# Birationally rigid Fano complete intersections

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**Abstract**

We prove that general Fano complete intersections $V = V_{d_1\cdot\ldots\cdot d_k} = F_1\cap\ldots\cap F_k \subset \mathbf{P}^{M+k}$, where $F_i\subset \mathbf{P}^{M+k}$ is a hypersurface of degree $d_i\geq 2$ and the integers $d_1,\ldots,d_k$ satisfy the relation $d_1+\ldots+d_k=M+k$ (so that $V$ is of index 1) are birationally superrigid for $M\geq 2k+1$, $k\geq 2$. In particular, they cannot be fibered by a rational map into uniruled varieties (over a positive-dimensional base), each birational map $\chi\colon V - - \to V'$ onto a Fano variety $V'$ with $\mathbf{Q}$-factorial terminal singularities is biregular in codimension one and the groups of birational and biregular self-maps coincide: $\operatorname{Bir} V = \operatorname{Aut} V$. The proof is based upon the method of maximal singularities and the techniques of hypertangent systems.

## Introduction

**0.1.** Recall that a smooth Fano variety $X$ of dimension $\geq 3$ with $\operatorname{Pic} X = \mathbf{Z}K_X$ is said to be *birationally superrigid*, if for each birational map $\chi\colon X - - \to X'$ onto a variety $X'$ of the same dimension, smooth in codimension one, and each linear system $\Sigma'$ on $X'$, free in codimension 1 (that is, $\operatorname{codim}\operatorname{Bs}\Sigma'\geq 2$), the inequality

$$c(\Sigma, X)\leq c(\Sigma', X')$$

holds, where $\Sigma = (\chi^{-1})_*\Sigma'$ is the proper inverse image of $\Sigma'$ on $X$ with respect to $\chi$, and $c(\Sigma, X) = c(D, X)$ stands for the *threshold of canonical adjunction*

$$c(D,X) = \sup\{b/a | b, a \in \mathbf{Z}_+ \setminus \{0\}, |aD + bK_X| \neq \emptyset\}$$

$D \in \Sigma$, and similarly for $\Sigma'$, $X'$.

If $X$ is birationally superrigid, then for each Fano variety $X'$ of the same dimension with $\mathbf{Q}$-factorial terminal singularities each birational map $\chi: X - - \to X'$ (provided there are such maps) is biregular in codimension 1. Moreover, if $\operatorname{Pic} X' \otimes \mathbf{Q} = \mathbf{Q}K_{X'}$, then $\chi, \chi^{-1}$ are isomorphisms in codimension 1. Furhermore, there exists no rational map $\psi: X - - \to S$ onto a positive dimensional algebraic variety $S$, $\dim S < \dim X$, a generic fiber of which, $\psi^{-1}(s)$, $s \in S$, is a uniruled variety. In particular, the variety $X$ is non-rational. Finally, the groups of birational and biregular self-maps coincide: $\operatorname{Bir} V = \operatorname{Aut} V$. These implications of birational rigidity are well known; see, for instance, [IM,IP,P2,CPR,...].

In the present paper we consider Fano varieties of the type

$$\begin{array}{ccc} V = & V_{d_1 \cdot d_2 \cdot \ldots \cdot d_k} & \subset \mathbf{P}^{M+k} \\ & \| & \\ & F_1 \cap F_2 \cap \ldots \cap F_k, & \end{array}$$

where $F_i \subset \mathbf{P}^{M+k}$ are hypersurfaces of degree $d_i \geq 2$. The integers $d_i$ satisfy the relation $d_1 + \ldots + d_k = M + k$, $k \geq 2$, and $V$ is assumed to be smooth. In other words, $V$ is a Fano complete intersection of index 1. The ground field is assumed to be the field of complex numbers $\mathbf{C}$. Our main result is the following

**Theorem.** *For $M \geq 2k+1$ a generic variety $V$ is birationally superrigid.*

Generic Fano hypersurfaces of index 1 (which correspond to the value $k = 1$ in the notations above, i.e. the hypersurfaces $V_{M+1} \subset \mathbf{P}^{M+1}$) were proved to be birationally superrigid in [P2] (for $M = 3$ it is the classical theorem of V.A.Iskovskikh and Yu.I.Manin on the three-dimensional quartic [IM], for $M = 4$ see [P1]).

For convenience we assume that $d_1 \leq d_2 \leq \ldots \leq d_k$. The condition $M \geq 2k+1$ implies the inequality $d_k \geq 4$. The dimension of the variety $V$ is equal to $M \leq 5$. For $M = 5$ there are two possible cases: $V_{2\cdot 5}$ and $V_{3\cdot 4}$.

In dimension three there are two families of complete intersections: $V_{2\cdot 3} \subset \mathbf{P}^5$ and $V_{2\cdot 2\cdot 2} \subset \mathbf{P}^6$. Birational rigidity of a general variety in the first family was proved in [IP], and its group of birational self-maps $\operatorname{Bir} V_{2\cdot 3} \neq \operatorname{Aut} V_{2\cdot 3}$ was described in [I,IP]. The variety $V_{2\cdot 2\cdot 2}$ has a lot of conic bundle structures, and geometry of its birational correspondences has not yet been studied.

**Conjecture.** *A smooth Fano complete intersection of index 1 and dimension $\geq 4$ in the projective space is birationally superrigid.*

There is an immense variety of families of Fano complete intersections with $\mathbf{Q}$-factorial terminal singularities in the weighted projective spaces [Fl]. Birational superrigidity of general varieties of the type

$$V_{m\cdot 2l} \subset \mathbf{P}(\underbrace{1, \ldots, 1}_{M+2}, l),$$

$m+l=M+1$, was proved in [P3] (these varieties are smooth). It seems that the Conjecture is true for the weighted complete intersections as well.

**0.2.** The theorem is proved in the following way. By the method of maximal singularities [IM,IP,P2,P3] we reduce the problem of whether a given variety is (super)rigid to behaviour of algebraic cycles on this variety. The following proposition gives a convenient sufficient condition of birational superrigidity.

**Proposition 1.** *Let $X$ be a smooth Fano variety with $\mathrm{Pic}\, X = \mathbf{Z}K_X$. Assume that for each point $x \in X$, each irreducible cycle $Y \subset X$ of codimension two and each linear system $\Sigma$ on $X$ without fixed components the following two properties are satisfied:*

(i) $\mathrm{mult}_Y \Sigma \leq n$;

(ii) $\mathrm{mult}_x Y \leq (4/\deg X) \deg Y$,

*where $n \in \mathbf{Z}_+$ is defined by the inclusion $\Sigma \subset |-nK_X|$, $\deg X = (-K_X)^{\dim X}$ and* $\deg Y = (Y \cdot (-K_X)^{\dim Y})$.

