# $p$- ADIC NEVANLINNA-CARTAN THEOREM


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# $p$-ADIC NEVANLINNA-CARTAN THEOREM

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## §1. INTRODUCTION

In recent years the Nevanlinna-Cartan theory of value distribution of holomorphic curves ([C]) is arousing increasing interest. By influence of Vojta's works establishing the deep relation between Nevanlinna theory and Diophantine approximation many results on finiteness of rational points on projective varieties are obtained (see [N1]-[N3], [R], [RW]). S Lang conjectured that a projective variety over a number field has only finitely many integral points if and only if the corresponding complex variety is hyperbolic ([L1], [L2], [L4], [V]). It is well-known that the Nevanlinna-Cartan theory is an effective tool in establishing the hyperbolicity of a complex variety. Very recently, by using the Nevanlinna-Cartan theory, K. Masuda and J. Noguchi ([MN]) proved the existence of smooth hyperbolic hypersurfaces of every large degree of the complex projective space $\mathbb{P}^n(\mathbb{C})$. Moreover, they give a partial answer to the Kobayashi conjecture which states that a generic hypersurface of large degree of the complex space $\mathbb{P}^n(\mathbb{C})$ is hyperbolic. J. Noguchi ([N4]) proved the Nevanlinna -Cartan theorem over function fields and, as a consequence, derived a version of "*abc* conjecture" in several variables over function

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fields. For more details on the subject we refer the reader to [N1]-[N4], [K1], [K2], [Z], [ZL].

The purpose of this note is to establish a $p$-adic version of the Nevanlinna-Cartan theorem. By using this $p$-adic Nevanlinna-Cartan theorem, as in the complex case, we can prove the existence of $p$-adic hyperbolic hypersurfaces. Notice that, the "error term" in $p$-adic Nevanlinna-Cartan theorem is more precise than the complex one, and then we obtain the hyperbolic hypersurfaces of smaller degree.

It is necessary to say a few words on "$p$-adic hyperbolicity". In the complex case, by Brody's theorem ([B], [L3]) for a compact manifold $X$ the Kobayashi hyperbolicity is equivalent to the property that $X$ does not contain any non-constant holomorphic curve. Because of the discontinuity of the $p$-adic plane it is difficult to construct an analogue of the Kobayashi semi-distance. Some people proposed different versions of non-archimedean Kobayashi distance (see [Ch], [N3]). So far as we know, however, there is no suitable analogue of the Kobayashi semi-distance. In this note, by "$p$-adic hyperbolicity" we mean "$p$-adic Brody hyperbolicity.

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## §2. Height of $p$-adic holomorphic functions

We recall some facts on heights of $p$-adic holomorphic functions for later use in this note. More details can be found in ([H1]-[H3]).

Let $p$ be a prime number, $Q_p$ the field of $p$-adic numbers, and $C_p$ the $p$-adic completion of the algebraic closure of $Q_p$. The absolute value in $Q_p$ is normalized so that $|p| = p^{-1}$. We further use the notion $v(z)$ for the additive valuation on $C_p$ which extends $\mathrm{ord}_p$.

Let $f(z)$ be a $p$-adic holomorphic function on $\mathbb{C}_p$ represented by a convergent series

$$f(z) = \sum_{n=0}^{\infty} a_n z^n.$$

Since we have

$$\lim_{n \to \infty} \{v(a_n) + nv(z)\} = \infty$$

for every $z \in \mathbb{C}_p$, it follows that for every $t \in \mathbb{R}$ there exists an $n$ for which $v(a_n) + nt$ is minimal.

**Definition 2.1.** The *height* of $f(z)$ is defined by

$$h(f, t) = \min_{0 \leq n < \infty} \{v(a_n) + nt\}.$$

Now let us give a geometric interpretation of height. For each $n$ we draw the graph $\Gamma_n$ which depicts $v(a_n z^n)$ as a function of $v(z)$. This graph is a straight line with slope n. Then $h(f, t)$ is the boundary of the intersection of all of the half-planes lying under the lines $\Gamma_n$. Then in any finite segment $[r, s], 0 < r, s < +\infty$, there are only finitely many $\Gamma_n$ which appear in $h(f, t)$. Thus, $h(f, t)$ is a polygonal line. The point $t$ at which $h(f, t)$ has vertices are called *the critical points* of $f(z)$. A finite segment $[r, s]$ contains only a finitely many critical points. It is clear that if $t$ is a critical point, then $v(a_n) + nt$ attains its minimum at least at two values of $n$.

