega(t[\eta]) \tag{1.7}$$

for all $\lambda \geq 1$ and all sufficiently large $|\eta|$, that is $\eta \mapsto \omega(t[\eta])$ is homogeneous of degree 0 for large $|\eta|$, we finally get this

1.11 Proposition. *For $Q \in As(\gamma, \Theta)$, the function $\eta \mapsto \omega(t[\eta])$ is an element of all the following spaces:*

$$S^0(\mathbb{R}^q_\eta;\, \mathcal{H}^{s,\gamma}, \mathcal{H}^{s,\varrho}) \quad \forall\, \varrho \leq \gamma$$

(here we also can replace $\mathcal{H}^{s,\gamma}$ by $\mathcal{K}^{s,\gamma}$, and $\mathcal{H}^{s,\varrho}$ by $\mathcal{K}^{s,\varrho}$) and

$$S^0(\mathbb{R}^q_\eta;\, \tilde{\mathcal{E}}_Q, E^k_Q), \quad S^0(\mathbb{R}^q_\eta;\, \mathcal{H}^{k,\gamma-\vartheta}, E^k_{\mathcal{O}}), \quad S^0(\mathbb{R}^q_\eta;\, E^k_Q, Z^k + \mathcal{E}_Q), \quad S^0(\mathbb{R}^q_\eta;\, Z^k + \mathcal{E}_Q, E^k_Q),$$

where $\mathcal{O}$ is the empty asymptotic type in $As(\gamma, \Theta)$. We also have

$$\eta \mapsto 1 - \omega(t[\eta]) \in S^0(\mathbb{R}^q_\eta;\, \mathcal{H}^{s,\gamma}, \mathcal{H}^{s,\gamma+\varepsilon}) \quad \forall\, \varrho \geq \gamma.$$

2 Green and smoothing Mellin symbols

2.1 Meromorphic Mellin symbols

2.1 Remark. For $p \in \mathbb{C}$ and $k \in \mathbb{N}_0$ set

$$\psi_{p,k}(z) := \mathcal{M}_{t\to z}(\omega(t)t^{-p}\log^k t)(z).$$

Then $\psi_{p,k}$ is a meromorphic function in $\mathbb{C}$ with exactly one pole in p of order $k+1$, admitting a decomposition

$$\psi_{p,k}(z) = (-1)^k k! \frac{1}{(z-p)^{k+1}} + g(z) \tag{2.1}$$

with a certain entire function g. If χ is a p-excision function then

$$\varrho \mapsto \chi(\beta + i\varrho)\psi_{p,k}(\beta + i\varrho) \in \mathcal{S}(\mathbb{R}_\varrho),$$

uniformly for real β in compact intervals.
As the notation implies, all the following constructions will be independent of the choice of ω.

2.2 Proposition. *(cf. [9], Proposition 7.5, Corollary 7.6) The weighted Mellin transform $\mathcal{M}_\gamma$ extends to isomorphisms $\mathcal{H}^{s,\gamma} \cap \mathcal{H}^{s,\gamma-\vartheta} \to \overline{\mathcal{A}}^{s,\gamma}_{\mathcal{O}}$, where $\mathcal{O}$ is the empty asymptotic type corresponding to (γ, Θ), $\Theta =]\vartheta, 0]$.*

2.3 Corollary. *For each $Q \in As(\gamma, \Theta)$ the Mellin transform $\mathcal{M}_\gamma$ induces isomorphisms*

$$\mathcal{M}_\gamma : \left(\mathcal{H}^{s,\gamma} \cap \mathcal{H}^{s,\gamma-\vartheta}\right) + \mathcal{E}_Q \to \overline{\mathcal{A}}^{s,\gamma}_Q.$$

From this it is clear, that $\overline{\mathcal{A}}^{s,\gamma}_Q$ is in fact a Hilbert space.

PROOF: Set $A_Q := \{\mathcal{M}f;\ f \in \mathcal{E}_Q\}$. This is a finite dimensional vector space of meromorphic functions. From Remark 2.1 we derive that $\overline{\mathcal{A}}^{s,\gamma}_Q = \overline{\mathcal{A}}^{s,\gamma}_{\mathcal{O}} + A_Q$ as a direct sum, and this immediately implies the assertion. ∎

2.4 Definition. A set P is called *discrete asymptotic type for Mellin symbols* if

$$P = \{(p_j, m_j) \in \mathbb{C} \times \mathbb{N}_0;\ j \in \mathbb{Z}\}, \quad \operatorname{Re} p_j \to \pm\infty \text{ for } j \to \mp\infty.$$

The projection of P to the complex plane is denoted by

$$\pi_{\mathbb{C}} P = \{p_j;\ j \in \mathbb{Z}\}.$$

To a given P we associate a space of formal series, namely

$$\mathcal{F}_P = \Big\{ \sum_{j\in\mathbb{Z}} \sum_{k=0}^{m_j} c_{jk} \psi_{p_j,k};\ c_{jk} \in \mathbb{C} \Big\}.$$

There is a canonical topology on $\mathcal{F}_P$ since it is isomorphic to $\mathbb{C}^{\mathbb{N}}$. For $f \in \mathcal{F}_P$ and real numbers $c_1 < c_2$ define

$$f_{[c_1,c_2]}(z) = \sum_{\{j;\, c_1 \le Re\, p_j \le c_2\}} \sum_{k=0}^{m_j} c_{jk} \psi_{p_j,k}(z).$$

2.5 Definition. For $\mu \in \mathbb{R}$ let M^μ_P denote the vector space of all functions h that are meromorphic in the complex plane with poles in $p_j \in \pi_{\mathbb{C}} P$ of order at most $m_j + 1$; further there exists an $f \in \mathcal{F}_P$ such that for all $c_1 < c_2$ holds

$$\varrho \mapsto h(\beta + i\varrho) - f_{[c_1,c_2]}(\beta + i\varrho) \in S^\mu(\mathbb{R}_\varrho),$$

uniformly for $\beta \in [c_1, c_2]$.

The element f associated to h in the latter definition is uniquely determined. In fact, if $\sum_{k=0}^{m_j} \sigma_{jk}(z - p_j)^{-(k+1)}$ is the principal part of the Laurent expansion of h in p_j, then f is obtained by setting

$$c_{jk} = \frac{(-1)^k}{k!} \sigma_{jk}, \qquad j \in \mathbb{Z},\ 0 \le k \le m_j.$$

Thus one can define a linear operator

$$T : M_P^\mu \to \mathcal{F}_P : h \mapsto Th := f. \tag{2.2}$$

Now we get a Fréchet topology on M_P^μ by a system of semi–norms consisting of that for the topology of $\mathcal{A}(\mathbb{C} \setminus \pi_{\mathbb{C}} P)$ and

$$\sup_{-c_1^k \leq \beta \leq c_2^k} q\left(h(\beta + i\cdot) - (Th)_{[-c_1^k, c_2^k]}(\beta + i\cdot)\right), \quad k \in \mathbb{N}, \tag{2.3}$$

where $q(\cdot)$ runs through a system of semi-norms of $S^\mu(\mathbb{R})$, and (c_1^k), (c_2^k) are sequences tending to infinity with k. Note that convergence of a sequence (h_n) in M_P^μ implies the convergence of the corresponding sequence of Laurent coefficients (σ_{jk}^n) in $\mathbb{C}$ (and therefore the operator T is continuous).

2.6 Lemma. *For $h \in M_P^\mu$ the function*

$$\beta \mapsto h(\beta + i\cdot) : \mathbb{R} \setminus \mathrm{Re}\,(\pi_{\mathbb{C}} P) \to S^\mu(\mathbb{R})$$

is continuous.

PROOF: By definition $h(\beta + i\cdot) \in S^\mu(\mathbb{R})$ uniformly in $\beta \in K$ for compact sets $K \subset \mathbb{R} \setminus \mathrm{Re}\,(\pi_{\mathbb{C}} P)$. Now the result follows by the fundamental theorem of calculus. ∎

2.7 Lemma. *For each $\gamma \in \mathbb{R}$ with $\pi_{\mathbb{C}} P \cap \Gamma_{1/2-\gamma} = \emptyset$ the mapping*

$$h \mapsto \mathrm{op}_M^\gamma(h) : M_P^\mu \to \mathcal{L}(\mathcal{H}^{s,\gamma}, \mathcal{H}^{s-\mu,\gamma})$$

is linear and continuous for all $s \in \mathbb{R}$.

PROOF: Clearly the mapping $h \mapsto h(1/2 - \gamma + i\cdot) : M_P^\mu \to S^\mu(\mathbb{R})$ is continuous. Now from (1.1) and (1.2) it is immediately seen that

$$\|\mathrm{op}_M^\gamma(h)u\|_{\mathcal{H}^{s-\mu,\gamma}} \leq c \sup_{\varrho \in \mathbb{R}} |h(1/2 - \gamma + i\varrho)\, \langle \varrho \rangle^{-\mu}|\, \|u\|_{\mathcal{H}^{s,\gamma}}.$$

This gives the desired result. ∎

2.8 Definition. For given types $P = \{(p_j, m_j);\ j \in \mathbb{Z}\}$, $P' = \{(p_j', m_j');\ j \in \mathbb{Z}\}$ define

$$P \cdot P' = \{(r_j, n_j);\ r_j \in \pi_{\mathbb{C}} P \cup \pi_{\mathbb{C}} P'\}, \quad n_j = \begin{cases} m_j & ;\text{if } r_j \in \pi_{\mathbb{C}} P \setminus \pi_{\mathbb{C}} P' \\ m_j' & ;\text{if } r_j \in \pi_{\mathbb{C}} P' \setminus \pi_{\mathbb{C}} P \\ m_j + m_j' + 1 & ;\text{if } r_j \in \pi_{\mathbb{C}} P \cap \pi_{\mathbb{C}} P' \end{cases}$$

Analogously we can associate to $Q \in \mathrm{As}(\gamma, \Theta)$ a type $P \cdot Q \in \mathrm{As}(\gamma, \Theta)$.

2.9 Lemma. *The mapping*

$$M_P^\mu \times M_{P'}^{\mu'} \to M_{P.P'}^{\mu+\mu'} : (h, h') \mapsto hh'$$

is bilinear and continuous. Here the product has to be understood as that of meromorphic functions in the complex plane.

PROOF: At first we verify that $hh' \in M_{P.P'}^{\mu+\mu'}$. Therefore we set

$$S(h,h')_{[c_1,c_2]} = h(Th')_{[c_1,c_2]} + (Th)_{[c_1,c_2]}h' + (Th)_{[c_1,c_2]}(Th')_{[c_1,c_2]}.$$

Then $\chi S(h,h')_{[c_1,c_2]}(\beta+i\cdot) \in \mathcal{S}(\mathbb{R})$ uniformly in $\beta \in [c_1,c_2]$ for each $\pi_{\mathbb{C}}P\cdot P'$–excision function χ. Further holds

$$hh' - T(hh')_{[c_1,c_2]} = (h-(Th)_{[c_1,c_2]})(h'-(Th')_{[c_1,c_2]}) - (T(hh')_{[c_1,c_2]} - S(h,h')_{[c_1,c_2]}).$$

Now from $T(hh')_{[c_1,c_2]} - S(h,h')_{[c_1,c_2]}$ being an entire function and $\left[T(hh')_{[c_1,c_2]} - S(h,h')_{[c_1,c_2]}\right](\beta+i\cdot) \in \mathcal{S}(\mathbb{R})$ uniformly in $\beta \in [c_1,c_2]$ we obtain that $(hh' - T(hh')_{[c_1,c_2]})(\beta+i\cdot) \in S^{\mu+\mu'}(\mathbb{R})$ uniformly in $\beta \in [c_1,c_2]$.
Since all the involved spaces are Fréchet spaces, it is sufficient to show that the mapping is separately continuous. To see this we use the closed graph theorem. Assume i) $h'_n \to h'$ in $M_{P'}^{\mu'}$, and ii) $hh'_n \to g$ in $M_{P.P'}^{\mu+\mu'}$. Now we have to show that $hh' = g$. But this is true since i) implies that $hh'_n \to hh'$ in $\mathcal{A}(\mathbb{C}\setminus\pi_{\mathbb{C}}P\cdot P')$ and ii) implies that $hh'_n \to g$ in $\mathcal{A}(\mathbb{C}\setminus\pi_{\mathbb{C}}P\cdot P')$. ∎

2.10 Lemma. *Let $Q \in As(\gamma,\Theta)$ and P an asymptotic type with $\pi_{\mathbb{C}}P \cap \Gamma_{1/2-\gamma} = \pi_{\mathbb{C}}P \cap \Gamma_{1/2-\gamma+\vartheta} = \emptyset$. Then the operator of multiplication*

$$h \mapsto \mathbf{M}_h : M_P^{\mu} \to \mathcal{L}(\overline{\mathcal{A}}_Q^{s,\gamma}, \overline{\mathcal{A}}_{P.Q}^{s-\mu,\gamma})$$

is linear and continuous.

PROOF: For $f \in \overline{\mathcal{A}}_P^{s,\gamma}$ obviously $hf \in \mathcal{A}(\{1/2-\gamma+\vartheta < \operatorname{Re} z < 1/2-\gamma\}\setminus\pi_{\mathbb{C}}P\cdot Q)$. Furthermore hf satisfies conditions i), i') from Section 1.1 (here formulated with respect to $\overline{\mathcal{A}}_{P.Q}^{s-\mu,\gamma}$) because of Lemma 2.6. If χ_Q and χ_P are $\pi_{\mathbb{C}}Q$– and $\pi_{\mathbb{C}}P$–excision functions, respectively, then for $\chi = \chi_Q\chi_P$ holds

$$\begin{aligned} &\sup\{\|\chi(hf)\|_{s-\mu,\beta};\ 1/2-\gamma+\vartheta < \beta < 1/2-\gamma\} \\ \leq\ &\sup\{|(\chi_P h)(\beta+i\varrho)\,\langle\varrho\rangle^{-\mu}|;\ 1/2-\gamma+\vartheta<\beta<1/2-\gamma,\ \varrho\in\mathbb{R}\} \\ &\cdot\sup\{\|\chi_Q f\|_{s,\beta};\ 1/2-\gamma+\vartheta<\beta<1/2-\gamma\} \end{aligned}$$

From Definition 2.5 it is clear that the first supremum on the right–hand side is finite. This shows $hf \in \overline{\mathcal{A}}_{P.Q}^{s-\mu,\gamma}$. Finally both the continuity of $\mathbf{M}_h$ and $h \mapsto \mathbf{M}_h$ follow easily by the closed graph theorem. ∎

2.11 Definition. For $\mu, m \in \mathbb{R}$ let $M_P^{\mu,m}$ denote the space of all functions $h \in C^\infty(\mathbb{R}^q, M_P^\mu)$ satisfying:

i) If $\sum_{k=0}^{m_j} \sigma_{jk}(y)(z-p_j)^{-(k+1)}$ is the principal part of the Laurent expansion of $h(y)$ in $p_j \in \pi_{\mathbb{C}}P$, then

$$\sup_{y\in\mathbb{R}^q} |\langle y\rangle^{|\beta|-m}\,\partial_y^\beta \sigma_{jk}(y)| < \infty \quad \forall\,\beta\in\mathbb{N}_0^q. \tag{2.4}$$

ii) For all $c_1 < c_2 \in \mathbb{R}$

$$(y,\varrho) \mapsto h(y,\beta+i\varrho) - (Th)_{[c_1,c_2]}(y,\beta+i\varrho) \in S^{\mu,m}(\mathbb{R}_y^q\times\mathbb{R}_\varrho)$$

uniformly in $\beta \in [c_1,c_2]$. Here T is the operator introduced in (2.2).