*Then the variety $X$ is birationally superrigid.*

For the proof, see [P2]. In that paper the proof is carried out by means of the method of maximal singularities, which goes back to the classical paper of V.A.Iskovskikh and Yu.I.Manin [IM]. Another proof, based upon Shokurov's connectedness theorem and inversion of adjunction [Kol] can be extracted from [C].

We prove the main theorem of the present paper, checking the conditions (i) and (ii) for a general complete intersection $V = V_{d_1 \cdot \ldots \cdot d_k}$. The property (i) obviously follows from the Lefschetz theorem, since $\dim V = M \geq 5$. The property (ii) is proved by means of the method of hypertangent systems, which was introduced in [P2]. This is the principal part of our arguments, making the contents of Sec. 1.

In order to apply the method of hypertangent systems, we need the complete intersection $V$ to satisfy a certain regularity condition (Sec. 1.1), which is analogous to the regularity condition of the papers [P2,P3]. Therefore it is necessary to verify that a general complete intersection satisfies the regularity condition at each point. This is done in Sec. 2, thus making the proof of our theorem complete.

**Remark.** Cheltsov has noted [Ch], that the method of maximal singularities gives not only a description of all the structures of Fano fibrations on a given variety, but also a description of all the structures of $K$-trivial fibrations. In particular, since a general complete intersection $V \subset \mathbf{P}^{M+k}$ with $M \geq 2k+1$ satisfies the property (ii), $K$–trivial fibrations on $V$ are determined by linear systems $\Sigma \subset |-nK_V|$, which either have a base cycle $Y \subset V$ of codimension 2 such that $\mathrm{mult}_Y \Sigma = n$, or have a base cycle $Y$ of codimension 3 such that $\mathrm{mult}_Y \Sigma = 2n$. The second case can be easily excluded by the arguments of Sec. 1. In the first case, it is easy to see that $Y$ is numerically equivalent to a section of $V$ by a plane of codimension 2 in $\mathbf{P}$, whereas the system $\Sigma$ defines a rational map $\varphi_\Sigma \colon V - - \to \mathbf{P}^1$. In other words, each structure of a $K$-trivial fibration on $V$ has one-dimensional base $\mathbf{P}^1$. It seems that all these structures are just pencils of hyperplane sections of $V$. For the Fano hypersurfaces $V_{M+1} \subset \mathbf{P}^{M+1}$ this is true [Ch] and the proof is obtained by the methods of [P2]. For the complete intersections, the techniques of [P2] should be improved.

### 0.3. Acknowledgements

I started to work with Fano complete intersections in 1997 during my stay at Max-Planck-Institut für Mathematik. In [P2] it was pointed out that the techniques developed for the Fano hypersurfaces should work for complete intersections, too. However, it was not until now that all the details have been clarified. The author thanks the Institute for the excellent conditions of work.

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# 1 Proof of the theorem

## 1.1 The regularity condition

Fix integers $k \geq 2$, $M \geq 2k+1$, $d_k \geq \ldots \geq d_1 \geq 2$, where $d_1 + \ldots + d_k = M + k$. The sympol $\mathbf{P}$ stands for the projective space $\mathbf{P}^{M+k}$ over the field $\mathbf{C}$ of complex numbers. Let

$$f_i \in H^0(\mathbf{P}, \mathcal{O}_{\mathbf{P}}(d_i)) \setminus \{0\}, \quad i = 1, \ldots, k,$$

be homogeneous polynomials on $\mathbf{P}$. The symbol $V(f_1, \ldots, f_k) = V(f_*)$ denotes the closed algebraic set $\{f_1 = \ldots = f_k = 0\}$. Let $\mathbf{C}^{M+k} \subset \mathbf{P}$ be an open affine set, equipped with a system of coordinates $(z_1, \ldots, z_{M+k})$ with the origin at the point $x \in \mathbf{P}$. Set

$$f_i = q_{i,0} + \ldots + q_{i,d_i},$$

where $q_{i,a}$ are homogeneous of degree $a$ in $(z_*)$. Obviously, $x \in V(f_*)$ if and only if $q_{1,0} = \ldots = q_{d_k,0} = 0$.

**Definition 1.** A point $x \in V(f_1, \ldots, f_k)$ is called a *regular point* of the complete intersection $V(f_*)$, if the set of polynomials

$$\{q_{i,j} | 1 \leq i \leq k, 1 \leq j \leq d_i, (i,j) \neq (k, d_k)\},$$

which consists of all the polynomials $q_{i,j}$, *but the last one* $q_{k,d_k}$, makes a regular sequence in $\mathcal{O}_{x,\mathbf{C}^{M+k}} = \mathcal{O}_{x,\mathbf{P}}$.

In other words, the system of equations

$$\{q_{i,j} = 0 | (i,j) \neq (k, d_k)\}$$

defines a curve in $\mathbf{C}^{M+k}$. Since the polynomials $q_{i,j}$ are homogeneous, this curve is a finite union of lines in $\mathbf{C}^{M+k}$, passing through the origin. In particular, on $V(f_*)$ there are at most a finite number of lines passing through a regular point.

Set

$$\mathcal{H} = \prod_{i=1}^{k} \left( H^0(\mathbf{P}, \mathcal{O}_{\mathbf{P}}(d_i)) \setminus \{0\} \right)$$

to be the set of sequences $(f_1, \ldots, f_k)$ of non-zero homogeneous polynomials of degrees $d_1, \ldots, d_k$, respectively.

**Proposition 2.** *There exists a non-empty Zariski open set $U_{\rm reg} \subset \mathcal{H}$ such that for $(f_1, \ldots, f_k) \in U_{\rm reg}$ the complete intersection*

$$V = V(f_1, \ldots, f_k) \subset \mathbf{P}$$

*is regular at each point.*

The **proof** is given in Sec. 2.

We will prove our theorem in the following form: for $(f_*) \in U_{\rm reg}$ the complete intersection $V = V(f_*)$ is birationally superrigid.

Let us fix the complete intersection $V$ and check that the conditions (i) and (ii) of Proposition 1 are satisfied for $V$. This is enough for the theorem.