If $v(z) = t$ is not a critical point, then $f(z) \neq 0$ and $|f(z)| = p^{-h(f,t)}$. The function $f(z)$ has zeros when $v(z) = t_i$, where $t_o > t_1 > \ldots$ is the sequence of critical points; and the number of zeros (counting multiplicity) for which $v(z) = t_i$ is equal to the difference $n_{i+1} - n_i$ between the slope of $h(f,t)$ at $t_i - 0$ and its slope at $t_i + 0$. It is easy to see that $n_i$ and $n_{i+1}$, respectively, are the smallest and the largest values of $n$ at which $v(n) + nt$ attains minimum.

**Lemma 2.2.** *Let $f(z)$ be a non-constant holomorphic function on $\mathbb{C}_p$. Then we have*

$$h(f',t) - h(f,t) \geq -t + O(1),$$

*where $O(1)$ is bounded when $t \to -\infty$*

**Lemma 2.3.** *A function $f(z)$ is a polynomial if and only if $h(f,t) = O(t)$ when $t \to -\infty$*

The proof of Lemmas 2.2, 2.3 follows immediately from Definition 2.1, and the geometric interpretation of height.

## §3. $p$-ADIC NEVANLINNA-CARTAN THEOREM

Let $f$ be a $p$-adic holomorphic curve in the projective space $\mathbb{P}^n(\mathbb{C}_p)$, i.e., a holomorphic map from $\mathbb{C}_p$ to $\mathbb{P}^n(\mathbb{C}_p)$. We identify $f$ with its representation by a collection of holomorphic functions on $\mathbb{C}_p$:

$$f = (f_1, f_2, \ldots, f_{n+1}),$$

where the functions $f_i$ have no common zeros. The curve $f$ is said to be *non-degenerate* if the image of $f$ is not contained in any linear subspace of $\mathbb{P}^n(\mathbb{C}_p)$ of dimension less than $n$.

**Definition 3.1.** The *height* of the holomorphic curve $f$ is defined by:

$$h(f,t) = \min_{1 \leq i \leq n+1} h(f_i, t).$$

We also use the notation

$$h^+(f,t) = -h(f,t).$$

**Definition 3.2.** *Counting function.* For every holomorphic function $g$ on $\mathbb{C}_p$, the following function is called *the counting function* of $g$

$$N(g,t) = \sum_{a_i} \{v(a_i) - t\},$$

where the sum is taken on all of zeros $a_i$ of $g(z)$ (counting multiplicity) with $v(a_i) \geq t$.

Notice that, for every $t$, the sum in Definition 3.2 is a finite sum.

Now we define the *truncated counting function* which is due to Cartan.

**Definition 3.3**. For every positive entire number $k$ denote by $N_k(g,t)$ the sum in Definition 3.2, where every zero $a_i$ is counted with multiplicity if its multiplicity less that $k$, and $k$ times otherwise. We call $N_k(g,t)$ *the k-truncated counting function of g.*

We have the following obvious lemma.

**Lemma 3.4.** *For every* $k \geq 1$

$$N_1(g,t) \leq N_k(g,t) \leq kN_1(g,t),$$

$$N_k(g,t) \leq N(g,t).$$

Now let $H_1, H_2, \ldots, H_q$ are $q$ hyperplanes in $\mathbb{P}^n(\mathbb{C}_p)$ in *general position.* This means that these hyperplanes are linearly independent if $q \leq n$, and any $n+1$ of these hyperplanes are linearly independent if $q \geq n+1$.

Suppose that $F_i = 0$ are the equations defining the hyperplanes $H_i$. Then we set:

$$h(f \circ H_i, t) = h(F_i \circ f, t),$$

$$N_k(f \circ H_i, t) = N_k(F_i \circ f, t).$$

The following theorem is a $p$-adic analogue of the Nevanlinna-Cartan theorem ([C]).