$M_P^{\mu,m}$ is a Fréchet space with the semi-norms of $C^\infty(\mathbb{R}^q, M_P^\mu)$, that from (2.4) and that induced from ii), cf. the construction in (2.3). As usual we define $M_P^{-\infty,m} = \cap_{\mu\in\mathbb{R}} M_P^{\mu,m}$ and analogously $M_P^{\mu,-\infty}$, $M_P^{-\infty,-\infty}$, equipped with the topology of projective limits.

2.12 Remark. Obviously (2.4) is equivalent to

$$\sup_{y\in\mathbb{R}^q} |\langle y\rangle^{|\beta|-m}\, \partial_y^\beta c_{jk}(y)| < \infty \quad \forall \beta \in \mathbb{N}_0^q,$$

if $c_{jk}(y)$ are the coefficients of $Th(y)$. Beside the coefficients of the principal part of $h(y)$, also each Laurent coefficient $\sigma(y)$ of $h(y)$ in some point $p \in \mathbb{C}$ satisfies the estimates (2.4). To verify this let first $p = p_j \in \pi_{\mathbb{C}} P$. If we choose c_1, c_2 such that $\operatorname{Re} p_j \in [c_1, c_2]$ then by (2.1)

$$\left(\frac{d}{dz}\right)^l \{h - (Th)_{[c_1,c_2]}\}(y, p_j) = (l-1)!\sigma_l(y) + \sum_{\{j;\, c_1 \le Re\, p_j \le c_2\}} \sum_{k=0}^{m_j} c_{jk}(y) g_{jkl}(p_j),$$

with certain entire functions g_{jkl}, and $\sigma_l(y)$ being the l–th coefficient of the holomorphic part of the Laurent expansion of $h(y)$ in p_j. Now both the left–hand side and the second term on the right–hand side are elements of $S^m(\mathbb{R}_y^q)$ and hence $\sigma_l(y)$ is. The case $p \in \mathbb{C} \setminus \pi_{\mathbb{C}} P$ can be treated similarly.

2.13 Proposition. *Let two asymptotic types P, P' be given. Then*

a) *$h \in M_P^{\mu,m}$ implies $\partial_y^\beta h \in M_P^{\mu,m-|\beta|}$.*

b) *The mapping*

$$M_P^{\mu,m} \times M_{P'}^{\mu',m'} \to M_{P.P'}^{\mu+\mu',m+m'} : (h, h') \mapsto hh',$$

is bilinear and continuous.

c) *If $h \in M_P^\mu$ and $\gamma \in \mathbb{R}$ with $\pi_{\mathbb{C}} P \cap \Gamma_{1/2-\gamma} = \emptyset$, then*

$$y \mapsto \mathrm{op}_M^\gamma(h)(y) \in C^\infty(\mathbb{R}^q, \mathcal{L}(\mathcal{H}^{s,\gamma}, \mathcal{H}^{s-\mu,\gamma})).$$

Further it holds

$$\partial_y^\beta \mathrm{op}_M^\gamma(h)(y) = \mathrm{op}_M^\gamma(\partial_y^\beta h)(y),$$

$$\|\partial_y^\beta \mathrm{op}_M^\gamma(h)(y)\|_{\mathcal{H}^{s,\gamma},\mathcal{H}^{s-\mu,\gamma}} \le c_\beta \langle y\rangle^{m-|\beta|} \quad \forall y \in \mathbb{R}^q. \tag{2.5}$$

d) *If $Q \in As(\gamma, \Theta)$ and $h \in M_P^{\mu,m}$ with $\pi_{\mathbb{C}} P \cap \Gamma_{1/2-\gamma} = \pi_{\mathbb{C}} P \cap \Gamma_{1/2-\gamma+\vartheta} = \emptyset$, then*

$$y \mapsto \mathbf{M}_{h(y)} \in C^\infty(\mathbb{R}^q, \mathcal{L}(\overline{\mathcal{A}}_Q^{s,\gamma}, \overline{\mathcal{A}}_{P.Q}^{s-\mu,\gamma})).$$

Further it holds

$$\partial_y^\beta \mathbf{M}_{h(y)} = \mathbf{M}_{\partial_y^\beta h(y)} \quad \textit{and} \quad \|\partial_y^\beta \mathbf{M}_{h(y)}\| \le c\, \langle y\rangle^{m-|\beta|} \quad \forall y \in \mathbb{R}^q.$$

PROOF: a) is elementary.
b) By Lemma 2.9 is $hh' \in C^\infty(\mathbb{R}^q, M^{\mu+\mu'}_{P.P'})$. Further hh' satisfies i) from Definition 2.11 by Remark 2.12. As in the proof of Lemma 2.9 it can be seen that hh' satisfies 2.11.ii) and that the mapping is separately continuous.
c) follows from Lemma 2.7 (including the norm estimate in the proof).
d) follows from Lemma 2.10 and the fact that

$$p(\partial_y^\beta h(y)) \leq c\, \langle y\rangle^{m-|\beta|} \quad \forall y \in \mathbb{R}^q$$

for each semi-norm $p(\cdot)$ of M^μ_P. ∎

We finish this section by stating three remarks which will be useful later on.

2.14 Remark. Let $h \in M^{\mu,m}_P$ and $\gamma, N \in \mathbb{R}$ with $\pi_{\mathbb{C}} P \cap \Gamma_{1/2-\gamma} = \emptyset$. Then

$$\mathrm{op}^\gamma_M(h)(y) t^N = t^N \mathrm{op}^{\gamma-N}_M (T^{-N} h)(y)$$

as operators $\mathcal{H}^{s,\gamma-N} \to \mathcal{H}^{s-\mu,\gamma}$. Here t^N has to be understood as the operator of multiplication $\mathbf{M}_{t^N} : \mathcal{H}^{s,\gamma} \to \mathcal{H}^{s,\gamma+N}$ and $(T^\sigma h)(y,z) = h(y, z+\sigma)$. (For a proof see [16], Part I, Remark 4.2.6).

2.15 Remark. For $h \in M^{\mu,m}_P$ set $h^{(*)}(y,z) = \overline{h(y, 1-\bar{z})}$ (the reason for the notation $(*)$ will become clear from Lemma 2.22). Then $h^{(*)} \in M^{\mu,m}_{P^{(*)}}$ with $P^{(*)} = \{(1-\overline{p_j}, m_j);\ j \in \mathbb{Z}\}$.

2.16 Remark. If $h \in M^\mu_P$ and $\pi_{\mathbb{C}} P \cap \Gamma_{1/2-\gamma} = \emptyset$ it is easy to verify that

$$\kappa_\lambda^{-1} \mathrm{op}^\gamma_M(h) \kappa_\lambda = \mathrm{op}^\gamma_M(h).$$

2.2 Associated operator valued symbols

2.17 Definition. For $\nu, m \in \mathbb{R} \cup \{-\infty\}$ and some given weight-data $\underline{g} = (\gamma, \delta, \Theta)$, let $R^{\nu,m}_G(\mathbb{R}^q \times \mathbb{R}^q, \underline{g})$ denote the space of *Green symbols*, i.e., all functions

$$g(y,\eta) \in \cap_{s\in\mathbb{R}} S^{\nu,m}(\mathbb{R}^q \times \mathbb{R}^q; \mathcal{K}^{s,\gamma}, \mathcal{K}^{\infty,\delta})$$

with the property

$$g \in \cap_{s\in\mathbb{R}} S^{\nu,m}(\mathbb{R}^q \times \mathbb{R}^q; \mathcal{K}^{s,\gamma}, \mathcal{S}^\delta_{Q_1}),$$
$$g^* \in \cap_{s\in\mathbb{R}} S^{\nu,m}(\mathbb{R}^q \times \mathbb{R}^q; \mathcal{K}^{s,-\delta}, \mathcal{S}^{-\gamma}_{Q_2}),$$

for certain asymptotic types $Q_1 \in \mathrm{As}(\delta,\Theta)$, $Q_2 \in \mathrm{As}(-\gamma,\Theta)$ (depending on g). Here * denotes the formal adjoint with respect to $(\cdot,\cdot)_{\mathcal{K}^{0,0}} : \mathcal{K}^{s,\varrho} \times \mathcal{K}^{-s,-\varrho} \to \mathbb{C}$.

2.18 Lemma. *Let weight-data $\underline{g} = (\gamma, \delta, \Theta)$, $\underline{g}' = (\delta, \varrho, \Theta)$, and $N \in \mathbb{R}$ be given. Then the following inclusions hold:*

a) $\partial_\eta^\alpha \partial_y^\beta R^{\nu,m}_G(\mathbb{R}^q \times \mathbb{R}^q, \underline{g}) \subset R^{\nu-|\alpha|, m-|\beta|}_G(\mathbb{R}^q \times \mathbb{R}^q, \underline{g})$,

b) $R^{\nu',m'}_G(\mathbb{R}^q \times \mathbb{R}^q, \underline{g}') \cdot R^{\nu,m}_G(\mathbb{R}^q \times \mathbb{R}^q, \underline{g}) \subset R^{\nu+\nu', m+m'}_G(\mathbb{R}^q \times \mathbb{R}^q, (\gamma, \varrho, \Theta))$,

c) *If* $\tilde{\gamma} \geq \gamma$ *and* $\tilde{\delta} \leq \delta$ *then* $R_G^{\nu,m}(\mathbb{R}^q \times \mathbb{R}^q, \underline{g}) \subset R_G^{\nu,m}(\mathbb{R}^q \times \mathbb{R}^q, (\tilde{\gamma}, \tilde{\delta}, \Theta))$,

d) $R_G^{\nu,m}(\mathbb{R}^q \times \mathbb{R}^q, \underline{g}) \subset R_G^{\nu,m}(\mathbb{R}^q \times \mathbb{R}^q, (\gamma, \delta,]\vartheta + \tau, 0]))\quad \forall 0 \leq \tau < -\vartheta$,

e) $t^N \omega(t[\eta]) R_G^{\nu,m}(\mathbb{R}^q \times \mathbb{R}^q, \underline{g}) \subset R_G^{\nu-N,m}(\mathbb{R}^q \times \mathbb{R}^q, (\gamma, \delta + N, \Theta))$ *and* $R_G^{\nu,m}(\mathbb{R}^q \times \mathbb{R}^q, \underline{g}) t^N \omega(t[\eta]) \subset R_G^{\nu-N,m}(\mathbb{R}^q \times \mathbb{R}^q, (\gamma - N, \delta, \Theta))$.

PROOF: a), b) are elementary.
c), d) follow from $\mathcal{K}^{s,\tilde{\gamma}} \hookrightarrow \mathcal{K}^{s,\gamma}$ and that for each $Q \in \mathrm{As}(\gamma, \Theta)$ there is a $\tilde{Q} \in \mathrm{As}(\tilde{\delta}, \Theta)$ $(\tilde{Q} \in \mathrm{As}(\delta,]\vartheta + \tau, 0]))$ such that $E_Q^k \hookrightarrow E_{\tilde{Q}}^k \quad \forall k \in \mathbb{N}$.
e) Using Proposition 1.11 and the notations of Definition 1.7, the claim follows from

$$t^N (Z_Q^k + \mathcal{E}_Q) = (\mathcal{H}^{k,(\gamma+N)-\vartheta-c_k} \cap \mathcal{H}^{k,\gamma+N}) + \mathcal{E}_{\tilde{Q}} = Z_{\tilde{Q}}^k + \mathcal{E}_{\tilde{Q}}$$

with $\tilde{Q} = \{(q_j - N, m_j);\, q_j \in Q\} \in \mathrm{As}(\gamma + N, \Theta)$, and the fact that $\kappa^{-1}(\eta) t^N \kappa(\eta) = [\eta]^{-N} t^N$. ∎

2.19 Proposition. *Let* $\gamma, \nu \in \mathbb{R}$ *and* $A(y,\eta) = t^{-\nu} \omega(t[\eta]) \mathrm{op}_M^\gamma(h)(y) \tilde{\omega}(t[\eta])$ *with* $h \in M_P^{-\infty,m}$ *and* $\pi_{\mathbb{C}} P \cap \Gamma_{1/2-\gamma} = \emptyset$. *Then we have*

$$A \in \cap_{s,r \in \mathbf{R}} S^{\nu,m}(\mathbb{R}^q \times \mathbb{R}^q; \mathcal{K}^{s,\gamma}, \mathcal{K}^{r,\gamma-\nu}).$$

Further for each $Q_1 \in As(\gamma, \Theta)$ *there is a* $Q_2 \in As(\gamma - \nu, \Theta)$ *such that*

$$A \in \cap_{k \in \mathbf{N}} S^{\nu,m}(\mathbb{R}^q \times \mathbb{R}^q; E_{Q_1}^k, E_{Q_2}^k).$$

