ALGEBRAIC SURFACES WITH EXTREME CHERN NUMBERS

(Report on the thesis of Th. Höfer, Bonn, 1984)

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F. H I R Z E B R U C H

(Lecture, Steklov Institute, Moscow, September 1984)

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For a smooth algebraic surface X the Chern numbers $c_1^2(X)$ and $c_2(X)$ are defined. Here $c_2(X)$ equals the Euler-Poincaré characteristic of X and $c_1^2(X)$ is the selfintersection number of a canonical divisor of X. For a surface of general type they satisfy the Miyaoka-Yau inequality $c_1^2 \leq 3c_2$ (see [Mi1] and [Y]) where the equality sign holds if and only if the universal cover of the surface is the unit ball $|z_1|^2 + |z_2|^2 < 1$ (see [Y] and [Mi2] § 2 for the difficult and [Hi1] for the easy direction of this equivalence). If $c_1^2 = 3c_2$, then automatically $c_2 > 0$, in fact c_2 is the volume of the surface (normalized by the Gauß-Bonnet form) with respect to the complex-hyperbolic metric induced from the ball.

We wish to construct surfaces of general type with extreme Chern numbers ($c_1^2 = 3c_2$) which are ramified covers of the complex projective plane branched along lines. Thus we continue the investigation of the paper [Hi2] . However, the much more developped theory and the new examples are due to Thomas Höfer (Bonn, dissertation in preperation, [Hö]). This note is a report on his work.

We have to omit many things. For example, we only consider the case where the surface is the quotient of the ball by a discrete group Γ of automorphisms, Γ operating freely with compact quotient. Höfer includes the case where Γ may have torsion and cusps (at infinity), and constructs many such ball quotients as ramified covers of the plane. His formulas (with suitable modifications) hold for this more general case.

1. Let S be an algebraic surface and $C_i (i = 1,\dots t)$ finitely many distinct smooth irreducible curves on S such that the curve $C = \sum C_i$ has only ordinary double points. Let Y be a smooth algebraic surface which is a Galois cover over S with covering map

$$\pi: Y \to S$$

which ramifies only over C. Then, for every point $p \in Y$ there exist local coordinates (u,v) on Y with center p and local coordinates (z,w) on S with center $f(p)$ such that π is given by functions

$$z = u^a \quad , \quad w = v^b \tag{1}$$

where a,b are positive integers. If a or b is ≥ 2, then $p \in \pi^{-1}(C)$. We can associate to each C_i a positive integer a_i such that for a smooth point q of C lying on C_i, the map π is given locally (for $p \in \pi^{-1}(q)$) by (1) with $a = a_i$ and $b = 1$. If $q \in C_i \cap C_j$ $(i \neq j)$, then, for $p \in \pi^{-1}(q)$, the map π is given locally by (1) with $a = a_i$, $b = a_j$, and, if d is the mapping degree of π, the number of inverse image points of q equals $d/(a_i a_j)$. A covering Y of S with the properties explained in this section will be called a good covering with respect to the curves C_i and the branching numbers a_i.

2. As in [Hi2] we consider the complex projective plane $P_2(\mathbb{C})$ with homogeneous coordinates $z_0 : z_1 : z_2$ and an arrangement of k distinct lines $L_1,\dots,L_k$ given by $l_i = 0 (i = 1,\dots k)$ where l_i ia a linear form in z_0, z_1, z_2. For a point p in the plane we let r_p be the number of lines in the arrangement passing through p and t_r (for $r \geq 2$) be the number of points p with $r_p = r$. Then we have

$$\text{(2)} \qquad \frac{k(k-1)}{2} = \sum_{r \geq 2} t_r \frac{r(r-1)}{2}$$

Given an arrangement, we blow up the points p_j in the plane with $r_{p_j} \geq 3$ to get an algebraic surface S in which we have a configuration of curves C_i as in 1., namely the strict transforms of the lines of the arrangement, (which we also call $L_1,\dots,L_k$ as smooth irreducible curves on S) and the curves $E_1,\dots,E_s$ obtained by blowing up the points p_j where $1 \leq j \leq s$ and $s = \sum_{r \geq 3} t_r$. (The number t of 1. equals $k + s$).

