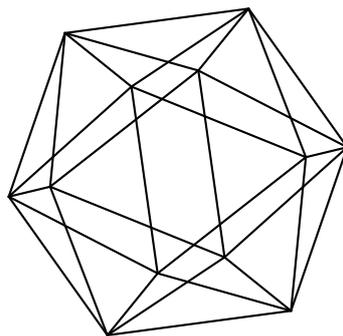


# Max-Planck-Institut für Mathematik Bonn

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and applications

by

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# The $\text{Spin}^c$ Dirac operator on hypersurfaces and applications

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# The $\text{Spin}^c$ Dirac Operator on Hypersurfaces and Applications

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## Abstract

We extend to the eigenvalues of the hypersurface  $\text{Spin}^c$  Dirac operator well known lower and upper bounds. Examples of limiting cases are then given. Furthermore, we prove a correspondence between the existence of a  $\text{Spin}^c$  Killing spinor on homogeneous 3-dimensional manifolds  $\mathbb{E}^*(\kappa, \tau)$  with 4-dimensional isometry group and isometric immersions of  $\mathbb{E}^*(\kappa, \tau)$  into the complex space form  $\mathbb{M}^4(c)$  of constant holomorphic sectional curvature  $4c$ , for some  $c \in \mathbb{R}^*$ . As applications, we show the non-existence of totally umbilic surfaces in  $\mathbb{E}^*(\kappa, \tau)$  and we give necessary and sufficient geometric conditions to immerse a 3-dimensional Sasaki manifold into  $\mathbb{M}^4(c)$ .

**Key words.**  $\text{Spin}^c$  structures, isometric immersions, spectrum of the Dirac operator, parallel and Killing spinors, manifolds with boundary and boundary conditions, Sasaki and Kähler manifolds.

## 1 Introduction

It is well known that the spectrum of the Dirac operator on hypersurfaces of a Spin manifold detects informations on the geometry of such manifolds and their hypersurfaces ([3, 4, 5, 16, 18, 19]). For example, O. Hijazi, S. Montiel and X. Zhang [16] proved that on the compact boundary  $M^n$  of a Riemannian Spin manifold  $\mathcal{Z}^{n+1}$  of

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dimension  $n + 1$  and with nonnegative scalar curvature, the first positive eigenvalue  $\lambda_1$  of the Dirac operator satisfies

$$\lambda_1 \geq \frac{n}{2} \inf_M H, \quad (1)$$

where  $H$  denotes the mean curvature of  $M$ , assumed to be nonnegative. Equality holds if and only if  $H$  is constant and every eigenspinor associated with  $\lambda_1$  is the restriction to  $M$  of a parallel spinor on  $\mathcal{Z}$  (and so  $\mathcal{Z}$  is Ricci-flat). As application of the limiting case, they gave an elementary proof of the famous Alexandrov theorem [16]: *the only compact embedded hypersurface in  $\mathbb{R}^{n+1}$  of constant mean curvature is the sphere  $\mathbb{S}^n$  of dimension  $n$ .*

Assume now that  $M^n$  is a closed hypersurface of  $\mathcal{Z}^{n+1}$ . Evaluating the Rayleigh quotient applied to a parallel or Killing spinor field coming from  $\mathcal{Z}$ , C. Bär [5] derived an upper bound for the eigenvalues of the Dirac operator on  $M$  by using the min-max principle. More precisely, there are at least  $\mu$  eigenvalues  $\lambda_1, \dots, \lambda_\mu$  of the Dirac operator on  $M$  satisfying

$$\lambda_j^2 \leq n^2 \alpha^2 + \frac{n^2}{4 \operatorname{vol}(M)} \int_M H^2 dv, \quad (2)$$

where  $\operatorname{vol}(M)$  is the volume of  $M$ ,  $dv$  is the volume form of the manifold  $M$ ,  $\alpha$  is the Killing number ( $\alpha = 0$  if the ambient spinor field is parallel) and  $\mu$  is the dimension of the space of parallel or Killing spinors. Moreover, it is easy to check that when  $\alpha = 0$ , equality holds in (2) if equality holds in (1).

Recently,  $\operatorname{Spin}^c$  geometry became a field of active research with the advent of Seiberg-Witten theory [22, 35, 31]. Applications of the Seiberg-Witten theory to 4-dimensional geometry and topology are already notorious ([9, 24, 25, 13]). From an intrinsic point of view,  $\operatorname{Spin}$ , almost complex, complex, Kähler, Sasaki and some classes of CR manifolds have a canonical  $\operatorname{Spin}^c$  structure. The complex projective space  $\mathbb{C}P^m$  is always  $\operatorname{Spin}^c$  but not  $\operatorname{Spin}$  if  $m$  is even. Nowadays, and from the extrinsic point of view, it seems that it is more natural to work with  $\operatorname{Spin}^c$  structures rather than  $\operatorname{Spin}$  structures. Indeed, O. Hijazi, S. Montiel and F. Urbano [20] constructed on Kähler-Einstein manifolds with positive scalar curvature,  $\operatorname{Spin}^c$  structures carrying Kählerian Killing spinors. The restriction of these spinors to minimal Lagrangian submanifolds provides topological and geometric restrictions on these submanifolds. In [30, 29], and via  $\operatorname{Spin}^c$  spinors, the authors gave an elementary proof for a *Lawson type correspondence* between constant mean curvature surfaces of 3-dimensional homogeneous manifolds with 4-dimensional isometry group. We point out that, using  $\operatorname{Spin}$  spinors, we cannot prove this *Lawson type correspondence*. Moreover, they characterized isometric immersions of a 3-dimensional almost contact metric manifold  $M$  into the complex space form by the existence of a  $\operatorname{Spin}^c$  structure on  $M$  carrying a special spinor called a generalized Killing spinor.

In the first part of this paper and using the  $\text{Spin}^c$  Reilly inequality, we extend the lower bound (1) to the first positive eigenvalue of the Dirac operator defined on the compact boundary of a  $\text{Spin}^c$  manifold. Examples of the limiting case are then given where the equality case cannot occur if we consider the  $\text{Spin}$  Dirac operator on these examples. Also, by restriction of parallel and Killing  $\text{Spin}^c$  spinors, we extend the upper bound (2) to the eigenvalues of the Dirac operator defined on a closed hypersurface of  $\text{Spin}^c$  manifolds. Examples of the limiting case are also given.

In the second part, we study  $\text{Spin}^c$  structures on 3-dimensional homogeneous manifolds  $\mathbb{E}^*(\kappa, \tau)$  with 4-dimensional isometry group. It is well known that the only complete simply connected  $\text{Spin}^c$  manifolds admitting real Killing spinor other than the  $\text{Spin}$  manifolds are the non-Einstein Sasakian manifolds endowed with their canonical or anti-canonical  $\text{Spin}^c$  structure [27]. Since  $\mathbb{E}^*(\kappa, \tau)$  are non-Einstein Sasakian manifolds [7], the canonical and the anti-canonical  $\text{Spin}^c$  structure carry real Killing spinors. In [30], the authors proved that this canonical (resp. this anti-canonical)  $\text{Spin}^c$  structure on  $\mathbb{E}(\kappa, \tau)$  is the lift of the canonical (resp. the anti-canonical)  $\text{Spin}^c$  structure on  $\mathbb{M}^2(\kappa)$  via the submersion  $\mathbb{E}(\kappa, \tau) \rightarrow \mathbb{M}^2(\kappa)$ , where  $\mathbb{M}^2(\kappa)$  denotes the simply connected 2-dimensional manifold with constant curvature  $\kappa$ . Moreover, they proved that the Killing constant of the real Killing spinor field is equal to  $\frac{\tau}{2}$ . Here, we reprove the existence of a Killing spinor for the canonical and the anti-canonical  $\text{Spin}^c$  structure. This proof is based on the existence of an isometric embedding of  $\mathbb{E}^*(\kappa, \tau)$  into the complex projective space or the complex hyperbolic space (see Proposition 4.1). Conversely, from the existence of a Killing spinor on  $\mathbb{E}^*(\kappa, \tau)$ , we prove the existence of an isometric immersion of  $\mathbb{E}^*(\kappa, \tau)$  into the complex space form  $\mathbb{M}^4(c)$  of constant holomorphic sectional curvature  $4c$ , for some  $c \in \mathbb{R}^*$  (see Proposition 4.2). Since every non-Einstein Sasaki manifold has a  $\text{Spin}^c$  structure with Killing spinors, it is natural to ask if this last result remains true for any 3-dimensional Sasaki manifold. Indeed, every simply connected non-Einstein Sasaki manifold can be immersed into  $\mathbb{M}^4(c)$  for some  $c \in \mathbb{R}^*$ , providing that the scalar curvature is constant (see Theorem 4.3). Finally, we make use of the existence of Killing spinors on  $\mathbb{E}^*(\kappa, \tau)$  to calculate some eigenvalues of Berger spheres endowed with different  $\text{Spin}^c$  structures. By restriction of this Killing spinor to any surface of  $\mathbb{E}^*(\kappa, \tau)$ , we give a  $\text{Spin}^c$  proof for the non-existence of totally umbilic surfaces in  $\mathbb{E}^*(\kappa, \tau)$  (see Theorem 4.4) proved already by R. Souam and E. Toubiana [32].