Let $\Sigma \subset |-nK_V|$ be a linear system of divisors without fixed components, $Y \subset \operatorname{Bs}\Sigma$ its base cycle of codimension 2. Take two general divisors, $D_1, D_2 \in \Sigma$ and consider the algebraic cycle of their scheme-theoretic intersection:

$$Z = (D_1 \bullet D_2) = aY + \sum a_i Y_i,$$

where $a_i \geq 1$, $Y_i \neq Y$ are some irreducible cycles of codimension 2. Obviously, $a \geq (\operatorname{mult}_Y \Sigma)^2$. Compute the degrees:

$$n^2 \deg V = \deg Z = a \deg Y + \sum a_i \deg Y_i,$$

where $\deg V = d_1 \ldots d_k$. By the Lefschetz theorem $Y$ is numerically equivalent to $mK_V^2$, $m \geq 1$, so that $\deg Y = m \deg V$, whence

$$(\operatorname{mult}_Y \Sigma)^2 \leq a \leq n^2/m \leq n^2,$$

which is what we need. Q.E.D. for the condition (i) of Proposition 1.

## 1.2 Hypertangent linear systems

Let us prove that for a regular point $x \in V$ and an irreducible cycle $C \subset V$ of codimension 2 the following estimate is true:

$$\frac{\operatorname{mult}_x}{\deg} C \leq \frac{4}{\deg V}. \tag{1}$$

**Remark.** The estimate (1) is exact: just take $C = H_1 \cap H_2 \cap V$, where $H_i$ are distinct tangent hyperplanes to $V$ at the point $x$. We get: $\deg C = \deg V$, $\operatorname{mult}_x C = 4$.

**Proof** of the inequality (1). Fix an open affine set $\mathbf{A} \cong \mathbf{C}^{M+k} \subset \mathbf{P}$ and a system of coordinates $(z_*)$ with the origin at the point $x$ on $\mathbf{A}$. Set

$$f_{i,\alpha} = q_{i,1} + \ldots + q_{i,\alpha}.$$

The symbol $s_a$, $a \in \mathbf{Z}_+$, stands for an arbitrary homogeneous polynomial of degree $a$ in $(z_*)$. Consider the linear systems

$$\Lambda^{\mathbf{A}}_{i,j} = \left| \sum_{\alpha=1}^{\min\{j, d_i - 1\}} f_{i,\alpha} s_{j-\alpha} \right|$$

on $\mathbf{A}$. Taking the closure of a general divisor of the system $\Lambda^{\mathbf{A}}_{i,j}$ in $\mathbf{P}$, we get the linear system $\Lambda^{\mathbf{P}}_{i,j}$ on $\mathbf{P}$. Obviously,

$$\operatorname{Bs}\Lambda^{\mathbf{A}}_{i,j}=\{q_{i,\alpha}=0|1\le\alpha\le\min\{j,d_i-1\}\},$$

$$\operatorname{Bs}\Lambda^{\mathbf{P}}_{i,j}=\overline{\operatorname{Bs}\Lambda^{\mathbf{A}}_{i,j}}.$$

Set

$$\Lambda^{\mathbf{P}}_j=\sum_{i=1}^k\Lambda^{\mathbf{P}}_{i,j}.$$

For this linear system we get

$$\operatorname{Bs}\Lambda^{\mathbf{P}}_j=\overline{\{q_{i,\alpha}=0|1\le\alpha\le\min\{j,d_i-1\}\}}$$

and

$$c(j)=\operatorname{codim}_{\mathbf{P}}\operatorname{Bs}\Lambda^{\mathbf{P}}_j=\sum_{i=1}^k\min\{j,d_i-1\}. \qquad (2)$$

We get $\operatorname{Bs}\Lambda^{\mathbf{P}}_{d_k-1}=\operatorname{Bs}\Lambda^{\mathbf{P}}_{d_k}=\ldots=\operatorname{Bs}^{\mathbf{P}}_{\infty}$ and $\operatorname{codim}_{\mathbf{P}}\operatorname{Bs}^{\mathbf{P}}_{\infty}=M$.

**Definition 2.** The linear system

$$\Lambda_j=\Lambda^{\mathbf{P}}_j|_V$$

is called the *$j$-th hypertangent system* at the point $x$.

Now we must estimate the codimension of the base set of hypertangent systems. A priori one can assert only that $\operatorname{codim}_V\operatorname{Bs}\Lambda_j\ge\operatorname{codim}_{\mathbf{P}}\operatorname{Bs}\Lambda^{\mathbf{P}}_j-k$: when a system is restricted from $\mathbf{P}$ to $V$, the codimension of the base set may decrease. However, in our case the regularity condition improves the matter.

**Lemma 1.** (i) $\operatorname{codim}_V\operatorname{Bs}\Lambda_1=\operatorname{codim}_{\mathbf{P}}\operatorname{Bs}\Lambda^{\mathbf{P}}_1$.

(ii) *For* $j\in\{2,\ldots,d_k-2\}$

$$\operatorname{codim}_V\operatorname{Bs}\Lambda_j=\operatorname{codim}_{\mathbf{P}}\operatorname{Bs}\Lambda^{\mathbf{P}}_j \qquad (3)$$

*in a neighborhood of the point $x$.*

(iii) *For each $j$ we have the estimate*

$$\operatorname{codim}_V\operatorname{Bs}\Lambda_j\ge\operatorname{codim}_{\mathbf{P}}\operatorname{Bs}\Lambda^{\mathbf{P}}_j-1.$$

**Proof.** It is easy to see that for $\alpha\le d_i-1$

$$f_{i,\alpha}|_{V\cap\mathbf{A}}=(q_{i,\alpha+1}+\ldots+q_{i,d_i})|_{V\cap\mathbf{A}}. \qquad (4)$$

Set $\mathbf{T}=T_xV\cong\mathbf{C}^M$. Now we introduce the auxiliary linear systems

$$\Lambda^{\mathbf{T}}_{i,j}=\left|\sum_{\alpha=1}^{\min\{j,d_i-1\}}q_{i,\alpha+1}s_{j-\alpha}\right|_{\mathbf{T}},$$

$$\Lambda^{\mathbf{T}}_j=\sum_{i=1}^k\Lambda^{\mathbf{T}}_{i,j}.$$

These are linear systems of cones of degree $j+1$ on the tangent space $\mathbf{T}$. By the regularity condition and (4)

$$\Lambda_j^{\mathbf{T}} = T_x\Lambda_j.$$

See Sec. 2.3 for the definitions and (elementary) facts of the theory of tangent cones to divisors, cycles and linear systems. (Here we need the regularity condition to conclude that the restriction $q_{i,\alpha+1}|_{\mathbf{T}}$ is a non-zero polynomial.) Obviously,

$$\operatorname{Bs}\Lambda_j^{\mathbf{T}} = \{q_{i,\alpha+1}|_{\mathbf{T}} = 0, 1 \leq \alpha \leq \min\{j, d_i - 1\}\} \subset \mathbf{T}.$$

