MODULAR OPERADS

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A family of Kähler-Einstein manifolds

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MODULAR OPERADS

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Introduction

Recently, there has been increased interest in applications of operads outside homotopy theory, much of it due to the relation between operads and moduli spaces of algebraic curves.

The formalism of operads is closely related to the combinatorics of trees [6], [8]. However, in dealing with moduli spaces of curves, one encounters general graphs, the case of trees corresponding to curves of genus 0.

This suggests considering a "higher genus" analogue of the theory of operads, in which graphs replace trees. We call the resulting objects modular operads: their systematic study is the purpose of this paper.

On the category of differential graded (dg) operads, there is a duality functor B, the cobar-construction, which is a sum over trees [8]. On the category of dg-modular operads, we construct an analogous duality functor F, the Feynman transform, which is a sum over arbitrary graphs (4.2). This functor is closely related to Kontsevich's graph complexes [14].

The behaviour of F is more mysterious than that of the cobar construction. For example, for such a simple operad as Com, describing commutative algebras, BCom is a resolution of the Lie operad. On the other hand, knowledge of the homology of FCom implies complete information on the dimensions of the spaces of Vassiliev invariants of knots (by a theorem of Kontsevich and Bar-Natan [2]; see (6.5)).

Our main result about the Feynman transform is the calculation of its Euler characteristic, using the theory of symmetric functions. As a model for this calculation, take the formula for the enumeration of graphs known in mathematical physics as Wick's theorem [3]. Consider the asymptotic expansion of the integral

$$W(\xi, \hbar, v_3, v_4, \dots) = \log \int \exp \frac{1}{\hbar}\Big(x\xi - \frac{x^2}{2} + \sum_{k=3}^{\infty} \frac{a_k x^k}{k!} \Big) \frac{dx}{\sqrt{2\pi\hbar}} \tag{0.1}$$

for small ξ and $\hbar$. (This expansion is independent of the domain of integration, provided it contains 0.) Wick's theorem gives

$$W \sim \sum_{g=0}^{\infty} \hbar^{g-1} \sum_{n=0}^{\infty} \frac{\xi^n}{n!} \sum_{G\in\Gamma(g,n)} \frac{1}{|\operatorname{Aut}(G)|} \prod_{v\in \mathsf{Vert}(G)} a_{|v|}, \tag{0.2}$$

where $\Gamma(g,n)$ is the set of connected graphs G with $\dim H_1(G) = g$, having exactly n legs numbered from 1 to n, $|v|$ is the valence of the vertex $v \in \operatorname{Vert}(G)$, and $|\operatorname{Aut}(G)|$ is the cardinality of the automorphism group of G.

The "classical limit" $W_0 = \lim_{\hbar\to 0} \hbar W$ is a sum over simply connected graphs, that is, trees. A formal application of the principle of stationary phase to (0.1) shows that

W_0 is the value of the function

$$x\xi - \frac{x^2}{2} + \sum_{k=3}^{\infty} \frac{a_k x^k}{k!}$$

at its critical value x_0: this is the same thing as the Legendre transform $\mathcal{L}f$ of the formal power series

$$f(x) = -\frac{x^2}{2} + \sum_{k=3}^{\infty} \frac{a_k x^k}{k!}.$$

The calculation of the Euler characteristic of F is a natural generalization of this, in which the coefficients a_k are replaced by representations of the symmetric groups $\mathbb{S}_k$, sums and products are replaced by the operations $\oplus$ and $\otimes$, and the weight $|\operatorname{Aut}(G)|^{-1}$ is replaced by taking the coinvariants with respect to a natural action of $\operatorname{Aut}(G)$. Up to isomorphism, a sequence $\mathcal{V} = \{\mathcal{V}(k) \mid k \geq 0\}$ of $\mathbb{S}_k$-modules is determined by its Frobenius characteristic $\operatorname{ch}(\mathcal{V})$, which is a symmetric function (power series) $f(x_1, x_2, \dots)$ in infinitely many variables. In Sections 7 and 8, we define analogues of the Legendre and Fourier transforms for symmetric functions. In this way, we obtain formulas for the characteristics of $\mathsf{B}\mathcal{A}$ and $\mathsf{F}\mathcal{A}$, where $\mathcal{A}$ is a cyclic, respectively modular, operad.

The use of symmetric functions in enumeration of graphs goes back to Pólya [20]. Our approach is slightly different: while he associates symmetric functions to permutations of vertices of the graph, we associate them to permutations of flags of the graph (pairs consisting of a vertex and an incident edge). The idea of attaching arbitrary representations of symmetric groups to vertices of a tree appears (under the name "blobs") in Hanlon-Robinson [9]; they obtain formulas resembling our formula for the characteristic of $\mathsf{B}\mathcal{A}$ (in Pólya's setting). The introduction of the Legendre transform in this problem leads to a new perspective on this class of problems by bringing out a hidden involutive symmetry, which is very natural from the point of view of operads.

Our analogue of Wick's theorem may be viewed as a fusion of the methods of graphical enumeration of quantum field theory with Pólya's ideas. Our formula for the character of $\mathsf{F}\mathcal{A}$ has another link to quantum field theory, since the space of symmetric functions is the Hilbert space for the basic representation of $\mathrm{GL}_{\mathrm{res}}(\infty)$ (Kac-Raina [12]); in this direction, we present a formal representation of the characteristic of $\mathsf{F}\mathcal{A}$ as a functional integral (8.17).

Acknowledgements

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1. Cyclic operads

In this section, we recall the definition of a cyclic operad — this will be useful later, since one way of looking at modular operads is as a special kind of cyclic operad.

Our presentation of the theory of cyclic operads is a little different from our previous account [7]; we need a non-unital version of the theory, due to Markl [16]. One advantage of this formulation is that the basic operations in an operad are bilinear. In any case, if one simply took the original definition of an operad (May [18]), and omitted the axioms involving the unit, one would not obtain the correct notion: the best justification for the definition which we present is that it leads to a simple construction of the free non-unital operad generated by an $\mathbb{S}$-module.

(1.1) $\mathbb{S}$-modules. Throughout this paper, we work over a fixed field k of characteristic 0.

A chain complex (dg-vector space) is a graded vector space $V_\bullet$ together with a differential $\delta : V_i \to V_{i-1}$, such that $\delta^2 = 0$. The suspension $\Sigma V_\bullet$ of a chain complex $V_\bullet$ has components $(\Sigma V)_n = V_{n+1}$, and differential equal to minus that of $V_\bullet$. By $\Sigma^n V_\bullet$, $n \in \mathbb{Z}$, we denote the n-fold iterated suspension of $V_\bullet$.

As in [7], denote by $\mathbb{S}_n$ the group $\mathrm{Aut}\{1, \dots, n\}$ and by $\mathbb{S}_{n+1}$ the group $\mathrm{Aut}\{0, 1, \dots, n\}$. An $\mathbb{S}$-module is a sequence of chain complexes $\mathcal{V} = \{\mathcal{V}(n) \mid n \geq 0\}$, together with an action of $\mathbb{S}_n$ on $\mathcal{V}(n)$ for each n.

A map of $\mathbb{S}$-modules is called a weak equivalence if it induces isomorphisms in homology.

(1.2) Operads. An operad is an $\mathbb{S}$-module $\mathcal{P}$ together with bilinear operations

$$\circ_i : \mathcal{P}(m) \otimes \mathcal{P}(n) \to \mathcal{P}(m+n-1), 1 \leq i \leq m,$$

satisfying the following axioms.

(1) If $\pi \in \mathbb{S}_m$ and $\rho \in \mathbb{S}_n$, let $\sigma \in \mathbb{S}_{m+n-1}$ be the permutation defined by the explicit formula

$$\sigma(j) = \begin{cases} \pi(j), & j < i, \\ \pi(i) + \rho(j-i+1), & i \leq j < i+n, \\ \pi(j-n+1), & i+n \leq j < m+n. \end{cases}$$

If $a \in \mathcal{P}(m)$ and $b \in \mathcal{P}(n)$, then

$$(\pi a) \circ_i (\rho b) = \sigma(a \circ_{\pi(i)} b).$$

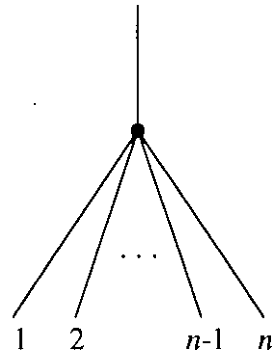
(2) For $a \in \mathcal{P}(k)$, $b \in \mathcal{P}(l)$ and $c \in \mathcal{P}(m)$, and $1 \leq i < j \leq m$,

$$(a \circ_i b) \circ_{j+l-1} c = (a \circ_j b) \circ_i c.$$

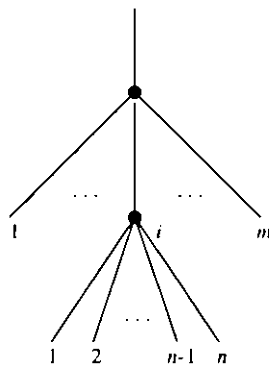
(3) For $a \in \mathcal{P}(k)$, $b \in \mathcal{P}(l)$ and $c \in \mathcal{P}(m)$, and $1 \leq i \leq k$, $1 \leq j \leq m$,

$$(a \circ_i b) \circ_{i+j-1} c = a \circ_i (b \circ_j c).$$

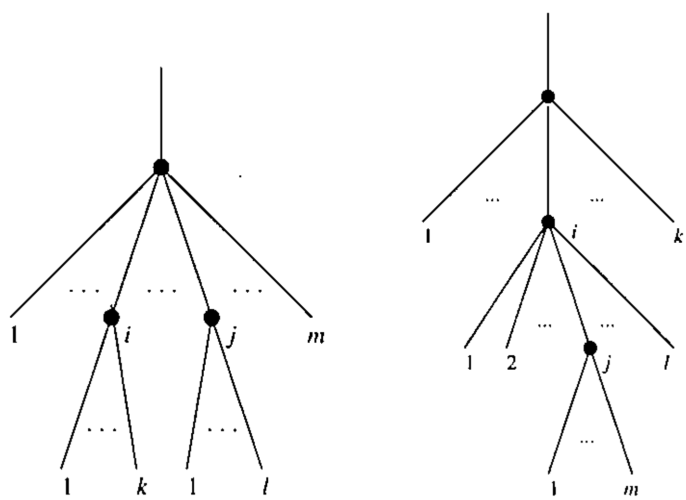
(1.3) Operads and trees. We think of an element of $\mathcal{P}(n)$ as corresponding to a rooted tree with one vertex, n inputs numbered from 1 up to n (and one output).



The compositions correspond to grafting two such trees together along the input of the first tree numbered i. Axiom (1) expresses the equivariance of this construction.



Axioms (2) and (3) mean that we can construct unambiguous compositions corresponding to the following two trees respectively.



In fact, the axioms imply that the products $\circ_i$ give rise to an unambiguous definition of composition for any rooted tree [6], [8]. This point of view will be explained in greater detail, in the context of modular operads, in Section 2.

(1.4) Cyclic $\mathbb{S}$-modules. A cyclic $\mathbb{S}$-module $\mathcal{V}$ is a sequence of vector space $\mathcal{V}(n)$, with action of $\mathbb{S}_{n+1}$ on $\mathcal{V}(n)$. In particular, each vector space $\mathcal{V}(n)$ is a module over the symmetric group $\mathbb{S}_n$, and over the cyclic group C_{n+1} generated by $\tau_n = (01\dots n)$.

If $\mathcal{V}$ is a cyclic $\mathbb{S}$-module, and I is a $(k+1)$-element set, define

$$\mathcal{V}((I)) = \Big(\bigoplus_{\substack{\text{bijections} \\ f:\{0,\dots,k\} \longrightarrow I}} \mathcal{V}(k) \Big)_{\mathbb{S}_{k+1}}.$$

This makes $\mathcal{V}$ into a functor from the category of nonempty finite sets and their bijections into the category of vector spaces. In the case when $k = n-1$ and $I = \{1, \dots, n\}$, we write $\mathcal{V}((n))$ instead of $\mathcal{V}((I))$. Note that $\mathcal{V}((n)) = \mathcal{V}(n-1)$.

(1.5) Cyclic operads. A cyclic operad [7] is a cyclic $\mathbb{S}$-module $\mathcal{P}$ whose underlying $\mathbb{S}$-module has the structure of an operad, such that

$$\tau_{m+n-1}(a \circ_m b) = (\tau_n b) \circ_1 (\tau_m a) \tag{1.6}$$

for any $a \in \mathcal{P}(m)$, $b \in \mathcal{P}(n)$. Here τ_n stands for the cycle $(01\ldots n) \in \mathbb{S}_{n+1}$.

(Cyclic) $\mathbb{S}$-modules may be defined, in exactly the same way, in any symmetric monoidal category. The most important case for us will be the category of chain complex. Other examples are the category of topological spaces, giving rise to topological $\mathbb{S}$-modules and operads, and the opposite category to the category of chain complexes, whose operads are called dg-cooperads.

In the remainder of this paper, unless otherwise specified, by an $\mathbb{S}$-module, operad or cooperad, we mean a differential graded $\mathbb{S}$-module, operad or cooperad. A map of operads is called a weak equivalence if it is a weak equivalence of the underlying $\mathbb{S}$-module.

(1.7) Example: endomorphism operads. Let V be a vector space with a symmetric scalar product $B(x,y)$. We define a cyclic operad $\mathcal{E}[V]$ by putting $\mathcal{E}[V](n) = V^{\otimes(n+1)}$. For $a = v_0 \otimes \ldots \otimes v_m \in \mathcal{E}[V](m)$ and $b = w_0 \otimes w_1 \otimes \ldots \otimes w_n \in \mathcal{E}[V](n)$ we put

$$a \circ_i b = B(v_i, w_0) \cdot v_0 \otimes v_1 \otimes \ldots \otimes v_{i-1} \otimes w_1 \otimes \ldots \otimes w_n \otimes v_{i+1} \otimes \ldots \otimes v_m.$$

If $\dim(V) < \infty$ and B is non-degenerate, we can identify $\mathcal{E}[V](n)$ with $\mathrm{Hom}(V^{\otimes n}, V)$ by means of B. In this way, the underlying operad of $\mathcal{E}[V]$ becomes identified with $\mathcal{E}_V$, the endomorphism operad of the space V [8], [18].