## 2 Preliminaries

In this section, we briefly introduce basic notions concerning the Dirac operator on  $\text{Spin}^c$  manifolds (with or without boundary) and their hypersurfaces. Details can be

found in [10], [26], [23], [15] and [5].

**The Dirac operator on  $\text{Spin}^c$  manifolds.** We consider an oriented Riemannian manifold  $(M^n, g)$  of dimension  $n$  with or without boundary and denote by  $\text{SOM}$  the  $\text{SO}_n$ -principal bundle over  $M$  of positively oriented orthonormal frames. A  $\text{Spin}^c$  structure of  $M$  is given by an  $\mathbb{S}^1$ -principal bundle  $(\mathbb{S}^1 M, \pi, M)$  of some Hermitian line bundle  $L$  and a  $\text{Spin}_n^c$ -principal bundle  $(\text{Spin}^c M, \pi, M)$  which is a 2-fold covering of the  $\text{SO}_n \times \mathbb{S}^1$ -principal bundle  $\text{SOM} \times_M \mathbb{S}^1 M$  compatible with the group covering

$$0 \longrightarrow \mathbb{Z}_2 \longrightarrow \text{Spin}_n^c = \text{Spin}_n \times_{\mathbb{Z}_2} \mathbb{S}^1 \longrightarrow \text{SO}_n \times \mathbb{S}^1 \longrightarrow 0.$$

The bundle  $L$  is called the auxiliary line bundle associated with the  $\text{Spin}^c$  structure. If  $A : T(\mathbb{S}^1 M) \longrightarrow i\mathbb{R}$  is a connection 1-form on  $\mathbb{S}^1 M$ , its (imaginary-valued) curvature will be denoted by  $F_A$ , whereas we shall define a real 2-form  $\Omega$  on  $\mathbb{S}^1 M$  by  $F_A = i\Omega$ . We know that  $\Omega$  can be viewed as a real valued 2-form on  $M$  [10, 21]. In this case,  $i\Omega$  is the curvature form of the auxiliary line bundle  $L$  [10, 21].

Let  $\Sigma M := \text{Spin}^c M \times_{\rho_n} \Sigma_n$  be the associated spinor bundle where  $\Sigma_n = \mathbb{C}^{2^{\lfloor \frac{n}{2} \rfloor}}$  and  $\rho_n : \text{Spin}_n^c \longrightarrow \text{End}(\Sigma_n)$  the complex spinor representation [10, 23, 29]. A section of  $\Sigma M$  will be called a spinor field. This complex vector bundle is naturally endowed with a Clifford multiplication, denoted by “ $\cdot$ ”,  $\cdot : \text{Cl}(TM) \longrightarrow \text{End}(\Sigma M)$  which is a fiber preserving algebra morphism and with a natural Hermitian scalar product  $\langle \cdot, \cdot \rangle$  compatible with this Clifford multiplication [26, 10, 15]. If  $n$  is even,  $\Sigma M = \Sigma^+ M \oplus \Sigma^- M$  can be decomposed into positive and negative spinors by the action of the complex volume element [10, 26, 15, 29]. If such data are given, one can canonically define a covariant derivative  $\nabla$  on  $\Sigma M$  given, for all  $X \in \Gamma(TM)$ , by [10, 23, 15, 29]:

$$\nabla_X \psi = X(\psi) + \frac{1}{4} \sum_{j=1}^n e_j \cdot \nabla_X e_j \cdot \psi + \frac{i}{2} A(s_*(X))\psi, \quad (3)$$

where the second  $\nabla$  is the Levi-Civita connection on  $M$ ,  $\psi = [\widetilde{b \times s}, \sigma]$  is a locally defined spinor field,  $b = (e_1, \dots, e_n)$  is a local oriented orthonormal tangent frame,  $s : U \longrightarrow \mathbb{S}^1 M$  is a local section of  $\mathbb{S}^1 M$ ,  $\widetilde{b \times s}$  is the lift of the local section  $b \times s : U \longrightarrow \text{SOM} \times_M \mathbb{S}^1 M$  to the 2-fold covering and  $X(\psi) = [\widetilde{b \times s}, X(\sigma)]$ . For any other connection  $A'$  on  $\mathbb{S}^1 M$ , there exists a real 1-form  $\alpha$  on  $M$  such that  $A' = A + i\alpha$  [10]. If we endow the  $\mathbb{S}^1$ -principal fiber bundle  $\mathbb{S}^1 M$  with the connection  $A'$ , there exists on  $\Sigma M$  a covariant derivative  $\nabla'$  given by

$$\nabla'_X \psi = \nabla_X \psi + \frac{i}{2} \alpha(X)\psi, \quad (4)$$

for all  $X \in \Gamma(TM)$  and  $\psi \in \Gamma(\Sigma M)$ . Moreover, the curvature 2-form of  $A'$  is given by  $F_{A'} = F_A + i d\alpha$ . But  $F_A$  (resp.  $F_{A'}$ ) can be viewed as an imaginary 2-form on

$M$  denoted by  $i\Omega$  (resp.  $i\Omega'$ ). Thus,  $i\Omega$  (resp.  $i\Omega'$ ) is the curvature of the auxiliary line bundle associated with the  $\mathbb{S}^1$ -principal fiber bundle  $\mathbb{S}^1M$  endowed with the connection  $A$  (resp.  $A'$ ) and we have  $i\Omega' = i\Omega + id\alpha$ .

The Dirac operator, acting on  $\Gamma(\Sigma M)$ , is a first order elliptic operator locally given by  $D = \sum_{j=1}^n e_j \cdot \nabla_{e_j}$ , where  $\{e_j\}_{j=1,\dots,n}$  is any orthonormal local basis tangent to  $M$ . An important tool when examining the Dirac operator on  $\text{Spin}^c$  manifolds is the Schrödinger-Lichnerowicz formula [10, 23]:

$$D^2 = \nabla^* \nabla + \frac{1}{4} S \text{Id}_{\Gamma(\Sigma M)} + \frac{i}{2} \Omega \cdot, \quad (5)$$

where  $S$  is the scalar curvature of  $M$ ,  $\nabla^*$  is the adjoint of  $\nabla$  with respect to the  $L^2$ -scalar product and  $\Omega \cdot$  is the extension of the Clifford multiplication to differential forms. The Ricci identity is given, for all  $X \in \Gamma(TM)$ , by

$$\sum_{j=1}^n e_j \cdot \mathcal{R}(e_j, X) \psi = \frac{1}{2} \text{Ric}(X) \cdot \psi - \frac{i}{2} (X \lrcorner \Omega) \cdot \psi, \quad (6)$$

for any spinor field  $\psi$ . Here  $\text{Ric}$  (resp.  $\mathcal{R}$ ) denotes the Ricci tensor of  $M$  (resp. the  $\text{Spin}^c$  curvature associated with the connection  $\nabla$ ) and  $\lrcorner$  the interior product.

A  $\text{Spin}$  structure can be seen as a  $\text{Spin}^c$  structure with trivial auxiliary line bundle  $L$  endowed with the trivial connection. Every almost complex manifold  $(M^{2m}, g, J)$  of complex dimension  $m$  has a canonical  $\text{Spin}^c$  structure. In fact, the complexified cotangent bundle  $T^*M \otimes \mathbb{C} = \Lambda^{1,0}M \oplus \Lambda^{0,1}M$  decomposes into the  $\pm i$ -eigenbundles of the complex linear extension of the complex structure. Thus, the spinor bundle of the canonical  $\text{Spin}^c$  structure is given by

$$\Sigma M = \Lambda^{0,*}M = \bigoplus_{r=0}^m \Lambda^{0,r}M,$$

where  $\Lambda^{0,r}M = \Lambda^r(\Lambda^{0,1}M)$  is the bundle of  $r$ -forms of type  $(0, 1)$ . The auxiliary line bundle of this canonical  $\text{Spin}^c$  structure is given by  $L = (K_M)^{-1} = \Lambda^m(\Lambda^{0,1}M)$ , where  $K_M$  is the canonical bundle of  $M$  [10, 26, 29]. Let  $\times$  be the Kähler form defined by the complex structure  $J$ , i.e.  $\times(X, Y) = g(X, JY)$  for all vector fields  $X, Y \in \Gamma(TM)$ . The auxiliary line bundle  $L = (K_M)^{-1}$  has a canonical holomorphic connection induced from the Levi-Civita connection whose curvature form is given by  $i\Omega = i\rho$ , where  $\rho$  is the Ricci 2-form given by  $\rho(X, Y) = \text{Ric}(X, JY)$ . For any other  $\text{Spin}^c$  structure the spinorial bundle can be written as [10, 20]:

$$\Sigma M = \Lambda^{0,*}M \otimes \mathcal{L},$$

where  $\mathcal{L}^2 = K_M \otimes L$  and  $L$  is the auxiliary bundle associated with this  $\text{Spin}^c$  structure. In this case, the 2-form  $\times$  can be considered as an endomorphism of  $\Sigma M$  via Clifford multiplication and we have the well-known orthogonal splitting  $\Sigma M = \bigoplus_{r=0}^m \Sigma_r M$ ,

where  $\Sigma_r M$  denotes the eigensubbundle corresponding to the eigenvalue  $i(m - 2r)$  of  $\kappa$ , with complex rank  $\binom{m}{k}$ . The bundle  $\Sigma_r M$  correspond to  $\Lambda^{0,r} M \otimes \mathcal{L}$ . For the canonical  $\text{Spin}^c$  structure, the subbundle  $\Sigma_0 M$  is trivial. Hence and when  $M$  is a Kähler manifold, this  $\text{Spin}^c$  structure admits parallel spinors (constant functions) lying in  $\Sigma_0 M$  [27]. Of course, we can define another  $\text{Spin}^c$  structure for which the spinor bundle is given by  $\Lambda^{*,0} M = \bigoplus_{r=0}^m \Lambda^r(T_{1,0}^* M)$  and the auxiliary line bundle by  $K_M$ . This  $\text{Spin}^c$  structure will be called the anti-canonical  $\text{Spin}^c$  structure.