**Theorem 3.5.** *Let $H_1, H_2, \ldots, H_q$ be $q$ hyperplanes in general position, and let $f$ be a non-degenerate holomorphic curve in $\mathbb{P}^n(C_p)$. Then we have*

$$(q-n-1)h^+(f,t) \leq \sum_{j=1}^{q} N_n(f \circ H_j, t) + \frac{n(n+1)}{2}t + O(1),$$

*where $O(1)$ is bounded when $t \to -\infty$*

The ideas of the proof of Theorem 3.5 are similar to ones in the complex case ([C]), where instead of the Nevanlinna-Cartan characteristic function we use the height function. However, there are some facts valid only in the $p$-adic case (for example, Lemma 3.8). So, it is necessary to give here a detail proof.

We first prove the following

**Lemma 3.6.** *Let $G_i(z) = F_i \circ f(z)$, $i = 1, 2, \ldots q$. Then for every $z \in \mathbb{C}_p$ there are at most $n$ functions $G_i$ such that $G_i(z) = 0$*

**Proof.** Assume that there are $z \in \mathbb{C}_p$ and $n+1$ functions $G_{\alpha_i}$, $i = 1, 2, \ldots, n+1$ such that $G_{\alpha_i}(z) = 0$. Then from the system of equations

$$G_{\alpha_i}(z) = \sum_{j=1}^{n+1} a_j^{\alpha_i} f_j(z), \quad i = 1, \ldots, n+1$$

and the hypothesis of general position it follows that $f_j(z) = 0$ for every $j = 1, \ldots, n+1$. This is a contradiction, since the functions $f_j$ have no common zeros.

Now let $\beta_1, \beta_2, \ldots, \beta_{q-n-1}$ be $(q-n-1)$ distinct numbers from the numbers $\{1, 2, \ldots, q\}$. We set

$$G = (\ldots, G_{\beta_1} \ldots G_{\beta_{q-n-1}}, \ldots),$$

where $(\beta_1, \ldots, \beta_{q-n-1})$ is taken by all possible choices. Then $G$ define a holomorphic curve of $\mathbb{P}^k(\mathbb{C}_p)$, $k = C_q^{q-n-1}$, since by Lemma 3.6 the functions $G_{\beta_1} \ldots G_{\beta_{q-n-1}}$ have no common zeros.

**Lemma 3.7.** *We have*

$$h(G,t) \leq (q-n-1)h(f,t) + O(1),$$

*where $O(1)$ does not depend on $t$*

**Proof**. By definition we have

$$h(G,t) = \min_{(\beta_1,\ldots,\beta_{q-n-1})} h(G_{\beta_1} \ldots G_{\beta_{q-n-1}}, t) = \min_{(\beta_1,\ldots,\beta_{q-n-1})} \sum_{j=1}^{q-n-1} h(G_{\beta_j}, t).$$

Now let for a fixed $t$ the following inequality holds

$$h(G_{\beta_1}, t) \leq h(G_{\beta_2}, t) \leq \cdots \leq h(G_{\beta_q}, t).$$

We then obtain

$$h(G,t) = h(G_{\beta_1}, t) + h(G_{\beta_2}, t) + \cdots + h(G_{\beta_{q-n-1}}, t).$$

On the other hand, since the hypothesis of general position, we can represent $f_i$ by a linear combination of $G_{\beta_{q-n}}, \ldots, G_{\beta_q}$:

$$f_i = \sum_{j=0}^{n} a_{ij} G_{\beta_{q-j}}.$$

From this it follows that

$$h(f_i, t) \geq \min_{0 \leq j \leq n} h(G_{\beta_{q-j}}, t) + O(1)$$

Then we have

$$h(f_i, t) \geq h(G_{\beta_j}, t) + O(1), \qquad j = 1, 2, \ldots, q-n-1.$$

Thus,

$$h(f,t) = \min h(f_i,t) \geq h(G_{\beta_j},t) + O(1), \quad j = 1,2,\ldots,q-n-1.$$

The Lemma then is proved by sumarizing $(q-n-1)$ inequalities.

By using Lemma 3.7, to prove Theorem 3.5 it remains to estimate $h(G,t)$.

Now for $(n+1)$ functions $\Phi_1, \Phi_2, \ldots, \Phi_{n+1}$ we denote by $||\Phi_1, \Phi_2, \ldots, \Phi_{n+1}||$ their Wronskian.