(1.8) Cyclic algebras. A cyclic algebra over a cyclic operad $\mathcal{P}$ is a vector space A together with a scalar product B and a morphism if cyclic operads $\mathcal{P} \to \mathcal{E}[A]$.

(1.9) Examples.

(1.9.1) ***Stable curves of genus*** 0**.** Define a topological cyclic operad $\mathcal{M}_0$ by letting $\mathcal{M}_0(n)$ be the moduli space $\mathcal{M}_{0,n+1}$ of stable $(n+1)$-pointed curves of genus 0 [13] (see also [8]). By definition, a point of $\mathcal{M}_{g,n}$ is a system $(C, x_1, \dots, x_n)$, where C is a projective curve of arithmetic genus 0, with possibly nodal singularities, x_i are distinct smooth points, and C has no infinitesimal automorphisms preserving the points x_i (this amounts to saying that each component of C minus its singularities and marked points has negative Euler characteristic). The $\mathbb{S}_n$-action on $\mathcal{M}_{0,n}$ is given by renumbering the punctures. The composition $\circ_i$ takes two pointed curves $(C, x_0, \dots, x_m)$ and $(D, y_0, \dots, y_n)$ into

$$\big(C \coprod D/(x_i \sim y_0), x_0, \dots, x_{i-1}, y_1, \dots, y_n, x_{i+1}, \dots, x_m\big).$$

(1.9.2) ***Spheres with holes.*** Define a topological cyclic operad $\widehat{\mathcal{M}}_0$ by letting $\widehat{\mathcal{M}}_0(n)$ be the moduli space of data $(C, f_0, \dots, f_n)$, where C is a complex manifold isomorphic to $\mathbb{C}P^1$, and f_i are biholomorphic maps of the unit disk

$$\Delta = \{z \in \mathbb{C} \mid |z| \leq 1\}$$

into C with disjoint images. The composition $\circ_i$ takes $(C, f_0, \dots, f_m)$ and $(D, g_0, \dots, g_n)$ into

$$\Big(\big(C \setminus f_i[\mathring{\Delta}]\big) \coprod\nolimits_{f_i(t) \sim g_0(t), t \in \partial\Delta} \big(D \setminus g_0[\mathring{\Delta}]\big), f_0, \dots, f_{i-1}, g_1, \dots, g_n, f_{i+1}, \dots, f_m\Big)$$

Note that by applying the total homology functor $H_\bullet(-, \mathsf{k})$ to the topological operads $\overline{\mathcal{M}}_0$ and $\widehat{\mathcal{M}}_0$, we obtain cyclic operads in the category of graded vector spaces.

2. Modular operads

(2.1) Stable $\mathbb{S}$-modules. A stable $\mathbb{S}$-module is a collection of chain complexes

$$\{\mathcal{V}((g, n)) \mid n, g \geq 0\}$$

with an action of $\mathbb{S}_n$ on $\mathcal{V}((g, n))$, such that $\mathcal{V}((g, n)) = 0$ if $2g + n - 2 \leq 0$.

A morphism $\mathcal{V} \to \mathcal{W}$ of stable $\mathbb{S}$-modules is a collection of equivariant maps of chain complexes $\mathcal{V}((g, n)) \to \mathcal{W}((g, n))$.

We have borrowed the term "stable" from the theory of moduli spaces of curves, since the condition of stability is the same in the two settings.

Any cyclic $\mathbb{S}$-module $\mathcal{V}$ may be regarded as a stable $\mathbb{S}$-module by setting:

$$\mathcal{V}((g, n+1)) = \begin{cases} \mathcal{V}(n), & g = 0, \\ 0, & g > 0. \end{cases} \tag{2.2}$$

In the other direction, we have the forgetful functor, which we denote by Cyc. If $\mathcal{V}$ is a stable $\mathbb{S}$-module, then $\mathrm{Cyc}(\mathcal{V})$ is a cyclic $\mathbb{S}$-module, and

$$\mathrm{Cyc}(\mathcal{V})((n)) = \mathcal{V}((0, n)). \tag{2.3}$$

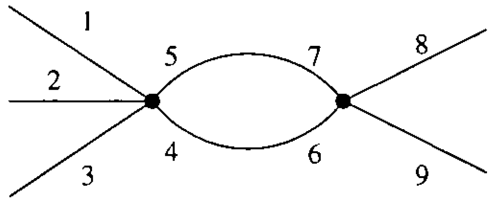
A stable $\mathbb{S}$-module $\mathcal{V}$ has a natural extension to all finite sets I:

$$\mathcal{V}((g, I)) = \Big(\bigoplus_{\substack{\text{bijections} \\ f:\{1,\dots,n\} \to I}} \mathcal{V}((g, n)) \Big)_{\mathbb{S}_n} \tag{2.4}$$

(2.5) Graphs. A graph G is a finite set $\mathrm{Flag}(G)$ (whose elements are called flags) together with an involution σ and a partition λ. (By a partition of a set, we mean a disjoint decomposition into several unordered, possibly empty, subsets.)

The vertices of G are the blocks of the partition λ, and the set of them is denoted $\mathrm{Vert}(G)$. The edges of G are the pairs of flags forming a two-cycle of σ, and the set of them is denoted $\mathrm{Edge}(G)$. The legs of G are the fixed-points of σ, and the set of them is denoted $\mathrm{Leg}(G)$. (Thus, each flag lies in either an edge or a leg.)

We may associate to a graph the finite one-dimensional cell complex $|G|$, obtained by taking one copy of $[0, \frac{1}{2}]$ for each flag, and imposing the following equivalence relation: the points $0 \in [0, \frac{1}{2}]$ are identified for all flags in a block of the partition λ, and the points $\frac{1}{2} \in [0, \frac{1}{2}]$ are identified for pairs of flags exchanged by the involution σ. For example, the following corresponds to the set of flags $\{1,\ldots,9\}$, the involution $\sigma = (46)(57)$ and the partition $\{1,2,3,4,5\} \cup \{6,7,8,9\}$.



A labelled graph is a connected graph G together with a map g from $\mathrm{Vert}(G)$ into the natural numbers. The value of this map at a given vertex v is called the genus of v

The genus $g(G)$ of a labelled graph G is the sum of $\dim H^1(|G|,\mathsf{k})$ (the number of circuits of G) and the numbers $g(v)$. It is given by the formula

$$g(G) = \sum_{v\in\mathrm{Vert}(G)} (g(v)-1) + |\,\mathrm{Leg}(G))| + |\text{components of } G|. \tag{2.6}$$

A forest is a (labelled) graph with genus 0; a tree is a connected forest.

This definition is slightly different from the definition of trees in [7]: unlike in that paper, we do not admit the tree with two legs and no vertices.

For any vertex $v \in \mathrm{Vert}(G)$, the set of flags incident with v is denoted $\mathrm{Leg}(v)$; the valence of the vertex v is the cardinality of $\mathrm{Leg}(v)$.

(2.7) Definition. A connected labelled graph is called stable if $2(g(v)-1)+|\,\mathrm{Leg}(v)| > 0$ for each vertex.

(2.8) The category of graphs. Let G_1, G_2 be two graphs. A morphism $f : G_1 \to G_2$ is an injective map $f^* : \mathrm{Flag}(G_2) \to \mathrm{Flag}(G_1)$ such that

(1) $f^* \circ \sigma_i = \sigma_2 \circ f^*$, where σ_i is the involution on $\mathrm{Flag}(G_i)$.
(2) The pullback with respect to f^* of the partition λ_1 is a refinement of λ_2.
(3) f^* defines a bijection of the fixed point sets

$$\mathrm{Flag}(G_2)^{\sigma_2} \to \mathrm{Flag}(G_1)^{\sigma_1}.$$

This definition is the same as that of Kontsevich-Manin [15].

A morphism $G_1 \to G_2$ defines a surjective cellular map $|G_1| \to |G_2|$ which is bijective on the legs. Every map obtained in this way is a composition of an isomorphism and contraction on a (possibly disconnected) subgraph which does not contain any legs.

If G_i are labelled graphs, we say that a morphism $f : G_1 \to G_2$ preserves the labelling if the genus of any vertex of G_2 is equal to the sum of the genera of all its preimages in G_1.

Let $\Gamma((g,n))$ be the (finite) set of representatives of isomorphism classes of stable graphs of genus g with a bijection between their set of legs and the set $\{1,\ldots,n\}$. We consider $\Gamma((g,n))$ as a small category, namely the full subcategory of the category of all labelled graphs and morphisms preserving the labellings. It is clear that $\Gamma((g,n))$ has a terminal object, consisting of the graph with a single vertex, of genus g, and no edges.

Denote by $\mathrm{Aut}(G)$ the automorphism group of a graph $G \in \Gamma((g,n))$: if G is a tree, $\mathrm{Aut}(G) \cong 1$.

(2.9) The determinant of a graph. For a finite-dimensional vector space V we denote $\det(V) = \lambda^{\mathrm{top}}(V)$ and call this 1-dimensional vector space the determinant of V. If G is a graph, we denote by $\det(G)$ the determinant of the vector space

$$\bigoplus_{e \in \mathrm{Edge}(G)} \mathrm{Or}(e),$$

where the sum is over all edges of G and $\mathrm{Or}(e)$ is the orientation line of the edge e. The complex of cellular cochains of Γ has the form

$$0 \to \mathsf{k}^{\mathrm{Vert}(G)} \to \bigoplus_{e \in \mathrm{Edge}(G)} \mathrm{Or}(e) \to 0,$$

giving the natural isomorphism

$$\det(G) \cong \det(\mathsf{k}^{\mathrm{Vert}(G)}) \otimes \det(H^1(|G|, \mathsf{k})).$$

It follows that our definition of $\det(G)$ agrees with Kontsevich's [14].

(2.10) The triple of stable graphs. Let $\mathcal{V}$ be a stable $\mathbb{S}$-module, and let G be a stable graph. Define $\mathcal{V}((G))$ to be the direct sum

$$\mathcal{V}((G)) = \bigotimes_{v \in \mathrm{Vert}(G)} \mathcal{V}((g(v), \mathrm{Leg}(v))).$$

We now define a functor $\mathbb{M}_+$ from the category of stable $\mathbb{S}$-modules to itself, by summing over isomorphism classes of graphs:

$$\mathbb{M}_+ \mathcal{V}((g,n)) = \bigoplus_{G \in \Gamma((g,n))} \mathcal{V}((G))_{\mathrm{Aut}(G)}. \tag{2.11}$$

Similarly, we define a functor $\mathbb{M}_-$ by

$$\mathbb{M}_- \mathcal{V}((g,n)) = \bigoplus_{G \in \Gamma((g,n))} \Big(\det(G) \otimes \mathcal{V}((G))\Big)_{\mathrm{Aut}(G)}.$$

On the category of cyclic $\mathbb{S}$-modules, there is a suspension functor Λ, given by the formula

$$\Lambda \mathcal{V}(n) = \Sigma^{1-n} \operatorname{sgn}_{n+1} \otimes \mathcal{V}(n),$$

where sgn_{n+1} is the alternating character of $\mathbb{S}_{n+1}$ [7]. The analogue of this in the category of graded stable $\mathbb{S}$-modules is given by the formula

$$\Lambda \mathcal{V}((g,n)) = \Sigma^{-2(g-1)-n} \operatorname{sgn}_n \otimes \mathcal{V}((g,n)).$$

(2.12) Proposition. *The functors $\mathbb{M}_+$ and $\mathbb{M}_-$ are related by the formula*

$$\mathbb{M}_\pm = \Lambda^{-1} \circ \mathbb{M}_\mp \circ \Lambda.$$

Grafting of graphs defines, for any stable $\mathbb{S}$-module $\mathcal{V}$, a natural map $\mathbb{M}_+\mathbb{M}_+\mathcal{V} \to \mathbb{M}_+\mathcal{V}$, giving to $\mathbb{M}_+$ the structure of a triple; the unit of the triple is given by the embeddings $\mathcal{V} \to \mathbb{M}_+\mathcal{V}$ induced by graphs with a single vertex.

By conjugation, we see that $\mathbb{M}_- = \Lambda^{-1} \circ \mathbb{M}_+ \circ \Lambda$ also has the structure of a triple.

(2.13) Definition. A modular operad is an algebra over the triple $\mathbb{M}_+$ in the category of stable $\mathbb{S}$-modules. An antimodular operad is an algebra over the triple $\mathbb{M}_-$.

For example, for any stable $\mathbb{S}$-module $\mathcal{V}$, $\mathbb{M}_+\mathcal{V}$ is a modular operad, called the free modular operad generated by $\mathcal{V}$. As for cyclic operads, modular operads may be considered in any symmetric monoidal categories.

(2.14) The structure map. If $\mathcal{A}$ is a modular operad and $G \in \Gamma((g,n))$, the structure map $\mathbb{M}_+\mathcal{A}((g,n)) \to \mathcal{A}((g,n))$ restricted to $\mathcal{A}((G)) \subset \mathbb{M}_+((g,n))$ is a $\mathbb{S}_n$-equivariant map $\mu_G : \mathcal{A}((G)) \to \mathcal{A}((g,n))$, which we will call composition along the graph G.

A modular operad determines a functor from the category $\Gamma((g,n))$ to the category of $\mathbb{S}_n$-modules. To a labelled graph G is associated the $\mathbb{S}_n$-module $\mathcal{A}((G))$. To define the action of morphisms, it suffices to describe the map $\mathcal{A}((G)) \to \mathcal{A}((G/H))$ where H is a subgraph of G with no legs; this is induced by the composition map μ_H.

A modular operad may be viewed as a cyclic operad with additional structure; this point of view will be explained in detail in Section 3.