We now associate weights $n_1,n_2,\dots,n_k$ (integers ≥ 2) to the "old" lines $L_1,L_2,\dots,L_k$ and weights $m_1,m_2,\dots,m_s$ (integers ≥ 1) to the "new lines" $E_1,E_2,\dots,E_s$ or equivalently to the points p_j.

We speak of a <u>weighted arrangement</u> of lines in the plane. (Each line L_i has a weight n_i and each intersection point p_j with $r_{p_j} \geq 3$ has a weight m_j. For a weighted arrangement we have the surface S and on it curves $L_1,\dots,L_k,E_1,\dots,E_s$ with branching numbers $n_1,\dots,n_k,m_1,\dots,m_s$ (which in 1. were called a_i). <u>Let</u> Y <u>be a good covering of</u> S <u>with respect to</u> $L_1,\dots,L_k,E_1,\dots,E_s$ <u>and the given branching numbers.</u> (Right now, we do not consider the difficult problem whether such a surface Y exists. For a partial result see [Ka]. For the very special case $n_i = m_j = n$ see [Hi2].)

It is possible and in principal not difficult to calculate the Chern numbers of Y in a similar and rather elementary way as it was done in [Hi2] in a special case. However, the formulas for arbitrary weights are not easy to handle. Höfer found several nice formulas to express $(3c_2(Y) - c_1^2(Y))/d$ in terms of the weights and the combinatorial features of the (unweighted) arrangement:

Let σ_i be the number of points p with $r_p \geq 3$ lying on the i^{th} line of the given arrangement of k lines in the plane. Consider the $k \times k$ symmetric matrix A with

$$(3) \qquad A_{ij} = \begin{cases} 3\sigma_i - 4 & (i = j) \\ 2 & (i \neq j,\ p = L_i \cap L_j \text{ with } r_p = 2) \\ -1 & (i \neq j,\ p = L_i \cap L_j \text{ with } r_p \geq 3) \end{cases}$$

Associate real variables x_i to the k lines and let x be the column vector $(x_1,\ldots,x_k)$. Associate real variables y_j to the s points p_j with $r_{p_j} \geq 3$. For the point p_j with $r_{p_j} \geq 3$ we consider the linear form

$$P_j(x,y) = 2y_j + \sum_{p_j \in L_i} x_i \quad \text{where} \quad y = (y_1,\ldots,y_s)$$

<u>Höfer's formula.</u>

<u>For the algebraic surface</u> Y (<u>good covering of</u> S <u>of degree</u> d <u>with respect to</u> $L_1,\ldots,L_k,E_1,\ldots,E_s$ <u>and the given branching numbers</u> $n_1,\ldots,n_k,m_1,\ldots,m_s$) <u>we have</u>

$$(4) \qquad (3c_2(Y) - c_1^2(Y))/d = \frac{1}{4}\Big(x^tAx + \sum_{j=1}^{s} P_j(x,y)^2\Big)$$

<u>if we put</u>

$$x_i = 1 - \frac{1}{n_i} \quad \text{and} \quad y_j = -1 - \frac{1}{m_j}$$

Thus Höfer's formula expresses $(3c_2(Y) - c_1^2(Y))/d$ as a quadratic form over $\mathbb{R}^{k+s}$ in the x_i and y_j. The quadratic form depends only on the unweighted arrangement.

The sum of the entries in each line of the matrix A equals $3\tau_i - (k + 3)$ where τ_i is the number of points p on L_i with $r_p \geq 2$. This follows from the equation

$$\sum_{p \in L_i} (r_p - 1) = k - 1$$

The formula (4) implies:

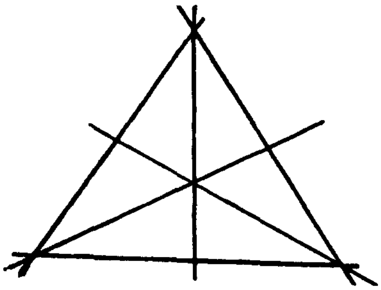
<u>If for all lines</u> $3\tau_i = k + 3$ <u>and if all weights</u> n_i <u>are equal</u> ($n_i = n$ for $1 \leq i \leq k$), <u>then</u>

$$(5) \qquad (3c_2(Y) - c_1^2(Y))/d = \frac{1}{4} \sum_{j=1} P_j(x,y)^2$$

<u>where</u> $x_i = 1 - \frac{1}{n}$ and $y_j = -1 - \frac{1}{m_j}$

<u>3.</u> I know only the following arrangements with $3\tau_i = k + 3$ for all lines L_i of the arrangement. We exclude the triangle $k = 3$, $t_2 = 3$. They all are related to unitary reflection groups acting on $\mathbb{C}^3$ (see [Hi2]).

a) The complete quadrilateral



$k = 6$, $t_2 = 3$, $t_3 = 4$, $t_r = 0$ otherwise

b) <u>The arrangements</u> $A_3^\circ m$, $m \geq 3$.

$k = 3m$, $t_2 = 0$, $t_3 = m^2$, $t_m = 3$, $t_r = 0$ otherwise,

(for $m = 3$, $t_2 = 0$, $t_3 = 12$) .

In homogeneous coordinates the $3m$ lines can be given by the equation

$$(z_0^m - z_1^m)(z_1^m - z_2^m)(z_2^m - z_0^m) = 0$$

c) <u>The arrangements</u> $A_3^3(m)$, $m \geq 2$.

$k = 3m + 3$, $t_2 = 3m$, $t_3 = m^2$, $t_{m+2} = 3$, $t_r = 0$ otherwise

In homogeneous coordinates the $3m + 3$ lines can be given by the equation

$$z_0 z_1 z_2 (z_0^m - z_1^m)(z_1^m - z_2^m)(z_2^m - z_0^m) = 0$$

d) <u>The Hesse arrangement</u>

$k = 12$, $t_2 = 12$, $t_4 = 9$, $t_r = 0$ otherwise

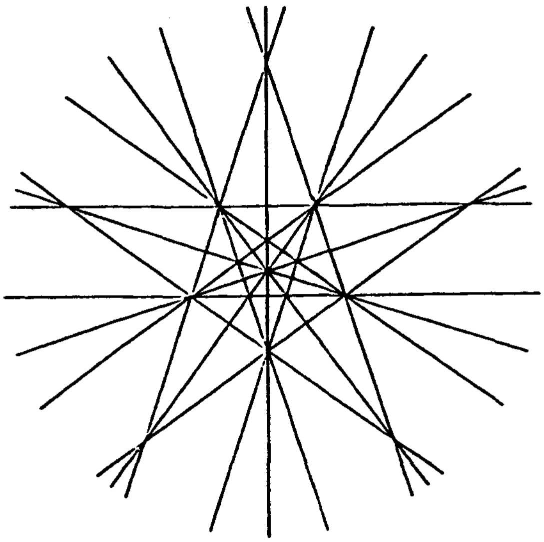
The Hense pencil of all cubics passing through the 9 inflection points of a smooth cubic has 4 singular cubics (triangles) which make up the 12 lines. These 12 lines are dual to the 12 triple points of $A_3^\circ(3)$.

e) <u>The extended Hesse arrangement</u> (see [Hi2])

$k = 21$, $t_2 = 36$, $t_4 = 9$, $t_5 = 12$, $t_r = 0$ otherwise

The extended Hesse arrangement contains the 12 lines of the Hesse arrangement and nine additional lines which make an arrangement $A_3^\circ(3)$ such that the 12 triple points of $A_3^\circ(3)$ coincide with the 12 double points of the Hesse arrangement.

f) <u>The icosahedral arrangement</u>



$k = 15$, $t_2 = 15$, $t_3 = 10$, $t_5 = 6$, $t_r = 0$ otherwise

g) <u>The</u> G_{168} - <u>arrangement</u>

The simple group or order 168 operates on the complex projective plane. It has 21 involutions with 21 fixed lines.