**Spin<sup>c</sup> hypersurfaces and the Gauss formula.** Let  $(M^n, g)$  be an  $n$ -dimensional oriented hypersurface isometrically immersed in a Riemannian  $\text{Spin}^c$  manifold  $(\mathcal{Z}^{n+1}, g_{\mathcal{Z}})$ . The hypersurface  $M$  inherits a  $\text{Spin}^c$  structure from that on  $\mathcal{Z}$ , and we have [26, 5, 29, 28]:

$$\begin{cases} \Sigma \mathcal{Z}|_M \simeq \Sigma M & \text{if } n \text{ is even,} \\ \Sigma^+ \mathcal{Z}|_M \simeq \Sigma M & \text{if } n \text{ is odd.} \end{cases}$$

Moreover Clifford multiplication by a vector field  $X$ , tangent to  $M$ , is given by

$$X \bullet \varphi = (X \cdot \nu \cdot \psi)|_M,$$

where  $\psi \in \Gamma(\Sigma \mathcal{Z})$  (or  $\psi \in \Gamma(\Sigma^+ \mathcal{Z})$  if  $n$  is odd),  $\varphi$  is the restriction of  $\psi$  to  $M$ , “ $\cdot$ ” is the Clifford multiplication on  $\mathcal{Z}$ , “ $\bullet$ ” that on  $M$  and  $\nu$  is the unit normal vector. When  $n$  is odd, we can also get  $\Sigma^- \mathcal{Z}|_M \simeq \Sigma M$ . In this case, the Clifford multiplication by a vector field  $X$  tangent to  $M$  is given by  $X \bullet \varphi = -(X \cdot \nu \cdot \psi)|_M$  and we have  $\Sigma \mathcal{Z}|_M \simeq \Sigma M \oplus \Sigma M$ . The connection 1-form defined on the restricted  $\mathbb{S}^1$ -principal bundle  $(\mathbb{S}^1 M =: \mathbb{S}^1 \mathcal{Z}|_M, \pi, M)$ , is given by

$$A = A^{\mathcal{Z}}|_M : T(\mathbb{S}^1 M) = T(\mathbb{S}^1 \mathcal{Z})|_M \longrightarrow i\mathbb{R}.$$

Then the curvature 2-form  $i\Omega$  on the  $\mathbb{S}^1$ -principal bundle  $\mathbb{S}^1 M$  is given by  $i\Omega = i\Omega^{\mathcal{Z}}|_M$ , which can be viewed as an imaginary 2-form on  $M$  and hence as the curvature form of the line bundle  $L$ , the restriction of the line bundle  $L^{\mathcal{Z}}$  to  $M$ . We denote by  $\nabla^{\mathcal{Z}}$  the spinorial Levi-Civita connection on  $\Sigma \mathcal{Z}$  and by  $\nabla$  that on  $\Sigma M$ . For all  $X \in \Gamma(TM)$  and for every spinor field  $\psi \in \Gamma(\Sigma \mathcal{Z})$  (or  $\psi \in \Gamma(\Sigma^+ \mathcal{Z})$  if  $n$  is odd), we consider  $\varphi = \psi|_M$  and we get the following  $\text{Spin}^c$  Gauss formula [26, 5, 28]:

$$(\nabla_X^{\mathcal{Z}} \psi)|_M = \nabla_X \varphi + \frac{1}{2} II(X) \bullet \varphi, \quad (7)$$

where  $II$  denotes the Weingarten map with respect to  $\nu$ . Moreover, Let  $D^{\mathcal{Z}}$  and  $D$  be the Dirac operators on  $\mathcal{Z}$  and  $M$ , after denoting by the same symbol any spinor and it's restriction to  $M$ , we have

$$\tilde{D}\varphi = \frac{n}{2} H\varphi - \nu \cdot D^{\mathcal{Z}}\varphi - \nabla_{\nu}^{\mathcal{Z}}\varphi, \quad (8)$$

$$\tilde{D}(\nu \cdot \varphi) = -\nu \cdot \tilde{D}\varphi, \quad (9)$$

where  $H = \frac{1}{n}\text{tr}(II)$  denotes the mean curvature and  $\tilde{D} = D$  if  $n$  is even and  $\tilde{D} = D \oplus (-D)$  if  $n$  is odd.

**Homogeneous 3-dimensional manifolds with 4-dimensional isometry group.**

We denote a 3-dimensional homogeneous manifold with 4-dimensional isometry group by  $\mathbb{E}(\kappa, \tau)$ ,  $\kappa - 4\tau^2 \neq 0$ . It is a Riemannian fibration over a simply connected 2-dimensional manifold  $\mathbb{M}^2(\kappa)$  with constant curvature  $\kappa$  and such that the fibers are geodesic. We denote by  $\tau$  the bundle curvature, which measures the default of the fibration to be a Riemannian product. Precisely, we denote by  $\xi$  a unit vertical vector field, that is tangent to the fibers. If  $\tau \neq 0$ , the vector field  $\xi$  is a Killing field and satisfies for all vector field  $X$ ,

$$\nabla_X \xi = \tau X \wedge \xi,$$

where  $\nabla$  is the Levi-Civita connection and  $\wedge$  is the exterior product. In this case  $\mathbb{E}(\kappa, \tau)$  is denoted by  $\mathbb{E}^*(\kappa, \tau)$ . When  $\tau$  vanishes, we get a product manifold  $\mathbb{M}^2(\kappa) \times \mathbb{R}$ . If  $\tau \neq 0$ , these manifolds are of three types: They have the isometry group of the Berger spheres if  $\kappa > 0$ , of the Heisenberg group  $\text{Nil}_3$  if  $\kappa = 0$  or of  $\text{PSL}_2(\mathbb{R})$  if  $\kappa < 0$ . Note that if  $\tau = 0$ , then  $\xi = \frac{\partial}{\partial t}$  is the unit vector field giving the orientation of  $\mathbb{R}$  in the product  $\mathbb{M}^2(\kappa) \times \mathbb{R}$ . The manifold  $\mathbb{E}^*(\kappa, \tau)$  admits a local direct orthonormal frame  $\{e_1, e_2, e_3\}$  with  $e_3 = \xi$ , and such that the Christoffel symbols  $\Gamma_{ij}^k = \langle \nabla_{e_i} e_j, e_k \rangle$  are given by

$$\begin{cases} \Gamma_{12}^3 = \Gamma_{23}^1 = -\Gamma_{21}^3 = -\Gamma_{13}^2 = \tau, \\ \Gamma_{32}^1 = -\Gamma_{31}^2 = \tau - \frac{\kappa}{2\tau}, \\ \Gamma_{ii}^i = \Gamma_{ij}^i = \Gamma_{ji}^i = \Gamma_{ii}^j = 0, \quad \forall i, j \in \{1, 2, 3\}. \end{cases} \quad (10)$$

We call  $\{e_1, e_2, e_3 = \xi\}$  the canonical frame of  $\mathbb{E}^*(\kappa, \tau)$ . Except the Berger spheres and with  $\mathbb{R}^3$ ,  $\mathbb{H}^3$ ,  $\mathbb{S}^3$  and the solvable group  $\text{Sol}_3$ , the manifolds  $\mathbb{E}(\kappa, \tau)$  define the geometry of Thurston. The authors [30] proved that there exists on  $\mathbb{E}^*(\kappa, \tau)$  a  $\text{Spin}^c$  structure (the canonical  $\text{Spin}^c$  structure) carrying a Killing spinor field  $\psi$  of Killing constant  $\frac{\tau}{2}$ , i.e., a spinor field  $\psi$  satisfying

$$\nabla_X \psi = \frac{\tau}{2} X \cdot \psi,$$

for all  $X \in \Gamma(T\mathbb{E}^*(\kappa, \tau))$ . Moreover,  $\xi \cdot \psi = i\psi$  and the curvature of the auxiliary line bundle is given by

$$i\Omega(e_1, e_2) = i(\kappa - 4\tau^2) \quad \text{and} \quad i\Omega(e_k, e_j) = 0, \quad (11)$$

elsewhere in the canonical frame  $\{e_1, e_2, \xi\}$ . There exists also another  $\text{Spin}^c$  structure (the anti-canonical  $\text{Spin}^c$  structure) carrying a Killing spinor field  $\psi$  of Killing constant  $\frac{\tau}{2}$  such that  $\xi \cdot \psi = -i\psi$  and the curvature of the auxiliary line bundle is given by

$$i\Omega(e_1, e_2) = -i(\kappa - 4\tau^2) \quad \text{and} \quad i\Omega(e_k, e_j) = 0, \quad (12)$$

elsewhere in the canonical frame  $\{e_1, e_2, \xi\}$ .