(2.15) Cyclic operads and the triple of trees. For a cyclic $\mathbb{S}$-module $\mathcal{V}$ we define a cyclic $\mathbb{S}$-module $\mathbb{T}_+\mathcal{V}$ by summing over trees:

$$\mathbb{T}_+\mathcal{V}((n)) = \bigoplus_{T\in\Gamma((0,n))} \mathcal{V}((T)).$$

The functor $\mathbb{T}_+$ is a triple and a cyclic operad $\mathcal{P}$ with $\mathcal{P}(0) = \mathcal{P}(1) = 0$ is the same as an algebra over $\mathbb{T}_+$. In particular, we have free cyclic operads $\mathbb{T}_+\mathcal{V}$, where $\mathcal{V}$ is a cyclic $\mathbb{S}$-module. Note that we have the following commutative diagram of triples:

$$\begin{array}{ccc}
\text{stable } \mathbb{S}\text{-modules} & \xrightarrow{\mathbf{M}_\pm} & \text{stable } \mathbb{S}\text{-modules} \\
{\scriptstyle \mathrm{Cyc}}\big\downarrow & & {\scriptstyle \mathrm{Cyc}}\big\downarrow \\
\text{cyclic } \mathbb{S}\text{-modules} & \xrightarrow{\mathbf{T}_\pm} & \text{cyclic } \mathbb{S}\text{-modules}
\end{array}$$

(2.16) Endomorphism operads. If V is a finite-dimensional vector space with a non-degenerate inner product $B(x,y)$, the endomorphism modular operad $\mathcal{E}[V]$ of V has stable $\mathbb{S}$-module

$$\mathcal{E}[V]((g,n)) = V^{\otimes n}$$

for all g such that $2(g-1)+n > 0$.

To define the composition maps $\mathcal{E}[V]((g,G)) \to \mathcal{E}[V]((g,n))$, we identify

$$\mathcal{E}[V]((G)) \cong V^{\otimes \operatorname{Flag}(G)}$$

with $(V \otimes V)^{\otimes \operatorname{Edge}(G)} \otimes V^{\otimes n}$, where $\operatorname{Edge}(G)$ is the set of edges of G, and pair it with the tensor power $B^{\otimes \operatorname{Edge}(G)}$.

The cyclic operad $\operatorname{Cyc}\mathcal{E}[V]$ is isomorphic to a non-unital version of the endomorphism operad of V:

$$\operatorname{Cyc}\mathcal{E}[V](n) \cong \begin{cases} \operatorname{Hom}(V^{\otimes n}, V), & n \geq 2, \\ 0, & n < 2. \end{cases}$$

(2.17) Modular algebras. An algebra over a modular operad $\mathcal{A}$ is a chain complex V with non-degenerate inner product B, together with a morphism of modular operads $\mathcal{A} \to \mathcal{E}[V]$.

If V has a non-degenerate graded antisymmetric bilinear form $B(x,y)$, then $\mathcal{E}[V]$ is an antimodular operad. An algebra over an antimodular operad $\mathcal{A}$ is a chain complex V with non-degenerate graded antisymmetric product B, together with a morphism of antimodular operads $\mathcal{A} \to \mathcal{E}[V]$.

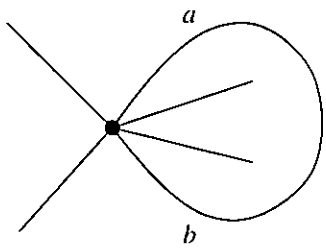
3. The structure of modular operads

If $\mathcal{A}$ is a modular operad, then the cyclic $\mathbb{S}$-module

$$\mathcal{A}((n)) = \bigoplus_g \mathcal{A}((g,n))$$

is a cyclic operad, such that the compositions are compatible with the grading by g. In this section, we explain what additional structure must be imposed on such a cyclic operad in order for it to be a modular operad, and derive the consistency conditions which all of these data must satisfy.

(3.1) The contraction maps. For a finite set I and distinct elements $a, b \in I$, define a labelled graph $G^I_{ab}(g)$ with one vertex, labelled with the genus g, such that $\mathrm{Flag}(G^I_{ab}) = I$ and there is one edge (a loop) joining the flags a and b.



If $\mathcal{A}$ is a modular operad, we have the composition map

$$\mu_{G^I_{ab}(g)} : \mathcal{A}((g,I)) = \mathcal{A}((G^I_{ab}(g))) \to \mathcal{A}((g+1, I \setminus \{a,b\}))$$

which we denote by ξ_{ab}, and call a contraction map.

(3.2) Example. Let V be a finite-dimensional vector space with a non-degenerate invariant scalar product $B(x,y)$. Let $\mathcal{E}[V]$ be its modular endomorphism operad (2.16); thus $\mathcal{E}[V]((g,n+1)) \cong \mathrm{Hom}(V^{\otimes n}, V)$ may be identified with the space of all n-linear operations from V to itself. If $\mu \in \mathcal{E}[V]((g,n+1))$ is such an operation, then $\xi_{ij}(\mu) \in \mathcal{E}[V]((g+1,n-1))$ is the $(n-2)$-linear operation

$$(x_1,\dots,x_{n-2}) \mapsto \sum_k \mu(x_1,\dots,x_{i-1},e_k,x_i,\dots,x_{j-2},f_k,x_{j-1},\dots,x_{n-2}),$$

where $\{e_k\}$ and $\{f_k\}$ are dual bases of V with respect to the scalar product B.

(3.3) Coherence for modular operads. A modular operad $\mathcal{A}$ can be regarded, by forgetting part of the structure, as a (non-unital) cyclic operad graded by genus:

$$\mathcal{A}((n)) = \bigoplus_{2(g-1)+n>0} \mathcal{A}((g,n)).$$

Note that any unlabelled graph of genus g can be obtained from a tree by joining g pairs of legs. This implies that the structure of a modular operad on $\mathcal{A}$ is uniquely defined by the structure of a cyclic operad, the grading by genus and the action of the contractions ξ_{ab}.

We must now to determine the relations between composition in $\mathcal{A}$ and the contractions ξ_{ab}.

(3.4) Theorem. *The data consisting of a cyclic operad $\mathcal{A}$, of $\mathbb{S}_n$-invariant decompositions $\mathcal{A}((n)) = \bigoplus \mathcal{A}((g,n))$ and of contraction maps*

$$\xi_{ab} : \mathcal{A}((g,I)) \longrightarrow \mathcal{A}((g+1, I \setminus \{a,b\})),$$

come from a modular operad if and only if the following coherence conditions are satisfied:

(1) *For any bijection $\sigma : I \longrightarrow J$ of finite sets and $a, b \in I$,*

$$\xi_{\sigma(a)\sigma(b)}(\sigma(\mu)) = \sigma(\xi_{ab}(\mu)), \quad \textit{for all } \mu \in \mathcal{A}((g,I)).$$

(2) *For any finite set I and distinct elements $a, b, c, d \in I$ we have*

$$\xi_{ab} \circ \xi_{cd} = \xi_{cd} \circ \xi_{ab}.$$

(3) *For any $\mu \in \mathcal{A}(n)$, $\nu \in \mathcal{A}(m)$ and $1 \leq k \leq n$, $1 \leq i < j \leq n$, we have*

$$\mu \circ_k \xi_{ij}(\nu) = \xi_{i+k-1, j+k-1}(\mu \circ_k \nu).$$

(4) *For any $\mu \in \mathcal{A}(n)$, $\nu \in \mathcal{A}(m)$ and $1 \leq a < k \leq n$, $1 \leq b \leq m$, we have*

$$\xi_{a,k+b-1}(\mu \circ_k \nu) = \xi_{a+m-k,k+m-1}(\mu \circ_a \tau^{-b}(\nu)),$$

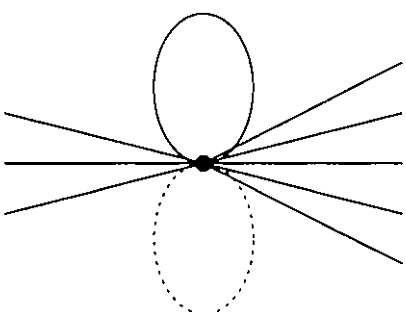
where $\tau = (012...n)$ is the cyclic rotation.

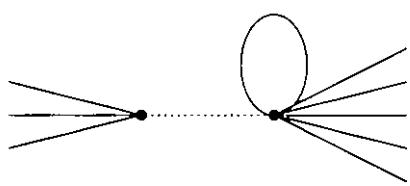
Proof. The compositions and contractions in a modular operad are precisely the composition maps

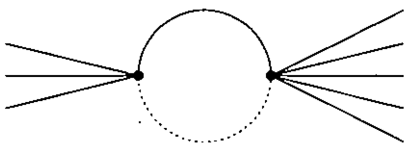
$$\mu_G : \mathcal{A}((G)) \longrightarrow \mathcal{A}((g,n))$$

associated with graphs $G \in \Gamma((g,n))$ with one edge. Indeed, such a graph has either two vertices (in which case μ_G is a composition $\circ_i$) or one vertex (in which case μ_G is a contraction).

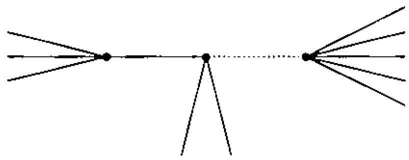
In a similar way, the relations among compositions and contractions in a modular operad are determined by the graphs with two edges: for such a graph, the relation is that the result of performing the compositions along the two edges may be done in either order without affecting the result. Omitting the numbering of legs, the graphs with two edges have the following form:


(a)


(b)



(c)



(d)

The relation associated to such a graph is that we composition gives the same result whether we compose along the solid or the dotted line first.

The equivariance condition (1) is certainly necessary. Together with the relations (2-4), it generates the relations coming from all possible labellings of graphs with two edges and one or two vertices. (Trees with two edges, as in (d), simply correspond to the relations saying that $\mathcal{A}$ is a cyclic operad whose compositions $\circ_i$ are compatible with the grading by genus g.) Graphs (a-c) now correspond respectively to conditions (2-4) of the theorem. □

4. The Feynman transform of a modular operad

In this section, we define a homotopy involution F of the category of dg-modular operads: that is, there is a natural transformation from FF to the identity functor such that if $\mathcal{A}$ is a modular operad, $\mathsf{FF}\mathcal{A} \to \mathcal{A}$ is a weak equivalence. Furthermore, F is a homotopy functor, in the sense that it maps weak equivalences to weak equivalence. We call F the Feynman transform, since $\mathsf{F}\mathcal{A}$ is a sum over graphs, as is Feynman's expansion for amplitudes in quantum field theory.

(4.1) Linear duality. In defining the Feynman transform, we restrict attention to stable $\mathbb{S}$-modules $\mathcal{V}$ such that $\mathcal{V}((g,n))$ is a complex which is finite-dimensional in each degree for each g and n. For such $\mathcal{V}$, we define its linear dual $\mathcal{V}^\vee$ by

$$\mathcal{V}^\vee((g,n)) = \Sigma^{3(g-1)+n} \operatorname{sgn}_n \otimes \mathcal{V}((g,n))^*.$$

It is clear that $(\mathcal{V}^\vee)^\vee$ is naturally isomorphic to $\mathcal{V}$.

(4.2) Definition of the transform F. Let $\mathcal{A}$ be a modular operad. As a stable $\mathbb{S}$-module, the Feynman tranform $\mathsf{F}\mathcal{A}$ equals $\mathbb{M}_+(\mathcal{A}^\vee)$, the free modular operad generated by $\mathcal{A}^\vee$. The differential $\delta_{\mathsf{F}\mathcal{A}}$ is the sum $\delta_{\mathsf{F}\mathcal{A}} = \delta_{\mathcal{A}^\vee} + \partial$, where $\delta_{\mathcal{A}^\vee}$ is the differential on $\mathbb{M}_+(\mathcal{A}^\vee)$ induced by the differential on $\mathcal{A}^\vee$, and ∂ is defined as follows.

If G is a stable graph, and e is an edge of G, let G/e be the graph obtained by contracting the edge e to a point. We define a map

$$d_{G,e} : \mathcal{A}((G)) \to \mathcal{A}((G/e))$$

of degree 0, in the following way:

(1) if the two ends $v_1, v_2 \in \operatorname{Vert}(G)$ of e are distinct, denote by v the vertex of G/e obtained by merging them; then $d_{G,e}$ is induced by the composition in the modular operad $\mathcal{A}$

$$\mathcal{A}((h_1, \operatorname{Leg}(v_1))) \otimes \mathcal{A}((h_2, \operatorname{Leg}(v_2))) \to \mathcal{A}((h_1 + h_2, \operatorname{Leg}(v)));$$

(2) if the edge e is a loop both of whose flags $\{f_+, f_-\}$ meet the vertex v, $d_{G,e}$ is induced by contraction in the modular operad $\mathcal{A}$

$$\mathcal{A}((h, \operatorname{Leg}(v))) \to \mathcal{A}((h+1, \operatorname{Leg}(v) \setminus \{f_+, f_-\})).$$

We now define

$$\partial_{G,e} = d^{\vee}_{G,e} : \mathcal{A}^{\vee}((G/e)) \to \mathcal{A}^{\vee}((G))$$

to be the adjoint map. Note that because of the suspensions involved in the definition of $\mathcal{A}^{\vee}$, the map $\partial_{G,e}$ has degree -1. (We leave to the reader the verification of this fact.)

For two stable graphs G and H, the matrix element

$$\partial_{H,G} : \mathcal{A}^{\vee}((H))_{\mathrm{Aut}(H)} \to \mathcal{A}^{\vee}((G))_{\mathrm{Aut}(G)}$$

is induced by the sum of $\partial_{G,e}$ over all edges e of G such that $H \cong G/e$.

(4.3) Theorem. *The map $\delta_{\mathsf{F}\mathcal{A}}$ has square zero and the pair $(\mathsf{F}\mathcal{A} = \mathbb{M}_{+}(\mathcal{A}^{\vee}))$ is a modular operad in the category of chain complexes.*

Proof. We must prove the following statements:

(a) $\partial \circ \delta_{\mathcal{A}^{\vee}} + \delta_{\mathcal{A}^{\vee}} \circ \partial = 0$;

(b) $\partial^2 = 0$;

(c) ∂ is compatible with compositions and contractions in $\mathbb{M}_{+}\mathcal{A}^{\vee}$.

Part (a) is obvious, because the differential in $\mathcal{A}$ which induces $\delta_{\mathcal{A}^{\vee}}$ is compatible with composition and contraction in $\mathcal{A}$.

(b) Observe that the matrix element of ∂^2 from $\mathcal{A}^{\vee}((H))_{\mathrm{Aut}(H)}$ to $\mathcal{A}^{\vee}((G))_{\mathrm{Aut}(G)}$ is a sum over pairs (e_1, e_2) of distinct edges of G such that $H \cong (G/e_1)/e_2$. The exchange $e_1 \leftrightarrow e_2$ is a fixed-point free involution on the set of such pairs, and the corresponding contributions cancel each other due to the presence of the sign character in the definition of $\mathcal{A}^{\vee}$.