$k = 21$, $t_3 = 28$, $t_4 = 21$, $t_5 = 0$ otherwise

h) <u>The</u> A_6 - <u>configuration</u>

The alternating group A_6 (of order 360) operates on the complex projective plane. It has 45 involutions with 45 fixed lines.

$k = 45$, $t_3 = 120$, $t_4 = 45$, $t_5 = 36$, $t_r = 0$ otherwise

<u>Is this a complete list of the arrangements with</u> $3\tau_i = k + 3$ <u>for all lines</u> L_i?

4. We wish to study good coverings Y of an arrangement with $3\tau_i = k + 3$ for all lines and the weights n_i along the lines all equal to n. We are looking for surfaces Y with $3c_2(Y) = c_1^2(Y)$. Then by (5) all $P_j(x,y)$ have to be 0 which means

$$2(-1 - \frac{1}{m_j}) + r(1 - \frac{1}{n}) = 0$$

where m_j is the branching number (weight) in the point p_j with $r_{p_j} = r \geq 3$. Thus we have to look at all triples n,r,m of natural numbers with $n \geq 2$, $r \geq 3$, $m \geq 1$ satisfying

$$\frac{2}{m} + \frac{r}{n} = r - 2 \tag{6}$$

There are exactly 11 possibilities

(7)

n	2	2	2	3	3	4	4	5	5	6	9
r	5	6	8	4	6	3	4	3	5	3	3
m	4	2	1	3	1	8	2	5	1	4	3

Now we can list good coverings Y for weighted arrangements with constant weight n for all lines L_i which satisfy $3c_1(Y) = c_1^2(Y)$. We list all such cases for the arrangements given in 3. where (5) gives the value 0. In all these cases Höfer shows that such good coverings Y of S exist (for some degree d), that these surfaces Y are of general type and therefore have the ball as universal covering.

For the complete quadrilateral we can take $n_i = n = 4,5,6,9$, the weight in each of the 4 triple points being 8,5,4,3 respectively.

For the following arrangements we indicate only the constant weight n

for the lines L_i , because the weight m_j in a multiple point p_j with $r_{p_j} = r \geq 3$ is determined by (6) and listed in (7) .

For the arrangement $A_3^\circ(3)$ we can take $n = 4,5,6,9,$ for $A_3^\circ(4)$ we can take $n = 4$, for $A_3^\circ(5)$ the constant weight $n = 5$ is possible. These are all cases among the $A_3^\circ(m)$.

For $A_3^3(2)$ we can take $n = 4$, for $A_3^3(3)$ we can take $n = 5$. These are all cases among the $A_3^3(m)$.

For the Hesse arrangement $n = 3$ and $n = 4$ is possible. For the icosahedral arrangement $n = 5$ gives a solution, and for the G_{168}-arrangement $n = 4$. There is no solution for the extended Hesse arrangement neither for the A_6-arrangement.

5. Höfer has associated to each arrangement a quadratic form over $\mathbb{R}^{k+s}$ in $k + s$ variables $x_1,\ldots,x_k,y_1,\ldots,y_s$ (see (4)) which he denotes by $\mathrm{Prop}(x,y)$ because it gives the deviation from the "proportionality" $3c_2 = c_1^2$. We have by definition

$$(8) \qquad \mathrm{Prop}(x,y) = \frac{1}{4}(x^tAx + \sum_{j=1}^{s} P_j(x,y)^2)$$

Höfer's formula (4) and the Miyaoka-Yau inequality could lead to the guess that this form is semidefinite. However, this is wrong in general (see [Hi2]). But, in some cases positive semidefiniteness can be shown. The matrix A can be written in the form

$$A = 3B - U$$

where U is the matrix with all entries equal to 1 and B has

$B_{ii} = \sigma_i - 1$ and $B_{ij} = 1$ if $i \neq j$ and $p = L_i \cap L_j$ is a double point $(r = 2)$. Otherwise $B_{ij} = 0$. If for all lines $3\tau_i = k + 3$ then the column vector $e = (1,\ldots,1) \in \mathbb{R}^k$ satisfies $Ae = 0$ and for every vector x orthogonal to e (in the standard metric of $\mathbb{R}^k$)

$$Ax = 3Bx$$

Therefore $\mathrm{Prop}(x,y)$ is positive semidefinite, if B is. For all the arrangements listed in 3. the matrix B is positive semi-definite and in cases b), d), f), g), h) positive definite which implies that $\mathrm{Prop}(x,y)$ (form in $k + s$ variables) is positive semidefinite with an eigenvalue 0 of multiplicity 1, and the good coverings Y with $3c_2(y) = c_1^2(Y)$ must have constant weights for all lines L_i of the arrangement.

