AN EISENSTEIN IDEAL FOR IMAGINARY QUADRATIC FIELDS AND THE BLOCH-KATO CONJECTURE FOR HECKE CHARACTERS

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Abstract. For certain algebraic Hecke characters χ of an imaginary quadratic field F we define an Eisenstein ideal in a p-adic Hecke algebra acting on cuspidal automorphic forms of GL_2/F. By finding congruences between Eisenstein cohomology classes (in the sense of G. Harder) and cuspforms we prove a lower bound for the index of the Eisenstein ideal in the Hecke algebra in terms of the special L-value $L(0,\chi)$. We further prove that its index is bounded from above by the p-valuation of the order of the Selmer group of the p-adic Galois character associated to χ^{-1}. This uses the work of R. Taylor *et al.* on attaching Galois representations to cuspforms of GL_2/F. Together these results imply a lower bound for the size of the Selmer group in terms of $L(0,\chi)$, coinciding with the value given by the Bloch-Kato conjecture.

1. Introduction

The aim of this work is to demonstrate the use of Eisenstein cohomology, as developed by Harder, in constructing elements of Selmer groups for Hecke characters of an imaginary quadratic field F. The strategy to first find congruences between Eisenstein series and cuspforms and then use the Galois representations attached to the cuspforms to prove lower bounds on the size of Selmer groups goes back to Ribet [Rib76], and has been applied and generalized in [Wil90], [HP92], [SU02], [BC04] amongst others. What is different in our situation is that the symmetric space associated to GL_2/F is not hermitian and that we therefore use for the congruences the integral structure coming from cohomology and the theory of Eisenstein cohomology classes.

We give a brief outline of the contents of the paper: Let p be a prime unramified in the extension $F/\mathbf{Q}$ and let $\mathfrak{p}$ be a prime of F dividing (p). Fix embeddings $\overline{F} \hookrightarrow \overline{F}_{\mathfrak{p}} \hookrightarrow \mathbf{C}$. Let $\phi_1, \phi_2 : F^* \backslash \mathbf{A}_F^* \to \mathbf{C}^*$ be two Hecke characters of infinity type z and z^{-1}, respectively. Let $\mathcal{R}$ be the ring of integers in a sufficiently large finite extension of $F_{\mathfrak{p}}$.

We start with the second step of the strategy sketched above. In §3 we recall proven and expected properties of the Galois representations attached to cuspforms of GL_2/F by the work of R. Taylor *et al.* Let $\mathbf{T}$ be the $\mathcal{R}$-algebra generated by Hecke operators acting on cuspidal automorphic forms of GL_2/F. For $\phi = (\phi_1, \phi_2)$ we define in §4 an Eisenstein ideal $\mathbf{I}_\phi$ in $\mathbf{T}$. Following previous work of Wiles and Urban we construct elements in the Selmer group of $\chi_{\mathfrak{p}}\epsilon$, where $\chi_{\mathfrak{p}}$ is the p-adic

Date: September 19, 2006.

2000 *Mathematics Subject Classification.* 11F80, 11F33, 11F75.

Key words and phrases. Selmer groups, Eisenstein cohomology, congruences of modular forms, Bloch-Kato conjecture.

Galois characters associated to $\chi := \phi_1/\phi_2$, and obtain a lower bound on its size in terms of that of the congruence module $\mathbf{T}/\mathbf{I}_\phi$. A complication that arises in the application of Taylor's theorem is that we need to work with cuspforms with cyclotomic central character. This is achieved by a twisting argument (see, in particular, Lemma 8).