(c) The compositions of the free operad $\mathbb{M}_{+}\mathcal{A}^{\vee}$ are induced by grafting of graphs. If a graph G is obtained by grafting graphs G_1 and G_2 along two legs, then each vertex of G either lies in G_1 or in G_2. The differential ∂ is induced by "splitting" vertices into edges. Thus $\partial(a_1 \circ_i a_2)$, $a_j \in \mathcal{A}((G_j))$, is the sum of two terms, one corresponding to splitting vertices of G_1, the other corresponding to splitting vertices of G_2. The first of these clearly equals $(\partial a_1) \circ_i a_2$, and the second is $(-1)^{|a_1|} a_1 \circ (\partial a_2)$. Thus ∂ satisfies the Leibniz formula for compositions. The compatibility of contractions is proved in the same way. $\square$

We now arrive at the main result of this section.

(4.4) Theorem. *If $\mathcal{A}$ is a modular operad, the canonical map $\mathsf{FF}\mathcal{A} \to \mathcal{A}$ is a weak equivalence, that is, induces an isomorphism on homology.*

Proof. The component $(\mathsf{FF}\mathcal{A})((g,n))$ is a sum over graphs in $\Gamma((g,n))$, in which the edges are colored black or white, depending on whether they represent an edge arising from the construction of the inner or the outer F. Denote a colored graph by (G,c), where $c : \mathrm{Edge}(G) \to \{\mathrm{black}, \mathrm{white}\}$, and denote the corresponding contribution to $(\mathsf{FF}\mathcal{A})((g,n))$ by $(\mathsf{FF}\mathcal{A})((g,G,c))$.

Filter the complex $(\mathsf{FF}\mathcal{A})((g,n))$ by the number of edges in the colored graph (G,c). On the associated graded complex $\mathrm{gr}(\mathsf{FF}\mathcal{A})((g,n))$, the differential is the sum of the internal differential induced by the differential of $\mathcal{A}$ and terms in which a white edge e

of the colored graph (G, c) is painted black. Denoting the resulting colored graph by (G, c_e), such a term reflects the isomorphism

$$\mathrm{gr}(\mathsf{FF}\mathcal{A})((G, c_e)) \cong \Sigma^{-1}\,\mathrm{gr}(\mathsf{FF}\mathcal{A})((G, c)).$$

Denote by (G, c^e) the colored graph in which a black edge e of the colored graph (G, c) is bleached, and let $h(G, c)$ be the map

$$h(G, c) : \mathrm{gr}(\mathsf{FF}\mathcal{A})((G, c)) \to \mathrm{gr}(\mathsf{FF}\mathcal{A})((g, n))$$

of degree $+1$ obtained by averaging the identifications

$$\mathrm{gr}(\mathsf{FF}\mathcal{A})((G, c^e)) \cong \Sigma\,\mathrm{gr}(\mathsf{FF}\mathcal{A})((G, c)).$$

over the edges of G. (Here, we make use of the assumption that the characteristic of k is zero.) The resulting map $h : \mathrm{gr}(\mathsf{FF}\mathcal{A}) \to \mathrm{gr}(\mathsf{FF}\mathcal{A})$ is a contracting homotopy from the complex $\mathrm{gr}(\mathsf{FF}\mathcal{A})$ to the subcomplex obtained by summing over graphs with no edges: but this subcomplex is isomorphic to $\mathcal{A}$. □

(4.5) Theorem. *The Feynman transform* F *is a homotopy functor: if* $f : \mathcal{A} \to \mathcal{B}$ *is a weak equivalence of modular operads, then so is* $\mathsf{F}f : \mathsf{F}\mathcal{A} \to \mathsf{F}\mathcal{B}$.

Proof. This is easily proved by considering a spectral sequence associated to the cone of the map $\mathsf{F}f$: we filter by number of edges, in such a way that the E^1-term of the spectral sequence equals the cone of the map $\mathsf{F}H_\bullet(f)$, which is zero by hypothesis. The convergence of the spectral sequence follows from the fact that $\mathsf{F}\mathcal{A}((g, n))$ and $\mathsf{F}\mathcal{B}((g, n))$ have contributions from a finite number of graphs, so that the spectral sequence is uniformly bounded in one direction. □

(4.6) The cobar operad of a cyclic operad. If $\mathcal{A}$ is a cyclic operad (regarded as a modular operad as in (2.2)), then $\mathrm{Cyc}(\mathsf{F}\mathcal{A})$ is the cobar operad $\mathsf{B}\mathcal{A}$ of $\mathcal{A}$ in the sense of Section 3.2 of [8].

5. Modular operads and moduli spaces of curves

In this section, we give some basic examples of modular operads, coming from the theory of moduli spaces of stable algebraic curves. Throughout this section, the base field is taken to be the field of complex numbers $\mathbb{C}$.

(5.1) Deligne-Mumford moduli spaces. Define a topological modular operad $\overline{\mathcal{M}}$ by letting $\overline{\mathcal{M}}((g, n))$ be the Deligne-Mumford moduli space (or stack) $\overline{\mathcal{M}}_{g,n}$ of stable n-pointed curves of genus g, as constructed by Knudsen [13]. If $G \in \Gamma((g, n))$ is a stable graph, then the structure map

$$\mu_G : \overline{\mathcal{M}}((G)) = \prod_{v \in \mathsf{Vert}(G)} \overline{\mathcal{M}}((g(v), \mathrm{Leg}(v))) \to \overline{\mathcal{M}}((g, n)) \tag{5.2}$$

is defined by gluing the marked points of the curves from $\overline{\mathcal{M}}((g(v), \mathrm{Leg}(v)))$, $v \in \mathrm{Vert}(g)$, according to the graph G (see [8], 1.4.3).

Taking homology, we obtain a modular operad $H_\bullet(\overline{\mathcal{M}}, \mathsf{k})$ in the category of graded vector spaces. An algebra over this operad is the same as a cohomological field theory in the sense of Kontsevich-Manin [15].

(5.3) Stratification of $\overline{\mathcal{M}}((g,n))$. For a stable curve $(C, x_1, \dots, x_n) \in \overline{\mathcal{M}}((g,n))$, define a labelled graph $G(C, x_1, \dots, x_n) \in \Gamma((g,n))$, the dual graph of $(C, x_1, \dots, x_n)$, as follows. Its flags are pairs (K, y) where K is an irreducible component of C and y is either a nodal point or one of the marked points x_i lying on K. Vertices of $G(C, x_1, \dots, x_n)$ correspond to components of C; each intersection point gives rise to an edge. Legs of the graph correspond to the points x_i and the i-th leg is attached to the vertex corresponding to the component containing x_i. If $v \in G(C, x_1, \dots, x_n)$ is the vertex corresponding to the component $K \in C$, label v by the genus $g(v)$ of the desingularization of K.

Given $G \in \Gamma((g,n))$, denote by $\mathcal{M}_G \subset \overline{\mathcal{M}}((g,n))$ the set of the stable curves whose dual graph is G. In this way, we obtain a stratification of $\overline{\mathcal{M}}((g,n))$ whose open stratum $\mathcal{M}_{g,n}$, consists of smooth curves and corresponds to the graph with no edges. Note that the closure $\overline{\mathcal{M}}_G$ of $\mathcal{M}_G$ is isomorphic to $\overline{\mathcal{M}}((G))/\operatorname{Aut}(G)$, and the map μ_G induces the embedding of this quotient as a closed stratum.

(5.4) Logarithmic forms and residues. We now recall some general facts on logarithmic forms [4]. Let X be a smooth algebraic variety, $D \subset X$ a divisor with normal crossings, $D^i \subset D$ the locus of i-fold self intersection (in particular, $D^0 = X, D^1 = D$), $\pi : \tilde{D} \to D$ the normalization morphism and $\epsilon^i = \pi_* \mathbb{C}|_{D^i}$ be the sheaf on D^i spanned by branches of D near D^i. The Poincaré residues are maps

$$\Omega^\bullet_X(\log D)\Big|_{D^i} \to \Omega^\bullet_{D^i}(\log D^{i+1}) \otimes \Sigma^i \det(\epsilon^i). \tag{5.5}$$

Using such maps, we obtain a resolution of $\Omega^\bullet_X$ on X:

$$\begin{aligned}(5.6)\quad 0 \to \Omega^\bullet_X \to\\ \Omega^\bullet_X(\log D) \to \Omega^\bullet_D(\log D^2) \otimes \Sigma^1 \det(\epsilon^1) \to \Omega^\bullet_{D^2}(\log D^3) \otimes \Sigma^2 \det(\epsilon^2) \to \dots,\end{aligned}$$

where sheaves on each D^i are regarded as sheaves on X with support in D^i.

(5.7) Logarithmic forms on $\overline{\mathcal{M}}((g,n))$. The compactification divisor $D((g,n)) = \overline{\mathcal{M}}((g,n)) \setminus \mathcal{M}((g,n))$ is a divisor with normal crossings. Let $\Omega^\bullet_{\overline{\mathcal{M}}((g,n))}(\log D((g,n)))$ be the corresponding logarithmic de Rham complex; this is considered as a chain complex by placing it in negative degree, so that Ω^i is placed in degree $-i$. Consider the collection of complexes of sheaves

$$\Omega^\bullet((g,n)) = \Sigma^{3(g-1)+n} \operatorname{sgn}_n \otimes \Omega^\bullet_{\overline{\mathcal{M}}((g,n))}(\log D((g,n)))$$

on $\overline{\mathcal{M}}((g,n))$.

For a finite set I with $|I| = n$, denote by $\Omega^\bullet((g,I))$ the sheaf on $\overline{\mathcal{M}}((g,I))$ associated to $\Omega^\bullet((g,n))$, as in (2.4).

Let $G \in \Gamma((g,n)$ be a stable graph, and let μ_G be the corresponding composition map (5.2). We define a morphism of complexes of sheaves

$$\lambda_G : \mu_G^* \Omega^\bullet((g,n)) \to \bigotimes_{v \in \operatorname{Vert}(G)} p_v^* \Omega^\bullet((g(v), \operatorname{Leg}(v))),$$

where p_v is the projection of $\overline{\mathcal{M}}((G)) = \prod_v \overline{\mathcal{M}}((g(v), \operatorname{Leg}(v)))$ to the factor labelled by v. Namely, $\overline{\mathcal{M}}((G))$ is the intersection of several branches of the divisor $D((G,n))$. These branches near $\overline{\mathcal{M}}((G))$ are parametrized by the edges of G. Therefore, the Poincaré residue (5.5) gives a map

$$\Omega^\bullet_{\overline{\mathcal{M}}((g,n))}(\log D((g,n))) \to \Sigma^{|\operatorname{Edge}(G)|} \det(G) \otimes \Omega^\bullet_{\overline{\mathcal{M}}((G))}(\log D((G))).$$

After rearranging the grading this gives precisely the desired map λ_G.

Denote by $R\Gamma(\overline{\mathcal{M}}((g,n)), \Omega^\bullet((g,n)))$ the complex obtained by taking global sections of the Dolbeault resolution of $\Omega^\bullet((g,n))$.

(5.8) Proposition. *The maps λ_G make the stable $\mathbb{S}$-module $R\Gamma\Big(\overline{\mathcal{M}}((g,n)), \Omega^\bullet((g,n))\Big)$ into a modular cooperad.*

The proof follows by checking an obvious associativity condition for any morphism $G_1 \to G_2$ in $\Gamma((g,n))$.

(5.9) The gravity operad. Denote by $\tilde{\mathcal{G}}$ the modular operad formed by taking the dual complexes to the above cooperad:

$$\tilde{\mathcal{G}}((g,n)) = R\Gamma\Big(\overline{\mathcal{M}}((g,n)), \Omega^\bullet((g,n))\Big)^*.$$

We call $\tilde{\mathcal{G}}$ the gravity operad.

The homology operad $\mathcal{G}$ of $\tilde{\mathcal{G}}$ has as underlying stable $\mathbb{S}$-module

$$\mathcal{G} = H_\bullet(\tilde{\mathcal{G}}) = H^\bullet(\mathcal{M}, \mathbb{C})^\vee.$$

(5.10) Remarks. (a) The cyclic operad $\operatorname{Cyc}(\mathcal{G})$ coincides with what was called the gravity operad in [8], but differs by a suspension from the definition in [5]. In the latter paper, an explicit presentation of the operad $\operatorname{Cyc}(\mathcal{G})$ is given.

(b) It is shown in [8] that the cyclic operads $\operatorname{Cyc}(\tilde{\mathcal{G}})$ and $\operatorname{Cyc}(\mathcal{G})$ are weakly equivalent (where the latter is understood to have zero differential). It is unknown whether this is true for $\tilde{\mathcal{G}}$ and $\mathcal{G}$.

(c) Note that the topological stable $\mathbb{S}$-module of open strata $\mathcal{M}((g,n))$ is not a topological operad, although its homology is, once suitable regraded.

(5.11) The gravity operad and $\mathsf{F}H_\bullet(\overline{\mathcal{M}})$. Let $\mathcal{E}_\bullet(\overline{\mathcal{M}}((g,n)))$ be the complex of de Rham currents on $\overline{\mathcal{M}}((G,n))$, that is, the topological dual of the complex of smooth differential forms. Obviously, these complexes form a modular operad $\mathcal{E}_\bullet(\overline{\mathcal{M}})$, whose homology is the operad $H_\bullet(\overline{\mathcal{M}})$.