Let $(\alpha_1, \alpha_2, \ldots, \alpha_{n+1})$ be distinct numbers from $\{1, 2, \ldots, q\}$ and $(\beta_1, \beta_2, \ldots, \beta_{q-n-1})$ be the rest ones. As it is mentioned above, the functions $\{f_i\}$ can be represented as linear combinations of $G_{\alpha_1}, \ldots, G_{\alpha_{n+1}}$. Then we have

$$||G_{\alpha_1} \ldots G_{\alpha_{n+1}}|| = \frac{1}{c(\alpha_1, \ldots, \alpha_{n+1})}||f_1 \ldots f_{n+1}||,$$

where $c(\alpha_1, \ldots, \alpha_{n+1})$ are constants depending only on $(\alpha_1, \ldots, \alpha_{n+1})$. For simplicity we denote

$$A(\alpha_1, \ldots, \alpha_{n+1}) = \frac{||G_{\alpha_1} \ldots G_{\alpha_{n+1}}||}{G_{\alpha_1} \ldots G_{\alpha_{n+1}}} = \begin{vmatrix} 1 & 1 & \ldots & 1 \\ \frac{G'_{\alpha_1}}{G_{\alpha_1}} & \frac{G'_{\alpha_2}}{G_{\alpha_2}} & \ldots & \frac{G'_{\alpha_{n+1}}}{G_{\alpha_{n+1}}} \\ \ldots & \ldots & \ldots & \ldots \\ \frac{G^{(n)}_{\alpha_1}}{G_{\alpha_1}} & \frac{G^{(n)}_{\alpha_2}}{G_{\alpha_2}} & \ldots & \frac{G^{(n)}_{\alpha_{n+1}}}{G_{\alpha_{n+1}}} \end{vmatrix}$$

From this it implies that

$$\begin{aligned} \frac{G_{\beta_1} \ldots G_{\beta_{q-n-1}}}{c(\alpha_1, \ldots, \alpha_{n+1})A(\alpha_1, \ldots, \alpha_{n+1})} &= \frac{G_{\alpha_1} \ldots G_{\alpha_{n+1}} G_{\beta_1} \ldots G_{\beta_{q-n-1}}}{c(\alpha_1, \ldots, \alpha_{n+1})A(\alpha_1, \ldots, \alpha_{n+1})} \\ &= \frac{G_1 \ldots G_q}{||f_1 \ldots f_{n+1}||} \end{aligned}$$

This means that we can denote by $R(z)$ the following function

$$R(z) = \frac{G_{\beta-1} \ldots G_{\beta_{q-n-1}}}{c(\alpha_1, \ldots, \alpha_{n+1})A(\alpha_1, \ldots, \alpha_{n+1})}, \tag{1}$$

which does not depend on choices of $(\alpha_1, \ldots, \alpha_{n+1}), (\beta_1, \ldots, \beta_{q-n-1})$. We then obtain

$$G_{\beta_1} \ldots G_{\beta_{q-n-1}} = c(\alpha_1, \ldots, \alpha_{n+1}) R(z) A(\alpha_1, \ldots, \alpha_{n+1}).$$

Thus, to estimate the height of $G(z)$, it suffices to estimate the heights of $R(z)$ and $A(\alpha_1, \ldots, \alpha_{n+1})$.

We first consider the function $A$. We have:

$$h(A, t) = \min_{(\alpha_1, \ldots, \alpha_{n+1})} h(\frac{G_{\alpha_1}^{(k_1)}}{G_{\alpha_1}} \ldots \frac{G_{\alpha_{n+1}}^{(k_{n+1})}}{G_{\alpha_{n+1}}}, t)$$

where the minimum is taken by all permutations $(k_1, \ldots, k_{n+1})$ of numbers $\{0, 1, \ldots, n\}$. On the other hand, we have:

$$h(\frac{G_{\alpha_i}^{(k_i)}}{G_{\alpha_i}}, t) = h(\frac{G_{\alpha_i}^{(k_i)}}{G_{\alpha_i}^{(k_i-1)}} . \frac{G_{\alpha_i}^{(k_i-1)}}{G_{\alpha_i}^{(k_i-2)}} \cdots \frac{G'_{\alpha_i}}{G_{\alpha_i}}, t)$$