Again by the regularity condition we get

$$\operatorname{codim}_{\mathbf{T}} \operatorname{Bs}\Lambda_j^{\mathbf{T}} = \operatorname{codim}_{\mathbf{P}} \operatorname{Bs}\Lambda_j^{\mathbf{P}}$$

for $j \in \{1, \ldots, d_k - 2\}$. The case $j = d_k - 1$ is excluded because $q_{k,d_k}$ is excluded from the regularity condition. Applying Lemma 5, we get both (ii) and the local version of (i): so far we can assert only that the equality $\operatorname{codim}_V \operatorname{Bs}\Lambda_1 = \operatorname{codim}_{\mathbf{P}} \operatorname{Bs}\Lambda_1^{\mathbf{P}} = k$ is true in a neighborhood of the point $x$.

The tangent hyperplane to the hypersurface $f_i = 0$ at the point $x$ is given by the equation $q_{i,1} = 0$. Denote it by $T_i$. Obviously,

$$\operatorname{Bs}\Lambda_1 = T_1 \cap \ldots \cap T_k \cap V \ni x,$$

where, as we have already established, this cycle is of codimension $k$ in a neighborhood of the point $x$. However, it is easy to see that for each $i$, $1 \leq i \leq k$, the cycle $T_1 \cap \ldots \cap T_i \cap V$ is irreducible by the Lefschetz theorem. Eventually we get that $\operatorname{Bs}\Lambda_1$ is an irreducible cycle of codimension $k$. The part (i) is completely proved.

Let us prove (iii). The base set of the system $\Lambda_j^{\mathbf{P}}$ is given by a system of $c(j)$ equations and has codimension $c(j)$, see (2). The base set of the system $\Lambda_j$ is given by the same equations, restricted to $V$, and $c(j) \leq \dim V$. This implies that $\operatorname{codim}_V \operatorname{Bs}\Lambda_j - \operatorname{codim}_{\mathbf{P}} \operatorname{Bs}\Lambda_j^{\mathbf{P}}$ is a non-increasing function of the parameter $j$. (For $a \geq b$ the codimension $\operatorname{codim}_V \operatorname{Bs}\Lambda_a$ is higher than $\operatorname{codim}_V \operatorname{Bs}\Lambda_b$ by, at most, the number of the new equations, that is, $c(a) - c(b)$.) Thus it is sufficient to prove (iii) for $j = \infty$: $\dim(\operatorname{Bs}_\infty^{\mathbf{P}})|_V \leq 1$.

Indeed, the base subscheme $(\operatorname{Bs}_\infty^{\mathbf{P}})|_V$ is given on $V$ by the system of equations

$$\{q_{i,\alpha} = 0 | 1 \leq \alpha \leq d_i - 1\}.$$

Therefore, as a subscheme of the projective space $\mathbf{P}$ $(\operatorname{Bs}_\infty^{\mathbf{P}})|_V$ is given by the system

$$\{q_{i,\alpha} = 0 | 1 \leq \alpha \leq d_i\}.$$

of $M + k$ equations. By the regularity condition $(\operatorname{Bs}_\infty^{\mathbf{P}})|_V \leq 1$, which is what we need. Q.E.D. for Lemma 1.

Let us come back to the proof of inequality (1). For $j \geq 1$ set $w^+(j) = \sharp\{i, 1 \leq i \leq k \,|\, j \leq d_i - 1\}$. Furthermore, set $w(j) = w^+(j) - 1$ for $j = 1, 2$ and $d_k - 1$, and $w(j) = w^+(j)$ for the other values of $j$. Obviously, $w^+(1) = k$ and

$$c(j) = \sum_{\alpha=1}^{j} w^+(\alpha)$$

for each $j \geq 1$. For $1 \leq j \leq d_k - 1$ we introduce the spaces

$$\Delta_j = \underbrace{\Lambda_j \times \ldots \times \Lambda_j}_{w(j)},$$

parametrizing the sets $(D_{j,1}, \ldots, D_{j,w(j)}) \in \Delta_j$ of divisors from the linear systems $\Lambda_j$. Set

$$\Delta = \prod_{j=2}^{d_k - 1} \Delta_j.$$

## 1.3 Degrees and multiplicities

Let $R \subset V$ be an irreducible cycle of codimension $k+1$.

**Lemma 2.** *For a generic set $(D_{*,*}) \in \Delta$ the set-theoretic intersection*

$$R^* = R \bigcap \left( \bigcap_{j=2}^{d_k-1} \bigcap_{a=1}^{w(j)} D_{j,a} \right)$$

*is one-dimensional in a neighborhood of the point $x$.*

**Proof.** First of all, note that the set $(D_{*,*})$ consists of $w(2) + \ldots + w(d_k - 1) = M - k - 2$ divisors, so that $\dim R^* \geq 1$. For $b \in \{2, \ldots, d_k - 1\}$ set

$$R_b = R \bigcap \left( \bigcap_{j=2}^{b} \bigcap_{a=1}^{w(j)} D_{j,a} \right).$$

Let us prove by (ascending) induction on $b$ that for a general set $(D_{*,*}) \in \prod_{j=2}^{b} \Delta_j$ we have

$$\operatorname{codim} R_b = 2 + \sum_{j=2}^{b} w(j) \tag{5}$$

in a neighborhood of the point $x$. In what follows we omit the words "in a neighborhood of the point $x$" and simply speak about codimension. However, during the proof of Lemma 2 we always mean the codimension near the point $x$.