(5.12) Theorem. *There is a weak equivalence of modular operads $\mathcal{E}_\bullet(\overline{\mathcal{M}})) \simeq \mathsf{F}(\tilde{\mathcal{G}})$.*

Proof. Specializing (5.6) to the case $X = \overline{\mathcal{M}}((g,n))$ and $D = D((g,n))$, we find that

$$D^i = \bigcup_{\substack{G \in \Gamma((g,n)) \\ |\operatorname{Edge}(G)| = i}} \overline{\mathcal{M}}((g)),$$

and

$$(5.13) \quad R\Gamma\Big(D^i, \Omega^\bullet_{D^i}(\log D^{i+1}) \otimes \Sigma^i \det(\epsilon^i)\Big) \simeq \bigoplus_{\substack{G \in \Gamma((g,n)) \\ |\operatorname{Edge}(G)| = i}} R\Gamma\Big(\overline{\mathcal{M}}((g)), \Omega^\bullet_{\overline{\mathcal{M}}((g))}(\log D((G))) \otimes \det(G)\Big).$$

By exactness of (5.6), we see that the right hand side of (5.13) is naturally weakly equivalent to $R\Gamma\left(\overline{\mathcal{M}}((g,n)), \Omega^\bullet_{\overline{\mathcal{M}}((g,n))}\right)$. But the linear dual of the right-hand side of (5.13) is $\mathsf{F}(\tilde{\mathcal{G}})((g,n))$, while the linear dual of $R\Gamma\left(\overline{\mathcal{M}}((g)), \Omega^\bullet_{\overline{\mathcal{M}}((g))}(\log D((G))\right)$ is $\mathcal{E}_\bullet(\overline{\mathcal{M}}((g,n))$. This completes the proof. $\square$

(5.14) Riemann surfaces with holes. There is another topological modular operad $\widehat{\mathcal{M}}$ related to moduli spaces, which is, in some sense, dual to $\overline{\mathcal{M}}$, and is defined as follows. The space $\widehat{\mathcal{M}}((g,n))$ is the moduli space of data $(C, f_1, \dots, f_n)$ where C is a Riemann surface (compact 1-dimensional complex manifold) of genus g, and $f_i : \Delta \to C$ are embedding of the unit disk with disjoint images (this is a generalization to higher genus of (1.9.2)). As in (5.1), the structure maps

$$\widehat{\mathcal{M}}((G)) = \prod_{v\in \mathrm{Vert}(G)} \widehat{\mathcal{M}}((g(v), \mathrm{Leg}(v))) \to \widehat{\mathcal{M}}((g,n)).$$

are induced by gluing Riemann surfaces.

6. Modular extension of cyclic operads

In this section, we construct a left adjoint functor to the functor

$$\{\text{modular operads}\} \xrightarrow{\mathrm{Cyc}} \{\text{cyclic operads}\}$$

of (2.3), which we call the modular extension and denote Mod:

$$\{\text{cyclic operads}\} \xrightarrow{\mathrm{Mod}} \{\text{modular operads}\}.$$

(6.1) Definition of Mod. Recall (2.14) that every modular operad $\mathcal{P}$ gives rise to a functor

$$\Gamma((g,n)) \to \{\mathbb{S}_n\text{-modules}\}.$$

If $\mathcal{P}$ is a cyclic operad, then we may define similar functors, but only on the subcategory $\Gamma_0((g,n)) \hookrightarrow \Gamma((g,n))$. This is the category $\Gamma((g,n))$ has a subcategory $\Gamma_0((g,n))$, whose objects are those labelled graphs of $\Gamma((g,n))$ such that $g(v) = 0$ for all vertices v, and whose morphisms are those morphisms of $\Gamma((g,n))$ such that the inverse image of each vertex in the target is a tree. Thus, if T is a subforest of G (a set of edges of G containing no circuits), we obtain a morphism $G \to G/T$ in $\Gamma_0(g,n)$.

If $\mathcal{A}$ is a cyclic operad, we define a stable $\mathbb{S}$-module $\mathrm{Mod}(\mathcal{A})$ by

$$\mathrm{Mod}(\mathcal{A})((g,n)) = \operatorname*{colim}_{G\in\Gamma_0(g,n)} \mathcal{A}((G)).$$

The category $\Gamma_0((g,n))$ has an object $G(g,n)$, consisting of the graph with n legs, g edges and a single vertex. Every other object has at least one morphism to $G(g,n)$. (This amounts to the fact that every connected graph has a spanning tree.) Since $\mathcal{A}((G(g,n))) \cong \mathcal{A}((0,2g+n))_{\mathrm{Aut}(G(g,n))}$, it follows that $\mathrm{Mod}(\mathcal{A})((g,n))$ is a quotient of $\mathcal{A}((0,2g+n)) \cong \mathcal{A}(2g+n-1)$.

(6.2) Proposition. *The stable* $\mathbb{S}$*-module* $\mathrm{Mod}(\mathcal{A})$ *has a natural modular operad structure.*

Proof. If we replace the colimit in the definition of $\mathrm{Mod}(\mathcal{A})((g,n))$ by the direct sum over objects of $\Gamma_0((g,n))$, we obtain $\mathbb{M}_+\mathcal{A}$, the free modular operad generated by $\mathcal{A}$. It is not hard to see that the identifications involved in taking the colimit preserve the operad structure. □

(6.3) The modular extension of a free cyclic operad. The following diagram commutes:

$$\begin{tikzcd} \text{cyclic operads} \arrow[r, hook] \arrow[d, "\mathsf{B}"'] & \text{modular operads} \arrow[d, "\mathsf{F}"] \\ \text{cyclic operads} \arrow[r, "\mathrm{Mod}"] & \text{modular operads} \end{tikzcd} \tag{6.4}$$

In particular, considering a cyclic operad with zero compositions, we find that the modular extension of a free cyclic operad is a free modular operad (with the same generators).

(6.5) Modular extension and Vassiliev invariants. Vassiliev has introduced a filtered space $V = \bigcup_{m=0}^{\infty} V_m$ of knot invariants of finite order (see [2]). The associated graded space $W = \operatorname{gr} V$ is a commutative cocommutative Hopf algebra. Let $P = \bigoplus P_m$ be its space of primitives. One of the chief results of Kontsevich and Bar-Natan identifies the quotient V_k/V_{k-1} with the zeroth homology group of a certain graph complex. In our language,

$$P_m \cong \bigoplus_{g=0}^{m} H_0(\mathsf{FCom})((g, m-g+1))_{\mathbb{S}_{m-g+1}}.$$

This may be neatly reformulated using the concept of modular extension. By (6.4), we have

$$\mathsf{FCom} = \mathrm{Mod}(\mathsf{BCom}),$$

where Com is the commutative operad.

If $\mathcal{A}$ is a cyclic operad concentrated in positive degree, then $H_0(\mathcal{A})$ is a cyclic operad concentrated in degree 0. It is easy to see that $\mathrm{Mod}(\mathcal{A})$ is also concentrated in positive degree, and that the modular operad $H_0(\mathrm{Mod}(\mathcal{A}))$ is naturally isomorphic to $\mathrm{Mod}(H_0(\mathcal{A}))$.

We may apply this observation to the cobar operad BCom of the commutative operad, which is concentrated in positive degree and is a resolution of the Lie operad (see [8]):

$$H_0(\mathsf{BCom}) \cong \mathsf{Lie}.$$

It follows that

$$H_0(\mathsf{FCom}) \cong H_0(\mathrm{Mod}(\mathsf{Lie})) \cong \mathrm{Mod}(\mathsf{Lie}).$$

We conclude that

$$P_m \cong \bigoplus_{g=0}^{m} \mathrm{Mod}(\mathsf{Lie})((g, m-g+1))_{\mathbb{S}_{m-g+1}},$$

which may be thought of as an expression, in the language of modular operads, of the relationship between Vassiliev invariants and Lie algebras.

(6.6) The derived functor of modular extension. The functor Mod, applied to cyclic operads in the category of chain complexes, is not homotopy invariant; it does not, in general, take weak equivalences to weak equivalences. This motivates the introduction of $\mathsf{L}\,\mathrm{Mod}$, the left-derived functor of Mod. To define $\mathsf{L}\,\mathrm{Mod}$, we consider the canonical free resolution $\mathsf{BB}\mathcal{A} \to \mathcal{A}$ and apply Mod:

$$\mathsf{L}\,\mathrm{Mod}(\mathcal{A}) = \mathrm{Mod}(\mathsf{BB}\mathcal{A}) = \mathsf{FB}\mathcal{A}.$$

It is clear that $\mathsf{L}\,\mathrm{Mod}$ is homotopy invariant (since B and F are). Also, for a free cyclic operad

$$\mathcal{A} = \mathbb{T}_+\mathcal{V} = \mathsf{B}\mathcal{V},$$

(where $\mathcal{V}$ is a cyclic operad with vanishing compositions), $\mathsf{L}\,\mathrm{Mod}(\mathcal{A}) = \mathsf{FBB}\mathcal{V}$ is weakly equivalent to $\mathsf{F}\mathcal{V} = \mathrm{Mod}(\mathcal{A})$.

7. Characteristics of cyclic operads

If $\mathcal{V}$ is a stable $\mathbb{S}$-module, we can associate to it a symmetric function $\mathrm{Ch}(\mathcal{V})$, called its characteristic. In this section and the next, we give formulas for $\mathrm{Ch}(\mathsf{B}\mathcal{A})$ in terms of $\mathrm{Ch}(\mathcal{A})$, where $\mathcal{A}$ is a cyclic operad, and for $\mathrm{Ch}(\mathsf{F}\mathcal{A})$ in terms of $\mathrm{Ch}(\mathcal{A})$, where $\mathcal{A}$ is a modular operad. The first of these formulas involves a generalization of the Legendre transform, and the second a generalization of the Fourier transform, from power series in one variable to the ring of symmetric functions. Here, symmetric functions arise because of well-known correspondence between the characters of the symmetric group and the ring of symmetric functions. For further details on the theory of symmetric functions, see Chapter 1 of Macdonald [17].

(7.1) Symmetric functions.
Consider the ring

$$\Lambda = \varprojlim \mathbb{Z}[\![x_1, \dots, x_k]\!]^{\mathbb{S}_k}$$

of symmetric functions (power series) in infinitely many variables. The following standard symmetric functions

$$h_n(x_i) = \sum_{i_1 \le \dots \le i_n} x_{i_1} \dots x_{i_n}, \quad e_n(x_i) = \sum_{i_1 < \dots < i_n} x_{i_1} \dots x_{i_n},$$
$$p_n(x_i) = \sum_{i=1}^{\infty} x_i^n,$$

are called respectively the complete symmetric functions, the elementary symmetric functions and the power sums. It is a basic fact that

$$\Lambda = \mathbb{Z}[\![h_1, h_2, \dots]\!] = \mathbb{Z}[\![e_1, e_2, \dots]\!],$$
$$\Lambda \otimes \mathbb{Q} = \mathbb{Q}[\![p_1, p_2, \dots]\!],$$

that is, that each of these three series of symmetric functions freely generates Λ (in the case of the power sums, over $\mathbb{Q}$). In particular, $h_1 = e_1 = p_1$, while $h_2 = \frac{1}{2}(p_1^2 + p_2)$ and $e_2 = \frac{1}{2}(p_1^2 - p_2)$.

Let σ be an element of the symmetric group $\mathbb{S}_n$, with cycles of length $a_1 \ge a_2 \ge \dots \ge a_\ell$; thus $n = a_1 + \dots + a_\ell$. The cycle index of σ is the symmetric function

$$\psi(\sigma) = p_{a_1} \dots p_{a_\ell} \in \Lambda.$$

The characteristic of a finite-dimensional $\mathbb{S}_n$-module V is the symmetric function

$$\operatorname{ch}_n(V) = \frac{1}{n!} \sum_{\sigma \in \mathbb{S}_n} \operatorname{Tr}_V(\sigma)\psi(\sigma).$$

It may be proved that $\operatorname{ch}_n(V)$ is in Λ, although it is only evident from its definition that it is in $\Lambda \otimes \mathbb{Q}$.

We extend the definition of ch_n to graded $\mathbb{S}_n$-modules by

$$\operatorname{ch}_n(V) = \sum_i (-1)^i \operatorname{ch}_n(V_i),$$

where V_i is the degree i component of V. Finally, the characteristic of a graded $\mathbb{S}$-module $\mathcal{V} = \{\mathcal{V}(n) \mid n \geq 0\}$ such that $\mathcal{V}(n)$ is finite-dimensional for all n is

$$\operatorname{ch}(\mathcal{V}) = \sum_{n=0}^{\infty} \operatorname{ch}_n(\mathcal{V}(n)).$$

We denote by $\operatorname{rk} : \Lambda \to \mathbb{Q}[\![x]\!]$ the ring homomorphism which sends

$$h_n \mapsto \frac{x^n}{n!},$$

or equivalently, $p_1 \mapsto x$ and $p_n \mapsto 0$, $n > 1$. If V is an $\mathbb{S}_n$-module,

$$\operatorname{rk}(\operatorname{ch}_n(V)) = \frac{\dim(V)x^n}{n!}.$$

For this reason, we call rk the rank homomorphism.

(7.2) Plethysm. Plethysm is the associative operation on Λ, denoted $f \circ g$, characterized by the formulas

(1) $(f_1 + f_2) \circ g = f_1 \circ g + f_2 \circ g$;
(2) $(f_1 f_2) \circ g = (f_1 \circ g)(f_2 \circ g)$;
(3) if $f = f(p_1, p_2, \ldots)$, then $p_n \circ f = f(p_n, p_{2n}, \ldots)$.

Note that under the rank homomorphism, plethysm is carried into composition of power series.

There is a monoidal structure on the category of $\mathbb{S}$-modules, with tensor product

$$(\mathcal{V} \circ \mathcal{W})(n) = \bigoplus_{k=0}^{\infty} \Big(\mathcal{V}(k) \otimes \bigoplus_{f:\{1,\ldots,n\} \to \{1,\ldots,k\}} \bigotimes_{i=1}^{k} \mathcal{W}(f^{-1}(i)) \Big)_{\mathbb{S}_k}.$$

(An operad $\mathcal{V}$ is just an $\mathbb{S}$-module with an associative composition $\mathcal{V} \circ \mathcal{V} \to \mathcal{V}$.)

(7.3) Proposition. $\operatorname{ch}(\mathcal{V} \circ \mathcal{W}) = \operatorname{ch}(\mathcal{V}) \circ \operatorname{ch}(\mathcal{W})$

When $\mathcal{V}$ and $\mathcal{W}$ are ungraded, this is proved in Macdonald [17]. In the general case, the proof depends on an analysis of the interplay between the minus signs in the Euler characteristic and the action of symmetric groups on tensor powers of graded vector spaces.

(7.4) Characteristic of $\mathbb{S}$-modules. If $\mathcal{V} = \{\mathcal{V}((n)) \mid n \geq 1\}$ is a cyclic $\mathbb{S}$-module, its characteristic is

$$\operatorname{Ch}(\mathcal{V}) = \sum_{n=1}^{\infty} \operatorname{ch}_n(\mathcal{V}((n))).$$

There is a forgetful functor from cyclic $\mathbb{S}$-modules to $\mathbb{S}$-modules, obtained by restricting the action of $\mathcal{V}(n) = \mathcal{V}((n+1))$ from $\mathbb{S}_{n+1}$ to the subgroup $\mathbb{S}_n$. The characteristics of $\mathcal{V}$ considered as a cyclic $\mathbb{S}$-module and an $\mathbb{S}$-module are related by

$$\mathrm{ch}(\mathcal{V}) = \frac{\partial\,\mathrm{Ch}(\mathcal{V})}{\partial p_1}. \tag{7.5}$$

(7.6) Examples of characteristics. To illustrate the above definitions, let us give some examples of characteristics of cyclic operads.