$$= \sum_{j=0}^{k_i-1} h(\frac{G_{\alpha_i}^{(k_i)}}{G_{\alpha_i}}, t).$$

From Lemma 2.2 it implies that

$$h(\frac{G_{\alpha_i}^{(k_i)}}{G_{\alpha_i}}, t) \geq -k_i.t + O(1).$$

Then we have

$$h(A, t) = \min_{(\alpha_1, \ldots, \alpha_{n+1})} \{\sum_{i=1}^{n+1} h(\frac{G_{\alpha_i}^{(k_i)}}{G_{\alpha_i}}, t)\} \geq$$

$$\min_{(\alpha_1, \ldots, \alpha_{n+1})} \sum_{i=1}^{n+1} (-k_i t) + O(1) = -\frac{n(n+1)}{2} t + O(1).$$

Now we consider the $h(R, t)$. We first prove the following

**Lemma 3.8.** *For every holomorphic function $\phi(z)$ on $\mathbb{C}_p$ we have*

$$-h(\phi,t) = N(\phi,t) + O(1),$$

*where $O(1)$ depends on $\phi$, but not on $t$*

**Proof.** To prove Lemma 3.8 we use the geometric interpretation of heights. Notice that for every $t$, there are only finitely many critical points of the function $\phi(z)$ with $v(z) \geq t$. Let $t_0 > t_1 > \cdots > t_m \geq t$ be all these critical points.

By definition of critical points, the height $h(\phi, s)$ is a linear function of $s$ in every segment $[t_{i+1}, t_i]$. Denote by $n_i^+$ and $n_i^-$ the slopes of $h(\phi,t)$ at $t_i + 0$ and $t_i - 0$, respectively. Then we have

$$h(f,t_0) - h(f,t) = n_o^-(t_o - t_1) + n_1^-(t_1 - t_2) + \cdots + n_m^-(t_m - t)$$

$$= n_o^- t_o + (n_1^- - n_o^-)t_1 + \cdots + (n_m^- - n_{m-1}^-)t_m + n_m^- t$$

$$= n_o^-(t_o - t_1) + (n_1^- - n_o^-)(t_1 - t) + \cdots + (n_m^- - n_{m-1}^-)(t_m - t)$$

We can see that $\phi(z) \neq 0$ when $v(z) > t_o$, and $\phi(z)$ has $n_o^+$ zeros (counted with multiplicity) with $v(z) \geq t_o$ , $(n_i^- - n_{i-1}^-)$ zeros with $v(z) = t_i$, $i = 1, 2, \ldots, m$. Then the sum in the right hand side is exactely $N(\phi,t)$. Lemma 3.8 is proved.

Now we can return to the proof of Theorem 3.5. By Lemma 3.8 we have

$$h(R,t) = -N(R,t) + O(1).$$

For every $z$ the function $R(z)$ can be represented in the form (1) where $G_{\beta_i}(z) \neq 0$, $i = 1, 2, \ldots, q-n-1$ (by Lemma 3.6). Thus, the zeros of $R$ are just only the poles of $A(\alpha_1, \ldots, \alpha_{n+1})$, i.e., the poles of $\dfrac{G_{\alpha_i}^{(k_i)}}{G_{\alpha_i}}$. From the definitions of the truncated counting function and the function $A$ it follows that

$$N(R,t) \leq \sum_{i=1}^{q} N_n(G_i,t) = \sum_{i=1}^{q} N_n(f \circ F_i, t).$$

Finally, we have

$$h(G,t) = \min_{(\beta_1,\ldots,\beta_{q-n-1})} \{h(G_{\beta_1}\ldots G_{\beta_{q-n-1}},t)\}$$

$$= h(R,t) + \min_{(\alpha_1,\ldots,\alpha_{n+1})} \{h(c(\alpha_1,\ldots,\alpha_{n+1})A(\alpha_1,\ldots,\alpha_{n+1}),t) \geq$$

$$\geq -N(R,t) - \frac{n(n+1)}{2}t + O(1).$$

Thus,

$$(q-n-1)h(f,t) \geq -N(R,t) - \frac{n(n+1)}{2}t + O(1).$$

This means that

$$(q-n-1)h^+(f,t) \leq N(R,t) + \frac{n(n+1)}{2}t + O(1)$$

$$\leq \sum_{i=1}^{q} N_n(f \circ F_i.t) + \frac{n(n+1)}{2}t + O(1).$$

Theorem 3.5 is proved.