(Here we assume that $c_{Q_1} = c_{P\cdot Q_1} = c_{Q_2}$, *cf. (1.3))*

PROOF: The first claim is an immediate consequence of Remark 2.16 and Propositions 1.11, 2.13.c). By Proposition 1.11 is $\tilde{\omega}(t[\eta]) \in S^0(\mathbb{R}_\eta^q; E_{Q_1}^k, Z_{Q_1}^k + \mathcal{E}_{Q_1})$. Now by Corollary 2.3 $\mathcal{M}_\gamma :$ $Z_{Q_1}^k + \mathcal{E}_{Q_1} \to \overline{\mathcal{A}}_{Q_1}^{k,\gamma-\vartheta-c_k}$ isomorphically. In view of $\mathrm{op}_M^\gamma(h)(y) = \mathcal{M}_\gamma^{-1} \mathrm{M}_{h(y)} \mathcal{M}_\gamma$, Proposition 2.13.d), and Remark 2.16, we obtain $\mathrm{op}_M^\gamma(h)(y) \in S^{0,m}(\mathbb{R}^q \times \mathbb{R}^q; Z_{Q_1}^k + \mathcal{E}_{Q_1}, Z_{P\cdot Q_1}^k + \mathcal{E}_{P\cdot Q_1})$. Furthermore, $\omega(t[\eta]) \in S^0(\mathbb{R}_\eta^q; Z_{P\cdot Q_1}^k + \mathcal{E}_{P\cdot Q_1}, E_{P\cdot Q_1}^k)$ by Proposition 1.11. Finally, as in the proof of 2.18.e), the factor $t^{-\nu}$ causes a translation of the type $P \cdot Q_1$, and the order ν (with respect to η) in the symbol estimates of $A(y,\eta)$. ∎

2.20 Definition. Let data $\underline{g} = (\gamma, \gamma - \mu, \Theta)$, $\Theta =]-k, 0]$, with $\gamma, \mu \in \mathbb{R}$ and $k \in \mathbb{N}$ be given. Further let $\nu \in \mathbb{R}$ with $\mu - \nu \in \mathbb{N}_0$. A function m on $\mathbb{R}^q \times \mathbb{R}^q$ is called a *smoothing Mellin symbol* (of order (ν, m) with respect to $\underline{g}$) if it has a representation

$$\begin{aligned} m(y,\eta) &= \omega(t[\eta]) \sum_{j=0}^{k+\nu-\mu-1} t^{-\nu+j} \sum_{|\alpha|\leq j} \mathrm{op}_M^{\gamma_{j\alpha}}(h_{j\alpha})(y) \eta^\alpha \tilde{\omega}(t[\eta]) \\ &= \sum_{0 \leq |\alpha| \leq j \leq k+\nu-\mu-1} A_{j\alpha}(y,\eta), \\ & \qquad A_{j\alpha}(y,\eta) = \omega(t[\eta]) t^{-\nu+j} \mathrm{op}_M^{\gamma_{j\alpha}}(h_{j\alpha})(y) \eta^\alpha \tilde{\omega}(t[\eta]) \end{aligned}$$

with

$$h_{j\alpha} \in M_{P_{j\alpha}}^{-\infty,m}, \quad \pi_{\mathbb{C}} P_{j\alpha} \cap \Gamma_{1/2-\gamma_{j\alpha}} = \emptyset, \quad \gamma - (\mu - \nu) - j \leq \gamma_{j\alpha} \leq \gamma. \tag{2.6}$$

Note that $m \equiv 0$ if $\mu - \nu \geq k$. The reason for taking $k + \nu - \mu - 1$ as upper summation bound is that terms corresponding to larger coefficients are Green symbols, as will be proved in Proposition 2.30.

2.21 Definition. Let ν, $\underline{g} = (\gamma, \gamma - \mu, \Theta)$, be as in Definition 2.20. The space of all functions $m + g$ with $g \in R_G^{\nu,m}(\mathbb{R}^q \times \mathbb{R}^q, \underline{g})$ and m as described in 2.20 is denoted by

$$R_{M+G}^{\nu,m}(\mathbb{R}^q \times \mathbb{R}^q, \underline{g}).$$

Note that in case of $\mu - \nu \geq k$ we have $R_{M+G}^{\nu,m}(\mathbb{R}^q \times \mathbb{R}^q, \underline{g}) \subset R_G^{\nu,m}(\mathbb{R}^q \times \mathbb{R}^q, \underline{g})$.
The *conormal symbol* of order $\nu - j$ of $m + g$ is defined by

$$\sigma_M^{\nu-j}(m+g)(y,z,\eta) = \sum_{|\alpha| \leq j} h_{j\alpha}(y,z)\eta^\alpha, \qquad 0 \leq j \leq k + \nu - \mu - 1.$$

From the cone calculus it is seen that the notion of conormal symbols is well defined (cf. [16], Part II, Proposition 3.1.27).

2.22 Lemma. *For $\nu, \underline{g}$ as in Definition 2.20 and $\underline{g}^{(*)} = (-\gamma + \mu, -\gamma, \Theta)$ is*

$$\{R_{M+G}^{\nu,m}(\mathbb{R}^q \times \mathbb{R}^q, \underline{g})\}^* = R_{M+G}^{\nu,m}(\mathbb{R}^q \times \mathbb{R}^q, \underline{g}^{(*)}),$$

that means $R_{M+G}^{\nu,m}(\mathbb{R}^q \times \mathbb{R}^q, \cdot)$ is 'closed' under pointwise formal adjoint.

PROOF: From the definition of Green symbols it is obvious that

$$\{R_G^{\nu,m}(\mathbb{R}^q \times \mathbb{R}^q, \underline{g})\}^* = R_G^{\nu,m}(\mathbb{R}^q \times \mathbb{R}^q, \underline{g}^{(*)}).$$

Thus we only have to consider the formal adjoint of an operator $A_{j\alpha}$, cf. Definition 2.20. Now it is known, see e.g., [16], Part I, Lemma 5.1.10, that

$$\begin{aligned} A_{j\alpha}(y,\eta)^* &= \tilde{\omega}(t[\eta])\mathrm{op}_M^{-\gamma_{j\alpha}}(h_{j\alpha}^{(*)})(y)t^{-\nu+j}\eta^\alpha\omega(t[\eta]) \\ &= \tilde{\omega}(t[\eta])t^{-\nu+j}\mathrm{op}_M^{-\gamma_{j\alpha}+\nu-j}(T^{\nu-j}h_{j\alpha}^{(*)})(y)\eta^\alpha\omega(t[\eta]). \end{aligned}$$

For the second equation we used Remark 2.14. In view of Remark 2.15 the desired result follows. ∎

2.3 The behaviour under weight shifts

Here we establish a number of lemmas concerning mainly (the interaction of) smoothing Mellin symbols and their behaviour under weight shifts.

2.23 Lemma. *Let $h \in M_P^{-\infty,m}$, $\gamma \in \mathbb{R}$ with $\pi_{\mathbb{C}} P \cap \Gamma_{1/2-\gamma} = \emptyset$ and $g_0' \in R_G^{\nu,m'}(\mathbb{R}^q \times \mathbb{R}^q, (\gamma, \delta, \Theta))$, $g_1' \in R_G^{\nu,m'}(\mathbb{R}^q \times \mathbb{R}^q, (\delta, \gamma, \Theta))$ be given and set*

$$g_0(y,\eta) = g_0'(y,\eta)\omega(t[\eta])\mathrm{op}_M^\gamma(h)(y)\tilde{\omega}(t[\eta]), \qquad g_1(y,\eta) = \omega(t[\eta])\mathrm{op}_M^\gamma(h)(y)\tilde{\omega}(t[\eta])g_1'(y,\eta).$$

Then $g_0 \in R_G^{\nu,m+m'}(\mathbb{R}^q \times \mathbb{R}^q, (\gamma, \delta, \Theta))$ and $g_1 \in R_G^{\nu,m+m'}(\mathbb{R}^q \times \mathbb{R}^q, (\delta, \gamma, \Theta))$.

PROOF: First, it is clear that $g_0 \in S^{\nu,m+m'}(\mathbb{R}^q \times \mathbb{R}^q; \mathcal{K}^{s,\gamma}, E^k_Q)$, $k \in \mathbb{N}$, for an appropriate type $Q \in \mathrm{As}(\delta, \Theta)$. Also $g_1 \in S^{\nu,m'}(\mathbb{R}^q \times \mathbb{R}^q; \mathcal{K}^{s,\delta}, E^k_{Q_1})$, $k \in \mathbb{N}$, for a certain $Q_1 \in \mathrm{As}(\gamma, \Theta)$ by Proposition 2.19. Thus the statement follows since g_j^* is of the same form as g_{1-j} $(j = 0, 1)$. ∎

2.24 Proposition. *Let $h \in M_P^{\mu,m}$ and $\delta, \gamma \in \mathbb{R}$ with $\pi_{\mathbb{C}} P \cap \Gamma_{1/2-\gamma} = \pi_{\mathbb{C}} P \cap \Gamma_{1/2-\delta} = \emptyset$. Further set $\tau = \min(\gamma, \delta)$, $\varrho = \max(\gamma, \delta)$, $\Theta =]-|\gamma - \delta|, 0]$, and*

$$G(y) = \mathrm{op}_M^{\gamma}(h)(y) - \mathrm{op}_M^{\delta}(h)(y).$$

Then there exist asymptotic types $R_1 \in As(\tau, \Theta)$ and $R_2 \in As(-\varrho, \Theta)$ such that

$$G \in S^{0,m}(\mathbb{R}^q \times \mathbb{R}^q; \mathcal{H}^{s,\delta} \cap \mathcal{H}^{s,\gamma}, \tilde{\mathcal{E}}_{R_1}), \qquad G^* \in S^{0,m}(\mathbb{R}^q \times \mathbb{R}^q; \mathcal{H}^{s,-\delta} \cap \mathcal{H}^{s,-\gamma}, \tilde{\mathcal{E}}_{R_2}).$$

If $\pi_{\mathbb{C}} P \cap \Gamma_{1/2-\beta} = \emptyset$ for all real β between γ and δ then $G \equiv 0$.

PROOF: (cf. [19] Theorem 1.1.55). Let $\delta \geq \gamma$ and $u \in C_0^\infty(\mathbb{R}_+)$. Let p_j, $j = 1, \ldots, m$, be those poles of h with $1/2 - \delta < \mathrm{Re}\, p_j < 1/2 - \gamma$, and $\sum_{k=0}^{m_j} \sigma_{jk}(y)(z - p_j)^{-(k+1)}$ be the principal part of the Laurent expansion of h in p_j, and $\sum_{k=0}^{m_j} \xi_{jk}(z - p_j)^k$ be a part of the expansion of $\mathcal{M}u$ in p_j. Now

$$G(y)u(t) = \frac{1}{2\pi i} \int_C t^{-z} h(y,z)(\mathcal{M}u)(z)\, dz$$

with a contour C surrounding the poles of h in the strip $\{1/2 - \delta < \mathrm{Re}\, z < 1/2 - \gamma\}$. From this we get

$$G(y)u(t) = \sum_{j=0}^{m} \sum_{l=0}^{m_j} \frac{(-1)^l}{l!} d_{jl}(y) t^{-p_j} \log^l t,$$

where

$$d_{jl}(y) = \sum_{m-k=l} \sigma_{jk}(y) \xi_{jm},$$

this is $G(y)u \in \tilde{\mathcal{E}}_{R_1}$ with $R_1 = \{(p_j, m_j);\ j = 1, \ldots, m\}$. From the continuity of the mapping $u \mapsto \xi_{jk} : \mathcal{H}^{s,\delta} \cap \mathcal{H}^{s,\gamma} \to \mathbb{C}$ we obtain

$$\|G(y)u\|_{\tilde{\mathcal{E}}_{R_1}} \sim \sum_{j=0}^{m} \sum_{l=0}^{m_j} \frac{1}{l!} |d_{jl}(y)| \leq c \langle y \rangle^m \|u\|_{\mathcal{H}^{s,\delta} \cap \mathcal{H}^{s,\gamma}}.$$

The derivatives can be treated in the same way, since

$$\partial^\beta G(y) = \mathrm{op}_M^{\gamma}(\partial^\beta h)(y) - \mathrm{op}_M^{\delta}(\partial^\beta h)(y).$$

Hence the result follows by density of $C_0^\infty(\mathbb{R}_+)$ in $\mathcal{H}^{s,\delta} \cap \mathcal{H}^{s,\gamma}$. For treating G^* note that $G^*(y) = \mathrm{op}_M^{-\gamma}(h^{(*)})(y) - \mathrm{op}_M^{-\delta}(h^{(*)})(y)$. ∎

For notational convenience we now set $\Theta_N =]-N, 0]$ for each $N > 0$.

2.25 Corollary. *Let $h \in M_P^{\mu,m}$ and $\delta, \gamma \in \mathbb{R}$ with $\pi_{\mathbb{C}} P \cap \Gamma_{1/2-\gamma} = \pi_{\mathbb{C}} P \cap \Gamma_{1/2-\delta} = \emptyset$. Then the function*

$$g(y,\eta) = \omega(t[\eta]) \left\{ \mathrm{op}_M^\gamma(h)(y) - \mathrm{op}_M^\delta(h)(y) \right\} \tilde{\omega}(t[\eta])$$

is for each $N > 0$ an element of

$$R_G^{0,m}(\mathbb{R}^q \times \mathbb{R}^q, (\max(\gamma,\delta), \min(\gamma,\delta), \Theta_N)).$$

If $\pi_{\mathbb{C}} P \cap \Gamma_{1/2-\beta} = \emptyset$ for all real β between γ and δ then $g \equiv 0$.

PROOF: With notations from Proposition 2.24, one obtains at first that for each $N > 0$ there is a $Q_1 \in \mathrm{As}(\tau, \Theta_N)$ such that $\omega(t[\eta]) \in S^0(\mathbb{R}^q_\eta; \tilde{\mathcal{E}}_{R_1}, E^k_{Q_1})$ for all $k \in \mathbb{N}_0$. Further $\tilde{\omega}(t[\eta]) \in S^0(\mathbb{R}^q_\eta; \mathcal{K}^{s,\delta}, \mathcal{H}^{s,\delta} \cap \mathcal{H}^{s,\gamma})$. Hence the result simply follows from $g(y,\eta) = \omega(t[\eta]) G(y) \tilde{\omega}(t[\eta])$. The formal adjoint g^* is treated analogously. ∎

2.26 Corollary. *Let $h \in M_P^{\mu,m}$ and $\delta, \gamma \in \mathbb{R}$ with $\pi_{\mathbb{C}} P \cap \Gamma_{1/2-\gamma} = \pi_{\mathbb{C}} P \cap \Gamma_{1/2-\gamma+\delta} = \emptyset$. Then the function*

$$g(y,\eta) = \omega(t[\eta]) \left\{ t^\delta \mathrm{op}_M^\gamma(h)(y) - \mathrm{op}_M^\gamma(T^\delta h)(y) t^\delta \right\} \tilde{\omega}(t[\eta])$$

is for each $N > 0$ an element of

$$R_G^{-\delta,m}(\mathbb{R}^q \times \mathbb{R}^q, (\max(\gamma-\delta,\gamma), \min(\gamma+\delta,\gamma), \Theta_N)).$$

If $\pi_{\mathbb{C}} P \cap \Gamma_{1/2-\beta} = \emptyset$ for all real β between γ and $\gamma - \delta$ then $g \equiv 0$.