(7.6.1) ***The commutative operad.*** For the commutative operad Com, $\mathsf{Com}((n))$ is the trivial representation of $\mathbb{S}_n$ for all $n \geq 3$. (Note that we work with the non-unital form of the commutative operad, in which $\mathsf{Com}((2)) = 0$.) It follows that $\mathrm{ch}_n(\mathsf{Com}((n))) = h_n$ for $n \geq 3$, and hence that

$$\mathrm{Ch}(\mathsf{Com}) = \exp\Big(\sum_{n=1}^{\infty} \frac{p_n}{n}\Big) - \big(1 + h_1 + h_2\big).$$

(7.6.2) ***The associative operad.*** The associative operad is a cyclic operad with

$$\mathsf{Ass}((n)) \cong \mathrm{Ind}_{C_n}^{\mathbb{S}_n} \mathbb{1};$$

here, $C_n \subset \mathbb{S}_n$ is a cyclic subgroup of order n and $\mathbb{1}$ is the trivial character of C_n. It follows that

$$\mathrm{ch}_n(\mathsf{Ass}((n))) = \sum_{d|n} \frac{\varphi(d)}{n} p_d^{n/d},$$

where $\varphi(d)$ is the Euler totient function. Summing over $n \geq 3$, we see that

$$\mathrm{Ch}(\mathsf{Ass}) = -\sum_{n=1}^{\infty} \frac{\varphi(n)}{n} \log(1 - p_n) - (h_1 + h_2).$$

(7.6.3) ***The Lie operad.*** In (7.24), we will prove that the characteristic of the Lie operad is

$$\mathrm{Ch}(\mathsf{Lie}) = (1 - p_1) \sum_{n=1}^{\infty} \frac{\mu(n)}{n} \log(1 - p_n) + p_1,$$

where $\mu(n)$ is the Möbius function.

(7.7) The Legendre transform. Classically, the Legendre transform of a convex function $f : \mathbb{R} \to \mathbb{R}$ is the function

$$(\mathfrak{L}f)(\xi) = g(\xi) = \max_x \big(x\xi - f(x)\big).$$

(See Section 3.3 of Arnold [1].) Setting $\xi = f'(x)$, we see that

$$g \circ f' + f = xf'. \tag{7.8}$$

Suppose that, instead of being a convex function, $f(x)$ is a formal power series of the form

$$f(x) = \sum_{n=2}^{\infty} \frac{a_n x^n}{n!} \in \mathbb{Q}[\![x]\!], \tag{7.9}$$

where $a_2 \neq 0$; we denote this set of power series by $\mathbb{Q}[\![x]\!]_*$. The equation (7.16) defines a unique power series $(\mathfrak{L}f)(\xi) = g(\xi) \in \mathbb{Q}[\![\xi]\!]_*$, which we again call the Legendre transform.

(7.10) Proposition. *If f and g are series of the form (7.9), then $\mathfrak{L}f = g$ if and only if f' and g' are inverse under composition, that is,*

$$g' \circ f' = x.$$

Proof. Taking the derivative of (7.8), we see that

$$(g' \circ f')f'' + f' = xf'' + f'.$$

Cancelling f' from each side and dividing by f'', which is invertible in $\mathbb{Q}[\![x]\!]$ by hypothesis, we find that $g' \circ f' = x$. The same reasoning proves the converse. □

As a consequence of this proposition, we see that $\mathfrak{L}$ is involutive: $\mathfrak{L}(\mathfrak{L}f) = f$.

(7.11) $\mathfrak{L}$ and trees. Let $\mathcal{V}$ be a cyclic $\mathbb{S}$-module with $\mathcal{V}((n)) = 0$ for $n \leq 2$. The cyclic $\mathbb{S}$-module $\mathbb{T}_+\mathcal{V}$ was defined in (2.15).

(7.12) Proposition. *Let $a_n = \chi(\mathcal{V}((n)))$ and $b_n = \chi(\mathbb{T}_+\mathcal{V}((n)))$ be the Euler characteristics of components $\mathcal{V}$ and $\mathbb{T}_+\mathcal{V}$. If*

$$f(x) = \frac{x^2}{2} - \sum_{n=3}^{\infty} \frac{a_n x^n}{n!} \quad \textit{and} \quad g(x) = \frac{x^2}{2} + \sum_{n=3}^{\infty} \frac{b_n x^n}{n!},$$

then $g = \mathfrak{L}f$.

Proof. It is a corollary of Theorem 3.3.2 of [8] that $g' \circ f' = x$. The results follows by (7.10). □

With the notation of the proposition,

$$b_n = \sum_{n\text{-trees } T} \prod_{v \in \mathrm{Vert}(T)} a_{|\mathrm{Leg}(v)|}. \tag{7.13}$$

In fact, the proposition remains true for an arbitrary sequence of rational numbers $\{a_3, a_4, \dots\}$, if we define $\{b_3, b_4, \dots\}$ by (7.13).

(7.14) The Legendre transform for symmetric functions. Denote by Λ_* the set of symmetric functions such that $\mathrm{rk}(f) \in \mathbb{Q}[\![x]\!]_*$.

(7.15) Theorem. *(a) If $f \in \Lambda_*$, there is a unique element $g = \mathfrak{L}f \in \Lambda_*$ such that*

$$g \circ \frac{\partial f}{\partial p_1} + f = p_1 \frac{\partial f}{\partial p_1}. \tag{7.16}$$

We call $\mathfrak{L} : \Lambda_* \to \Lambda_*$ the Legendre transform.
(b) The Legendre transform of symmetric functions is compatible with that of power series, in the sense that the following diagram commutes:

$$\begin{array}{ccc} \Lambda_* & \xrightarrow{\mathfrak{L}} & \Lambda_* \\ {\scriptstyle \mathrm{rk}}\downarrow & & \downarrow{\scriptstyle \mathrm{rk}} \\ \mathbb{Q}[\![x]\!]_* & \xrightarrow{\mathfrak{L}} & \mathbb{Q}[\![x]\!]_* \end{array}$$

(c) The symmetric functions

$$\frac{\partial(\mathfrak{L}f)}{\partial p_1} \text{ and } \frac{\partial f}{\partial p_1}$$

are plethystic inverses. (Note that, unlike for power series, this equation does not determine $\mathfrak{L}f$.)

(d) The transformation $\mathfrak{L}$ is an involution, that is, $\mathfrak{L}\mathfrak{L} = \mathrm{Id}$.

Proof. If $f \in \Lambda_*$, then $\partial f/\partial p_1$ is invertible with respect to plethysm. Thus (7.16) defines $g \in \Lambda_*$ uniquely, proving (a). Part (b) is obvious, since rk transforms plethysm into composition.

In proving (c), we need an analogue of the chain rule for $\partial/\partial p_1$ acting on Λ:

$$\frac{\partial}{\partial p_1}(u \circ v) = \Big(\frac{\partial u}{\partial p_1} \circ v\Big)\frac{\partial v}{\partial p_1}.$$

This formula is proved by checking that both sides are compatible with the rules (1-3) defining plethysm (7.2).

Using this, the reasoning needed to prove (c) is formally identical to that in the proof of (7.10).

To prove (d), we note that (c) implies

$$p_1\frac{\partial f}{\partial p_1} = \Big(p_1\frac{\partial g}{\partial p_1}\Big) \circ \frac{\partial f}{\partial p_1}.$$

This shows that

$$g \circ \frac{\partial f}{\partial p_1} \circ \frac{\partial g}{\partial p_1} + f \circ \frac{\partial g}{\partial p_1} = \Big(p_1\frac{\partial f}{\partial p_1}\Big) \circ \frac{\partial g}{\partial p_1} = \Big(p_1\frac{\partial g}{\partial p_1}\Big) \circ \frac{\partial f}{\partial p_1} \circ \frac{\partial g}{\partial p_1}.$$

Cancellation proves that

$$g + f \circ \frac{\partial g}{\partial p_1} = p_1\frac{\partial g}{\partial p_1},$$

and hence that $f = \mathfrak{L}g$. $\square$

For example, $\mathfrak{L}h_2 = e_2$ and vice versa.

The following theorem is related to results of Otter [19] and Hanlon-Robinson [9] on the enumeration of unrooted trees.

(7.17) Theorem. *Let $\mathcal{V}$ be a cyclic $\mathbb{S}$-module such that $\mathcal{V}((n)) = 0$ for $n \le 2$ and $\mathcal{V}((n))$ is finite dimensional for all n. Define the elements of Λ_**

$$f = e_2 - \mathrm{Ch}(\mathcal{V}) \text{ and } g = h_2 + \mathrm{Ch}(\mathbb{T}_+\mathcal{V}).$$

Then $g = \mathfrak{L}f$.

Proof. Recall (7.5) that $\mathrm{ch}(\mathcal{V}) = \partial\,\mathrm{Ch}(\mathcal{V})/\partial p_1$. By definition of $\mathfrak{L}$, we must prove that

$$\Big(h_2 + \mathrm{Ch}(\mathbb{T}_+\mathcal{V})\Big) \circ \Big(p_1 - \mathrm{ch}(\mathcal{V})\Big) + e_2 - \mathrm{Ch}(\mathcal{V}) = p_1\Big(p_1 - \mathrm{ch}(\mathcal{V})\Big).$$

Since $h_2 = e_2 + p_1^2$, this may be rewritten as

$$\Big(h_2 + \mathrm{Ch}(\mathbb{T}_+\mathcal{V})\Big) \circ \Big(p_1 - \mathrm{ch}(\mathcal{V})\Big) = h_2 + \mathrm{Ch}(\mathcal{V}) - p_1\Big(p_1 - \mathrm{ch}(\mathcal{V})\Big).$$

By the formula

$$h_2 \circ \Big(p_1 - \mathrm{ch}(\mathcal{V})\Big) = h_2 - p_1\,\mathrm{ch}(\mathcal{V}) + h_2 \circ \Big(-\,\mathrm{ch}(\mathcal{V})\Big),$$

we see that this is equivalent to

$$\mathrm{Ch}(\mathbb{T}_+\mathcal{V}) \circ \big(p_1 - \mathrm{ch}(\mathcal{V})\big) = \mathrm{Ch}(\mathcal{V}) - h_2 \circ \big(-\mathrm{ch}(\mathcal{V})\big). \tag{7.18}$$

We prove this formula by constructing a differential graded $\mathbb{S}$-module $C = \{C((n))\}$ such that $\mathrm{ch}(C)$ equals the left-hand side of (7.18), and $\mathrm{ch}(H_\bullet(C))$ equals the right-hand side. Define the $\mathbb{S}$-module underlying C to be the plethysm $\mathcal{X} \circ \mathcal{W}$, where the $\mathbb{S}$-modules $\mathcal{X}$ and $\mathcal{W}$ are defined by

$$\mathcal{X}((n)) = \begin{cases} 0, & n \leq 2, \\ (\mathbb{T}_+\mathcal{V})((n)), & n \geq 3, \end{cases}$$

$$\mathcal{W}((n)) = \begin{cases} 0, & n = 1, \\ \mathsf{k}, & n = 2, \\ \Sigma \operatorname{Res}^{\mathbb{S}_{n+1}}_{\mathbb{S}_n} \mathcal{V}((n+1)), & n \geq 3. \end{cases}$$

Here, Σ is the suspension functor on graded $\mathbb{S}_n$-modules. It follows from (7.3) that $\mathrm{ch}(C)$ equals the right-hand side of (7.18).

We now construct a differential δ on $(\mathcal{X} \circ \mathcal{W})(n)$. A vertex v of a tree T is a boundary vertex if exactly one of its flags forms part of an edge. Define a colouring of a tree to be an assignment of colours black and white to its boundary vertices; we denote a coloured tree by (T, B), where B is the set of black boundary vertices. Then

$$(\mathcal{X} \circ \mathcal{W})(n) = \bigoplus_{\substack{\text{coloured } n\text{-trees} \\ (T,B)}} \Big(\bigotimes_{v \in \mathrm{Vert}(T) \setminus B} \mathcal{V}((\mathrm{Leg}(v))) \otimes \bigotimes_{v \in B} \Sigma\mathcal{V}((\mathrm{Leg}(v))) \Big). \tag{7.19}$$

On the summand of (7.19) corresponding to the coloured tree (T, B), we define

$$\delta = \sum_{v \in B} \delta_v,$$

where δ_v is the natural identification, of degree -1, between this summand and the summand corresponding to $(T, B \setminus \{v\})$.

Clearly $(\mathcal{X} \circ \mathcal{W})(n)$ splits into a direct sum of subcomplexes C_T corresponding to all the colourings of the tree T. If T has at least one non-boundary vertex, the complex C_T is contractible, since it is the tensor product of the graded vector space $\mathcal{V}((T))$ and the augmented chain complex of the simplex whose vertices are the boundary vertices of T. There are two remaining cases:

(1) the contribution of C_T from all trees with one vertex is $\mathrm{Ch}(\mathcal{V})$;
(2) the contribution of C_T from all trees with two vertices is $h_2 \circ \big(-\mathrm{ch}(\mathcal{V})\big)$.

Implicit here is the observation (Jordan [11]) that the centre of a tree, the remnant obtained by repeatedly stripping away boundary vertices, has either one vertex or one edge. □

(7.20) The characteristic of the cobar operad of a cyclic operad. Using this theorem, we will now write a formula for $\mathrm{Ch}(\mathsf{B}\mathcal{A})$, where $\mathcal{A}$ is a cyclic operad. Up to differential, $\mathsf{B}\mathcal{A}$ is the cyclic operad $\mathbb{T}_+\mathcal{A}^\vee$, and thus $\mathrm{Ch}(\mathsf{B}\mathcal{A}) = \mathrm{Ch}(\mathbb{T}_+\mathcal{A}^\vee)$. Thus, it suffices to give a formula for $\mathrm{Ch}(\mathcal{A}^\vee)$.