### 3 Lower and upper bounds for the eigenvalues of the hypersurface Dirac operator

We will extend the lower bound (1) and the upper bound (2) to the eigenvalues of the hypersurface  $\text{Spin}^c$  Dirac operator  $\tilde{D}$ . Examples of the limiting cases are then given.

#### 3.1 Lower bounds for the eigenvalues of the hypersurface Dirac operator

We assume that the manifold  $\mathcal{Z}^{n+1}$  is a  $\text{Spin}^c$  manifold having a compact domain  $\mathbb{D}$  with compact boundary  $M = \partial\mathbb{D}$ . Using suitable boundary conditions for the Dirac operator  $D^{\mathcal{Z}}$ , we extend the lower bound (1) to the first positive eigenvalue of the extrinsic hypersurface Dirac operator  $\tilde{D}$  on  $M$  endowed with the induced  $\text{Spin}^c$  structure.

Since  $M$  is compact, the Dirac operator  $\tilde{D}$  has a discrete spectrum and we denote by  $\pi_+ : \Gamma(\Sigma M) \rightarrow \Gamma(\Sigma M)$  the projection onto the subspace of  $\Gamma(\Sigma M)$  spanned by eigenspinors corresponding to the nonnegative eigenvalues of  $\tilde{D}$ . This projection provides an Atiyah-Patodi-Singer type boundary conditions for the Dirac operator  $D^{\mathcal{Z}}$  of the domain  $\mathbb{D}$ . It has been proved that this is a global self-adjoint elliptic condition [17, 16].

It is not difficult to extend the Spin Reilly inequality (see [17], [16], [18], [19]) to  $\text{Spin}^c$  manifolds. Indeed, for all spinor fields  $\psi \in \Gamma(\Sigma\mathbb{D})$ , we have

$$\int_{\partial\mathbb{D}} \left( \langle \tilde{D}\varphi, \varphi \rangle - \frac{n}{2}H|\varphi|^2 \right) ds \geq \int_{\mathbb{D}} \left( \frac{1}{4}S^{\mathcal{Z}}|\psi|^2 + \langle \frac{i}{2}\Omega^{\mathcal{Z}} \cdot \psi, \psi \rangle - \frac{n}{n+1}|D^{\mathcal{Z}}\psi|^2 \right) dv, \quad (13)$$

where  $dv$  (resp.  $ds$ ) is the Riemannian volume form of  $\mathbb{D}$  (resp.  $\partial\mathbb{D}$ ). Moreover equality occurs if and only if the spinor field  $\psi$  is a twistor-spinor, i.e., if and only if it satisfies  $P^{\mathcal{Z}}\psi = 0$ , where  $P^{\mathcal{Z}}$  is the twistor operator acting on  $\Sigma\mathcal{Z}$  locally given, for all  $X \in \Gamma(T\mathcal{Z})$ , by  $P_X^{\mathcal{Z}}\psi = \nabla_X^{\mathcal{Z}}\psi + \frac{1}{n+1}X \cdot D^{\mathcal{Z}}\psi$ . Now, we can state the main theorem of this section:

**Theorem 3.1** *Let  $\mathcal{Z}^{n+1}$  be a Riemannian  $\text{Spin}^c$  manifold such that the operator  $S + 2i\Omega \cdot$  is nonnegative. We consider  $M^n$  a compact hypersurface with nonnegative mean curvature  $H$  and bounding a compact domain  $\mathbb{D}$  in  $\mathcal{Z}$ . Then, the first positive eigenvalue  $\lambda_1$  of  $\tilde{D}$  satisfies*

$$\lambda_1 \geq \frac{n}{2} \inf_M H. \quad (14)$$

*Equality holds if and only if  $H$  is constant and the eigenspace corresponding to  $\lambda_1$  consists of the restrictions to  $M$  of parallel spinors on the domain  $\mathbb{D}$ .*

**Proof.** Let  $\varphi$  is an eigenspinor on  $M$  corresponding to the first positive eigenvalue  $\lambda_1 > 0$  of  $\tilde{D}$ , i.e.,  $\tilde{D}\varphi = \lambda_1\varphi$  and  $\pi_+\varphi = \varphi$ . The following boundary problem has a unique solution (see [17], [16], [18] and [19])

$$\begin{cases} D^{\mathcal{Z}}\psi = 0 & \text{on } \mathbb{D} \\ \pi_+\psi = \pi_+\varphi = \varphi & \text{on } M = \partial\mathbb{D}. \end{cases}$$

From the Reilly inequality (13), we get

$$\int_M (\lambda_1 - \frac{n}{2}H)|\psi|^2 ds \geq \int_{\mathbb{D}} (\frac{1}{4}S^{\mathcal{Z}}|\psi|^2 + \frac{i}{2} \langle \Omega^{\mathcal{Z}} \cdot \psi, \psi \rangle) dv \geq 0,$$

which implies (14). If the equality case holds in (14), then  $\psi$  is a harmonic spinor and a twistor spinor, hence parallel. Since  $\pi_+\psi = \varphi$  along the boundary,  $\psi$  is a non-trivial parallel spinor and  $\lambda_1 = \frac{n}{2}H$ . Furthermore, since  $\psi$  is parallel, we deduce by (8) that  $\tilde{D}\varphi = \frac{n}{2}H\varphi$ . Hence we have  $\varphi = \pi_+\psi = \psi$ . Conversely if  $H$  is constant, the fact that the restriction to  $M$  of a parallel spinor on  $\mathbb{D}$  is an eigenspinor with eigenvalue  $\frac{n}{2}H$  is a direct consequence of (8).

**Examples 3.1** *A complete simply connected Riemannian  $\text{Spin}^c$  manifold  $\mathcal{Z}^{n+1}$  carrying a parallel spinor field is isometric to the Riemannian product of a simply connected Kähler manifold  $\mathcal{Z}_1^{n_1}$  of complex dimension  $m_1$  ( $n_1 = 2m_1$ ) and a simply connected  $\text{Spin}$  manifold  $\mathcal{Z}_2^{n_2}$  of dimension  $n_2$  ( $n + 1 = n_1 + n_2$ ) carrying a parallel spinor and the  $\text{Spin}^c$  structure of  $\mathcal{Z}$  is the product of the canonical  $\text{Spin}^c$  structure of  $\mathcal{Z}_1$  and the  $\text{Spin}$  structure of  $\mathcal{Z}_2$  [27]. Moreover, if we assume that  $\mathcal{Z}_1$  is Einstein, then*

$$i\Omega^{\mathcal{Z}}(X, Y) = i\rho^{\mathcal{Z}_1}(X_1, Y_1) = i\text{Ric}^{\mathcal{Z}_1}(X_1, JY_1) = i\frac{S^{\mathcal{Z}_1}}{n_1} \times (X_1, Y_1), \quad (15)$$

*for every  $X = X_1 + X_2, Y = Y_1 + Y_2 \in \Gamma(T\mathcal{Z})$  and where  $J$  denotes the complex structure on  $\mathcal{Z}_1$ . Moreover, if the Einstein manifold  $\mathcal{Z}_1$  is of positive scalar curvature, we have, for any spinor field  $\psi \in \Gamma(\Sigma\mathcal{Z})$ ,*

$$\begin{aligned} S^{\mathcal{Z}}|\psi|^2 + 2i \langle \Omega \cdot \psi, \psi \rangle &= S^{\mathcal{Z}_1}|\psi|^2 + \frac{i}{m_1}S^{\mathcal{Z}_1} \langle \times \cdot \psi, \psi \rangle \\ &= S^{\mathcal{Z}_1} \sum_{r=0}^{m_1} (1 - \frac{m_1 - 2r}{m_1}) |\psi_r|^2 = S^{\mathcal{Z}_1} \sum_{r=0}^{m_1} \frac{2r}{m_1} |\psi_r|^2 \geq 0. \end{aligned}$$