6. The quadratic form $\mathrm{Prop}(x,y)$ (see (8)) can be written as

$$(9) \qquad \mathrm{Prop}(x,y) = \frac{1}{2}\left(\sum_{i=1}^{k} x_i \frac{\partial}{\partial x_i}\mathrm{Prop}(x,y) + \sum_{j=1}^{s} y_j \frac{\partial}{\partial y_j}\mathrm{Prop}(x,y)\right)$$

where (see (4))

$$\frac{\partial}{\partial y_\beta}\mathrm{Prop}(x,y) = P_\beta(x,y) \quad , \quad 1 \leqq \beta \leqq s$$

We define

$$\frac{\partial}{\partial x_\alpha}\mathrm{Prop}(x,y) = Q_\alpha(x,y) \quad , \quad 1 \leqq \alpha \leqq k$$

The $k + s$ homogeneous linear equations in the $k + s$ real variables $x_1,\ldots,x_k,y_1,\ldots,y_s$

$$Q_\alpha(x,y) = 0 \quad , \quad P_\beta(x,y) = 0$$

or equivalently

$$(10) \qquad Ax = 0 \quad , \quad P_\beta(x,y) = 0 \qquad (1 \leqq \beta \leqq s)$$

define the nullspace of Prop(x,y) whose dimension equals the corank of A . If (10) holds, then Prop(x,y) = 0 . The converse is true, if Prop(x,y) is positive semidefinite. Hence, for all the arrangements listed in 3., an algebraic surface Y (good covering of S of degree d with respect to $L_1,\dots,L_k,E_1,\dots,E_s$ and the given branching numbers $n_1,\dots,n_k,m_1,\dots,m_s$) satisfies $3c_2(Y) = c_1^2(Y)$ if and only if (10) holds with

$$(11) \qquad x_i = 1 - \frac{1}{n_i} \quad \text{and} \quad y_j = -1 + \frac{1}{m_j} \quad ; \quad n_i \geq 2 \ , \ m_j \geq 1$$

For the complete quadrilateral the corank of A is 4 , there are finitely many solutions of (10) , (11) . This corresponds to the theory of the hypergeometric differential equation [DM] . We come back to this later.

For the extended Hesse arrangement (see 3.e)) the corank of A is 2 . The x_i have to be constant for the lines of the Hesse arrangement and also constant for the additional 9 lines. Höfer shows that there are exactly 3 solutions of (10) , (11) . The weights n_i are:

Hesse lines	additional lines
3	9
4	2
4	6

The weights m_j are determined by $P_j(x,y) = 0$. Höfer shows that such coverings exist.

<u>7.</u> The ball $|z_1|^2 + |z_2|^2$ is embedded in $P_2(\mathbb{C})$. The automorphisms of the ball are exactly the projective isomorphisms of $P_2(\mathbb{C})$ which map

the ball to itself. The ball carries the invariant complex hyperbolic metric. The totally geodesic smooth curves (totally geodesic as 2-dimensional surfaces in a 4-dimensional Riemannian manifold) are the intersections of lines of $P_2(\mathbb{C})$ with the ball.

Let Y be an algebraic surface whose universal cover is the ball. Then Y inherits the complex hyperbolic metric from the ball. Up to a constant factor, this the unique Einstein-Kähler metric of Y. Every automorphism of Y is an isometry. The totally geodesic curves of Y are those curves which when lifted to the ball become lines. If a curve is pointwise fixed under an automorphism of Y (different from the identity), then this curve is totally geodesic. Therefore, if Y is in addition a good covering of S as in 1. , then all curves $\pi^{-1}(C_i)$ (they are smooth, but not necessarily connected) are totally-geodesic, if the branching number a_i is greater than 1 .