PROOF: Follows from Remark 2.14, Corollary 2.25 and Lemma 2.18.e). ∎

2.27 Lemma. *Let $h \in M_P^{-\infty,m}$ and $\gamma \in \mathbb{R}$ with $\pi_{\mathbb{C}} P \cap \Gamma_{1/2-\gamma} = \emptyset$ and $\phi \in C_0^\infty(\mathbb{R}_+)$. Then the functions*

$$g_0(y,\eta) = \phi(t[\eta]) \mathrm{op}_M^\gamma(h)(y) \omega(t[\eta]), \qquad g_1(y,\eta) = \omega(t[\eta]) \mathrm{op}_M^\gamma(h)(y) \phi(t[\eta])$$

are for each $N > 0$ elements of

$$R_G^{0,m}(\mathbb{R}^q \times \mathbb{R}^q, (\gamma, \gamma, \Theta_N)).$$

PROOF: Because of Propositions 1.11, 2.19 and Lemma 1.6, it is obvious that $g_0 \in S^{0,m}(\mathbb{R}^q \times \mathbb{R}^q; \mathcal{K}^{s,\gamma}, E^k_{\mathcal{O}})$, $\quad k \in \mathbb{N}$, where $\mathcal{O}$ is the empty assymptotic type. Choosing $\tilde{\omega}$ such that $\tilde{\omega}\phi = \phi$, we have a decomposition of g_1 as

$$g_1(y,\eta) = \omega(t[\eta]) \left\{ \mathrm{op}_M^\gamma(h)(y) - \mathrm{op}_M^{\gamma+N}(h)(y) \right\} \tilde{\omega}(t[\eta]) \phi(t[\eta]) + \omega(t[\eta]) \mathrm{op}_M^{\gamma+N}(h)(y) \tilde{\omega}(t[\eta]) \phi(t[\eta]).$$

Here we can without loss of generality assume that $\pi_{\mathbb{C}} P \cap \Gamma_{1/2-\gamma-N} = \emptyset$; otherwise we could replace N bei $N + \varepsilon$ with an appropriate $\varepsilon > 0$ and use Lemma 2.18.d). Now, by Lemma 1.6 and Corollary 2.25, the first term on the right-hand side is an element of $S^{0,m}(\mathbb{R}^q \times \mathbb{R}^q; \mathcal{K}^{s,\gamma}, E^k_Q)$, $\quad k \in \mathbb{N}$, with a certain $Q \in \mathrm{As}(\gamma, \Theta_N)$. By Lemma 1.6 and Propositions 2.13.c), 1.11 the second term is an element of $S^{0,m}(\mathbb{R}^q \times \mathbb{R}^q; \mathcal{K}^{s,\gamma}, E^k_{\mathcal{O}})$, $\quad k \in \mathbb{N}$. This shows $g_1 \in S^{0,m}(\mathbb{R}^q \times \mathbb{R}^q; \mathcal{K}^{s,\gamma}, E^k_Q)$, $\quad k \in \mathbb{N}$. Finally note that g_j^* is an operator of the form g_{1-j} $(j = 0, 1)$. ∎

2.28 Lemma. *Let $h \in M_P^{-\infty,m}$, $h' \in M_{P'}^{-\infty,m'}$ and $\gamma \le \delta \in \mathbb{R}$ with $\pi_{\mathbb{C}} P \cap \Gamma_{1/2-\gamma} = \pi_{\mathbb{C}} P' \cap \Gamma_{1/2-\delta} = \emptyset$. Then for each choice of γ_0, $\delta_0 \in \mathbb{R}$ with $\gamma \le \gamma_0 \le \delta_0 \le \delta$ and $\pi_{\mathbb{C}} P \cap \Gamma_{1/2-\gamma_0} = \pi_{\mathbb{C}} P' \cap \Gamma_{1/2-\delta_0} = \emptyset$ the function*

$$g(y,\eta) = \omega(t[\eta]) \left\{ \mathrm{op}_M^\gamma(h)(y)\omega_1(t[\eta])\mathrm{op}_M^\delta(h')(y) - \mathrm{op}_M^{\gamma_0}(h)(y)\omega_1(t[\eta])\mathrm{op}_M^{\delta_0}(h')(y) \right\} \tilde{\omega}(t[\eta])$$

is for each $N > 0$ an element of

$$R_G^{0,m+m'}(\mathbb{R}^q \times \mathbb{R}^q, (\delta, \gamma, \Theta_N)).$$

PROOF: Choose ω_2 such that $\omega_2\omega_1 = \omega_1$. Then the statement follows from the decomposition

$$\begin{aligned} g(y,\eta) &= \omega(t[\eta])\mathrm{op}_M^\gamma(h)(y)\omega_2(t[\eta]) \left[\omega_1(t[\eta]) \{ \mathrm{op}_M^\delta(h')(y) - \mathrm{op}_M^{\delta_0}(h')(y) \} \tilde{\omega}(t[\eta]) \right] + \\ &+ \left[\omega(t[\eta]) \{ \mathrm{op}_M^\gamma(h)(y) - \mathrm{op}_M^{\gamma_0}(h)(y) \} \omega_1(t[\eta]) \right] \omega_2(t[\eta])\mathrm{op}_M^{\delta_0}(h')(y)\tilde{\omega}(t[\eta]) \end{aligned}$$

and application of Corollary 2.25, Lemma 2.23, and Lemma 2.18.c). ∎

2.29 Lemma. *Let $h \in M_P^{-\infty,m}$, $h' \in M_{P'}^{-\infty,m'}$ and $\gamma \in \mathbb{R}$ with $\pi_{\mathbb{C}} P \cap \Gamma_{1/2-\gamma} = \pi_{\mathbb{C}} P' \cap \Gamma_{1/2-\gamma} = \emptyset$. Then the function*

$$g(y,\eta) = \omega(t[\eta])\mathrm{op}_M^\gamma(h)(y)(1 - \omega_1(t[\eta]))\mathrm{op}_M^\gamma(h')(y)\omega_2(t[\eta])$$

is for each $N > 0$ an element of

$$R_G^{0,m+m'}(\mathbb{R}^q \times \mathbb{R}^q, (\gamma, \gamma, \Theta_N)).$$

PROOF: Consider the decomposition

$$\begin{aligned} g(y,\eta) &= \omega(t[\eta]) \{ \mathrm{op}_M^\gamma(h)(y) - \mathrm{op}_M^{\gamma+N}(h)(y) \} (1 - \omega_1(t[\eta]))\mathrm{op}_M^\gamma(h')(y)\omega_2(t[\eta]) \\ &+ \omega(t[\eta])\mathrm{op}_M^{\gamma+N}(h)(y)(1 - \omega_1(t[\eta]))\mathrm{op}_M^\gamma(h')(y)\omega_2(t[\eta]). \end{aligned}$$

By using Propositions 1.11, 2.24 it can be verified, in analogy to the proofs of the latter propositions of this section, that the first term is an element of $S^{0,m+m'}(\mathbb{R}^q \times \mathbb{R}^q; \mathcal{K}^{s,\gamma}, E_Q^k)$, $k \in \mathbb{N}$, for a certain $Q \in \mathrm{As}(\gamma, \Theta_N)$, and the second one of $S^{0,m+m'}(\mathbb{R}^q \times \mathbb{R}^q; \mathcal{K}^{s,\gamma}, E_{\mathcal{O}}^k)$, $k \in \mathbb{N}$. Also g^* can be handled in this way since it is of the same type as g. ∎

2.30 Proposition. *Let data $\underline{g} = (\gamma, \gamma - \mu, \Theta)$, $\Theta =]-k, 0]$ with $k \in \mathbb{N}$, and $h \in M_P^{-\infty,m}$ be given. Further let $\delta, N \in \mathbb{R}$ with $\pi_{\mathbb{C}} P \cap \Gamma_{1/2-\delta} = \emptyset$, $k \le \mu + N \in \mathbb{N}_0$, and $\gamma - \mu - N \le \delta \le \gamma$. Then*

$$g(y,\eta) := \omega(t[\eta])t^N \mathrm{op}_M^\delta(h)(y)\tilde{\omega}(t[\eta]) \in R_G^{-N,m}(\mathbb{R}^q \times \mathbb{R}^q, \underline{g}).$$

PROOF: First we will show that $g \in S^{-N,m}(\mathbb{R}^q \times \mathbb{R}^q; \mathcal{K}^{s,\gamma}, \mathcal{S}_Q^{\gamma-\mu})$ for a certain type $Q \in \mathrm{As}(\gamma - \mu, \Theta)$. Consider the case of $\delta = \gamma$. Then $t^N \mathrm{op}_M^\delta(h)(y)\tilde{\omega}(t[\eta]) \in S^{-N,m}(\mathbb{R}^q \times \mathbb{R}^q; \mathcal{K}^{s,\gamma}, \mathcal{H}^{r,\gamma+N})$ for all $r \in \mathbb{R}$. Since $N + \mu \ge k$, Proposition 1.11 shows that $g \in S^{-N,m}(\mathbb{R}^q \times \mathbb{R}^q; \mathcal{K}^{s,\gamma}, E_{\mathcal{O}}^l)$, $l \in \mathbb{N}$. If $\delta < \gamma$, we find an $\varepsilon_0 > 0$ such that $\delta < \gamma - \varepsilon_0$ and $\pi_{\mathbb{C}} P \cap \Gamma_{1/2-\gamma+\varepsilon} = \emptyset$ for all $0 < \varepsilon \le \varepsilon_0$. By Corollary 2.25 and Lemma 2.18.c), e)

$$g_1(y,\eta) = \omega(t[\eta])t^N \{ \mathrm{op}_M^{\gamma-\varepsilon}(h)(y) - \mathrm{op}_M^\delta(h)(y) \} \tilde{\omega}(t[\eta])$$

is an element of $R_G^{-N,m}(\mathbb{R}^q \times \mathbb{R}^q, (\gamma-\varepsilon, \gamma-\mu, \Theta))$ (note that $\delta + N \geq \gamma - \mu$) and is independent of ε (as an operator family on $\mathcal{K}^{s,\gamma}$). But this implies $g_1 \in S^{-N,m}(\mathbb{R}^q \times \mathbb{R}^q; \mathcal{K}^{s,\gamma}, \mathcal{S}_Q^{\gamma-\mu})$ for a certain $Q \in \mathrm{As}(\gamma-\mu, \Theta)$. Furthermore,

$$g_2(y,\eta) = \omega(t[\eta]) t^N \mathrm{op}_M^{\gamma-\varepsilon}(h)(y) \tilde{\omega}(t[\eta])$$

is in $S^{-N,m}(\mathbb{R}^q \times \mathbb{R}^q; \mathcal{K}^{s,\gamma}, \mathcal{K}^{r,\gamma+N-\varepsilon})$, $r \in \mathbb{R}$, and is independent of ε. This shows $g_2 \in S^{-N,m}(\mathbb{R}^q \times \mathbb{R}^q; \mathcal{K}^{s,\gamma}, \mathcal{S}_O^{\gamma-\mu})$. Hence the statement at the beginning of the proof is true since $g = g_1 + g_2$. The formal adjoint $g(y,\eta)^* = \tilde{\omega}(t[\eta]) t^N \mathrm{op}_M^{-\delta-N}(T^{-N} h^{(*)})(y) \omega(t[\eta])$ is treated analogously, by distinguishing the cases of $\delta = \gamma - \mu - N$ and $\delta > \gamma - \mu - N$. ∎

2.4 Differentiation and composition

2.31 Theorem. *Let weight-data $\underline{g} = (\gamma, \gamma-\mu, \Theta)$, $\Theta =]-k, 0]$, be given. Then*

$$\partial_\eta^\alpha \partial_y^\beta R_{M+G}^{\nu,m}(\mathbb{R}^q \times \mathbb{R}^q, \underline{g}) \subset R_{M+G}^{\nu-|\alpha|,m-|\beta|}(\mathbb{R}^q \times \mathbb{R}^q, \underline{g}).$$

In case of $|\alpha| \geq k + \nu - \mu$ we even have

$$\partial_\eta^\alpha \partial_y^\beta R_{M+G}^{\nu,m}(\mathbb{R}^q \times \mathbb{R}^q, \underline{g}) \subset R_{G}^{\nu-|\alpha|,m-|\beta|}(\mathbb{R}^q \times \mathbb{R}^q, \underline{g}).$$

PROOF: Since $\partial_y^\beta M_P^{-\infty,m} \subset M_P^{-\infty,m-|\beta|}$ we can assume that $\beta = 0$. By Lemma 2.18.a), we only have to take a look at smoothing Mellin symbols. With the notations from Definition 2.20 we get

$$\begin{aligned}
\partial_{\eta_l} m(y,\eta) &= \omega(t[\eta]) t^{-\nu} \sum_{j=0}^{k+\nu-\mu-1} \partial_{\eta_l} m_j(y,\eta) \tilde{\omega}(t[\eta]) + \\
&\quad + (\partial_{\eta_l}[\eta])(\partial_t \omega)(t[\eta]) t^{-(\nu-1)} \sum_{j=0}^{k+\nu-\mu-1} m_j(y,\eta) \tilde{\omega}(t[\eta]) + \\
&\quad + (\partial_{\eta_l}[\eta]) \omega(t[\eta]) t^{-\nu} \sum_{j=0}^{k+\nu-\mu-1} m_j(y,\eta) (\partial_t \tilde{\omega})(t[\eta]) t,
\end{aligned}$$

where $m_j(y,\eta) = t^j \sum_{|\alpha| \leq j} \mathrm{op}_M^{\gamma_{j\alpha}}(h_{j\alpha})(y) \eta^\alpha$. Now $[\eta] \in S^1(\mathbb{R}_\eta^q)$ and we obtain by Lemma 2.27 and Lemma 2.18.c), e), that the second and third term on the right-hand side are elements of $R_G^{\nu-1,m}(\mathbb{R}^q \times \mathbb{R}^q, \underline{g})$. Now an induction shows that $\partial_\eta^\alpha m(y,\eta)$ equals

$$\omega(t[\eta]) \sum_{j=0}^{k+(\nu-|\alpha|)-\mu-1} t^{-(\nu-|\alpha|)+j} \sum_{|\sigma| \leq j} \mathrm{op}_M^{\tilde{\gamma}_{j\sigma}}(\tilde{h}_{j\sigma})(y) \eta^\sigma \tilde{\omega}(t[\eta])$$

modulo a remainder in $R_G^{\nu-|\alpha|,m}(\mathbb{R}^q \times \mathbb{R}^q, \underline{g})$, with $\tilde{\gamma}_{j\sigma} = \gamma_{j+|\alpha|,\sigma+\alpha}$, and $\tilde{h}_{j\sigma} = \frac{(\alpha+\sigma)!}{\alpha!} h_{j+|\alpha|,\sigma+\alpha}$. Hence $\partial_\eta^\alpha m \in R_{M+G}^{\nu-|\alpha|,m}(\mathbb{R}^q \times \mathbb{R}^q; \underline{g})$, cf. Definition 2.20. ∎

2.32 Theorem. *Let* $m+g \in R^{\nu,m}_{M+G}(\mathbb{R}^q \times \mathbb{R}^q, \underline{g})$ *with* $\underline{g} = (\gamma, \gamma-\mu, \Theta)$, $\Theta =]-k,0]$, *and* $m'+g' \in R^{\nu',m'}_{M+G}(\mathbb{R}^q \times \mathbb{R}^q, \underline{g}')$ *with* $\underline{g}' = (\gamma-\mu, \gamma-\mu-\mu', \Theta)$. *Then with* $\underline{g}_0 = (\gamma, \gamma-\mu-\mu', \Theta)$ *the pointwise product satisfies*

$$(m'+g')(m+g) \in R^{\nu+\nu',m+m'}_{M+G}(\mathbb{R}^q \times \mathbb{R}^q, \underline{g}_0).$$

If one of the factors is a Green symbol, so is the product.