**Remark 3.9**. The "error term" $\frac{n(n+1)}{2}t$ in Theorem 3.5 is more precise than one in the complex case, since in the $p$-adic case, the "theorem on logarithmic derivative"(Lemma 2.3) is very simple.

Now we apply Theorem 3.5 for giving a "defect relation".

We need some notation. Let $H$ be a hyperplane of $\mathbb{P}^n(\mathbb{C}_p)$ such that the image of $f$ is not contained in $H$. We say that $f$ *ramifies at least $d$ ($d > 0$) over $H$* if for all $z \in f^{-1}H$ the degree of the pull-back divisor $f^*H$, $\deg_z f^*H \geq d$. This means that if $H$ is defined by the equation $F = 0$, then every zero of the function $F \circ f$ has multiplicity at least $d$. In the case $f^{-1}H = \emptyset$ we set $d = \infty$.

**Theorem 3.10.** *(Defect relation). Assume $f$ is linearly non-degenerate and ramifies at least $d_j$ over $H_j$, $j = 1, 2, \ldots, q$. Then*

$$\sum_{j=1}^{q}(1 - \frac{n}{d_j}) \leq n + 1.$$

*Moreover, if $f$ is a rational curve of degree $e$ (all $f_i(z)$, $i = 1, \ldots, n+1$ are polynomials of degree $e_j$ and $\min e_j = e$), then*

$$\sum_{j=1}^{q}(1 - \frac{n}{d_j}) \leq n + 1 - \frac{n(n+1)}{2e}.$$

**Proof.** Notice that from the geometric interpretation of height it follows that if $h^+(f,t)$ is bounded when $t \to -\infty$, then $f$ is a constant map. Thus, by the hypothesis of non-degeneracy, $h^+(f,t)$ is unbounded when $t \to -\infty$. From Theorem 3.5 we obtain:

$$\text{(2)} \qquad \sum_{i=1}^{q}\{1 - \frac{N_n(f \circ F_i, t)}{h^+(f,t)}\} \leq (n+1) + \frac{n(n+1)}{2}.\frac{t}{h^+(f,t)} + \frac{O(1)}{h^+(f,t)}.$$

On the other hand, from Lemma 3.4 we have

$$\text{(3)} \qquad 1 - \frac{N_n(f \circ F_i, t)}{h^+(f,t)} = 1 - \frac{N_n(f \circ F_i, t)}{N(f \circ F_i, t)}.\frac{N(f \circ F_i, t)}{h^+(f,t)} \geq$$

$$\geq 1 - \frac{nN_1(f \circ F_i, t)}{N(f \circ F_i, t)} + \frac{O(1)}{h^+(f,t)}.$$

By the hypothesis, $N(f \circ F_i, t) \geq d_i N_1(f \circ F_i, t)$. Theorem 3.10 is proved by using (2), (3), and the remark that if $f$ is a rational curve of degree $e$, then $h^+(f,t) = -e.t + O(1)$ when $t \to -\infty$.

In this section by using Theorem 3.5 we give a $p$-adic version of Borel's Lemma, and use it to construct some examples of $p$-adic hyperbolic hypersurfaces.

We follow the ideas of Masuda and Noguchi [MN], where istead of the Nevanlinna-Cartan theorem we use Theorem 3.5.

Let $(z_1, \ldots, z_{n+1})$ be a homogeneuos cordinate system of $\mathbb{P}^n(\mathbb{C}_p)$. Let $M_j = z_1^{\alpha_{j,1}} \ldots z_{n+1}^{\alpha_{j,n+1}}$ $1 \leq j \leq s$ be monomials of degree $l$ with non-negative integral exponents $\alpha_{j\nu} \in Z$. Let $X$ be a hypersurface of degree $dl$ of $\mathbb{P}^n(\mathbb{C}_p)$ defined by

$$X: \quad c_1 M_1^d + \cdots + c_s M_s^d = 0, \tag{4}$$

where $c_j \in \mathbb{C}_p^*$ are non-zero constants.