Set for $e \in \{0, 1, \ldots, w(b)\}$

$$R_{b,e} = R \bigcap \left( \bigcap_{j=2}^{b-1} \bigcap_{a=1}^{w(j)} D_{j,a} \right) \bigcap \left( \bigcap_{a=1}^{e} D_{b,a} \right).$$

In particular, $R_{b,w(b)} = R_{b+1,0} = R_b$. Let us prove by induction on $e$, that for a generic collection $(D_{*,*})$ the following equality

$$\operatorname{codim} R_{b,e} = k + 1 + \sum_{j=2}^{b-1} w(j) + e \tag{6}$$

holds. For $b = 2$, $e = 0$ (6) is true by assumption. Take $e \leq w(b) - 1$, and assume that (6) is true. By (3),

$$\operatorname{codim}_V \operatorname{Bs} \Lambda_b = c(b) = \sum_{j=1}^{b} w^+(j) = k + 1 + \sum_{j=2}^{b-1} w(j) + w(b),$$

so that a generic divisor $D_{b,e+1} \in \Lambda_b$ contains none of the components of the set $R_{b,e}$, which contain the point $x$. This proves (6).

The equality (6) implies (5) automatically. Q.E.D. for the lemma.

Fix an arbitrary collection $(D_{*,*}) \in \Delta$, satisfying the claim of Lemma 2. Let $R_b$ and $R_{b,e}$ mean the same as above (in the proof of the lemma).

**Lemma 3.** *In each of the closed algebraic sets $R_{b,e}$ there is an irreducible component $Y_{b,e}$ such that the sequence of irreducible cycles $\{Y_{b,e} | 2 \leq b \leq d_k - 1, 0 \leq e \leq w(b)\}$ satisfies the following properties*:

(i) $Y_{2,0} = R$,

(ii) $Y_{b,e} \ni x$ *and, in particular,*

$$\operatorname{codim}_V Y_{b,e} = k + 1 + \sum_{j=2}^{b-1} w(j) + e,$$

(iii) *for* $e \leq w(b) - 1$

$$\frac{\operatorname{mult}_x}{\deg} Y_{b,e+1} \geq \frac{b+1}{b} \cdot \frac{\operatorname{mult}_x}{\deg} Y_{b,e}. \tag{7}$$

**Proof.** Assume that the cycles $Y_{\alpha,\beta}$, $2 \leq \alpha \leq b-1$, and $Y_{b,\beta}$, $0 \leq \beta \leq e \leq w(b) - 1$, have already been constructed. Obviously,

$$\mathcal{O}_V(D_{b,e+1}) \cong \mathcal{O}_V(b)$$

in the sense of the inclusion $V \hookrightarrow \mathbf{P}$, and by (4)

$$\operatorname{mult}_x \Lambda_b \geq b + 1.$$

Since $Y_{b,e} \ni x$ is an irreducible cycle, $Y_{b,e} \not\subset D_{b,e+1}$. Therefore,

$$\frac{\operatorname{mult}_x}{\deg} (Y_{b,e} \bullet D_{b,e}) \geq \frac{b+1}{b} \cdot \frac{\operatorname{mult}_x}{\deg} Y_{b,e}.$$

Consequently, there is an irreducible component $Y_{b,e+1}$ of the cycle $(Y_{b,e} \bullet D_{b,e})$, satisfying the inequality (7). Q.E.D. for the lemma.

**Corollary.** *The following estimate holds*

$$\frac{\operatorname{mult}_x}{\deg} R \leq \frac{3d_k}{2d_k - 2} \cdot 2^{w^+(2)} \cdot \left( \prod_{d_i \geq 3} d_i \right)^{-1}.$$

**Proof.** The last cycle in the sequence $Y_{b,e}$ is an irreducible curve. Its multiplicity at the point $x$ does not exceed its degree. Applying part (iii) of Lemma 3, we get the estimate

$$\frac{\operatorname{mult}_x}{\deg} R \cdot \prod_{b=2}^{d_k - 1} \left( \frac{b+1}{b} \right)^{w(b)} \leq 1.$$

Recalling the definition of the integers $w(b)$ and $w^+(b)$, we get:

$$\prod_{b=2}^{d_k - 1} \left( \frac{b+1}{b} \right)^{w(b)} = \frac{2}{3} \cdot \frac{d_k - 1}{d_k} \cdot \prod_{b=2}^{d_k - 1} \left( \frac{b+1}{b} \right)^{w^+(b)} =$$

$$= \frac{2d_k - 2}{3d_k} \cdot 2^{-w^+(2)} \cdot \left( \prod_{b=3}^{d_k} b^{w^+(b-1)-w^+(b)} \right).$$

However, $w^+(b-1) - w^+(b) = \sharp\{i | 1 \leq i \leq k, d_i = b\}$. Consequently, the product in brackets is equal to

$$\prod_{d_i \geq 3} d_i,$$

which is what we need.

Now everything is ready for the proof of inequality (1). By Lemma 1 $\operatorname{codim} \operatorname{Bs} \Lambda_1 = k$. This implies that for generic divisors $D_{1,e}$, $e = 1, \ldots, k-2$, the scheme-theoretic intersection

$$C_e = C \cap D_{1,1} \cap \ldots \cap D_{1,e}$$

is of pure codimension $e+2$. Furthermore, all the divisors $D_{1,e}$ are hyperplane sections. Besides, $\operatorname{mult}_x \Lambda_1 = 2$. Therefore each of the closed sets $C_e$ has an irreducible component $Y_e \ni x$ such that

$$\frac{\operatorname{mult}_x}{\deg} Y_{e+1} \geq 2 \frac{\operatorname{mult}_x}{\deg} Y_e$$

for all $e = 1, \ldots, k-3$. Let $Y = Y_{k-2}$ be the last cycle in this sequence. We have

$$\frac{\operatorname{mult}_x}{\deg} Y \geq 2^{k-2} \cdot \frac{\operatorname{mult}_x}{\deg} C.$$

The cycle $Y$ is of the same codimension with respect to $V$ as $\operatorname{Bs} \Lambda_1$. We mentioned above that $\operatorname{Bs} \Lambda_1$ is an irreducible cycle by the Lefschetz theorem. Its degree is equal to $\deg V$, and by the regularity condition $\operatorname{mult}_x \operatorname{Bs} \Lambda_1 = 2^k$.

If $Y = \operatorname{Bs} \Lambda_1$, then

$$\frac{\operatorname{mult}_x}{\deg} C \leq 2^{2-k} \cdot \frac{2^k}{\deg V} = \frac{4}{\deg V},$$

as required.