PROOF: In view of Lemma 2.23 and Lemma 2.18.c), e), we have $g'm,\ m'g \in R^{\nu+\nu',m+m'}_{G}(\mathbb{R}^q \times\mathbb{R}^q, \underline{g}_0)$. The same is true for $g'g$ because of Lemma 2.18.b). Thus we only have to consider the term $m'm$. Using notations as in Definition 2.20, we have to take a look at products

$$\begin{aligned} A'_{l\beta}(y,\eta)A_{j\alpha}(y,\eta) &= \omega_0(t[\eta])t^{-(\nu+\nu')+(j+l)}\mathrm{op}_M^{\gamma'_{l\beta}+\nu-j}(T^{\nu-j}h'_{l\beta})(y)\eta^{\beta}\tilde{\omega}_0(t[\eta])\cdot \\ &\quad \cdot\omega(t[\eta])\mathrm{op}_M^{\gamma_{j\alpha}}(h_{j\alpha})(y)\eta^{\alpha}\tilde{\omega}(t[\eta]) \end{aligned}$$

(here Remark 2.14 was used). Now from the conditions (2.6) we obtain

$$\gamma-[(\mu+\mu')-(\nu+\nu')]-(l+j) \le \gamma'_{l\beta}+\nu-j \le \gamma-(\mu-\nu)-j \le \gamma_{j\alpha} \le \gamma.$$

In view of Lemma 2.28 (and Lemma 2.18.c), e)) we can assume that $\gamma_{j\alpha} = \gamma'_{l\beta}+\nu-j = \sigma$ with some $\gamma-[(\mu+\mu')-(\nu+\nu')]-(l+j) \le \sigma \le \gamma$, since changing of weights causes only a remainder $g_1 \in R_G^{\nu+\nu'-(j+l)+|\alpha|+|\beta|,m+m'}(\mathbb{R}^q \times \mathbb{R}^q, \underline{g}_0)$. Also only a remainder $g_2 \in R_G^{\nu+\nu'-(j+l)+|\alpha|+|\beta|,m+m'}(\mathbb{R}^q \times \mathbb{R}^q, \underline{g}_0)$ arises if we omit $\tilde{\omega}_0(t[\eta])\omega(t[\eta])$, cf. Lemma 2.29. Hence $A'_{l\beta}(y,\eta)A_{j\alpha}(y,\eta)$ equals

$$\omega_0(t[\eta])t^{-(\nu+\nu')+(j+l)}\mathrm{op}_M^{\sigma}((T^{\nu-j}h'_{l\beta})h_{j\alpha})(y)\eta^{\alpha+\beta}\tilde{\omega}(t[\eta]) + g_1 + g_2$$

and this finishes the proof. ∎

2.33 Remark. With the notations and the proof of Theorem 2.32 we obtain a formula for the behaviour of conormal symbols under multiplication:

$$\begin{aligned} \sigma_M^{\nu+\nu'-j}((m'+g')(m+g))(y,z,\eta) &= \sum_{p+q=j}\left[(T^{\nu-q}\sigma_M^{\nu'-p}(m'+g'))\sigma_M^{\nu-q}(m+g)\right](y,z,\eta) \quad (2.7) \\ &= \sum_{|\sigma|\le j}\left(\sum_{p+q=j}\ \sum_{\substack{|\alpha|\le q, |\beta|\le p \\ \alpha+\beta=\sigma}}(T^{\nu-q}h'_{p\beta})h_{q\alpha}\right)(y,z)\eta^{\sigma}, \end{aligned}$$

for $0 \le j \le k+\nu+\nu'-\mu-\mu'-1$. From this it is seen that for given $m \in R^{\nu,m}_{M+G}(\mathbb{R}^q \times \mathbb{R}^q, \underline{g})$, $m'' \in R^{\nu+\nu',m+m'}_{M+G}(\mathbb{R}^q \times \mathbb{R}^q, \underline{g}_0)$, there exists (for fixed cut-off functions) at most one $m' \in R^{\nu',m'}_{M+G}(\mathbb{R}^q \times \mathbb{R}^q, \underline{g}')$ such that

$$m'm = m'' \quad \text{modulo } R_G^{\nu+\nu',m+m'}(\mathbb{R}^q \times \mathbb{R}^q, \underline{g}_0).$$

In fact the conormal symbols of m' can be discovered from solving the equations

$$\sigma_M^{\nu+\nu'-j}(m'm) = \sigma_M^{\nu+\nu'-j}(m'')$$

with help of the above multiplication rule (2.6).

3 A subalgebra of global boundary value problems

3.1 Green and smoothing Mellin operators

3.1 Theorem. *Let the notations be as in Theorem 2.32. Then there exists an* $m_0 + g_0 \in R^{\nu+\nu',m+m'}_{M+G}(\mathbb{R}^q \times \mathbb{R}^q, \underline{g}_0)$ *such that*

$$\mathrm{op}(m' + g')\mathrm{op}(m + g) = \mathrm{op}(m_0 + g_0).$$

Further $m \equiv 0$ *or* $m' \equiv 0$ *implies* $m_0 \equiv 0$.

PROOF: From Proposition 2.19 we know that for each $Q_1 \in \mathrm{As}(\gamma, \Theta)$ there exists an $Q_2 \in \mathrm{As}(\gamma-\mu, \Theta)$ such that $m \in S^{\nu,m}(\mathbb{R}^q \times \mathbb{R}^q; E^k_{Q_1}, E^k_{Q_2})$, $k \in \mathbb{N}$. An analogous statement holds for m'. Hence from Theorem 1.4 it is immediately seen that $\mathrm{op}(g')\mathrm{op}(g) = \mathrm{op}(h_1)$, $\mathrm{op}(m')\mathrm{op}(g) = \mathrm{op}(h_2)$, $\mathrm{op}(g')\mathrm{op}(m) = \mathrm{op}(h_3)$ with certain $h_j \in R^{\nu+\nu',m+m'}_G(\mathbb{R}^q \times \mathbb{R}^q, \underline{g}_0)$ (note that * can be pulled under the integral). For considering $m' \# m$ we use Theorem 1.4 (with $N \geq k + \nu' - \mu'$) and derive from Theorem 2.31 that $\partial^\sigma_\eta m' \in R^{\nu'-N,m'}_G(\mathbb{R}^q \times \mathbb{R}^q, \underline{g}')$. Thus from Theorems 2.31, 2.32 it follows that $m' \# m \in R^{\nu+\nu',m+m'}_{M+G}(\mathbb{R}^q \times \mathbb{R}^q, \underline{g}_0)$. ∎

3.2 Theorem. *Let* $m + g \in R^{\nu,m}_{M+G}(\mathbb{R}^q \times \mathbb{R}^q, \underline{g})$ *with* $\underline{g} = (\gamma, \gamma - \mu, \Theta)$, $\Theta =]-k, 0]$. *By* $\langle \cdot, \cdot \rangle$ *denote the scalar product in* $L^2(\mathbb{R}^q, \mathcal{K}^{0,0})$. *Then there exists an* $m_0 + g_0 \in R^{\nu,m}_{M+G}(\mathbb{R}^q \times \mathbb{R}^q, \underline{g}^{(*)})$, *where* $\underline{g}^{(*)} = (-\gamma + \mu, -\gamma, \Theta)$, *such that*

$$\langle \mathrm{op}(m + g)u, v \rangle = \langle u, \mathrm{op}(m_0 + g_0)v \rangle$$

for all $u \in \mathcal{S}(\mathbb{R}^q, \mathcal{K}^{s,\gamma})$, $v \in \mathcal{S}(\mathbb{R}^q, \mathcal{K}^{r,-\gamma+\mu})$ *and arbitrary* $r, s \in \mathbb{R}$.

PROOF: Writing $m(y, \eta)$ as in Definition 2.20, we set

$$m^*(y, \eta) = \sum_{0 \leq |\alpha| \leq j \leq k+\nu-\mu-1} A_{j\alpha}(y, \eta)^*$$

with $A_{j\alpha}(y, \eta)^*$ as in the proof of Lemma 2.22. Now, using standard techniques for calculating formal adjoints of global pseudo-differential operators (for further details see, e.g. [4]), we obtain

$$(m_0 + g_0)(y, \eta) = \sum_{|\alpha| < N} \frac{1}{\alpha!} \partial^\alpha_\eta D^\alpha_y (m^* + g^*)(y, \eta) \quad + \quad r^{(*)}_N(y, \eta)$$

for each $N \in \mathbb{N}$, with a remainder

$$r^{(*)}_N(y, \eta) = N \sum_{|\sigma| = N} \int \frac{(1-\theta)^{N-1}}{\sigma!} \iint e^{-ix\xi} \partial^\sigma_\eta D^\sigma_y (m^* + g^*)(x + y, \eta + \theta\xi) \, dx đ\xi d\theta.$$

Choosing $N > k + \nu - \mu$, the presence of the differentiations $\partial^\sigma_\eta D^\sigma_y$ in connection with Theorem 2.31 then implies that $r^{(*)}_N \in R^{\nu-N,m-N}_G(\mathbb{R}^q \times \mathbb{R}^q, \underline{g}^{(*)})$. Hence $m_0 + g_0 \in R^{\nu,m}_{M+G}(\mathbb{R}^q \times \mathbb{R}^q, \underline{g}^{(*)})$ by Lemma 2.22 and Theorem 2.31. ∎

3.2 Block matrices including trace and potential conditions

To deal with boundary value problems it is necessary to introduce matrices of operator functions with Green and smoothing Mellin symbols as entries in the left upper corner.

3.3 Definition. Let $\nu, m \in \mathbb{R} \cup \{-\infty\}$, $N_+, N_- \in \mathbb{N}_0$ and $\underline{g} = (\gamma, \delta, \Theta)$. A function $\mathbf{g}(y,\eta) \in \cap_{s\in\mathbf{R}} S^{\nu,m}(\mathbb{R}^q \times \mathbb{R}^q; \mathcal{K}^{s,\gamma} \oplus \mathbb{C}^{N_-}, \mathcal{K}^{\infty,\delta} \oplus \mathbb{C}^{N_+})$ satisfying

$$\mathbf{g} \in \cap_{s\in\mathbf{R}} S^{\nu,m}(\mathbb{R}^q \times \mathbb{R}^q; \mathcal{K}^{s,\gamma} \oplus \mathbb{C}^{N_-}, \mathcal{S}^{\delta}_{Q_1} \oplus \mathbb{C}^{N_+}),$$

$$\mathbf{g}^* \in \cap_{s\in\mathbf{R}} S^{\nu,m}(\mathbb{R}^q \times \mathbb{R}^q; \mathcal{K}^{s,-\delta} \oplus \mathbb{C}^{N_+}, \mathcal{S}^{-\gamma}_{Q_2} \oplus \mathbb{C}^{N_-})$$

for certain asymptotic types $Q_1 \in \mathrm{As}(\delta, \Theta)$, $Q_2 \in \mathrm{As}(-\gamma, \Theta)$ depending on $\mathbf{g}$ is called a *Green symbol with trace and potential part.* As a rule, for a space E with group action $\{\kappa_\lambda\}$ we associate with $E \oplus \mathbb{C}^N$ the action $\{\kappa_\lambda \oplus 1\}$. Further * means the pointwise formal adjoint in the sense of

$$(gu, v)_{\mathcal{K}^{0,0}\oplus\mathbb{C}^{N_+}} = (u, g^*v)_{\mathcal{K}^{0,0}\oplus\mathbb{C}^{N_-}}$$

for all $u \in C_0^\infty(\mathbb{R}_+) \oplus \mathbb{C}^{N_-}$ and $v \in C_0^\infty(\mathbb{R}_+) \oplus \mathbb{C}^{N_+}$. The space of all such functions $\mathbf{g}$ is denoted by

$$R_G^{\nu,m}(\mathbb{R}^q \times \mathbb{R}^q, \underline{g}; N_-, N_+).$$

3.4 Remark. Each element of $R_G^{\nu,m}(\mathbb{R}^q \times \mathbb{R}^q, \underline{g}; N_-, N_+)$ can be written as

$$\mathbf{g}(y,\eta) = \begin{pmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{pmatrix}(y,\eta) : \begin{matrix} \mathcal{K}^{s,\gamma} \\ \oplus \\ \mathbb{C}^{N_-} \end{matrix} \longrightarrow \begin{matrix} \mathcal{S}^{\delta}_{Q_1} \\ \oplus \\ \mathbb{C}^{N_+} \end{matrix}.$$

We call g_{12} a potential and g_{21} a trace symbol. Clearly $g_{11} \in R_G^{\nu,m}(\mathbb{R}^q \times \mathbb{R}^q, \underline{g})$ and g_{22} is an $(N_+ \times N_-)$-matrix with entries from $S^{\nu,m}(\mathbb{R}^q \times \mathbb{R}^q)$.