Denote by $\omega : \Lambda \to \Lambda$ the ring homomorphism such that $\omega(h_n) = e_n$, $n \geq 1$. If V is a finite-dimensional $\mathbb{S}_n$-module,

$$\omega(\mathrm{ch}_n(V)) = \mathrm{ch}_n(\mathrm{sgn}_n \otimes V),$$

and thus ω is an involution. Note also that $\omega(p_n) = (-1)^{n-1} p_n$.

We also need a modified involution $\tilde{\omega}$, defined by $\tilde{\omega}(h_n) = (-1)^n e_n$, or equivalently $\tilde{\omega}(p_n) = -p_n$. Thus, if $\mathcal{V}$ is a cyclic $\mathbb{S}$-module such that $\mathcal{V}(\!(n)\!)$ is finite-dimensional for each n,

$$\mathrm{ch}(\mathcal{V}^\vee) = -\tilde{\omega}(\mathrm{Ch}(\mathcal{V})). \tag{7.21}$$

(7.22) Corollary. *Let $\mathcal{A}$ be a cyclic operad such that $\mathcal{A}(\!(n)\!) = 0$ for $n \leq 2$ and $\mathcal{A}(\!(n)\!)$ is finite-dimensional for each n, and let $\mathsf{B}\mathcal{A}$ be its cobar operad. Then*

$$h_2 + \mathrm{Ch}(\mathsf{B}\mathcal{A}) = \mathfrak{L}\tilde{\omega}(h_2 + \mathrm{Ch}(\mathcal{A})).$$

Recall [8] that $\mathsf{BB}\mathcal{A}$ is weakly equivalent to B, which suggests that the transform $\mathfrak{L}\tilde{\omega} : \Lambda_* \to \Lambda_*$ should be an involution. This follows from the following result.

(7.23) Proposition. *If $f \in \Lambda_*$, $-\mathfrak{L}\tilde{\omega}f = \mathfrak{L}(-f)$.*

Proof. By Ex. 8.1 of Macdonald [17], if u and v are symmetric functions, $u \circ (-v) = (\tilde{\omega}u) \circ v$. If $g = \mathfrak{L}(\tilde{\omega}f)$, we see that the defining equation

$$(\tilde{\omega}f) \circ \frac{\partial g}{\partial p_1} + g = x\frac{\partial g}{\partial p_1}$$

is equivalent to

$$-f \circ \frac{\partial(-g)}{\partial p_1} + (-g) = x\frac{\partial(-g)}{\partial p_1},$$

that is, $-g = \mathfrak{L}(-f)$. □

(7.24) Example: the Lie operad. The Lie operad Lie is weakly equivalent to the cobar operad BCom of the commutative operad, and thus $h_2 + \mathrm{Ch}(\mathsf{Lie})$ is the Legendre transform of

$$\begin{aligned}\tilde{\omega}(h_2 + \mathrm{Ch}(\mathsf{Lie})) &= \tilde{\omega}\Big(\exp\Big(\sum_{n=1}^{\infty} \frac{p_n}{n}\Big) - \big(1 + h_1\big)\Big) \\ &= \exp\Big(\sum_{n=1}^{\infty} -\frac{p_n}{n}\Big) - \big(1 - p_1\big).\end{aligned}$$

The Legendre transform of this symmetric function is

$$(1 - p_1)\sum_{n=1}^{\infty} \frac{\mu(n)}{n} \log(1 - p_n) + p_1.$$

It follows that

$$\mathrm{Ch}(\mathsf{Lie}) = (1 - p_1)\sum_{n=1}^{\infty} \frac{\mu(n)}{n} \log(1 - p_n) + h_1 - h_2.$$

(8.1) The ring $\Lambda((\hbar))$. Consider the ring $\lambda((\hbar))$ of Laurent series with coefficients in Λ. This ring has a descending filtration

$$F^m\Lambda((\hbar)) = \Big\{\sum f_i\hbar^i \,\Big|\, f_i \in F^{m-2i}\Lambda\Big\},$$

inducing a topology on $\Lambda((\hbar))$. If $f \in \Lambda$, the plethysm $f \circ (-) : \Lambda \to \Lambda$ extends to $\Lambda((\hbar))$ by retaining axioms (1) and (2) of (7.2) and replacing (3) by

(3′) $p_n \circ f(\hbar, p_1, p_2, \dots) = f(\hbar^n, p_n, p_{2n}, \dots)$.

(8.2) The characteristic of a stable $\mathbb{S}$-module. The characteristic of a stable $\mathbb{S}$-module $\mathcal{V}$ is the element of $\Lambda((\hbar))$ given by the formula

$$\mathrm{Ch}(\mathcal{V}) = \sum_{2(g-1)+n>0} \hbar^{g-1}\,\mathrm{ch}_n(\mathcal{V}((g,n))).$$

The stability condition ensures that $\mathrm{Ch}(\mathcal{V}) \in F^1\Lambda((\hbar))$. Our goal is to present a formula for $\mathrm{Ch}(\mathsf{F}\mathcal{A})$ in terms of $\mathrm{Ch}(\mathcal{A})$.

Note that this definition is consistent with the earlier definition of the characteristic of a cyclic $\mathbb{S}$-module (7.4), provided we set $\hbar = 1$.

For $f \in F^1\Lambda((\hbar))$, let

$$\mathrm{Exp}(f) = \Big(\sum_{n=0}^{\infty} h_n\Big) \circ f = \exp\Big(\sum_{n=1}^{\infty} \frac{p_n}{n}\Big) \circ f.$$

Note that

$$\mathrm{Exp}(f+g) = \mathrm{Exp}(f)\,\mathrm{Exp}(g),$$

and that under specialization $\mathrm{rk} : \Lambda((\hbar)) \to \mathbb{Q}[\![x]\!]((\hbar))$, the map Exp goes into exponentiation

$$f(\hbar, x) \mapsto e^{f(\hbar,x)}.$$

(8.3) Proposition. *If $\mathcal{V}$ is a stable $\mathbb{S}$-module, let $\mathrm{Exp}_n(\mathcal{V})$ be the stable $\mathbb{S}$-module such that*

$$\mathrm{Exp}_n(\mathcal{V})((g,n)) = \Bigg(\bigoplus_{\substack{f:I\to\{1,\dots,n\}\\ g_1+\dots+g_n=g}} \mathrm{Ind}_{\mathrm{Aut}(f)}^{\mathrm{Aut}(I)}\Big(\bigotimes_{i=1}^{n} \mathcal{V}((g_i, f^{-1}(i)))\Big)\Bigg)_{\mathbb{S}_n},$$

where $\mathrm{Aut}(f) = \mathrm{Aut}(f^{-1}(1)) \times \dots \times \mathrm{Aut}(f^{-1}(n))$. *Then*

$$\mathrm{Exp}(\mathrm{Ch}(\mathcal{V})) = \sum_{n=0}^{\infty} \hbar^{-n}\,\mathrm{Ch}(\mathrm{Exp}_n(\mathcal{V})).$$

Proof. This follows from (7.3) and the definition of $\mathrm{Exp}(f)$, $f \in \Lambda((\hbar))$. □

Informally, the stable $\mathbb{S}$-module $\mathrm{Exp}_n(\mathcal{V})$ may be thought of as representing disconnected graphs with n vertices and no edges: all of its flags are legs.

(8.4) Proposition. *The map* $\mathrm{Exp} : F^1\Lambda((\hbar)) \to 1 + F^1\Lambda((\hbar))$ *is invertible over* $\mathbb{Q}$, *with inverse*

$$\mathrm{Log}(f) = \sum_{n=1}^{\infty} \frac{\mu(n)}{n}\log(p_n) \circ f = \sum_{n=1}^{\infty} \frac{\mu(n)}{n}\log(p_n \circ f).$$

Proof.

$$\begin{aligned}
\operatorname{Log}(\operatorname{Exp}(f)) &= \sum_{n=1}^{\infty} \frac{\mu(n)}{n} \log(p_n) \circ \exp\Big(\sum_{m=1}^{\infty} \frac{p_m}{m}\Big) \circ f \\
&= \sum_{n=1}^{\infty} \frac{\mu(n)}{n} \log\exp\Big(\sum_{m=1}^{\infty} \frac{p_{nm}}{m}\Big) \circ f \\
&= \sum_{n=1}^{\infty} \sum_{d|n} \frac{\mu(d) p_n \circ f}{n} = f. \quad \square
\end{aligned}$$

(8.5) The inner product on Λ. To a partition $\lambda = (1^{m_1} 2^{m_2} \dots)$, where $m_k = 0$ for $k \gg 0$, is associated a monomial

$$p_\lambda = p_1^{m_1} p_2^{m_2} \dots .$$

These monomials form a topological basis of Λ. Let Λ_{alg} be the space of finite linear combinations of the p_λ. The standard inner product on Λ_{alg} is determined by the formula

$$\langle p_\lambda, p_\mu \rangle = \prod_{i=1}^{\infty} i^{m_i} m_i!$$

Note in particular that $\langle p_i, p_j \rangle = i\delta_{ij}$; the inner product on Λ_{alg} is the standard extension of the inner product on a vector space to its symmetric algebra (Fock space).

(8.6) Proposition. *If V and W are $\mathbb{S}_n$-modules,*

$$\langle \mathrm{ch}_n(V), \mathrm{ch}_n(W) \rangle = \dim \operatorname{Hom}_{\mathbb{S}_n}(V, W).$$

Proof. This statement is well-known in the theory of symmetric functions: it follows from the fact that the Schur functions form an orthonormal basis of Λ_{alg}. $\square$

We extend the inner product on Λ_{alg} to a $\mathbb{Q}((\hbar))$-valued inner product on $\Lambda_{\mathrm{alg}}((\hbar))$ by $\mathbb{Q}((\hbar))$-bilinearity. If $f \in \Lambda_{\mathrm{alg}}((\hbar))$, let $D(f) : \Lambda((\hbar)) \to \Lambda((\hbar))$ be the adjoint of multiplication by f with respect to this inner product. The following proposition is Ex. 5.3 of Macdonald [17].

(8.7) Proposition. *If $f = f(\hbar, p_1, p_2, \dots) \in \Lambda_{\mathrm{alg}}((\hbar))$, then*

$$D(f) = f\Big(\hbar, \tfrac{\partial}{\partial p_1}, 2\tfrac{\partial}{\partial p_2}, 3\tfrac{\partial}{\partial p_3}, \dots\Big).$$

(8.8) Proposition. *Let $k \leq n$, V be an $\mathbb{S}_k$-module, and W be an $\mathbb{S}_n$-module. Then*

$$D(\mathrm{ch}_k(V)) \, \mathrm{ch}_n(W) = \mathrm{ch}_{n-k} \operatorname{Hom}_{\mathbb{S}_k}\Big(V, \operatorname{Res}^{\mathbb{S}_n}_{\mathbb{S}_k \times \mathbb{S}_{n-k}} W\Big).$$

Proof. This follows by taking adjoints on both sides of the formula

$$\mathrm{ch}_j(U) \, \mathrm{ch}_k(V) = \mathrm{ch}_{j+k} \operatorname{Ind}^{\mathbb{S}_{j+k}}_{\mathbb{S}_j \times \mathbb{S}_k}(U \otimes V). \quad \square$$

(8.9) A Laplacian on $\Lambda((\hbar))$. We now introduce an analogue of the Laplacian on $\Lambda((\hbar))$, given by the formula

$$\Delta = \sum_{n=1}^{\infty} \hbar^n \left(\frac{n}{2} \frac{\partial^2}{\partial p_n^2} + \frac{\partial}{\partial p_{2n}} \right).$$

Note that Δ is homogeneous of degree zero, and thus preserves the filtration of $\Lambda((\hbar))$. Under specialization $\mathrm{rk} : \Lambda((\hbar)) \to \mathbb{Q}[\![x]\!]((\hbar))$, the operator Δ corresponds to the Laplacian $\frac{\hbar}{2}\frac{d}{dx^2}$ on the line.

(8.10) Proposition. $D(\mathrm{Exp}(\hbar h_2)) = \exp(\Delta)$

Proof. By (8.7), it suffices to substitute $n\partial/\partial p_n$ for p_n on the right-hand side of

$$\mathrm{Exp}(\hbar h_2) = \exp\Big(\sum_{n=1}^{\infty}\frac{p_n}{n}\Big) \circ \Big(\frac{\hbar}{2}(p_1^2+p_2)\Big) = \exp\Big(\sum_{n=1}^{\infty}\frac{\hbar^n}{2n}(p_n^2+p_{2n})\Big). \quad \square$$

(8.11) Theorem. *If $\mathcal{V}$ is a stable $\mathbb{S}$-module, then*

$$\mathrm{Ch}(\mathbb{M}_+\mathcal{V}) = \mathrm{Log}\Big(\exp(\Delta)\,\mathrm{Exp}(\mathrm{Ch}(\mathcal{V}))\Big).$$

Proof. Let us first neglect $\hbar$ and explain the appearance of the sum over graphs. on the right-hand side of the formula. Formally, we set $\hbar = 1$; this is legitimate if $\mathcal{V}((g,n)) = 0$ for $g \gg 0$.

Applying Exp to $\mathrm{Ch}(\mathcal{V})$, we obtain the stable $\mathbb{S}$-module representing possibly disconnected graphs each component of which has one vertex. Applying $D(h_2)$ to $\mathrm{Exp}(\mathrm{Ch}(\mathcal{V}))$ gives the sum over all ways of joining two legs (or flags) of such a graph; h_2 arises because the two ends of an edge are indistinguishable. (If edges carried a direction, we would replace h_2 by p_1^2, the characteristic of the regular representation of $\mathbb{S}_2$.)

Similarly, applying $D(\mathrm{Exp}(\hbar h_2))$ to $\mathrm{Exp}(\mathrm{Ch}(\mathcal{V}))$ gives the sum over all ways of joining together any number N of pairs of legs by edges. In this way, we see (recall that $\hbar$ temporarily equals equals 1) that

$$\exp(\Delta)\,\mathrm{Exp}(\mathrm{Ch}(\mathcal{V})) = \mathrm{Ch}(\mathcal{W}),$$

where $\mathcal{W}$ is the stable $\mathbb{S}$-module such that

$$\mathcal{W}((g,n)) = \bigoplus_{G} \mathcal{V}((G))_{\mathrm{Aut}(G)}, \tag{8.12}$$

where G runs over all possibly disconnected, labelled n-graphs such that each component is stable. But $\mathcal{W} = \mathrm{Exp}(\mathbb{M}_+\mathcal{V})$, since $\mathbb{M}_+\mathcal{V}$ is defined in a similar way, but summing only over connected graphs.