**Theorem 4.1.** *($p$-adic Borel's Lemma). Let* $f = (f_1, \ldots, f_{n+1}) : \mathbb{C}_p \to X$ *be a non-constant holomorphic curve such that any* $f_j \not\equiv 0$. *Assume that*

$$l \geq s(s-2).$$

*Then there is a decomposition of indices,* $\{1, 2, \ldots, s\} = \cup I_\nu$, *such that:*

*i) every* $I_\nu$ *contains at least 2 indices.*

*ii) the ratio of* $M_j^d \circ f(z)$ *and* $M_k^d \circ f(z)$ *is constant for* $j, k \in I_\nu$.

*iii)* $\sum_{j \in I_\nu} c_j M_j^d \circ f(z) \equiv 0$ *for all* $\nu$

**Proof.** We use the induction on the number $s$ of the monomials. The case of $s = 2$ is trivial. Assume that the statement for the number less or equal to $s - 1$ holds.

We first claim that $M_j^d \circ f$, $1 \leq j \leq s-1$, are linearly dependent over $\mathbb{C}_p$. Asuume that $M_j^d \circ f$, $1 \leq j \leq s-1$ are linearly independent. We define a holomorphic curve $g$ in $\mathbb{P}^{s-2}(\mathbb{C}_p)$ by

$$g : z \in \mathbb{C}_p \mapsto (M_1^d \circ f(z), \ldots, M_{s-1}^d \circ f(z)) \in \mathbb{P}^{s-2}(\mathbb{C}_p).$$

Take the following hyperplanes in general position

$$H_1 = \{z = 0\}, \ldots, H_{s-1} = \{z_{s-1} = 0\},$$

$$H_s = \{c_1 z_1 + \cdots + c_{s-1} z_{s-1} = 0\}.$$

Then by Theorem 3.5 we have

$$(s - (s-2) - 1)h^+(g,t) \leq \sum_{j=1}^{s} N_{s-2}(g \circ H_j, t) + \frac{n(n+1)}{2}t + O(1).$$

On the other hand, we have

$$h^+(g,t) = d \max_{1 \leq j \leq s-1} h^+(M_j \circ f, t),$$

$$N_{s-2}(g \circ H_j, t) \leq (s-2) N_1(M_j^d \circ f, t) = (s-2) N_1(M_j \circ f, t)$$

Hence, by Lemma (3.8) we obtain

$$N_{s-2}(g \circ H_j, t) \leq (s-2) h^+(M_j \circ f, t) + O(1).$$

Thus we have

$$(5) \quad N_{s-2}(g \circ H_j, t) \leq (s-2) \max_{1 \leq k \leq s-1} h^+(M_k \circ f, t) + O(1), \quad j = 0, \ldots, s-1.$$

From definitions of $g$, $H_j$ and the equation defining $X$ we can see that the inequality (5) holds also for $j = s$. Thus we have

$$d \max_{1 \leq j \leq s-1} h^+(M_j \circ f, t) \leq s(s-2) \max_{1 \leq j \leq s-1} h^+(M_j \circ f, t) + \frac{n(n+1)}{2}t + O(1).$$

By the hypothesis of $d \geq s(s-2)$ we have a contradiction as $t \to -\infty$.

Thus there is a non-trivial linear relation

$$c'_1 M_1^d \circ f + \cdots + c'_{s-1} M_{s-1}^d \circ f = 0, \quad c'_j \in \mathbb{C}_p. \tag{6}$$

Ignoring the terms with $c'_j = 0$ in (6), we apply the induction hypothesis to (6). Here it is clear that the assumption which the induction hypothesis requires is fulfilled. Thus one reduces the number $s$ in (4) to $s-1$ or less. Applying the induction hypothesis to the reduced equation again, we obtain our assertion.

As in the complex case, from Borel's Lemma (Theorem 4.1) we can derive many results on $p$-adic hyperbolicity. Let us mention here some of them.

We recall that *the Fermat variety* $X$ in $\mathbb{P}^n(\mathbb{C}_p)$ of degree $d$ is defined by the equation:

$$z_1^d + \cdots + z_{n+1}^d = 0.$$

The following theorem is a $p$-adic version of Green's theorem ([G], [L3]).