Finally, the first positive eigenvalue of the Dirac operator  $\tilde{D}$  of any compact hypersurface with nonnegative constant mean curvature  $H$  and bounding a compact domain  $\mathbb{D}$  in  $\mathcal{Z} = \mathcal{Z}_1 \times \mathcal{Z}_2$  satisfies the equality case in (14) for the restricted  $\text{Spin}^c$  structure. Next, we will give some explicit examples. The Alexandrov theorem for  $\mathbb{S}_+^2 \times \mathbb{R}$  says that the only embedded compact surface with constant mean curvature  $H > 0$  in  $\mathcal{Z} = \mathcal{Z}_1 \times \mathcal{Z}_2 = \mathbb{S}_+^2 \times \mathbb{R}$  is the standard rotational sphere described in [1, 2, 8]. Hence, the first positive eigenvalue of the Dirac operator  $\tilde{D}$  on the rotational sphere satisfies the equality case in (14). We consider the complex projective space  $\mathbb{C}P^m$  ( $\mathcal{Z}_2 = \{\emptyset\}$ ) endowed with the Einstein Fubini-Study metric and the canonical  $\text{Spin}^c$  structure. The first positive eigenvalue of the Dirac operator  $\tilde{D}$  of any compact hypersurface  $M$  with nonnegative constant mean curvature  $H$  and bounding a compact domain  $\mathbb{D}$  in  $\mathbb{C}P^m$  satisfies the equality case in (14). Compact embedded hypersurfaces in  $\mathbb{C}P^m$  are examples of manifolds viewed as a boundary of some enclosed domain in  $\mathbb{C}P^m$ . As an example, we know that there exists an isometric embedding of  $\mathbb{E}^*(\kappa, \tau)$  into  $\mathbb{M}^4(\frac{\kappa}{4} - \tau^2)$  of constant mean curvature  $H = \frac{\kappa - 16\tau^2}{12\tau}$  [34]. Here  $\mathbb{M}^4(\frac{\kappa}{4} - \tau^2)$  denotes the complex space form of constant holomorphic sectional curvature  $\kappa - 4\tau^2$ . We choose  $\kappa > 16\tau^2$  and  $\tau > 0$ , then  $H$  is positive. In this case,  $\mathbb{E}^*(\kappa, \tau)$  are Berger spheres (compact) and  $\mathbb{M}^4$  is the complex projective space  $\mathbb{C}P^2$  of constant holomorphic sectional curvature  $\kappa - 4\tau^2 > 0$ . The canonical  $\text{Spin}^c$  structure on  $\mathbb{M}^4$  carries a parallel spinor and hence the equality case in (14) is satisfied for the first positive eigenvalue of  $\tilde{D}$  defined on Berger spheres. Finally, we recall that  $\mathbb{S}_+^2 \times \mathbb{R}$  and  $\mathbb{C}P^m$  (when  $m$  is odd), have also a unique  $\text{Spin}$  structure. Hence, Inequality (14) holds for the first positive eigenvalue of the  $\text{Spin}$  Dirac operator  $\tilde{D}$  defined on the rotational sphere or on any compact embedded hypersurface in  $\mathbb{C}P^m$  (when  $m$  is odd). But, equality cannot occur since this unique  $\text{Spin}$  structure on  $\mathbb{S}_+^2 \times \mathbb{R}$  and on  $\mathbb{C}P^m$  does not carry a parallel spinor.

### 3.2 Upper bounds for the eigenvalues of the Dirac operator

A spinor field  $\psi$  on a Riemannian  $\text{Spin}^c$  manifold  $\mathcal{Z}^{n+1}$  is called a real Killing spinor with Killing constant  $\alpha \in \mathbb{R}$  if

$$\nabla_X^{\mathcal{Z}} \psi = \alpha X \cdot \psi, \quad (16)$$

for all  $X \in \Gamma(T\mathcal{Z})$ . When  $\alpha = 0$ , the spinor field  $\psi$  is a parallel spinor. We define

$$\mu = \mu(\mathcal{Z}, \alpha) := \dim_{\mathbb{C}}\{\psi, \psi \text{ is a Killing spinor on } \mathcal{Z} \text{ with Killing constant } \alpha\}$$

**Theorem 3.2** *Let  $M$  be an  $n$ -dimensional closed oriented hypersurface isometrically immersed in a Riemannian  $\text{Spin}^c$  manifold  $\mathcal{Z}$ . We endow  $M$  with the induced  $\text{Spin}^c$  structure. For any  $\alpha \in \mathbb{R}$ , there are at least  $\mu(\mathcal{Z}, \alpha)$  eigenvalues  $\lambda_1, \dots, \lambda_\mu$  of the Dirac operator  $\tilde{D}$  on  $M$  satisfying*

$$\lambda_j^2 \leq n^2 \alpha^2 + \frac{n^2}{4\text{vol}(M)} \int_M H^2 dv, \quad (17)$$

where  $H$  denotes the mean curvature of  $M$ . If equality holds, then  $H$  is constant.

**Proof.** First, note that the set of Killing spinors with Killing constant  $\alpha$  is a vector space. Moreover, linearly independent Killing spinors are linearly independent at every point, the space of restrictions of Killing spinors on  $\mathcal{Z}$  to  $M$ , i.e.,

$$\{\psi|_M, \psi \text{ is a spinor on } \mathcal{Z} \text{ satisfying } \nabla_X^{\mathcal{Z}}\psi = \alpha X \cdot \psi, \quad \forall X \in \Gamma(T\mathcal{Z})\}$$

is also  $\mu$ -dimensional. Now let  $\psi$  be a Killing spinor on  $\mathcal{Z}$  with Killing constant  $\alpha \in \mathbb{R}$ . Killing spinors have constant length so we can assume that  $|\psi| \equiv 1$ . By definition, we have  $D^{\mathcal{Z}}\psi = -(n+1)\alpha\psi$ , and hence using (8) we get  $\tilde{D}\varphi = n\alpha\nu \cdot \varphi + \frac{n}{2}H\varphi$ . We denote by  $(\cdot, \cdot) = \operatorname{Re} \int_M \langle \cdot, \cdot \rangle$  the real part of the  $L^2$ -scalar product. Now, we compute the Rayleigh quotient of  $\tilde{D}^2$

$$\frac{(\tilde{D}^2\psi, \psi)}{(\psi, \psi)} = \frac{(\tilde{D}\psi, \tilde{D}\psi)}{\operatorname{vol}(M)} = \frac{(n\alpha\nu \cdot \psi + \frac{n}{2}H\psi, n\alpha\nu \cdot \psi + \frac{n}{2}H\psi)}{\operatorname{vol}(M)} = n^2\alpha^2 + \frac{n^2}{4} \frac{\int_M H^2}{\operatorname{vol}(M)},$$

i.e., the Rayleigh quotient of  $\tilde{D}^2$  is bounded by  $n^2\alpha^2 + \frac{n^2}{4} \frac{\int_M H^2}{\operatorname{vol}(M)}$  on a  $\mu$ -dimensional space of spinors on  $M$ . Hence, the Min-Max principle implies the assertion. If equality holds, then the restriction to  $M$  of every Killing spinor  $\psi$  of Killing constant  $\alpha$  satisfies  $\tilde{D}^2\varphi = \lambda_1^2\varphi$ . But, it is known that [11]

$$\tilde{D}^2\varphi = \hat{D}^2\varphi + \frac{n}{2}dH \cdot \nu \cdot \varphi + \frac{n^2 H^2}{4}\varphi, \quad (18)$$

where  $\hat{D}$  is the Dirac-Witten operator given by  $\hat{D} = \sum_{j=1}^n e_j \cdot \nabla_{e_j}^{\mathcal{Z}}$ . Hence, using that  $\hat{D}\psi = -n\alpha\psi$  and (18), we get

$$\lambda_1^2\varphi = n^2\alpha^2\varphi + \frac{n}{2}dH \cdot \nu \cdot \varphi + \frac{n^2}{4}H^2\varphi.$$

Considering the real part of the scalar product of the last equality by  $\varphi$  implies that  $\lambda_1^2 = n^2\alpha^2 + \frac{n^2 H^2}{4}$ . Hence,  $H$  is constant.

**Examples 3.2** *Simply connected complete Riemannian  $\operatorname{Spin}^c$  manifolds carrying parallel spinors were described in Examples 3.1. The only  $\operatorname{Spin}^c$  structures on an irreducible Kähler not Ricci-flat manifold  $\mathcal{Z}$  which carry parallel spinors are the canonical and the anti-canonical one. In both cases,  $\mu(\mathcal{Z}, 0) = 1$  [27]. Hence, Inequality (17) holds for the first eigenvalue of the Dirac operator  $\tilde{D}$  defined on any compact Riemannian hypersurface endowed with the restricted  $\operatorname{Spin}^c$  structure. The complex projective space  $\mathbb{C}P^m$  or the complex hyperbolic space  $\mathbb{C}H^m$  with the Fubini-Study metric are examples of irreducible Kähler not Ricci-flat manifolds. By (8), the first eigenvalue of the Dirac operator  $\tilde{D}$  defined on any CMC (constant mean curvature) compact Riemannian hypersurface endowed with the restricted  $\operatorname{Spin}^c$*

structure, satisfies the equality case in (17). Berger spheres are examples of CMC compact hypersurfaces of  $\mathbb{C}P^2$  or  $\mathbb{C}H^2$  (see Examples 3.1 and Section 4). By (8), the first eigenvalue  $\lambda_1$  of the Dirac operator  $\tilde{D} = D^M$  of any compact Riemannian hypersurface of  $\mathbb{S}^2 \times \mathbb{R}$  or  $\mathbb{H}^2 \times \mathbb{R}$  satisfies the equality case in (17) if we assume that  $H$  is constant. Rotational spheres are CMC compact Riemannian hypersurfaces of  $\mathbb{S}^2 \times \mathbb{R}$  and  $\mathbb{H}^2 \times \mathbb{R}$ . For  $\mathbb{H}^2 \times \mathbb{R}$  and if  $H > \frac{1}{2}$ , rotational constant mean curvature  $H$  spheres are described in [1, 2, 8].