If a smooth curve C on Y is totally geodesic, then

$$e(C) = 2CC$$

where $e(C)$ is the Euler-Poincaré characteristic of C and CC the selfintersection number. This follows from a relative version of the "proportionality principle" [Hi1] , because it is true in $P_2(\mathbb{C})$ where the totally geodesic curves are the lines $(e(L) = 2 , LL = 1)$.

Enoki has proved [E] that for every smooth curve on Y (the universal cover of Y is still supposed to be the ball)

$$(12) \qquad e(C) \leq 2CC$$

and that C is totally geodesic if and only if the equality sign holds in (12) .

As deviation from proportionality we define

$$(13) \qquad \mathrm{prop}(C) = 2CC - e(C)$$

for a smooth curve C on an algebraic surface Y. If the universal cover of Y is the ball, then $\mathrm{prop}(C) \geq 0$ according to Enoki's observation.

8. Consider again a weighted arrangement of lines in the plane (as in 2.) and let Y be a good covering of S with respect to $L_1,\dots,L_k,E_1,\dots,E_s$ and the branching numbers $n_1,\dots,n_k$ $(n_i \geq 2)$ and $m_1,\dots,m_s$ $(m_j \geq 1)$. Let d be the degree of $\pi\colon Y \to S$. Then $\pi^{-1}(L_i)$ and $\pi^{-1}(E_j)$ are smooth curves in Y (generally not connected). The partial derivatives of $\mathrm{Prop}(x,y)$ introduced in 6. have, as Höfer shows, a geometric meaning, namely

$$(14)\qquad Q_\alpha(x,y) = \frac{n_\alpha}{d}\,\mathrm{Prop}(\pi^{-1}(L_\alpha)) \quad , \quad P_\beta(x,y) = \frac{m_\beta}{d}\,\mathrm{Prop}\,(\pi^{-1}(E_\beta))$$

$$\text{if}\qquad x_\alpha = 1 - \frac{1}{n_\alpha} \quad , \quad y_\beta = -1 - \frac{1}{m_\beta} .$$

According to (9) we have

$$(15)\qquad 3c_2(Y) - c_1^2(Y) = \frac{1}{2}\Big(\sum(n_i - 1)\,\mathrm{prop}(\pi^{-1}L_i) - \sum(m_j + 1)\,\mathrm{prop}(\pi^{-1}E_j)\Big)$$

Actually, this is a formula of a rather elementary nature to be obtained directly, but is useful to recognize the <u>prop</u> of the lifted ramification locus as partial derivatives of the quadratic form $\mathrm{Prop}(x,y)$.
If Y has the ball as universal cover (equivalently, Y of general type and $3c_2(Y) = c_1^2(Y)$), then $\mathrm{prop}(\pi^{-1}L_i) = 0$ and $\mathrm{prop}(\pi^{-1}E_j) \geq 0$, see 7. Therefore, by (15) also the $\mathrm{prop}(\pi^{-1}E_j)$ vanish. Thus we get a result which we cannot formally obtain from section 6. because we do not know in general that $\mathrm{Prop}(x,y)$ is positive semidefinite.

<u>Suppose</u> Y <u>is obtained as in the beginning of this section. Assume it is of general type. Then the universal cover of</u> Y <u>is the ball if and only if all</u> $prop(\pi^{-1}L_i)$ <u>and</u> $prop(\pi^{-1}E_j)$ <u>vanish.</u>

<u>9.</u> As an illustration let us look at $prop(\pi^{-1}E_j))$.

The curve E_j on S arose from blowing up the point p_j in the plane. Put $r_{p_j} = r$ and let $L_1,\ldots,L_r$ be the lines passing through p_j with weights $n_1,\ldots,n_r$.