From Section 5 on we impose some additional conditions on the prime p involving the class number and discriminant of F; we refer to the beginning of that section for the exact statement. To prove the lower bound on the congruence module in terms of the special L-value (the first step described above), we use the Eisenstein cohomology class $\mathrm{Eis}(\phi)$ constructed in [Ber06a] in the cohomology of a symmetric space S associated to GL_2/F. The class is an eigenvector for the Hecke operators at almost all places with eigenvalues corresponding to the generators of $\mathbf{I}_\phi$, and its restriction to the boundary of the Borel-Serre compactification of S is integral. The main result of [Ber06a], which we recall in §5, is that the denominator δ of $\mathrm{Eis}(\phi) \in H^1(S, \overline{F}_\mathfrak{p})$ is bounded from below by $L^{\mathrm{alg}}(0,\chi) \in \mathcal{R}$. We prove in Theorem 11 the existence of a cuspidal cohomology class congruent to $\delta \cdot \mathrm{Eis}(\phi)$ modulo the L-value supposing that there exists an integral cohomology class with the same restriction to the boundary as $\mathrm{Eis}(\phi)$. In §6 we prove that this hypothesis is satisfied for unramified χ. We achieve this by a careful analysis of the restriction map to the boundary $\partial\overline{S}$ of the Borel-Serre compactification. Starting with a group cohomological result for $\mathrm{SL}_2(\mathcal{O})$ due to Serre [Ser70] (which we extend to all maximal arithmetic subgroups of $\mathrm{SL}_2(F)$) we define an involution on $H^1(\partial\overline{S}, \mathcal{R})$ such that the restriction map

$$H^1(S,\mathcal{R}) \overset{\mathrm{res}}{\twoheadrightarrow} H^1(\partial\overline{S}, \mathcal{R})^-,$$

surjects onto the -1-eigenspace. We apply the resulting criterion to $\mathrm{res}(\mathrm{Eis}(\phi))$ to deduce the existence of a lift to $H^1(S,\mathcal{R})$.

Combining the two steps we obtain in §7 a lower bound for the size of the Selmer group of $\chi_\mathfrak{p}\epsilon$ in terms of $L^{\mathrm{alg}}(0,\chi)$ (unconditional for split p and unramified χ). To conclude, we relate this result to the Bloch-Kato conjecture. This conjecture has been proven in our case (at least for class number 1) starting from the Main Conjecture of Iwasawa theory for imaginary quadratic fields (see [Han97], [Guo93]). However, the method presented here, constructing elements in Selmer groups using cohomological congruences, is very different. Our hope is that it generalizes to higher rank groups.

These results generalize part of my thesis [Ber05] with Chris Skinner at the University of Michigan. The author would like to thank Trevor Arnold, Kris Klosin, Chris Skinner, and Eric Urban for helpful discussions.

2. Notation and Definitions

2.1. **General notation.** Let $F/\mathbf{Q}$ be an imaginary quadratic extension and d_F its absolute discriminant. Denote the classgroup by $\mathrm{Cl}(F)$ and the ray class group modulo a fractional ideal $\mathfrak{m}$ by $\mathrm{Cl}_\mathfrak{m}(F)$. For a place v of F let F_v be the completion of F at v. We write $\mathcal{O}$ for the ring of integers of F, $\mathcal{O}_v$ for the closure of $\mathcal{O}$ in F_v, $\mathfrak{P}_v$ for the maximal ideal of $\mathcal{O}_v$, π_v for a uniformizer of F_v, and $\hat{\mathcal{O}}$ for $\prod_{v \text{ finite}} \mathcal{O}_v$. We use the notations $\mathbf{A}, \mathbf{A}_f$ and $\mathbf{A}_F, \mathbf{A}_{F,f}$ for the adeles and finite adeles of $\mathbf{Q}$ and F, respectively, and write $\mathbf{A}^*$ and $\mathbf{A}_F^*$ for the group of ideles. Let p be a prime of $\mathbf{Z}$ that does not ramify in F, and let $\mathfrak{p} \subset \mathcal{O}$ be a prime dividing (p).