The only complete simply connected  $\text{Spin}^c$  manifolds admitting real Killing spinors other than the  $\text{Spin}$  manifolds are the non-Einstein Sasakian manifolds endowed with their canonical or anti-canonical  $\text{Spin}^c$  structure [27]. The manifolds  $\mathbb{E}^*(\kappa, \tau)$  are  $\text{Spin}^c$  manifolds carrying a Killing spinor  $\psi$  of Killing constant  $\frac{\tau}{2}$ . By (8) and (9), the restriction of  $\psi$  to any compact hypersurface gives a spinor field  $\varphi$  satisfying  $\tilde{D}\varphi = H\varphi + \tau\nu \cdot \varphi$ . If  $H$  is constant,  $(\tilde{D})^2\varphi = (H^2 + \tau^2)\varphi$ . Then, the equality case in (17) is satisfied. Rotational spheres are examples of compact Riemannian hypersurfaces of constant mean curvature and immersed into  $\mathbb{E}^*(\kappa, \tau)$  [1, 2, 8].

## 4 $\text{Spin}^c$ structures on $\mathbb{E}^*(\kappa, \tau)$ and applications

In this section, we make use of the existence of a  $\text{Spin}^c$  Killing spinor to immerse  $\mathbb{E}^*(\kappa, \tau)$  into complex space forms, to calculate some eigenvalues of the Dirac operator on Berger spheres and to prove the non-existence of totally umbilic surfaces in  $\mathbb{E}^*(\kappa, \tau)$ .

### 4.1 Isometric immersions of $\mathbb{E}^*(\kappa, \tau)$ into complex space forms

From the existence of an isometric embedding of  $\mathbb{E}^*(\kappa, \tau)$  into  $\mathbb{M}^4(\frac{\kappa}{4} - \tau^2)$ , we reprove that the only  $\text{Spin}^c$  structures on  $\mathbb{E}^*(\kappa, \tau)$  carrying a Killing spinor are the canonical and the anti-canonical one. Conversely, the existence of a  $\text{Spin}^c$  Killing spinor allows to immerse  $\mathbb{E}^*(\kappa, \tau)$  in  $\mathbb{M}^4(\frac{\kappa}{4} - \tau^2)$ . More generally, we give necessary and sufficient geometric conditions to immerse any 3-dimensional Sasaki manifold into  $\mathbb{M}^2(c)$  for some  $c \in \mathbb{R}^*$ .

**Proposition 4.1** *The only  $\text{Spin}^c$  structures on  $\mathbb{E}^*(\kappa, \tau)$  carrying a real Killing spinor are the canonical and the anti-canonical one. Moreover, the Killing constant is given by  $\frac{\tau}{2}$ .*

**Proof.** It is known that there exists an isometric embedding of  $\mathbb{E}^*(\kappa, \tau)$  into  $\mathbb{M}^4(\frac{\kappa}{4} - \tau^2)$  of constant mean curvature  $H = \frac{\kappa - 16\tau^2}{12\tau}$  [34]. Moreover, the second fundamental form

is given by

$$II(X) = -\tau X - \frac{4\tau^2 - \kappa}{\tau} g_{\mathbb{M}^4}(X, \xi)\xi,$$

for every  $X \in \Gamma(T\mathbb{E}^*(\kappa, \tau))$ . Here, we recall that the normal vector of the immersion is given by  $\nu := J\xi$  and  $\{e_1, e_2, \xi, \nu = J\xi\}$  is a local orthonormal basis tangent to  $\mathbb{M}^4$  where  $\{e_1, e_2, \xi\}$  is the canonical frame of  $\mathbb{E}^*(\kappa, \tau)$ . We denote by  $\eta$  the real 1-form associated with  $\xi$ , i.e.,  $\eta(X) = g(X, \xi)$  for any  $X \in \Gamma(T\mathbb{E}^*(\kappa, \tau))$ . The restriction of the canonical  $\text{Spin}^c$  structure on  $\mathbb{M}^4(\frac{\kappa}{4} - \tau^2)$  induces a  $\text{Spin}^c$  structure on  $\mathbb{E}^*(\kappa, \tau)$  and by the  $\text{Spin}^c$  Gauss formula (7), the restriction of the parallel spinor on  $\mathbb{M}^4(\frac{\kappa}{4} - \tau^2)$  induces a spinor field  $\varphi$  on  $\mathbb{E}^*(\kappa, \tau)$  satisfying, for all  $X \in \Gamma(T\mathbb{E}^*(\kappa, \tau))$ ,

$$\nabla_X \varphi = \frac{\tau}{2} X \bullet \varphi + \frac{4\tau^2 - \kappa}{8\tau} \eta(X) \xi \bullet \varphi.$$

Moreover, the spinor field  $\varphi$  satisfies  $\xi \bullet \varphi = -i\varphi$  [30, Theorem 3] and the curvature of the auxiliary line bundle  $L$  associated with the induced  $\text{Spin}^c$  structure is given by [30, Theorem 3]

$$i\Omega(e_1, e_2) = -6i\left(\frac{\kappa}{4} - \tau^2\right), \quad \text{and} \quad i\Omega(e_i, e_j) = 0, \quad (19)$$

elsewhere in the basis  $\{e_1, e_2, \xi\}$ . We deduce that, for all  $X \in \Gamma(T\mathbb{E}^*(\kappa, \tau))$ ,

$$\nabla_X \varphi = \frac{\tau}{2} X \bullet \varphi - i \frac{4\tau^2 - \kappa}{8\tau} g(X, \xi) \varphi.$$

The connection  $A$  on the  $\mathbb{S}^1$ -principal fiber bundle  $\mathbb{S}^1(\mathbb{E}^*(\kappa, \tau))$  associated with the induced  $\text{Spin}^c$  structure is the restriction to  $\mathbb{E}^*(\kappa, \tau)$  of the connection on the  $\mathbb{S}^1$ -principal fiber bundle  $\mathbb{S}^1\mathbb{M}^4$  associated with the canonical  $\text{Spin}^c$  structure on  $\mathbb{M}^4(\frac{\kappa}{4} - \tau^2)$ , i.e., the connection  $A$  on  $\mathbb{S}^1(\mathbb{E}^*(\kappa, \tau))$  is the restriction to  $\mathbb{E}^*(\kappa, \tau)$  of the connection on  $\mathbb{S}^1(\mathbb{M}^4(\frac{\kappa}{4} - \tau^2))$  defined by the Levi-Civita connection. Let  $\alpha$  be the real 1-form on  $\mathbb{E}^*(\kappa, \tau)$  defined by

$$\alpha(X) = \frac{4\tau^2 - \kappa}{4\tau} g(X, \xi),$$

for any  $X \in \Gamma(T\mathbb{E}^*(\kappa, \tau))$ . We endow the  $\mathbb{S}^1$ -principal fiber bundle  $\mathbb{S}^1(\mathbb{E}^*(\kappa, \tau))$  with the connection  $A' = A + i\alpha$ . From (4), there exists on  $\Sigma\mathbb{E}^*(\kappa, \tau)$  a covariant derivative  $\nabla'$  such that

$$\nabla'_X \varphi = \nabla_X \varphi + \frac{i}{2} \alpha(X) \varphi = \frac{\tau}{2} X \bullet \varphi,$$

for all  $X \in \Gamma(T\mathbb{E}^*(\kappa, \tau))$ . Hence, we obtain a  $\text{Spin}^c$  structure on  $\mathbb{E}^*(\kappa, \tau)$  carrying a Killing spinor field and whose  $\mathbb{S}^1$ -principal fiber bundle  $\mathbb{S}^1(\mathbb{E}^*(\kappa, \tau))$  has a connection given by  $A'$ . Now, we should prove that this  $\text{Spin}^c$  structure is the canonical one or the anti-canonical one. First, we calculate the curvature  $i\Omega' = i\Omega + i d\alpha$  of  $A'$ . It is easy to check that  $\xi \lrcorner d\alpha = 0$  and  $d\alpha(e_1, e_2) = -\frac{4\tau^2 - \kappa}{2}$ . Hence, using (19), we get

$$\Omega'(e_1, e_2) = -(\kappa - 4\tau^2) \quad \text{and} \quad \xi \lrcorner \Omega' = 0.$$

The curvature form  $i\Omega'$  is the same as the curvature form associated with the connection on the auxiliary line bundle of the anti-canonical  $\text{Spin}^c$  structure on  $\mathbb{E}^*(\kappa, \tau)$ . Since  $\mathbb{E}^*(\kappa, \tau)$  is a simply connected manifold, we deduce that the  $\mathbb{S}^1$ -principal fiber bundle  $\mathbb{S}^1(\mathbb{E}^*(\kappa, \tau))$  endowed with the connection  $A'$  is the auxiliary line bundle of the anti-canonical  $\text{Spin}^c$  structure on  $\mathbb{E}^*(\kappa, \tau)$ . Hence, we have on  $\mathbb{E}^*(\kappa, \tau)$  two  $\text{Spin}^c$  structures with the same auxiliary line bundle (the anti-canonical one and the one obtained by restriction of the canonical one on  $\mathbb{M}^4$ ). But, on a Riemannian manifold  $M$ ,  $\text{Spin}^c$  structures having the same auxiliary line bundle are parametrized by  $H^1(M, \mathbb{Z}_2)$  [26], which is trivial in our case since  $\mathbb{E}^*(\kappa, \tau)$  is simply connected. To get the canonical  $\text{Spin}^c$  structure on  $\mathbb{E}^*(\kappa, \tau)$ , we restrict the anti-canonical  $\text{Spin}^c$  structure on  $\mathbb{M}^4$ . In this case,  $\xi \bullet \varphi = i\varphi$ ,  $\Omega(e_1, e_2) = 6(\frac{\kappa}{4} - \tau^2)$ ,  $\xi \lrcorner \Omega = 0$  and we choose the real 1-form  $\alpha$  to be  $\alpha(X) = -\frac{4\tau^2 - \kappa}{4\tau}g(X, \xi)$ .