If $Y \neq \operatorname{Bs} \Lambda_1$, then there is a divisor $D_{1,k-1} \in \Lambda_1$, which does not contain $Y$, so that $\operatorname{codim}_V (Y \cap D_{1,k-1}) = k+1$. Again we take a component $R$ of the cycle $(Y \bullet D_{1,k-1})$, which satisfies the inequality

$$\frac{\operatorname{mult}_x}{\deg} R \geq 2 \cdot \frac{\operatorname{mult}_x}{\deg} Y \geq 2^{k-1} \cdot \frac{\operatorname{mult}_x}{\deg} C.$$

The irreducible cycle $R$ is of codimension $k+1$. Applying the corollary above to this cycle, we get

$$\frac{\operatorname{mult}_x}{\deg} C \leq \frac{3d_k}{2d_k - 2} \cdot 2^{w^+(2)+1-k} \cdot \left( \prod_{d_i \geq 3} d_i \right)^{-1}.$$

By definition $w^+(2) = \sharp\{i | 1 \leq i \leq k, d_i \geq 3\}$. Thus $k - w^+(2) = \sharp\{i | 1 \leq i \leq k, d_i = 2\}$. Finally, we get

$$\frac{\operatorname{mult}_x}{\deg} C \leq \frac{3d_k}{4d_k - 4} \cdot \frac{4}{\deg V}.$$

However, as we mentioned above, $d_k \geq 4$, so that the first factor in the right-hand part does not exceed one.

Q.E.D. for the inequality (1) and thus for Proposition 1.

# 2 Regular complete intersections

## 2.1 Non-regular sets of polynomials

Let us consider the general problem of estimating the codimension of non-regular sets of polynomials, that is, those sets, the zeros of which have a "wrong" dimension.

Let $z_1, \ldots, z_{N+1}$ be a set of variables. The symbol $\mathcal{P}_a$ stands for the space of homogeneous polynomials in the variables $z_*$. Set

$$\mathcal{L} = \prod_{i=1}^{k+1} \mathcal{P}_{m_i} = \{(p_1, \ldots, p_{k+1})\}$$

to be the set all collections of homogeneous polynomials in the variables $z_*$. Here $0 \leq k \leq N-1$ is an arbitrary integer (which is not connected with the number of equations, defining the complete intersection $V$: in Sec. 2.1 we do not deal with $V$). With each collection $(p_*) \in \mathcal{L}$ we associate the projectivized set of its zeros

$$Z(p_*) = \{p_1 = \ldots = p_{k+1} = 0\} \subset \mathbf{P}^N = \mathbf{P}(\mathbf{C}^{N+1}) = X.$$

Let

$$Y = \{(p_*) \in \mathcal{L} | \operatorname{codim}_X Z(p_*) \leq k\}$$

be the set of "irregular" collections. Assume that

$$m = \min\{m_1, \ldots, m_{k+1}\} \geq 2.$$

**Proposition 3.** *For $k \leq N-2$ the following estimate holds:*

$$\operatorname{codim}_{\mathcal{L}} Y \geq mN + m - 1. \tag{8}$$

*For $k = N-1$*

$$\operatorname{codim}_{\mathcal{L}} Y \geq \min\{mN + m - 1, \sum_{i=1}^{N}(m_i - 1) + 2\}.$$

**Proof.** Set

$$\mathcal{L}_a = \prod_{i=1}^{a} \mathcal{P}_{m_i} = \{(p_1, \ldots, p_a)\}.$$

For each irregular collection $(p_1, \ldots, p_{k+1})$ fix the first (counting from the left to the right) moment when the codimension of the zero set $p_1 = \ldots = p_a = 0$ fails to take the correct value. Consider the sets

$$Y_a = \{(p_*) \in \mathcal{L}_a | \operatorname{codim}_X Z(p_1, \ldots, p_a) = \operatorname{codim}_X Z(p_1, \ldots, p_{a-1}) = a - 1\}.$$

Obviously,

$$Y = \bigvee_{a=1}^{k+1} Y_a \times \mathcal{L}^{a+1},$$

where we use the notation

$$\mathcal{L}^a = \prod_{i=a}^{k+1} \mathcal{P}_{m_i}.$$

In particular,

$$\operatorname{codim}_{\mathcal{L}} Y = \min\{\operatorname{codim}_{\mathcal{L}_a} Y_a | 1 \le a \le k+1\}.$$

The inequality (8) does not depend on $k$. Thus to prove Proposition 3 it is sufficient to show that

$$\operatorname{codim}_{\mathcal{L}} Y_{k+1} \ge mN + m - 1 \tag{9}$$

for $k \le N-2$, and

$$\operatorname{codim}_{\mathcal{L}} Y_N \ge \min\{mN + m - 1, \sum_{i=1}^{N}(m_i - 1) + 2\}.$$

For any $k \in \{1, \dots, N-1\}$ set $Y^* = Y_{k+1}$. For each collection $(p_1, \dots, p_{k+1}) \in Y^*$ the set $Z(p_1, \dots, p_k)$ has the correct codimension $k$, but there exists an irreducible component $B \subset Z(p_1, \dots, p_k)$, on which $p_{k+1}$ vanishes. Let $\langle B \rangle$ be the linear span of $B$, $\operatorname{codim}\langle B \rangle = b \le k$. We denote by $Y^*(b)$ the set of all those collections $(p_*) \in Y^*$, for which there exists a component $B \subset Z(p_1, \dots, p_k)$ such that

$$\operatorname{codim}_X \langle B \rangle = b, \quad p_{k+1}|_B \equiv 0.$$

For distinct $b \ne a$ the sets $Y^*(a)$ and $Y^*(b)$ can have a non-empty intersection. Obviously,

$$Y^* = \bigcup_{b=0}^{k} Y^*(b).$$

Thus the inequality (9) follows from the estimate

$$\operatorname{codim}_{\mathcal{L}} Y^*(b) \ge mN + m - 1, \tag{10}$$

which is what we will actually prove.

**The case $\mathbf{b = 0}$.** Here $\langle B \rangle = \mathbf{P}^N$ and therefore each non-zero monomial in the linear forms in $z_1, \dots, z_{N+1}$ of degree $d_{k+1} \ge d$ does not vanish on $B$. The space of monomials

$$\left\{ \prod_{i=1}^{m_{k+1}} (a_{i,1}z_1 + \ldots + a_{i,N+1}z_{N+1}) \right\} \subset \mathcal{P}_{m_{k+1}}$$

is closed. Its dimension is equal to $\ge m(N+1) - 1 = mN + m - 1$. On the other hand, the set of polynomials $p_{k+1} \in \mathcal{P}_{m_{k+1}}$ that vanish on $B$ is closed. Therefore its codimension is no smaller than $\ge mN + m - 1$. This gives (10) for $b = 0$.