To finish the proof, we must account for the powers of $\hbar$ in each term of (8.12). Each term $\mathrm{ch}_n(\mathcal{V}((g,n)))$ in $\mathrm{Ch}(\mathcal{V})$ comes with a factor of $\hbar^{g-1}$. The term of $\mathrm{Exp}(\mathrm{Ch}(\mathcal{V}))$ corresponding to a labelled graph G (with each component having one vertex) comes with a factor of $\hbar$ raised to the power

$$-|\operatorname{Vert}(G)| + \sum_{v\in\operatorname{Vert}(G)} g(v).$$

Each new edge introduced by the action of $D(\mathrm{Exp}(\hbar h_2))$ contributes a factor of $\hbar$. Therefore, the term in (8.12) corresponding to a labelled graph G comes with a factor of $\hbar$ raised to the power

$$-\chi(G) + \sum_{v\in\operatorname{Vert}(G)} g(v).$$

Applying Log has the effect of discarding all the disconnected graphs G. If G is connected, the power of $\hbar$ in question equals $g(G)-1$, where $g(G)$ is defined in (2.6). $\square$

(8.13) Corollary. *If $\mathcal{A}$ is a modular operad with Feynman transform $\mathsf{F}\mathcal{A}$, then*

$$\mathrm{Ch}(\mathsf{F}\mathcal{A}) = \mathrm{Log}\Big(\exp(\Delta)\,\mathrm{Exp}(\tilde{\omega}\,\mathrm{Ch}(\mathcal{V}))\Big),$$

where $\tilde{\omega} : \Lambda((\hbar)) \to \Lambda((\hbar))$ is the ring homomorphism such that $\tilde{\omega}(p_n) = -p_n$ and $\tilde{\omega}(\hbar) = -\hbar$.

Proof. This follows from the fact that $\mathrm{Ch}(\mathcal{A}^\vee) = \tilde{\omega}\,\mathrm{Ch}(\mathcal{A})$, where $\mathcal{A}^\vee$ is the linear dual (4.1). (See (7.21) for the cyclic case.) □

There is an analogue of (8.11) for $\mathrm{Ch}(\mathbb{M}_- \mathcal{V})$:

$$\mathrm{Ch}(\mathbb{M}_- \mathcal{V}) = \mathrm{Log}\Big(\exp(\bar{\Delta})\,\mathrm{Exp}(\mathrm{Ch}(\mathcal{V}))\Big),$$

where

$$\bar{\Delta} = \sum_{n=1}^{\infty} \hbar^n \left(\frac{n}{2}\frac{\partial^2}{\partial p_n^2} - \frac{\partial}{\partial p_{2n}} \right).$$

The proof is the same as that of (8.11), except that the representation of $\mathbb{S}_2$ associated to an edge is e_2 instead of h_2.

(8.14) Plethystic Fourier transform. Let us give a formal interpretation of the previous theorem in terms of the Fourier transform on the infinite-dimensional vector space $\mathrm{Spec}(\Lambda_{\mathbf{R}}) \cong \mathbb{R}^\infty$, with coordinates $p_1, p_2, \dots$, where $\Lambda_{\mathbf{R}} = \Lambda_{\mathrm{alg}} \otimes \mathbb{R}$. This space has a translation invariant Riemannian metric

$$\langle p_i, p_j \rangle = i\delta_{ij}, \tag{8.15}$$

by means of which we may identify the vector space $\mathrm{Spec}(\Lambda_{\mathbf{R}}) \times \mathrm{Spec}(\Lambda_{\mathbf{R}})^*$ with $\mathrm{Spec}(\Lambda_{\mathbf{R}} \otimes \Lambda_{\mathbf{R}})$. We denote the function $p_n \otimes 1$ by p_n, and the function $1 \otimes p_n$ by q_n.

We may now rewrite (8.11) in the following formal way as a Fourier transform. Let $d\mu$ be the Gaussian measure on $\mathrm{Spec}(\Lambda_{\mathbb{R}})$

$$d\mu = \prod_{n \text{ odd}} \exp(-p_n^2/2n\hbar^n) \frac{dp_n}{\sqrt{2\pi n\hbar^n}} \prod_{n \text{ even}} \exp(-(p_n^2 - 2p_n)/2n\hbar^n) \frac{dp_n}{e^{1/2n}\sqrt{2\pi n\hbar^n}}.$$

Up to an infinite constant, the measure $d\mu$ has density $\mathrm{Exp}(-\hbar^{-1}e_2)$, and is the translate of the Gaussian measure associated to the metric (8.15) by the vector $(p_1, p_2, \dots) = (0, 1, 0, 1, \dots)$.

(8.16) Theorem.

$$\mathrm{Ch}(\mathbb{M}_+ \mathcal{V}) = \mathrm{Log} \int_{\mathbf{R}^\infty} \mathrm{Exp}(\hbar^{-1} p_1 q_1 + \mathrm{Ch}(\mathcal{V}))\, d\mu$$

Proof. Using the formula

$$\exp\left(\frac{t}{2}\frac{\partial^2}{\partial q^2}\right) f(q) = \int_{-\infty}^{\infty} \frac{\exp(-(p-q)^2/2t)}{\sqrt{2\pi t}} f(p)\, dp,$$

we see that

$$\begin{aligned}\exp(\Delta) f(\hbar, q_1, q_2, \dots) &= \int_{\mathbf{R}^\infty} \exp\left(\sum_{n=1}^{\infty} \hbar^n \frac{\partial}{\partial p_{2n}}\right) f(\hbar, p) \prod_{n=1}^{\infty} \frac{\exp(-(p_n - q_n)^2/2n\hbar^n)}{\sqrt{2\pi n\hbar^n}} dp_n \\ &= \int_{\mathbf{R}^\infty} f(\hbar, p) \exp\left(-\sum_{n=1}^{\infty} \hbar^n \frac{\partial}{\partial p_{2n}}\right) \prod_{n=1}^{\infty} \frac{\exp(-(p_n - q_n)^2/2n\hbar^n)}{\sqrt{2\pi n\hbar^n}} dp_n.\end{aligned}$$

We see that

$$\begin{aligned}
&\exp(\Delta)f(\hbar, q_1, q_2, \dots) \\
&= \int_{\mathbf{R}^\infty} f(p) \prod_{n=1}^{\infty} \frac{\exp(-(p_n - q_n - c(n)\hbar^{n/2})^2/2n\hbar^n)\, dp_n}{\sqrt{2\pi n \hbar^n}} \\
&= \prod_{n=1}^{\infty} \exp(-(q_n^2 + q_{2n})/2n\hbar^n) \int_{\mathbf{R}^\infty} f(p) \prod_{n=1}^{\infty} \frac{\exp((-p_n^2 - p_{2n} + 2p_n q_n)/2n\hbar^n)\, dp_n}{e^{c(n)/2n}\sqrt{2\pi n \hbar^n}},
\end{aligned}$$

where $c(n)$ equals 0 if n is odd, and 1 if n is even. □

A corollary of (8.13) is a particularly appealing formula for $\mathrm{Ch}(\mathsf{F}\mathcal{A})$.

(8.17) Corollary.

$$\hbar^{-1}h_2 + \mathrm{Ch}(\mathsf{F}\mathcal{A}) = \mathrm{Log} \int_{\mathbb{R}^\infty} \mathrm{Exp}\Big(\hbar^{-1}p_1q_1 + \tilde{\omega}(\hbar^{-1}h_2 + \mathrm{Ch}(\mathcal{A}))\Big) \prod_{n=1}^{\infty} dp_n - C,$$

where C is the divergent constant

$$C = \frac{1}{2}\sum_{n=1}^{\infty}\Big(\frac{1}{2n} + \log(2\pi n) + n\log(\hbar)\Big).$$

In this form, the resemblance of our theorem to Wick's theorem is clear. Although it is possible that the Legendre transform for symmetric functions can be obtained from (8.16) by the method of stationary phase, we do not know how to do this.

(8.18) The characteristic of FAss. Let us illustrate our formula (8.17) for $\mathrm{Ch}(\mathsf{F}\mathcal{A})$, in the case that the operad $\mathcal{A} = \mathsf{Ass}$ is the associative operad. The components of the Feynman transform $\mathsf{FAss}(\!(g,n)\!)$ may be identified in different regions of g and n:

(1) if $g = 0$, then as is shown in [8],

$$H_i(\mathsf{FAss}(\!(0,n)\!)) = \begin{cases} \mathsf{Ass}(\!(0,n)\!), & i = 0, \\ 0, & i \neq 0; \end{cases}$$

(2) if $n = 0$, then Kontsevich [14] shows that

$$H_i(\mathsf{FAss}(\!(g,0)\!)) = \bigoplus_{\{\gamma\geq 0, \nu>0 | 2(\gamma-1)+\nu=g-1\}} H^{3(g-1)-i}(\mathcal{M}_{\gamma,\nu}/\mathbb{S}_\nu, \mathsf{k}), \tag{8.19}$$

where on the right-hand side we have the singular cohomology of the quotient of the coarse moduli space $\mathcal{M}_{\gamma,\nu}$ by $\mathbb{S}_\nu$;

(3) if $n > 0$, (8.19) may be generalized — the relevant moduli spaces are then moduli spaces of compact Riemann surfaces with boundary, together with n points on the boundary.

In particular, it follows that the power series $\Psi(\hbar) = \mathrm{Ch}(\mathsf{FAss})(\hbar, p_1 = p_2 = \cdots = 0)$ has the following geometrical interpretation,

$$\Psi(\hbar) = \sum_{g=2}^{\infty}(-\hbar)^{g-1} \bigoplus_{\{\gamma\geq 0, \nu>0 | 2(\gamma-1)+\nu=g-1\}} e(\mathcal{M}_{\gamma,\nu}/\mathbb{S}_\nu), \tag{8.20}$$

where $e(\mathcal{M}_{\gamma,\nu}/\mathbb{S}_\nu)$ is the Euler characteristic of the topological space $\mathcal{M}_{\gamma,\nu}/\mathbb{S}_\nu$. Our formula for $\mathrm{Ch}(\mathsf{FAss})$ permits us to calculate this series, a problem which was left open in [14].

The reason that we can calculate Ch(FAss) explicitly is that the characteristic of Ass has such a simple form: it is a sum of terms the n-th of which is a function only of p_n:

$$\mathrm{Ch}(\mathsf{Ass}) = \left(-p_1 - \tfrac{1}{2}p_1^2 - \log(1-p_1)\right) + \tfrac{1}{2}\left(-p_2 - \log(1-p_2)\right) + \sum_{n=3}^{\infty} \frac{\varphi(n)}{n}\log(1-p_n).$$

This means that the integral (8.17) factors into a product of integrals, each of which is over one variable p_n. It is quite simple to calculate asymptotic expansions of these integrals in $\hbar$, and the result is as follows.

(8.21) Theorem. *Let $\alpha_n(\hbar)$ be the Laurent polynomial*

$$\alpha_n(\hbar) = \frac{1}{n}\sum_{d|n} \frac{\varphi(n/d)}{\hbar^d}.$$

Let $\Psi_n(\hbar)$ be the power series

$$\Psi_n(\hbar) = \sum_{k=1}^{\infty} \frac{\zeta(-k)}{-k}\alpha_n^{-k} + (\alpha_n + 1/2)\log(n\hbar^n\alpha_n) + \frac{1}{n\hbar^n} - \alpha_n - \frac{c(n)}{2n}.$$

(The role of the last three terms is to cancel the coefficients of $\hbar^i$, $i \leq 0$.) Then

$$\mathrm{Ch}(\mathsf{FAss}) = -(\hbar^{-1}+1)\sum_{n=1}^{\infty}\frac{\varphi(n)}{n}\log(1-p_n) - \hbar^{-1}(h_1+h_2) + \Psi(\hbar),$$

where

$$\Psi(\hbar) = \sum_{n=1}^{\infty}\sum_{\ell=1}^{\infty}\frac{\mu(\ell)}{\ell}\Psi_n(\hbar^\ell).$$

(This sum is convergent, since $\Psi_n = O(\hbar^{n/6})$.)

As expected, the coefficient of $\hbar^{-1}$ in Ch(FAss) is just Ch(Ass), consistent with the fact that Cyc(FAss) = BAss $\simeq$ Ass. The fact that

$$\mathrm{ch}(\mathsf{FAss}((1,n))) = \mathrm{ch}(\mathsf{Ass}((n))), n \geq 3,$$

is a little surprising. We expect that the vanishing of $\mathrm{ch}(\mathsf{FAss}((g,n)))$ if $g \geq 2$ and $n \geq 1$ is explained by the existence of a free circle action on the relevant moduli spaces.

It is quite easy to calculate the first few terms of $\Psi(\hbar)$:

$$\Psi(\hbar) = 2\hbar + 2\hbar^2 + 4\hbar^3 + 2\hbar^4 + 6\hbar^5 + 6\hbar^6 + 6\hbar^7 + \hbar^8 + O(\hbar^9).$$

The formula (8.20) for the power series $\Psi(\hbar)$ is a sum of contributions from different values of $\nu \geq 1$. The contribution of $\nu = 1$ is calculated in Harer-Zagier ([10], page 482):

$$\sum_{\gamma=1}^{\infty} e(\mathcal{M}_{\gamma,1})\hbar^{2\gamma-1} = \sum_{n=1}^{\infty}\frac{\varphi(n)}{n}\sum_{\ell=1}^{\infty}\mu(\ell)\Psi_{n,\ell}(\hbar),$$

where

$$\Psi_{n,\ell}(\hbar) = \sum_{k=1}^{\infty}\zeta(-k)\alpha_{n,\ell}^{-k} + \alpha_{n,\ell}\log(n\hbar^n\alpha_n) + \frac{1}{n\hbar^n} - \alpha_{n,\ell}.$$

Here, $\alpha_{n,\ell}$ is the Laurent polynomial

$$\alpha_{n,\ell}(\hbar) = \frac{1}{n}\sum_{d|n}\mu(d/(d,\ell))\frac{\varphi(n/d)}{\varphi(\ell/(d,\ell))}\hbar^{-d}.$$

There is a striking formal similarity between this formula and our formula for $\Psi(\hbar)$.

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