3.5 Definition. Let $\underline{g} = (\gamma, \gamma - \mu, \Theta)$, $\Theta =]-k, 0]$, with $\gamma, \mu \in \mathbb{R}$ and $k \in \mathbb{N}$. Further let $\nu \in \mathbb{R}$ with $\mu - \nu \in \mathbb{N}_0$ and $N_+, N_- \in \mathbb{N}_0$. Now

$$R_{M+G}^{\nu,m}(\mathbb{R}^q \times \mathbb{R}^q, \underline{g}; N_-, N_+)$$

denotes the space of all functions

$$\mathbf{m} + \mathbf{g} = \begin{pmatrix} m & 0 \\ 0 & 0 \end{pmatrix} + \mathbf{g}$$

with a smoothing Mellin symbol m corresponding to ν and the data $\underline{g}$, cf. Definition 2.20, and $\mathbf{g} \in R_G^{\nu,m}(\mathbb{R}^q \times \mathbb{R}^q, \underline{g}; N_-, N_+)$. The corresponding conormal symbols are given by

$$\sigma_M^{\nu-j}(\mathbf{m} + \mathbf{g}) = \sigma_M^{\nu-j}(m).$$

Now it is straightforward to generalize the Theorems 2.31, 2.32 and 3.1 to the block matrix situation, especially we have the following

3.6 Theorem. *Let* $\mathbf{m}+\mathbf{g} \in R^{\nu,m}_{M+G}(\mathbb{R}^q \times \mathbb{R}^q, \underline{g}; N_-, M)$ *with* $\underline{g} = (\gamma, \gamma-\mu, \Theta)$, $\Theta =]-k, 0]$, *and* $\mathbf{m}'+\mathbf{g}' \in R^{\nu',m'}_{M+G}(\mathbb{R}^q \times \mathbb{R}^q, \underline{g}'; M, N_+)$ *with* $\underline{g}' = (\gamma-\mu, \gamma-\mu-\mu', \Theta)$. *Then with* $\underline{g}_0 = (\gamma, \gamma-\mu-\mu', \Theta)$, *there is an* $\mathbf{m}_0+\mathbf{g}_0 \in R^{\nu+\nu',m+m'}_{M+G}(\mathbb{R}^q \times \mathbb{R}^q, \underline{g}_0; N_-, N_+)$ *such that*

$$\mathrm{op}(\mathbf{m}'+\mathbf{g}')\mathrm{op}(\mathbf{m}+\mathbf{g}) = \mathrm{op}(\mathbf{m}_0+\mathbf{g}_0).$$

We have $\mathbf{m}_0+\mathbf{g}_0 = (\mathbf{m}'+\mathbf{g}')(\mathbf{m}+\mathbf{g})+\mathbf{r}$ *with a certain remainder* $\mathbf{r} \in R^{\nu+\nu'-1,m+m'-1}_{M+G}(\mathbb{R}^q \times \mathbb{R}^q, \underline{g}_0; N_-, N_+)$. *Furthermore, if* $\mathbf{m} \equiv 0$ *or* $\mathbf{m}' \equiv 0$ *then* $\mathbf{m}_0 \equiv 0$.

3.3 Ellipticity and the nature of parametrices

In the following let $\underline{g} = (0, 0, \Theta)$, $\Theta =]-k, 0]$, be fixed.

3.7 Definition. Let 1 denote the identity operator $E \to E$ and $\mathbf{1} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$, viewed as an operator $E \oplus F \to E \oplus G$ for various spaces E, F, G.

In view of Theorem 3.6 the operators

$$\mathrm{op}(\mathbf{1}+\mathbf{m}+\mathbf{g}), \quad \mathbf{m}+\mathbf{g} \in R^{0,0}_{M+G}(\mathbb{R}^q \times \mathbb{R}^q, \underline{g}; N_-, N_+),$$

form a subalgebra of operators $\mathcal{S}(\mathbb{R}^q, L^2(\mathbb{R}_+) \oplus \mathbb{C}^{N_-}) \to \mathcal{S}(\mathbb{R}^q, L^2(\mathbb{R}_+) \oplus \mathbb{C}^{N_+})$. The aim of this paragraph is to find a notion of ellipticity which is equivalent to the existence of a parametrix.

3.8 Definition. Let $\mathbf{m}+\mathbf{g} \in R^{0,0}_{M+G}(\mathbb{R}^q \times \mathbb{R}^q, \underline{g}; N_-, N_+)$. The symbol $\mathbf{1}+\mathbf{m}+\mathbf{g}$ is called *elliptic* if

i) there exists an asymptotic type P with $\pi_{\mathbb{C}} P \cap \Gamma_{1/2} = \emptyset$ and

$$(1+\sigma^0_M(\mathbf{m}+\mathbf{g}))^{-1} \in M^{0,0}_P,$$

ii) for large $|(y,\eta)|$

$$(\mathbf{1}+\mathbf{m}+\mathbf{g})(y,\eta) : \begin{matrix} L^2(\mathbb{R}_+) \\ \oplus \\ \mathbb{C}^{N_-} \end{matrix} \longrightarrow \begin{matrix} L^2(\mathbb{R}_+) \\ \oplus \\ \mathbb{C}^{N_+} \end{matrix}$$

is invertible and the inverse is uniformly bounded in (y,η).

3.9 Remark. a) Condition i) of the ellipticity actually contains three assumptions on the conormal symbol of $\mathbf{1}+\mathbf{m}+\mathbf{g}$. At first, the independence of the asymptotic type P of y, which does not hold in general. It should be mentioned that SCHULZE developed a calculus allowing non–constant asymptotic types (see e.g. [17], keyword: continuous asymptotics), to which we plan to extend the present results. Second, the y–independence of the poles induces that $(1+\sigma^0_M(\mathbf{m}+\mathbf{g}))^{-1} \in C^\infty(\mathbb{R}^q, M^0_P)$ for an appropriate type P, but we will need additional conditions on the functions $(y,\varrho) \mapsto 1+\sigma^0_M(\mathbf{m}+\mathbf{g})(y, \beta+i\varrho)$ in order to ensure $(1+\sigma^0_M(\mathbf{m}+\mathbf{g}))^{-1} \in M^{0,0}_P$. Finally, the poles must stay away from the critical weight–line $\Gamma_{1/2}$ to allow the construction of $\mathrm{op}^0_M((1+\sigma^0_M(\mathbf{m}+\mathbf{g}))^{-1})$.

b) Condition ii) of the ellipticity guarantees that

$$(\mathbf{1}+\mathbf{m}+\mathbf{g})^{-1} \in S^{0,0}(\mathbb{R}^q \times \mathbb{R}^q; L^2(\mathbb{R}_+) \oplus \mathbb{C}^{N_-}, L^2(\mathbb{R}_+) \oplus \mathbb{C}^{N_+}) \quad \text{for large } |(y,\eta)|,$$

cf. the notations from Definition 1.3. This simply follows by the chain rule, and the unitarity of the standard group action on $L^2(\mathbb{R}_+)$.

3.10 Theorem. *For* $\mathbf{m}+\mathbf{g} \in R^{0,0}_{M+G}(\mathbb{R}^q \times \mathbb{R}^q, \underline{g}; N_-, N_+)$ *are equivalent:*

i) $\mathbf{1}+\mathbf{m}+\mathbf{g}$ *is elliptic.*

ii) There is a symbol $\mathbf{m}'+\mathbf{g}' \in R^{0,0}_{M+G}(\mathbb{R}^q \times \mathbb{R}^q, \underline{g}; N_+, N_-)$ *such that*

$$(\mathbf{1}+\mathbf{m}+\mathbf{g})\#(\mathbf{1}+\mathbf{m}'+\mathbf{g}') - 1 \in R_G^{-\infty,-\infty}(\mathbb{R}^q \times \mathbb{R}^q, \underline{g}; N_+, N_+),$$

$$(\mathbf{1}+\mathbf{m}'+\mathbf{g}')\#(\mathbf{1}+\mathbf{m}+\mathbf{g}) - 1 \in R_G^{-\infty,-\infty}(\mathbb{R}^q \times \mathbb{R}^q, \underline{g}; N_-, N_-).$$

Here 1 *is the identity operator on* $L^2(\mathbb{R}_+) \oplus \mathbb{C}^{N_+}$ *and* $L^2(\mathbb{R}_+) \oplus \mathbb{C}^{N_-}$*, respectively.*

The rest of this section is devoted to the proof of the above Theorem.

3.11 Proposition. *Let* $\mathbf{1}+\mathbf{m}+\mathbf{g}$ *be elliptic. Then there exists a symbol* $\mathbf{m}_1+\mathbf{g}_1 \in R^{0,0}_{M+G}(\mathbb{R}^q \times \mathbb{R}^q, \underline{g}; N_+, N_-)$ *such that*

$$(\mathbf{1}+\mathbf{m}+\mathbf{g})^{-1}(y,\eta) = (\mathbf{1}+\mathbf{m}_1+\mathbf{g}_1)(y,\eta) \quad \text{for large } |(y,\eta)|,$$

$$(\mathbf{1}+\mathbf{m}+\mathbf{g})(\mathbf{1}+\mathbf{m}_1+\mathbf{g}_1) - 1 \in R_G^{-\infty,-\infty}(\mathbb{R}^q \times \mathbb{R}^q, \underline{g}; N_+, N_+),$$

$$(\mathbf{1}+\mathbf{m}_1+\mathbf{g}_1)(\mathbf{1}+\mathbf{m}+\mathbf{g}) - 1 \in R_G^{-\infty,-\infty}(\mathbb{R}^q \times \mathbb{R}^q, \underline{g}; N_-, N_-).$$

PROOF: In view of Remark 2.33 and condition i) of the ellipticity, we find smoothing Mellin symbols $\mathbf{m}_1, \mathbf{m}_2 \in R^{0,0}_{M+G}(\mathbb{R}^q \times \mathbb{R}^q, \underline{g}; N_+, N_-)$ such that

$$(\mathbf{1}+\mathbf{m}_1)(\mathbf{1}+\mathbf{m}+\mathbf{g}) = 1 - \mathbf{g}' \tag{1}$$

$$(\mathbf{1}+\mathbf{m}+\mathbf{g})(\mathbf{1}+\mathbf{m}_2) = 1 - \mathbf{g}'' \tag{2}$$

with certain $\mathbf{g}' \in R_G^{0,0}(\mathbb{R}^q \times \mathbb{R}^q, \underline{g}; N_-, N_-)$, $\mathbf{g}'' \in R_G^{0,0}(\mathbb{R}^q \times \mathbb{R}^q, \underline{g}; N_+, N_+)$. Note that it is possible to write on the right–hand side 1 instead of $\mathbf{1}$, since $1 \in S^{0,0}(\mathbb{R}^q \times \mathbb{R}^q; \mathbb{C}^N, \mathbb{C}^N)$ and thus $\mathbf{1} - 1 \in R_G^{0,0}(\mathbb{R}^q \times \mathbb{R}^q; \mathbb{C}^N, \mathbb{C}^N)$. Equations (1) and (2) yield

$$\begin{aligned}(\mathbf{1}+\mathbf{m}+\mathbf{g})^{-1} &= \mathbf{1}+\mathbf{m}_1+\mathbf{g}'(\mathbf{1}+\mathbf{m}_2)+\mathbf{g}'(\mathbf{1}+\mathbf{m}+\mathbf{g})^{-1}\mathbf{g}'' \qquad (3)\\ &= \mathbf{1}+\mathbf{m}_2+(\mathbf{1}+\mathbf{m}_1)\mathbf{g}''+\mathbf{g}'(\mathbf{1}+\mathbf{m}+\mathbf{g})^{-1}\mathbf{g}''. \qquad (4)\end{aligned}$$

Now choose some $\phi \in C_0^\infty(\mathbb{R}^{2q})$ such that $\mathbf{1}+\mathbf{m}+\mathbf{g}$ is invertible in $\operatorname{supp}(1-\phi)$ and set

$$\begin{aligned}\mathbf{g}_1 &= (1-\phi)\mathbf{g}'(\mathbf{1}+\mathbf{m}_2)+\mathbf{g}'(1-\phi)(\mathbf{1}+\mathbf{m}+\mathbf{g})^{-1}\mathbf{g}'',\\ \mathbf{g}_2 &= (1-\phi)(\mathbf{1}+\mathbf{m}_1)\mathbf{g}''+\mathbf{g}'(1-\phi)(\mathbf{1}+\mathbf{m}+\mathbf{g})^{-1}\mathbf{g}'',\end{aligned}$$

which are elements of $R_G^{0,0}(\mathbb{R}^q \times \mathbb{R}^q, \underline{g}; N_+, N_-)$ in view of Remark 3.9.b). Now writing $\mathbf{1} + \mathbf{m}_1 + \mathbf{g}_1 = (1-\phi)\{\mathbf{1} + \mathbf{m}_1 + \mathbf{g}'(\mathbf{1}+\mathbf{m}_1) + \mathbf{g}'(\mathbf{1}+\mathbf{m}+\mathbf{g})^{-1}\mathbf{g}''\} + \phi(\mathbf{1}+\mathbf{m}_1)$, we get from (1) and (3)

$$(\mathbf{1}+\mathbf{m}_1+\mathbf{g}_1)(\mathbf{1}+\mathbf{m}+\mathbf{g}) = 1 - \phi + \phi(1-\mathbf{g}') = 1 - \phi\mathbf{g}'$$

and analogously from (2) and (4)

$$(\mathbf{1}+\mathbf{m}+\mathbf{g})(\mathbf{1}+\mathbf{m}_2+\mathbf{g}_2) = 1 - \phi\mathbf{g}''.$$

Multiplying the latter two equations with $(\mathbf{1}+\mathbf{m}_2+\mathbf{g}_2)$ and $(\mathbf{1}+\mathbf{m}_1+\mathbf{g}_1)$ from the right and the left, respectively, and then subtracting the resulting equations yields

$$(\mathbf{1}+\mathbf{m}_1+\mathbf{g}_1) = (\mathbf{1}+\mathbf{m}_2+\mathbf{g}_2) + \phi\mathbf{g}'''$$

with a certain remainder $\mathbf{g}''' \in R_G^{0,0}(\mathbb{R}^q \times \mathbb{R}^q, \underline{g}; N_+, N_-)$. Hence we obtain

$$(\mathbf{1}+\mathbf{m}+\mathbf{g})(\mathbf{1}+\mathbf{m}_1+\mathbf{g}_1) = 1 - \phi(\mathbf{g}'' - (\mathbf{1}+\mathbf{m}+\mathbf{g})\mathbf{g}'''),$$

and this finishes the proof, since ϕ is compactly supported in (y,η). ∎

3.12 Lemma. *Let $\mathbf{g}_j \in R_G^{\nu-j,m-j}(\mathbb{R}^q \times \mathbb{R}^q, \underline{g}; N_-, N_+)$, $j \in \mathbb{N}_0$, be a sequence of Green symbols where the involved asymptotic types are independent of j. Then there is a $\mathbf{g} \in R_G^{\nu,m}(\mathbb{R}^q \times \mathbb{R}^q, \underline{g}; N_-, N_+)$ satisfying*

$$\mathbf{g} - \sum_{j=0}^{N} \mathbf{g}_j \ \in R_G^{\nu-(N+1),m-(N+1)}(\mathbb{R}^q \times \mathbb{R}^q, \underline{g}; N_-, N_+) \quad \forall N \in \mathbb{N}_0 .$$