**The case $\mathbf{b \ge 1}$.** Here $\langle B \rangle = \mathbf{P}^{N-b}$. Let us restrict the polynomials $p_i$ onto $\langle B \rangle$. Since $B$ is an irreducible component of the closed set $Z(p_1, \dots, p_k)$, one can find $(k-b)$ polynomials among $p_1, \dots, p_k$ — let these be $p_1, \dots, p_{k-b}$ — such that there exists a chain of irreducible cycles $R_j$ of codimension $j$ in $\langle B\rangle$, $j=0,1,\dots,k-b$, $R_0=\langle B\rangle$, $B_{j+1}$ is an irreducible component of the set $\{p_{j+1}=0\}\cap B_j$ and $B_{k-b}=B$. Consequently, the polynomials

$$p_{k-b+1}|_{\langle B\rangle},\dots,p_{k+1}|_{\langle B\rangle}$$

vanish on $B$. Fix the linear space $\langle B\rangle\subset \mathbf{P}^N$. The arguments given above for the case $b=0$ show that the condition

$$\left(p_i|_{\langle B\rangle}\right)|_B\equiv 0$$

for all $i\in\{k-b+1,\dots,k+1\}$ gives not less than

$$(b+1)(m(N-b)+m-1)$$

independent conditions on $(p_{k-b+1},\dots,p_{k+1})$. Taking into account that the Grassmanian has the dimension $\dim G(N+1-b,N+1)=b(N+1-b)$, we get finally

$$\begin{array}{rcc} \operatorname{codim} Y^*(b) & \geq & (b+1)(m(N-b)+m-1)-b(N+1-b) \\ & & \| \\ & & g(b)+mN+m-1, \end{array}$$

where $g(t)=-(m-1)t^2+(N(m-1)-2)t$ is a quadratic function of the real variable $t$. The function $g$ has exactly two roots, $t=0$ and $t=\gamma=N-2/(m-1)$. It is nonnegative on the segment $[0,\gamma]$. The parameter $b$ takes integral values $0,\dots,k\leq N-1$. This proves (10) in all the cases but one: when $m=2$, $b=k=N-1$, since only in that case $g(b)<0$.

Now we have only one case to consider, that is, when $B=\langle B\rangle$ is a line in $\mathbf{P}^N$, on which vanish all the polynomials $p_1,\dots,p_N$. For a fixed line $B$ the condition $p|_B\equiv 0$ for a polynomial $p$ of degree $e\geq 1$ defines a closed algebraic set of polynomials of codimension $e+1$ in $\mathcal{P}_e$. Therefore,

$$\operatorname{codim}_{\mathcal{L}} Y^*(N-1)\geq \sum_{i=1}^{N}(m_i+1)-2(N-1)=\sum_{i=1}^{N}(m_i-1)+2,$$

which is what we need. Q.E.D. for Proposition 3.

## 2.2 Proof of Proposition 2.

Recall that $\mathcal{H}$ stands for the set of all collections $(f_1,\dots,f_k)$ of non-zero homogeneous polynomials on $\mathbf{P}$ of degrees $d_1,\dots,d_k$, respectively.

To begin with, there is a Zariski open set $U_{\mathrm{sm}}\subset\mathcal{H}$ such that for $(f_1,\dots,f_k)\in U_{\mathrm{sm}}$ the complete intersection $V(f_*)$ is a smooth subvariety of codimension $k$ in $\mathbf{P}$. Furthermore, let us consider the subset

$$Y=\{(x,(f_*))|x\in V(f_*)\text{ is a non-regular point}\}$$

in $\mathbf{P} \times U_{\mathrm{sm}}$. We claim that the closure

$$\overline{\pi(Y)} \subset U_{\mathrm{sm}}$$

is a proper closed subset of positive codimension in $U_{\mathrm{sm}}$. Here $\pi\colon \mathbf{P} \times U_{\mathrm{sm}} \to U_{\mathrm{sm}}$ is a projection onto the second factor. Indeed, set

$$\begin{array}{l} Y(x) = Y \cap (\{x\} \times U_{\mathrm{sm}}) \subset \mathbf{P} \times U_{\mathrm{sm}}, \\ I = \{(x, (f_*)) | x \in V(f_*)\} \subset \mathbf{P} \times U_{\mathrm{sm}}, \\ I(x) = I \cap (\{x\} \times U_{\mathrm{sm}}) \subset \mathbf{P} \times U_{\mathrm{sm}}. \end{array}$$

Identifying $\{x\} \times U_{\mathrm{sm}} \cong U_{\mathrm{sm}}$, we can naturally look at $Y(x)$ and $I(x)$ as subsets in $U_{\mathrm{sm}}$.

**Lemma 4.** *The following estimate holds*

$$\operatorname{codim}_{I(x)} Y(x) \geq M + k + 1. \tag{11}$$

**Proof.** Indeed, the system of equations

$$q_{1,1} = \ldots = q_{1,k} = 0$$

defines the tangent space $T_x V(f_*)$, which is of dimension exactly $M$. Projectivizing, we get the set of polynomials

$$\mathcal{Q} = \{q_{i,j} | 1 \leq i \leq k, 2 \leq j \leq d_i, (i,j) \neq (k, d_k)\}$$

of degree $\geq 2$ on $\mathbf{P}^N$, $N = M - 1$. Obviously, $\sharp \mathcal{Q} = N$. Applying Proposition 3 to the set $\mathcal{Q}$ with $q_{1,1}, \ldots, q_{1,k}$ fixed, we see that

$$\begin{array}{rcl} \operatorname{codim}_{I(x)} Y(x) & \geq & \min\{2M-1, \displaystyle\sum_{i=1}^{k-1} \sum_{j=2}^{d_i} (j-1) + \sum_{j=2}^{d_k - 1} (j-1) + 2\} \\ & & \qquad\qquad\qquad\qquad \| \\ & & \displaystyle\frac{1}{2} \sum_{i=1}^{k-1} d_i(d_i - 1) + \frac{1}{2}(d_k - 1)(d_k - 2) + 2. \end{array}$$

Denote the last expression by $\xi = \xi(d_1, \ldots, d_k)$. Since $M \geq 2k + 1$, the inequality (11) follows from the estimate

$$\xi(d_1, \ldots, d_k) \geq M + k + 1.$$

Let us compute the minimum of the function $\xi$ in the real arguments $(a_1, \ldots, a_k) \in \mathbf{R}_+^k$, satisfying the relation $a_1 + \ldots + a_k = M + k$. Obviously, the minimum is attained at $a_1 = \ldots = a_{k-1} = s \geq 0$, $a_k = M + k - (k-1)s \geq 0$. It is easy to compute that

$$\xi(s, \ldots, s, M + k - (k-1)s) =$$

$$= \frac{1}{2}(k-1)(ks^2 - 2(M + k - 1)s) + \frac{1}{2}(M + k - 1)(M + k - 2) + 2 = \zeta(s).$$

The function $\zeta(s)$ attains its minimum at $s_* = (M+k-1)/k$. This value satisfies all the restrictions. We get

$$\zeta(s_*) = \frac{(M+k-1)(M-1)}{2k} + 2.$$

Since $M \geq 2k+1$, $\zeta(s_*) \geq M+k+1$, which is what we need.