**Theorem 4.2.** *Let* $f = (f_1, \ldots, f_{n+1}) : \mathbb{C}_p \longrightarrow X$ *be a holomorphic curve such that any* $f_j \not\equiv 0$. *Define an equivalence* $i \approx f$ *if* $f_i/f_j$ *is constant. If* $d \geq n^2 - 1$ *then for each equivalence class* $I$ *we have*

$$\sum_{i \in I} f_i^d = 0.$$

Theorem 4.2 is a corollary of Theorem 4.1 with $s = n+1$. Notice that, in the complex case, the hypothesis is $d \geq n^2$.

When $n = 2$ we obtain the following corollary of Theorem 4.2.

**Corollary 4.3.** *Let* $f, g, h,$ *be* $p$*-adic holomorphic functions on* $\mathbb{C}_p$*, and let for some* $d \geq 3$ *we have*

$$f^d + g^d = h^d.$$

*Then the functions* $f, g, h$ *are different each from other only by a multiplicative constant.*

A similar statement for polynomials is a corollary of Mason's thorem ([M], [L4]).

To give the examples of $p$-adic hyperbolic hypersurfaces let us make the following remark. R. Brody and M. Green ([BG]) first constructed a smooth hyperbolic hypersurface of $\mathbb{P}^3(\mathbb{C})$ of even degree $\geq 50$. K. Masuda and J. Noguchi proved the existence of hyperbolic hypersurfaces of large degree of $\mathbb{P}^n(\mathbb{C})$ for any $n$ ([MN]). Both in [BG] and [NM] the main tool are Borel's Lemma and purely linear algebraic arguments. These algebraic arguments can apply without changes to the $p$-adic case. Then by using Theorems 3.5, 4.1 and Masuda-Noguchi's algebraic Lemmas, as well as the computation of Noguchi and Masuda-Noguchi, we can give the following examples of $p$-adic hyperbolic hypersurfaces. Notice that, the difference of the $p$-adic and complex cases is in degree of hypersurfaces (it follows from the difference of the hypothesis on the degree $d$ in Theorem 4.1 and in Borel's Lemmas in complex case.

In $\mathbb{P}^3(\mathbb{C}_p)$, we have $p$-adic hyperbolic hypersurfaces $X$ given by the following equations:

*Example 1* (Brody-Green).

$$z_1^d + \cdots + z_4^d + (z_1 z_2)^{d/2} + (z_1 z_3)^{d/2} = 0, \quad d \text{ even}, d \geq 48.$$

*Example 2* (J. Noguchi).

$$z_1^{3d} + \cdots + z_4^{3d} + t(z_1 z_2 z_3)^d = 0, \ d \geq 7, (\deg\ X = 3d \geq 21), t \in \mathbb{C}_p^*.$$

*Example 3* (J. Noguchi).

$$z_1^{4d} + \cdots + z_4^{4d} + t(z_1 z_2 z_3 z_4)^d = 0,\ d \geq 6(\deg\ X = 4d \geq 24), t \in \mathbb{C}_p^*$$

*Example 4* (K. Masuda and J. Noguchi).

$$z_1^{4d} + \cdots + z_4^{4d} + t(z_1^2 z_2 z_3)^d = 0,\ d \geq 6, \quad t \in \mathbb{C}_p^*.$$

K. Masuda and J. Noguchi give several examples also for $n = 4, 5$. We can copy their examples to the $p$-adic case (with suitable change of the degree), say, for $n = 5$, we have the following $p$-adic hypersurface:

*Example 4* (K. Masuda and J. Noguchi).

$$z_1^d + \cdots + z_6^d + t_1(z_1 z_2^4)^{d/5} + t_2(z_2 z_3^4)^{d/5} + t_3(z_3 z_4^4)^{d/5}$$

$$+t_4(z_4 z_5^4)^{d/5} + t_5(z_5 z_1^4)^{d/5}$$

$$+t_6(z_1^2 z_3^3)^{d/5} + t_7(z_2^2 z_4^3)^{d/5} + t_8(z_3^2 z_5^3)^{d/5}$$

$$+t_9(z_4^2 z_1^3)^{d/5} + t_{10}(z_5^2 z_2^3)^{d/5} = 0,$$

$$t_j \in \mathbb{C}_p^*, \quad d = 5e \geq 1120.$$

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