The element $\mathbf{g}$ is uniquely defined modulo $R_G^{-\infty,-\infty}(\mathbb{R}^q \times \mathbb{R}^q, \underline{g}; N_-, N_+)$ and we write

$$\mathbf{g} \sim \sum_{j=0}^{\infty} \mathbf{g}_j.$$

PROOF: The proof is completely analogous to that for scalar valued (global) pseudo–differential operators, i.e., we obtain $\mathbf{g}$ by

$$\mathbf{g}(y,\eta) = \sum_{j=0}^{\infty} \chi(y/c_j, \eta/c_j)\mathbf{g}_j(y,\eta)$$

with a $\chi \in C^\infty(\mathbb{R}^{2q})$ being zero in a neighborhood of 0 and identically 1 near infinity, and real numbers c_j tending to infinity with j sufficiently fast. ∎

PROOF OF THEOREM 3.10, i) ⇒ ii): By Proposition 3.11 there exists an $\mathbf{m}_1 + \mathbf{g}_1 \in R_{M+G}^{0,0}(\mathbb{R}^q \times \mathbb{R}^q, \underline{g}; N_+, N_-)$ such that

$$(\mathbf{1}+\mathbf{m}+\mathbf{g})\#(\mathbf{1}+\mathbf{m}_1+\mathbf{g}_1) = 1 - \mathbf{r}$$

with $\mathbf{r} \in R^{-1,-1}_{M+G}(\mathbb{R}^q \times \mathbb{R}^q, \underline{g}; N_+, N_+)$. For $j \geq k$ we have $\mathbf{r}^{\#j} \in R_G^{-j,-j}(\mathbb{R}^q \times \mathbb{R}^q, \underline{g}; N_+, N_+)$, moreover, the corresponding assymptotic types are independent of j. Thus by Lemma 3.12 there is a $\mathbf{g}_2 \in R_G^{-k,-k}(\mathbb{R}^q \times \mathbb{R}^q, \underline{g}; N_+, N_+)$ with

$$\mathbf{g}_2 \sim \sum_{j \geq k} \mathbf{r}^{\#j}.$$

Now we define $\mathbf{m}' + \mathbf{g}'$ by

$$\mathbf{1} + \mathbf{m}' + \mathbf{g}' = (\mathbf{1} + \mathbf{m}_1 + \mathbf{g}_1)\#(\sum_{j=0}^{k-1} \mathbf{r}^{\#j} + \mathbf{g}_2).$$

Then clearly $(\mathbf{1} + \mathbf{m} + \mathbf{g})\#(\mathbf{1} + \mathbf{m}' + \mathbf{g}') - 1 \in R_G^{-\infty,-\infty}(\mathbb{R}^q \times \mathbb{R}^q, \underline{g}; N_+, N_+)$. Analogously we obtain $\mathbf{m}'' + \mathbf{g}''$ with $(\mathbf{1} + \mathbf{m}'' + \mathbf{g}'')\#(\mathbf{1} + \mathbf{m} + \mathbf{g}) - 1 \in R_G^{-\infty,-\infty}(\mathbb{R}^q \times \mathbb{R}^q, \underline{g}; N_-, N_-)$. But then

$$(\mathbf{1} + \mathbf{m}' + \mathbf{g}') = (\mathbf{1} + \mathbf{m}' + \mathbf{g}')\#(\mathbf{1} + \mathbf{m} + \mathbf{g})\#(\mathbf{1} + \mathbf{m}'' + \mathbf{g}'') = (\mathbf{1} + \mathbf{m}'' + \mathbf{g}'')$$

modulo $R_G^{-\infty,-\infty}(\mathbb{R}^q \times \mathbb{R}^q, \underline{g}; N_+, N_-)$. ∎

PROOF OF THEOREM 3.10, ii) ⇒ i): By Theorem 3.6 there are certain remainders $\mathbf{r}_1, \tilde{\mathbf{r}}_1 \in R^{-1,-1}_{M+G}(\mathbb{R}^q \times \mathbb{R}^q, \underline{g}; N_+, N_+)$ such that

$$(\mathbf{1} + \mathbf{m} + \mathbf{g})(\mathbf{1} + \mathbf{m}' + \mathbf{g}') + \mathbf{r}_1 = (\mathbf{1} + \mathbf{m} + \mathbf{g})\#(\mathbf{1} + \mathbf{m}' + \mathbf{g}') = 1 + \tilde{\mathbf{r}}_1.$$

Now $\mathbf{r} := \mathbf{r}_1 - \tilde{\mathbf{r}}_1 \in S^{-1,-1}(\mathbb{R}^q \times \mathbb{R}^q; L^2(\mathbb{R}_+) \oplus \mathbb{C}^{N_+}, L^2(\mathbb{R}_+) \oplus \mathbb{C}^{N_+})$ shows the invertibility of $(1 - \mathbf{r})(y, \eta) : L^2(\mathbb{R}_+) \oplus \mathbb{C}^{N_+} \to L^2(\mathbb{R}_+) \oplus \mathbb{C}^{N_+}$ for large $|(y, \eta)|$. Hence from

$$(\mathbf{1} + \mathbf{m} + \mathbf{g})(\mathbf{1} + \mathbf{m}' + \mathbf{g}') = 1 - \mathbf{r}$$

we derive the surjectivity of $\mathbf{1} + \mathbf{m} + \mathbf{g}$ as an operator $L^2(\mathbb{R}_+) \oplus \mathbb{C}^{N_-} \to L^2(\mathbb{R}_+) \oplus \mathbb{C}^{N_+}$ for large $|(y, \eta)|$. Interchanging the roles of $(\mathbf{1} + \mathbf{m} + \mathbf{g})$ and $(\mathbf{1} + \mathbf{m}' + \mathbf{g}')$ yields the injectivity, hence invertibility of $\mathbf{1} + \mathbf{m} + \mathbf{g}$ for large $|(y, \eta)|$ and

$$(\mathbf{1} + \mathbf{m} + \mathbf{g})^{-1} = (\mathbf{1} + \mathbf{m}' + \mathbf{g}')(1 - \mathbf{r})^{-1}.$$

Clearly the righthand side is a uniformly bounded family of operators $L^2(\mathbb{R}_+) \oplus \mathbb{C}^{N_+} \to L^2(\mathbb{R}_+) \oplus \mathbb{C}^{N_-}$ (for large $|(y, \eta)|$). Hence condition ii) of the ellipticity is fulfilled. Finally from $\sigma_M^0(\mathbf{r}) = 0$ and $\sigma_M^0((\mathbf{m} + \mathbf{g})(\mathbf{m}' + \mathbf{g}')) = \sigma_M^0(\mathbf{m} + \mathbf{g})\sigma_M^0(\mathbf{m}' + \mathbf{g}')$ we deduce

$$\sigma_M^0(\mathbf{m} + \mathbf{g})(1 + \sigma_M^0(\mathbf{m} + \mathbf{g}))^{-1} = -\sigma_M^0(\mathbf{m}' + \mathbf{g}')$$

and obtain condition i) of the ellipticity by the following Lemma 3.13. ∎

3.13 Lemma. *If $h \in M_P^{-\infty,0}$ and $h' \in M_{P'}^{-\infty,0}$ and $h'(1 + h')^{-1} = h$, then $(1 + h')^{-1} \in M_P^{0,0}$.*

PROOF: This statement is a simple consequence of $(1 + h')^{-1} = 1 - h'(1 + h')^{-1} = 1 - h \in 1 + M_P^{-\infty,0} \subset M_P^{0,0}$. ∎

4 The algebra on the infinite wedge

Here we show how to modify the operators considered before to obtain an algebra of global wedge pseudo-differential operators on the infinite (open stretched) wedge $\mathbb{R}^q \times X^\wedge$. Here $X^\wedge = \mathbb{R}_+ \times X$ is the open (stretched) cone with base X, and X is a closed compact smooth manifold of dimension $\dim X = n$. We will state explicitely the generalized versions of all the objects (i.e., underlying Hilbert spaces, symbols, and so on) involved in Sections 1 to 3. Then all results obtained for the half-space situation carry over to the general case $\mathbb{R}^q \times X^\wedge$. Proofs are dropped, since they are completely analogous to those given before.
As usual, $L^\mu(X)$ denotes the pseudo-differential operators of order μ on X, and $L^\mu(X;\mathbb{R})$ the parameter-dependent ones with parameter $\varrho \in \mathbb{R}$. Under the identification $\Gamma_\beta \to \mathbb{R} : \beta + i\varrho \mapsto \varrho$ we also consider $L^\mu(X;\Gamma_\beta)$.

4.1 Remark. For each $\mu \in \mathbb{R}$ and reals $c_1 < c_2$ there exists a function $R^\mu(z) \in \mathcal{A}(\{z \in \mathbb{C};\, c_1 < \operatorname{Re} z < c_2\}, L^\mu(X))$ such that

i) $R^\mu(\beta + i\varrho) \in L^\mu(X;\mathbb{R}_\varrho)$ continuously in $\beta \in \,]c_1, c_2[$,

ii) $R^\mu(z)$ induces isomorphisms $H^s(X) \to H^{s-\mu}(X)$ for all $s \in \mathbb{R}$ and each z.

Such an R^μ is called (holomorphic) order-reduction of order μ.

We now turn to the definition of the underlying distribution spaces on $X^\wedge$. For $s, \gamma \in \mathbb{R}$ let $\mathcal{H}^{s,\gamma}(X^\wedge)$ be the completion of $C_0^\infty(X^\wedge) = C_0^\infty(\mathbb{R}_+, C^\infty(X))$ with respect to the norm

$$\|u\|^2_{s,\gamma} = \int_{\Gamma_{\frac{n+1}{2}-\gamma}} \|R^s(z)\mathcal{M}u(z)\|^2_{L^2(X)}\,|dz|.$$

To an $f \in L^\mu(X;\Gamma_{1/2-\gamma})$ we associate a Mellin pseudo-differential operator, defined on $C_0^\infty(X^\wedge)$ by

$$\left[\operatorname{op}_M^\gamma(f)u\right](t) = \frac{1}{2\pi i}\int_{\Gamma_{1/2-\gamma}} t^{-z} f(z)(\mathcal{M}u)(z)\,dz.$$

This extends by continuity to operators

$$\operatorname{op}_M^\gamma(f) : \mathcal{H}^{s,\gamma+n/2}(X^\wedge) \to \mathcal{H}^{s-\mu,\gamma+n/2}(X^\wedge) \quad \forall s \in \mathbb{R} \tag{4.1}$$

A distribution $u \in \mathcal{D}'(X^\wedge)$ is said to be an element of $H^s_{cone}(X^\wedge)$ if for each diffeomorphism $\kappa : U \subset X \to V \subset S^n$, where S^n is the unit sphere in $\mathbb{R}^{1+n}$, and each $\phi \in C_0^\infty(U)$ the push-forward of ϕu to $\mathbb{R}^{1+n} \setminus \{0\}$ under

$$\tilde\kappa : \mathbb{R}_+ \times U \to \mathbb{R}^{1+n} \setminus \{0\} : (t,x) \mapsto t\kappa(x)$$

belongs to $H^s(\mathbb{R}^{1+n} \setminus \{0\})$, i.e., the closure of $C_0^\infty(\mathbb{R}^{1+n} \setminus \{0\})$ in $H^s(\mathbb{R}^{1+n})$. A (Hilbert) norm on $H^s_{cone}(X^\wedge)$ is obtained by a standard construction using a partition of unity. Now the cone Sobolev spaces are defined as

$$\mathcal{K}^{s,\gamma}(X^\wedge) = [\omega]\mathcal{H}^{s,\gamma}(X^\wedge) + [1-\omega]H^s_{cone}(X^\wedge).$$

For $\gamma \in \mathbb{R}$, $\Theta =]\vartheta, 0]$, we write $Q \in \mathrm{As}(\gamma, \Theta)$ if

$$Q = \left\{(q_j, m_j) \in \mathbb{C} \times \mathbb{N}_0;\ \tfrac{n+1}{2} - \gamma + \vartheta < \operatorname{Re} q_j < \tfrac{n+1}{2} - \gamma,\ j = 0, \ldots, N\right\}$$

for some $N \in \mathbb{N}_0$. To such a type associate spaces

$$\begin{aligned} \tilde{\mathcal{E}}_Q^s(X^\wedge) &= \left\{(t,x) \mapsto \sum_{j=0}^{N} \sum_{k=0}^{m_j} \xi_{jk}(x) t^{-q_j} \log^k t;\ \xi_{jk} \in H^s(X)\right\}, \\ \tilde{\mathcal{E}}_Q(X^\wedge) &= \cap_{s \in \mathbb{R}} \tilde{\mathcal{E}}_Q^s(X^\wedge), \end{aligned}$$

which are canonically isomorphic to a finite product of $H^s(X)$ and $C^\infty(X)$, respectively. Writing $\mathcal{K}_\Theta^{s,\gamma}(X^\wedge) = \cap_{\varepsilon>0} \mathcal{K}^{s,\gamma-\vartheta-\varepsilon}(X^\wedge)$ we then set

$$\mathcal{K}_Q^{s,\gamma}(X^\wedge) = \mathcal{K}_\Theta^{s,\gamma}(X^\wedge) + [\omega]\tilde{\mathcal{E}}_Q(X^\wedge), \qquad \mathcal{K}_Q^{\infty,\gamma}(X^\wedge) = \cap_{s\in\mathbb{R}} \mathcal{K}_Q^{s,\gamma}(X^\wedge),$$

which are Fréchet spaces. Finally, we define the space of rapidly decreasing functions on $X^\wedge$ as $\mathcal{S}(X^\wedge) = \mathcal{S}(\mathbb{R}, C^\infty(X))|_{\mathbb{R}_+}$ and set

$$\mathcal{S}_Q^\gamma(X^\wedge) = [\omega]\mathcal{K}_Q^{\infty,\gamma}(X^\wedge) + [1-\omega]\mathcal{S}(X^\wedge),$$

which is a projective limit of the Hilbert spaces

$$E_Q^k(X^\wedge) = [\omega]\{\mathcal{K}^{k,\gamma-\vartheta-c_k}(X^\wedge) + \mathcal{E}_Q^k(X^\wedge)\} + [1-\omega]\,\langle t\rangle^{-k}\, H_{cone}^k(X^\wedge), \quad k \in \mathbb{N},$$

where $c_k = c_Q/k$ and c_Q is chosen in a way that $\operatorname{Re} q_j > \frac{n+1}{2} - \gamma + \vartheta + c_Q$ for all j.
The standard group action, cf. Example 1.2, now is the restriction of the mapping $\mathcal{D}'(X^\wedge) \to \mathcal{D}'(X^\wedge)$ defined by

$$\langle \kappa_\lambda u, s\rangle = \langle u, \lambda^{\frac{n-1}{2}} s(\lambda^{-1}t, x)\rangle,$$

where $s(t,x)$ is a smooth, compactly supported section in the density bundle over $X^\wedge$. In particular, this induces on $\mathcal{K}^{0,0}(X^\wedge)$ a group of unitary operators.
Proposition 1.11 concerning the parameter-dependent cut-off operator now takes the following form:

4.2 Proposition. *Let $Q \in As(\gamma, \Theta)$ and $Z_Q^k(X^\wedge) = \mathcal{H}^{\gamma-\vartheta-c_k}(X^\wedge) \cap \mathcal{H}^{s,\gamma}(X^\wedge)$. Then the function $\eta \mapsto \omega(t[\eta])$ is an element of*

$$S^0(\mathbb{R}^q_\eta;\ \mathcal{H}^{s,\gamma}(X^\wedge), \mathcal{H}^{s,\varrho}(X^\wedge)) \quad \forall\, \varrho \le \gamma$$

(here we also can replace $\mathcal{H}^{s,\gamma}(X^\wedge)$ by $\mathcal{K}^{s,\gamma}(X^\wedge)$, and $\mathcal{H}^{s,\varrho}(X^\wedge)$ by $\mathcal{K}^{s,\varrho}(X^\wedge)$) and

$$S^0(\mathbb{R}^q_\eta; \tilde{\mathcal{E}}_Q^k(X^\wedge), E_Q^k(X^\wedge)), \quad S^0(\mathbb{R}^q_\eta;\ \mathcal{H}^{k,\gamma-\vartheta}(X^\wedge), E_{\mathcal{O}}^k(X^\wedge)),$$

$$S^0(\mathbb{R}^q_\eta;\ E_Q^k(X^\wedge), Z^k(X^\wedge) + \mathcal{E}_Q^k(X^\wedge)), \quad S^0(\mathbb{R}^q_\eta;\ Z^k(X^\wedge) + \mathcal{E}_Q^k(X^\wedge), E_Q^k(X^\wedge)).$$

Here $\mathcal{O}$ is the empty asymptotic type in $As(\gamma, \Theta)$. We also have

$$\eta \mapsto 1 - \omega(t[\eta]) \in S^0(\mathbb{R}^q_\eta;\ \mathcal{H}^{s,\gamma}(X^\wedge), \mathcal{H}^{s,\varrho}(X^\wedge)) \quad \forall\, \varrho \ge \gamma.$$

To prove that a Mellin pseudo-differential operator with meromorphic symbol, cf. the definitions below, preserves asymptotics, one needs the spaces $\overline{\mathcal{A}}_Q^{s,\gamma}(X^\wedge)$, which are defined as follows:

4.3 Definition. Set $S_{\gamma,\Theta} = \{z \in \mathbb{C};\ \frac{n+1}{2} - \gamma + \vartheta < \operatorname{Re} z < \frac{n+1}{2} - \gamma\}$. Further let $R^s(z)$ be a holomorphic order-reduction of order s defined on a strip containing the closure of $S_{\gamma,\Theta}$. Then $\overline{\mathcal{A}}_Q^{s,\gamma}(X)$ is the space of all meromorphic functions $f \in \mathcal{A}(S_{\gamma,\Theta} \setminus \pi_{\mathbb{C}}Q, H^s(X))$ with poles in $q_j \in \pi_{\mathbb{C}}Q$ of order at most $m_j + 1$, that additionally satisfy

i) both $\lim_{\delta\to 0^+} R^s(\frac{n+1}{2} - \gamma - \delta + i\varrho) f(\frac{n+1}{2} - \gamma - \delta + i\varrho)$ and $\lim_{\delta\to 0^+} R^s(\frac{n+1}{2} - \gamma + \vartheta + \delta + i\varrho) f(\frac{n+1}{2} - \gamma + \vartheta + \delta + i\varrho)$ exist in $L^2(\mathbb{R}_\varrho, L^2(X))$,

ii) for arbitrary $\pi_{\mathbb{C}}Q$-excision function χ_Q is

$$\sup\Big\{\Big(\frac{1}{2\pi}\int_{\Gamma_\beta} \|R^s(z)\{\chi_Q(z) f(z)\}\|^2_{L^2(X)}\,|dz|\Big)^{1/2};\ \beta \in S_{\gamma,\Theta}\Big\} < \infty.$$

A Fréchet topology (in fact a Hilbert topology) is established by taking the semi-norms of $\mathcal{A}(S_{\gamma,\Theta} \setminus \pi_{\mathbb{C}}Q, H^s(X))$ and that given in ii).

Now we pass to the symbols of the pseudo-differential operators on $\mathbb{R}^q \times X^\wedge$. Again, we start with discrete asymptotic types for Mellin symbols, cf. Definition 2.4. For a given type P define a space of formal series

$$\mathcal{F}_P(X) = \Big\{\sum_{j\in\mathbf{Z}}\sum_{k=0}^{m_j} c_{jk}\psi_{p_j,k};\ c_{jk} \in L^{-\infty}(X)\Big\},$$

with $\psi_{p_j,k}$ as in Remark 2.1. This space is canonically isomorphic to a countable product of $L^{-\infty}(X)$. For $f \in \mathcal{F}_P(X)$ and real numbers $c_1 < c_2$ set

$$f_{[c_1,c_2]}(z) = \sum_{\{j;\, c_1 \le \operatorname{Re} p_j \le c_2\}} \sum_{k=0}^{m_j} c_{jk}\psi_{p_j,k}(z).$$

4.4 Definition. For $\mu \in \mathbb{R}$ let $M_P^\mu(X)$ denote the space of all meromorphic functions $h \in \mathcal{A}(\mathbb{C} \setminus \pi_{\mathbb{C}}P, L^\mu(X))$, where the poles in $p_j \in \pi_{\mathbb{C}}P$ are at most of order $m_j + 1$, and the Laurent coefficients of the principal part of h in p_j being elements of $L^{-\infty}(X)$. Further one asks that there exists an $f \in \mathcal{F}_P(X)$ such that for all $c_1 < c_2$

$$h(\beta + i\varrho) - f_{[c_1,c_2]}(\beta + i\varrho) \in L^\mu(X; \mathbb{R}_\varrho)$$

continuously in $\beta \in [c_1, c_2]$. The element f is uniquely determined. As in (2.2) we can define an operator $T : M_P^\mu(X) \to \mathcal{F}_P(X) : h \mapsto Th = f$, and then equip $M_P^\mu(X)$ with a Fréchet topology given by the following semi-norm systems:

i) that from $\mathcal{A}(\mathbb{C} \setminus \pi_{\mathbb{C}}P, L^\mu(X))$,

ii) that induced by the mappings

$$h \mapsto \frac{\partial^k}{\partial z^k}\{(z-p_j)^{m_j+1}h(z)\}|_{z=p_j} : M_P^\mu(X) \to L^{-\infty}(X)$$

for all $j \in \mathbb{Z}$, $0 \le k \le m_j$,

iii) those given by

$$\sup_{-c_1^k \le \beta \le c_2^k} q\left(h(\beta+i\varrho) - (Th)_{[-c_1^k, c_2^k]}(\beta+i\varrho)\right),$$

where $q(\cdot)$ runs through a system of semi-norms of $L^\mu(X;\mathbb{R}_\varrho)$ and $c_1^k, c_2^k \to \infty$.

4.5 Definition. The space $M_P^{\mu,m}(X)$, $\mu, m \in \mathbb{R}$, consists of all $h \in C^\infty(\mathbb{R}^q, M_P^\mu(X))$ satisfying:

i) If $\sum_{k=0}^{m_j} \sigma_{jk}(y)(z-p_j)^{-(k+1)}$ is the principal part of the Laurent expansion of $h(y)$ in $p_j \in \pi_{\mathbb{C}}P$, then

$$\sup_{y\in\mathbb{R}^q} \langle y\rangle^{|\alpha|-m} q\left(\partial_y^\alpha \sigma_{jk}(y)\right) < \infty$$

for all $\alpha \in \mathbb{N}_0^q$ and semi-norms $q(\cdot)$ of $L^{-\infty}(X)$.

ii) For all $c_1 < c_2 \in \mathbb{R}$, each semi-norm $q(\cdot)$ of $L^\mu(X;\mathbb{R}_\varrho)$, and all $\alpha \in \mathbb{N}_0^q$ is

$$\sup\left\{\langle y\rangle^{|\alpha|-m} q\left(\partial_y^\alpha[h(y,\beta+i\varrho) - (Th)_{[c_1,c_2]}(y,\beta+i\varrho)]\right);\, c_1 \le \beta \le c_2,\, y \in \mathbb{R}^q\right\} < \infty.$$

$M_P^{\mu,m}(X)$ is a Fréchet space if equipped with the semi-norms of $C^\infty(\mathbb{R}^q, M_P^\mu(X))$ and those from i), ii).

Using the same techniques as in Section 2.1 it is then not difficult to verify the analog of Proposition 2.13 for the case $X^\wedge$ instead of $\mathbb{R}_+$. The only modification is that in 2.13.c) we consider the $\mathrm{op}_M^{\gamma-n/2}$ instead of op_M^γ, and in 2.13.d) we replace condition $\pi_{\mathbb{C}}P \cap \Gamma_{1/2-\gamma} = \pi_{\mathbb{C}}P \cap \Gamma_{1/2-\gamma+\vartheta} = \emptyset$ by $\pi_{\mathbb{C}}P \cap \Gamma_{\frac{n+1}{2}-\gamma} = \pi_{\mathbb{C}}P \cap \Gamma_{\frac{n+1}{2}-\gamma+\vartheta} = \emptyset$.
Finally we consider the associated Green and smoothing Mellin symbols. The definition of the Green symbols is completely analogous to Definition 2.17 and Definition 3.3, respectively. Concerning the smoothing Mellin symbols, cf. Definition 2.20, there appears a slight modification, caused by the fact that the dimension of the base X enters in the weights and weight-lines. Thus, in Definition 2.20 we have to replace conditions (2.6) by

$$h_{j\alpha} \in M_{P_{j\alpha}}^{-\infty,m}(X), \quad \pi_{\mathbb{C}}P_{j\alpha} \cap \Gamma_{1/2-\gamma_{j\alpha}} = \emptyset, \quad \gamma - n/2 - (\mu-\nu) - j \le \gamma_{j\alpha} \le \gamma - n/2.$$

The important Proposition 2.24, dealing with the behaviour of smoothing Mellin operators under weight shifts, generalizes in the following way:

4.6 Proposition. *Let $h \in M_P^{\mu,m}(X)$ and $\delta, \gamma \in \mathbb{R}$ with $\pi_{\mathbb{C}}P \cap \Gamma_{\frac{n+1}{2}-\gamma} = \pi_{\mathbb{C}}P \cap \Gamma_{\frac{n+1}{2}-\delta} = \emptyset$. Further set $\tau = \min(\gamma,\delta)$, $\varrho = \max(\gamma,\delta)$, $\Theta =]-|\gamma-\delta|, 0]$, and*

$$G(y) = \mathrm{op}_M^{\gamma-n/2}(h)(y) - \mathrm{op}_M^{\delta-n/2}(h)(y).$$

Then there exist asymptotic types $R_1 \in As(\tau, \Theta)$ *and* $R_2 \in As(-\varrho, \Theta)$ *such that*

$$\begin{aligned} G &\in \cap_{s,r\in\mathbb{R}} S^{0,m}(\mathbb{R}^q \times \mathbb{R}^q; \mathcal{H}^{s,\delta}(X^\wedge) \cap \mathcal{H}^{s,\gamma}(X^\wedge), \tilde{\mathcal{E}}^r_{R_1}(X^\wedge)), \\ G^* &\in \cap_{s,r\in\mathbb{R}} S^{0,m}(\mathbb{R}^q \times \mathbb{R}^q; \mathcal{H}^{s,-\delta}(X^\wedge) \cap \mathcal{H}^{s,-\gamma}(X^\wedge), \tilde{\mathcal{E}}^r_{R_2}(X^\wedge)). \end{aligned}$$

If $\pi_{\mathbb{C}} P \cap \Gamma_{\frac{n+1}{2}-\beta} = \emptyset$ *for all real* β *between* γ *and* δ *then* $G \equiv 0$.

Now all the material from Sections 2 and 3 can be verified for the case $X^\wedge$. For completness we finally restate the ellipticity, cf. Definition 3.8.

4.7 Definition. Let $\mathbf{m} + \mathbf{g} \in R^{0,0}_{M+G}(\mathbb{R}^q \times \mathbb{R}^q, \underline{g}; N_-, N_+)$. The symbol $\mathbf{1} + \mathbf{m} + \mathbf{g}$ is called *elliptic* if

i) there exists an asymptotic type P with $\pi_{\mathbb{C}} P \cap \Gamma_{\frac{n+1}{2}} = \emptyset$ and

$$(1 + \sigma^0_M(\mathbf{m} + \mathbf{g}))^{-1} \in M^{0,0}_P(X),$$

ii) for large $|(y, \eta)|$

$$(\mathbf{1} + \mathbf{m} + \mathbf{g})(y, \eta) : \begin{array}{c} \mathcal{K}^{0,0}(X^\wedge) \\ \oplus \\ \mathbb{C}^{N_-} \end{array} \longrightarrow \begin{array}{c} \mathcal{K}^{0,0}(X^\wedge) \\ \oplus \\ \mathbb{C}^{N_+} \end{array}$$

is invertible and the inverse is uniformly bounded in (y, η).

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