Next, we want to prove the converse. Indeed, we have

**Proposition 4.2** *The manifolds  $\mathbb{E}^*(\kappa, \tau)$  are isometrically immersed into  $\mathbb{M}^4(c)$  for some  $c$ . Moreover,  $\mathbb{E}^*(\kappa, \tau)$  are of constant mean curvature and  $\eta$ -umbilic.*

**Proof.** We recall that the 3-dimensional homogeneous manifolds  $\mathbb{E}^*(\kappa, \tau)$  have a  $\text{Spin}^c$  structure (the anti-canonical  $\text{Spin}^c$  structure) carrying a Killing spinor field  $\varphi$  of Killing constant  $\frac{\tau}{2}$ . Moreover  $\xi \bullet \varphi = -i\varphi$  and

$$\Omega(e_1, e_2) = -(\kappa - 4\tau^2) \text{ and } \Omega(e_i, e_j) = 0, \quad (20)$$

in the basis  $\{e_1, e_2, e_3 = \xi\}$ . We denote by  $A$  the connection on the auxiliary line bundle defining the anti-canonical  $\text{Spin}^c$  structure. Let  $\alpha$  be the real 1-form on  $\mathbb{E}^*(\kappa, \tau)$  defined by  $\alpha(X) = -\frac{4\tau^2 - \kappa}{4\tau}g(X, \xi)$ , for any  $X \in \Gamma(T\mathbb{E}^*(\kappa, \tau))$ . We endow the  $\mathbb{S}^1$ -principal fiber bundle  $\mathbb{S}^1(\mathbb{E}^*(\kappa, \tau))$  with the connection  $A' = A + i\alpha$ . Then, there exists on  $\Sigma\mathbb{E}^*(\kappa, \tau)$  a covariant derivative  $\nabla'$  such that

$$\begin{aligned} \nabla'_X \varphi &= \frac{\tau}{2}X \bullet \varphi + \frac{i}{2}\alpha(X)\varphi \\ &= \frac{\tau}{2}X \bullet \varphi + \frac{4\tau^2 - \kappa}{8\tau}\eta(X)\xi \bullet \varphi, \end{aligned} \quad (21)$$

for all  $X \in \Gamma(T\mathbb{E}^*(\kappa, \tau))$ . Hence, we obtain a  $\text{Spin}^c$  structure on  $\mathbb{E}^*(\kappa, \tau)$  carrying a spinor field  $\varphi$  satisfying (21) and whose  $\mathbb{S}^1$ -principal fiber bundle  $\mathbb{S}^1(\mathbb{E}^*(\kappa, \tau))$  has a connection given by  $A'$ . We calculate the curvature  $i\Omega' = i\Omega + id\alpha$  of  $A'$ . It is easy to check that  $\xi \lrcorner d\alpha = 0$  and  $d\alpha(e_1, e_2) = \frac{4\tau^2 - \kappa}{2}$ . Hence,

$$\Omega'(e_1, e_2) = -6(\frac{\kappa}{4} - \tau^2) \text{ and } \xi \lrcorner \Omega' = 0.$$

Since  $\mathbb{E}^*(\kappa, \tau)$  are Sasakian, by [30, Theorem 4], we get an isometric immersion of  $\mathbb{E}^*(\kappa, \tau)$  into  $\mathbb{M}^4(c)$  for  $c = \frac{\kappa}{4} - \tau^2$ . Moreover,  $\mathbb{E}^*(\kappa, \tau)$  are of constant mean

curvature and  $\eta$ -umbilic (see [30]).

More general, we have:

**Theorem 4.3** *Every simply connected non-Einstein 3-dimensional Sasaki manifold  $M^3$  of constant scalar curvature can be immersed into  $\mathbb{M}^4(c)$  for some  $c \in \mathbb{R}^*$ . Moreover,  $M$  is  $\eta$ -umbilic and of constant mean curvature.*

**Proof.** We recall that a Sasaki structure on a 3-dimensional manifold  $M^3$  is given by a Killing vector field  $\xi$  of unit length such that the tensors  $\mathfrak{X} := \nabla\xi$  and  $\eta := g(\xi, \cdot)$  are related by

$$\mathfrak{X}^2 = -\text{Id} + \eta \otimes \xi.$$

We know that a non-Einstein Sasaki manifold has a  $\text{Spin}^c$  structure carrying a Killing spinor field  $\varphi$  of Killing constant  $\beta$ . By rescaling the metric, we can assume that  $\beta = -\frac{1}{2}$ . Because the dimension of  $M$  is 3, the Killing vector field  $\xi$  defining the Sasaki structure satisfies  $\xi \bullet \varphi = -i\varphi$  (see [14, Theorem 2.5]). The Ricci tensor on  $M$  is given by

$$\text{Ric}(e_j) = \frac{S-2}{2}e_j, j = 1, 2 \quad \text{and} \quad \text{Ric}(\xi) = 2\xi,$$

where  $\{e_1, e_2, \xi\}$  denotes a local orthonormal frame of  $M$ . Because we assumed that  $M$  is non-Einstein, we have  $S \neq 6$  and hence we can find  $c \in \mathbb{R}^*$  such that  $S = 8c + 6$ . The Ricci identity (6) in  $X = \xi$  gives that  $\xi \lrcorner \Omega = 0$  and by the Schrödinger-Lichnerowicz formula, it follows that  $\Omega(e_1, e_2) = \frac{6-S}{2}$ . Let  $\alpha$  be the real 1-form on  $M$  defined by  $\alpha(X) = -cg(X, \xi)$ , for any  $X \in \Gamma(TM)$ . We endow the  $\mathbb{S}^1$ -principal fiber bundle  $\mathbb{S}^1M$  with the connection  $A' = A + i\alpha$ , where  $A$  denotes the connection on  $\mathbb{S}^1M$  whose curvature form is given by  $i\Omega$ . From (4), there exists on  $\Sigma M$  a covariant derivative  $\nabla'$  such that

$$\nabla'_X \varphi = -\frac{1}{2}X \bullet \varphi - \frac{i}{2}cg(X, \xi)\varphi.$$

Now, we calculate the curvature  $i\Omega' = i\Omega + id\alpha$  of  $A'$ . It is easy to check that  $\xi \lrcorner d\alpha = 0$  and  $d\alpha(e_1, e_2) = -2c$ . Hence,

$$\xi \lrcorner \Omega' = 0, \quad \Omega'(e_1, e_2) = \frac{6-S}{2} - 2c = -6c.$$

By [30, Theorem 4],  $M$  is immersed into  $\mathbb{M}^4(c)$ . Additionally,  $M$  is  $\eta$ -umbilic and of constant mean curvature.

## 4.2 Totally umbilic surfaces in $\mathbb{E}^*(\kappa, \tau)$

By restriction of the Killing spinor of Killing constant  $\frac{\tau}{2}$  on  $\mathbb{E}^*(\kappa, \tau)$  to a surface  $M^2$ , the authors characterized isometric immersions into  $\mathbb{E}(\kappa, \tau)$  by the existence of a  $\text{Spin}^c$

structure carrying a special spinor field [30]. More precisely, consider  $(M^2, g)$  a Riemannian surface. We denote by  $E$  a field of symmetric endomorphisms of  $TM$ , with trace equal to  $2H$ . The vertical vector field can be written as  $\xi = dF(T) + f\nu$ , where  $\nu$  is the unit normal vector to the surface,  $f$  is a real function on  $M$  and  $T$  the tangential part of  $\xi$ . The isometric immersion of  $(M^2, g)$  into  $\mathbb{E}(\kappa, \tau)$  with shape operator  $E$ , mean curvature  $H$  is characterized by a  $\text{Spin}^c$  structure on  $M$  carrying a non-trivial spinor field  $\varphi$  satisfying, for all  $X \in \Gamma(TM)$ ,

$$\nabla_X \varphi = -\frac{1}{2}EX \bullet \varphi + i\frac{\tau}{2}X \bullet \bar{\varphi}.$$

Moreover, the auxiliary bundle has a connection of curvature given, in any local orthonormal frame  $\{t_1, t_2\}$ , by  $i\Omega(t_1, t_2) = -i(\kappa - 4\tau^2)f = -i(\kappa - 4\tau^2)\frac{\langle \varphi, \bar{\varphi} \rangle}{|\varphi|^2}$ . The vector  $T$  is given by

$$g(T, t_1) = \langle it_2 \bullet \varphi, \frac{\varphi}{|\varphi|^2} \rangle \quad \text{and} \quad g(T, t_2) = -\langle it_1 \bullet \varphi, \frac{\varphi}{|\varphi|^2} \rangle.$$

Here and also by restriction of the Killing spinor, we gave an elementary  $\text{Spin}^c$  proof of the following result proved by R. Souam and E. Toubiana in [32].