Put $E_j = E$ and $m_j = m$. Then

$$EE = -1 \quad \text{and} \quad \pi^{-1}(E)\cdot\pi^{-1}(E) = -\frac{d}{m^2}$$

For the Euler-Poincaré characteristic we get

$$e(\pi^{-1}E) = \frac{d}{m}(2 - r) + \sum_{i=1}^{r} \frac{d/m}{n_i}$$

Thus

$$prop(\pi^{-1}E) = \frac{d}{m}\left(-\frac{2}{m} - \sum_{i=1}^{r}\frac{1}{n_i} + r - 2\right)$$

$$= \frac{d}{m}\left(2y + \sum_{i=1}^{r} x_i\right)$$

if $x_1 = 1 - \frac{1}{n_i}$ and $y = -1 - \frac{1}{m}$

This checks (14) , see the definition of the linear form $P_j(x,y)$ in 2. Thus $prop(\pi^{-1}E)$ vanishes if and only if

$$(15) \qquad \frac{2}{m} + \sum_{i=1}^{r} \frac{1}{n_i} = r - 2 .$$

Höfer gives a complete list of the $(m; n_1,\ldots,n_r)$ with $m \geq 1$; $n_1 \geq n_2 \geq \ldots \geq n_r \geq 2$ satisfying (15) . Let N_r be the number of the solutions for given r .

We have

r	3	4	5	6	7	8
N_r	87	27	150	18	3	1

Included are the 11 cases with constant n_i (see (7)).

10. Consider the complete quadrilateral as in 3.a). We have to blow up the four triple points to get the surface S . It is a Del Pezzo surface on which we have 10 exceptional curves $L_1,\ldots,L_6,E_1,\ldots,E_4$. But the configuration of these 10 curves on S is very symmetric. In this special case, the L_i and E_j do not play seperate roles. We can index the 10 curves by the 10 subsets of $\{0,1,2,3,4\}$ of cardinality 2 , in such a way that two curves intersect if and only if their indexing subsets are disjoint. We denote the 10 curves by E_{ij} (with $i,j \in \{0,1,2,3,4\}$) . We see that the configuration of our 10 curves admits S_5 as symmetry group. We can choose E_1,E_2,E_3,E_4 as $E_{01},E_{02},E_{03},E_{04}$. The weights n_i,m_j are now denoted by n_{ij} , in particular $n_{0j} = m_j$.

We have

$$\operatorname{prop}(\pi^{-1}E_{01}) = \frac{d}{n_{01}}\left(-\frac{2}{n_{01}} - \frac{1}{n_{23}} - \frac{1}{n_{24}} - \frac{1}{n_{34}} + 1\right)$$

Therefore, to find surfaces Y whose universal cover is the ball we have to look at weights n_{ij} satisfying

$$\frac{2}{n_{01}} + \frac{1}{n_{23}} + \frac{1}{n_{24}} + \frac{1}{n_{34}} = 1 \tag{16}$$

and all permutations of (16) . We must have $n_{ij} \geq 2$. Up to permutation there are 7 solutions, the 4 solutions mentioned in 4. and 3 others. The table in [DM] has 27 cases due to the fact that ramified covers Y of the plane are admitted which are related to ball quotients for groups Γ which do not operate freely or are not cocompact and have cusps. Höfer's theory includes these cases for the complete quadrilateral and for all other arrangements. Refinements of Yau's theorem due to Miyaoka [Mi3] and R. Kobayashi [Ko] are needed.

If, in the case of the complete quadrilateral,

$$\frac{1}{n_{ij}} = 1 - \mu_i - \mu_j \quad \text{and} \quad \sum_{i=0}^{4} \mu_i = 2$$

then (16) and permutations hold. This is the notation of [DM]. The affine space $\sum\mu_i = 2$ corresponds to our 4-dimensional nullspace of the quadratic form Prop(x,y).

<u>11.</u> The ramified covers of the plane with respect to the complete quadrilateral correspond to the theory of the hypergeometric differential equation dating back to E. Picard (see [DM]). The question arises whether such differential equations whose monodromy gives our coverings exist also for other arrangements. This difficult question has been successfully treated by Masaaki Yoshida in two papers ([Yo1], [Yo2]).

The work of Holzapfel on Picard modular surfaces (see for example [Hol] and the references given there) has many connections with this paper.

R E F E R E N C E S

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