Since obviously $\operatorname{codim}_{U_{\rm sm}} I(x) = k$, we get the estimate

$$\operatorname{codim}_{U_{\rm sm}} Y(x) \geq M + k + (k+1).$$

But the set $\pi(Y)$ is covered by the subsets $Y(x)$ for all $x \in \mathbf{P}$, from which we get that

$$\operatorname{codim}_{U_{\rm sm}} \overline{\pi(Y)} \geq k+1,$$

so that $\overline{\pi(Y)}$ is a proper subset of $U_{\rm sm}$. In particular, the open set

$$U_{\rm reg} = U_{\rm sm} \setminus \overline{\pi(Y)}$$

is non-empty. Q.E.D. for Proposition 2.

## 2.3 Tangent cones

Let $x \in X$ be a non-singular germ, $C \ni x$ a germ of an irreducible cycle of codimension $a \geq 1$,

$$\begin{array}{cccc} \varphi\colon & \widetilde{X} & \to & X \\ & \bigcup & & \\ & E & \to & x \end{array}$$

the blow-up of the point $x$, $E \cong \mathbf{P}^{\dim X - 1}$ the exceptional divisor, $\widetilde{C} \subset \widetilde{X}$ the proper inverse image of $C$ on $\widetilde{X}$. The algebraic cycle $\widetilde{C}_E = (\widetilde{C} \bullet E)$, the support of which is contained in $E$, is called the (*algebraic*) *projectivized tangent cone to $C$ at the point* $x$. Denote by $\pi\colon T_x X \setminus \{0\} \to E$ the natural projection. The algebraic cycle

$$\overline{\pi^{-1}(\widetilde{C}_E)},$$

the support of which lies in $T_x X$ ($\overline{\phantom{x}}$ stands for the closure), is called the (*algebraic*) *tangent cone to $C$ at the point $x$*. We denote it by $T_x C$. Accordingly, the cycle $\widetilde{C}_E$ on $E$ will be also written as $\mathbf{P}(T_x C)$.

Let $D \ni x$ be a prime Weil divisor on $X$. Assume, moreover, that none of the components of the cycle $T_x D$ contains a component of the cycle $T_x C$. Then obviously $C \not\subset D$, $(C \bullet D) \subset X$ is a cycle of codimension $a+1$. Moreover, we have

**Proposition 4.** *$\mathbf{P}(T_x(C \bullet D)) = (\widetilde{C}_E \bullet \widetilde{D}_E)_E$ and thus* $\operatorname{mult}_x(C \bullet D) = \operatorname{mult}_x C \cdot \operatorname{mult}_x D$.

**Proof.** This is obvious by the general intersection theory [F]. If $T_x D$ contains certain components of the cycle $T_x C$, then the multiplicity $\operatorname{mult}_x(C \bullet D)$ is strictly higher than the product of multiplicities of $C$ and $D$, see [P2].

If $\Sigma$ is a non-empty linear system on $X$ and $x \in \operatorname{Bs}\Sigma$, then we define the corresponding *tangent linear system* $T_x\Sigma$ on $T_xX \cong \mathbf{C}^{\dim X}$, setting a generic divisor $R \in T_x\Sigma$ to be $R = T_xD$, where $D \in \Sigma$ is a generic divisor. Projectivizing, we get the linear system $\mathbf{P}T_x\Sigma = \widetilde{\Sigma}|_E$, which consists of projectivized tangent cones $\mathbf{P}T_xD = \widetilde{D}|_E$, where $D \in \Sigma$ is a *generic* divisor. Obviously, $T_x\Sigma = \overline{\pi^{-1}(\mathbf{P}T_x\Sigma)}$. Note that the tangent linear system consists of cones of degree $\operatorname{mult}_x \Sigma$ with the vertex at the point $0 \in \mathbf{C}^{\dim X}$.

Defining the tangent linear system, we must take generic, and not arbitrary, divisors $D \in \Sigma$, because for special divisors, corresponding to subsystems $\Sigma' \subset \Sigma$, their multiplicity at the point $x$ can jump. More exactly, let $\Sigma' = \{D \in \Sigma | \operatorname{mult}_x D \geq \operatorname{mult}_x \Sigma + 1\} \subset \Sigma$ be a proper subsystem of the system $\Sigma$. The tangent map $T_x$ from $\Sigma$ to $T_x\Sigma$ is well defined outside $\Sigma'$:

$$\begin{aligned} T_x &\colon \Sigma \setminus \Sigma' \to T_x\Sigma, \\ T_x &\colon D \mapsto T_xD. \end{aligned}$$

Obviously, $T_x$ is a usual linear projection from $\mathbf{P}^{\dim\Sigma}$ to $\mathbf{P}^{\dim T_x\Sigma}$.

In the present paper we consider tangent linear systems for *hyper*tangent linear systems in the sense of Sec. 1.2. Thus we somewhat abuse our terminology, but hope that this will cause no misunderstanding.

Let $x \in \operatorname{Bs}\Sigma$ be a base point of the linear system $\Sigma$. The following claim is obvious.

**Lemma 5.** *Set-theoretically*

$$T_x \operatorname{Bs}\Sigma \subset \operatorname{Bs} T_x\Sigma.$$

*In particular, in a neighborhood of the point $x$*

$$\operatorname{codim}_X \operatorname{Bs}\Sigma \geq \operatorname{codim}_{T_xX} \operatorname{Bs} T_x\Sigma.$$

It is this simple fact that we use in the proof of Proposition 1; we need the tangent cones only for this purpose.

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