**Theorem 4.4** *There are no totally umbilic surfaces in  $\mathbb{E}^*(\kappa, \tau)$ .*

**Proof.** Assume that  $M$  is a totally umbilical surface in  $\mathbb{E}^*(\kappa, \tau)$ , i.e.  $E = H \text{Id}$ . Then  $d^\nabla E(e_1, e_2) = (\nabla_{t_1} E)t_2 - (\nabla_{t_2} E)t_1 = J(dH)$ . The  $\text{Spin}^c$  curvature  $\mathcal{R}$  on the spinor field  $\varphi$  is given by [30]:

$$\mathcal{R}(t_1, t_2)\varphi = -\frac{1}{2}J(dH) \bullet \varphi + i\frac{H^2}{2}\bar{\varphi} + i\frac{\tau^2}{2}\bar{\varphi}.$$

The  $\text{Spin}^c$  Ricci identity (6) on the surface  $M$  implies

$$t_1 \bullet \mathcal{R}(t_1, t_2)\varphi = \frac{1}{2}\text{Ric}(t_2) \bullet \varphi - \frac{i}{2}(t_2 \lrcorner \Omega) \bullet \varphi$$

Hence,

$$-\frac{1}{2}t_1 \bullet J(dH) \bullet \varphi + \frac{i}{2}H^2 t_1 \bullet \bar{\varphi} + \frac{i}{2}\tau^2 t_1 \bullet \bar{\varphi} = \frac{1}{2}\text{Ric}(t_2) \bullet \varphi + \frac{i}{2}\Omega(t_1, t_2)t_1 \bullet \varphi$$

Consider the real part of the scalar product of the last identity by  $\varphi$ , we get

$$g(t_1, J(dH)) = \Omega(t_1, t_2) \langle it_1 \bullet \varphi, \frac{\varphi}{|\varphi|^2} \rangle = -\Omega(t_1, t_2)g(T, t_2).$$

Finally,  $-g(t_2, dH) = (\kappa - 4\tau^2)fg(T, t_2)$ . The same holds for  $t_1$ . Then,

$$dH = -(\kappa - 4\tau^2)fT,$$

which gives the contradiction. The last identity is the same obtained by R. Souam and E. Toubiana in [32].

### 4.3 Spectrum of the Spin<sup>c</sup> Dirac operator on Berger spheres

In this subsection, we apply a method of C. Bär [6, 12] to find explicitly some eigenvalues of the Spin<sup>c</sup> Dirac operator on Berger spheres, i.e., on  $\mathbb{E}^*(\kappa, \tau)$  with  $\kappa > 0$ .

**Lemma 4.5** *Let  $(M^n, g)$  be a Riemannian Spin<sup>c</sup> manifold carrying a Killing spinor  $\varphi$  of Killing number  $\alpha \in \mathbb{R}^*$ . Then,  $(\lambda_k(\Delta) + (\frac{n-1}{2})^2)_{k \in \mathbb{N}}$  are some eigenvalues of  $(D + \frac{\alpha}{2}\text{Id})^2$ . Here  $\lambda_k(\Delta)$ ,  $k = 0, 1, \dots$  denote the eigenvalue of the Laplacian  $\Delta$ .*

**Proof:** We have  $D\varphi = -\frac{n\alpha}{2}\varphi$ . For every  $f \in C^\infty(M, \mathbb{R})$ , we can easily check that

$$D^2(f\varphi) = \left(\frac{n^2}{4} - \frac{n}{2}\right)f\varphi - \alpha D(f\varphi) + (\Delta f)\varphi,$$

Hence,  $(D + \frac{\alpha}{2}\text{Id})^2(f\varphi) = (\Delta f + (\frac{n-1}{2})^2 f)\varphi$ . Now, if  $\{f_k\}_{k \in \mathbb{N}}$  denotes a  $L^2$ -orthonormal basis of eigenfunctions of  $\Delta$  of  $M$ , then for every  $k \in \mathbb{N}$ , we get

$$(D + \frac{\alpha}{2}\text{Id})^2(f_k\varphi) = (\lambda_k(\Delta) + (\frac{n-1}{2})^2)f_k\varphi,$$

where  $\lambda_k(\Delta)$  is the eigenvalue of  $\Delta$  whose eigenfunction is  $f_k$ . So,  $(\lambda_k(\Delta) + (\frac{n-1}{2})^2)_{k \in \mathbb{N}}$  are some eigenvalues of  $(D + \frac{\alpha}{2}\text{Id})^2$ .

**Spectrum of Berger spheres endowed with the canonical Spin<sup>c</sup> structure.** We consider Berger spheres with Berger metrics  $g_{\kappa, \tau}$ ,  $\kappa > 0$  and  $\tau \neq 0$  defined by

$$g_{(\kappa, \tau)}(X, Y) = \frac{\kappa}{4} \left( g(X, Y) + \left( \frac{4\tau^2}{\kappa} - 1 \right) g(X, \xi)g(Y, \xi) \right),$$

where  $g$  is the standard metric on  $\mathbb{S}^3$  of constant curvature 1. For simplicity, we can assume that  $\kappa = 4$  ( $\tau \neq \pm 1$ ). For any function  $f$ , the Laplacian  $\Delta_{4, \tau}$  with respect to  $g_{4, \tau}$  is related to the Laplacian  $\Delta$  with respect to  $g$  by [33]

$$\Delta_{4, \tau} f = \Delta f - (1 - \tau^{-2})\xi(\xi(f)).$$

It is known that each eigenfunction  $f_k$  of  $\Delta$  corresponding to  $\lambda_k(\Delta) = k(2+k)$  ( $k \in \mathbb{N}$ ) is also an eigenfunction of  $\Delta_{4, \tau}$  [33] corresponding to

$$\lambda_k(\Delta) - (1 - \tau^{-2})(k - 2p)^2, \quad 0 \leq p \leq \left\lfloor \frac{k}{2} \right\rfloor.$$

Moreover, each eigenvalue of  $\Delta_{4, \tau}$  takes the above form. We recall that the eigenspace of  $\Delta$  corresponding to  $\lambda_k(\Delta)$  is the restriction to the sphere  $\mathbb{S}^3$  of the set of harmonic homogeneous polynomial on  $\mathbb{R}^4$  of degree  $k$ . When we consider Berger spheres endowed with the canonical Spin<sup>c</sup> structure, we get by Lemma 4.5

$$\left( D + \frac{\tau}{2}\text{Id} \right)^2 (f_k\varphi) = \left[ 2 + k(2+k) - (1 - \tau^{-2})(k - 2p) \right] f_k\varphi,$$

where  $\varphi$  is the Killing spinor field of Killing constant  $\frac{\tau}{2}$ . Hence,

$$\mu_{k,p} = -\frac{\tau}{2} \pm \sqrt{2 + k(2 + k) - (1 - \tau^{-2})(k - 2p)}$$

are some eigenvalues of the Dirac operator on Berger spheres with  $-1 < \tau < 1$  and endowed with the canonical  $\text{Spin}^c$  structure.

**Spectrum of Berger spheres endowed with the  $\text{Spin}^c$  structure induced from the canonical one on  $\mathbb{M}^4(1 - \tau^2)$ .** On Berger spheres, we have shown that the  $\text{Spin}^c$  structure induced from the canonical one on  $\mathbb{M}^4(1 - \tau^2)$  carries a spinor field  $\varphi$  satisfying

$$\nabla_X \varphi = \frac{\tau}{2} X \bullet \varphi - i \frac{\tau^2 - 1}{2\tau} g(X, \xi) \varphi = \nabla'_X \varphi - i \frac{\tau^2 - 1}{2\tau} g(X, \xi) \varphi$$

Then, denoting by  $D$  (resp.  $D'$ ) the Dirac operator associated with the restricted  $\text{Spin}^c$  structure (resp. with the canonical  $\text{Spin}^c$  structure), we get  $D\varphi = D'\varphi - \frac{\tau^2 - 1}{2\tau} \varphi$ . for any function  $f$ , we have

$$D(f\varphi) = \text{grad} f \cdot \varphi + fD\varphi = D'(f\varphi) - fD'\varphi + fD\varphi = D'(f\varphi) - \left(\frac{\tau^2 - 1}{2\tau}\right) f\varphi.$$

Hence, we have  $D(f_k\varphi) = \left(\mu_{k,p} - \frac{\tau^2 - 1}{2\tau}\right) f_k\varphi$  and  $\mu_{k,p} - \frac{\tau^2 - 1}{2\tau}$  are some eigenvalues of the Dirac operator on Berger spheres endowed with the  $\text{Spin}^c$  structure induced from the canonical one on  $\mathbb{M}^4(1 - \tau^2)$ ,  $-1 < \tau < 1$ .

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