Denote by G_F the absolute Galois group of F. For Σ a finite set of places of F let G_Σ be the Galois group of the maximal extension of F unramified at all places not in Σ. We fix an embedding $\overline{F} \hookrightarrow \overline{F}_v$ for each place v of F. Denote the corresponding decomposition and inertia groups by G_v and I_v, respectively. Let $g_v = G_v/I_v$ be the Galois group of the maximal unramified extension of F_v. For each finite place v we also fix an embedding $\overline{F}_v \hookrightarrow \mathbf{C}$ that is compatible with the fixed embeddings $i_v : \overline{F} \hookrightarrow \overline{F}_v$ and $i_\infty : \overline{F} \hookrightarrow \mathbf{C}(= \overline{F}_\infty)$. For a discrete G_F-module (resp. G_v-module) M write $H^1(F, M)$ for the Galois cohomology group $H^1(G_F, M)$, and $H^1(F_v, M)$ for $H^1(G_v, M)$.

2.2. **Hecke characters.** A Hecke character of F is a continuous group homomorphism $\lambda : F^* \backslash \mathbf{A}_F^* \to \mathbf{C}^*$. Such a character corresponds uniquely to a character on ideals prime to the conductor, which we also denote by λ. Define the character λ^c by $\lambda^c(x) = \lambda(\overline{x})$.

Lemma 1 (Lemma 3.1 of [Ber05])**.** *If λ is an unramified Hecke character then $\lambda^c = \overline{\lambda}$.* □

For Hecke characters λ of type (A_0), i.e., with infinity type $\lambda_\infty(z) = z^m \overline{z}^n$ with $m, n \in \mathbf{Z}$ we define (following Weil) a p-adic Galois character

$$\lambda_{\mathfrak{p}} : G_F \to \overline{F}_{\mathfrak{p}}^*$$

associated to λ by the following rule: For a finite place v not dividing p or the conductor of λ, put $\lambda_{\mathfrak{p}}(\mathrm{Frob}_v) = i_{\mathfrak{p}}(i_\infty^{-1}(\lambda(\pi_v)))$ where Frob_v is the *arithmetic* Frobenius at v. It takes values in the integer ring of a finite extension of $F_{\mathfrak{p}}$.

Let $\epsilon : G_F \to \mathbf{Z}_p^*$ be the p-adic cyclotomic character defined by the action of G_F on the p-power roots of unity: $g.\xi = \xi^{\epsilon(g)}$ for ξ with $\xi^{p^m} = 1$ for some m. Our convention is that the Hodge-Tate weight of ϵ at $\mathfrak{p}$ is 1.

Write $L(0, \lambda)$ for the Hecke L-function of λ. Let λ a Hecke character of infinity type $z^a \left(\frac{z}{\overline{z}}\right)^b$ with conductor prime to p. Assume $a, b \in \mathbf{Z}$ and $a > 0$ and $b \geq 0$. Put

$$L^{\mathrm{alg}}(0, \lambda) := \Omega^{-a-2b} \left(\frac{2\pi}{\sqrt{d_F}} \right)^b \Gamma(a+b) \cdot L(0, \lambda).$$

In most cases, this normalization is integral, i.e., lies in the integer ring of a finite extension of $F_{\mathfrak{p}}$. See [Ber06a] Theorem 3 for the exact statement.

2.3. **Selmer groups.** Let $\rho : G_F \to \mathcal{R}^*$ be a continuous Galois character taking values in the ring of integers $\mathcal{R}$ of a finite extension L of $F_{\mathfrak{p}}$. Write $\mathfrak{m}_{\mathcal{R}}$ for its maximal ideal and put $\mathcal{R}^\vee = L/\mathcal{R}$. Let $\mathcal{R}_\rho$, L_ρ, and $\mathcal{R}_\rho^\vee$ be the free rank one modules on which G_F acts via ρ.

Following Bloch and Kato *et al.* we define the following Selmer groups: Let

$$H^1_f(F_v, L_\rho) = \begin{cases} \ker(H^1(F_v, L_\rho) \to H^1(I_v, L_\rho)) & \text{for } v \nmid p, \\ \ker(H^1(F_v, L_\rho) \to H^1(F_v, B_{\mathrm{cris}} \otimes L_\rho)) & \text{for } v \mid p, \end{cases}$$

where B_{cris} denotes Fontaine's ring of p-adic periods. Put $H^1_f(F_v, \mathcal{R}_\rho^\vee) = \mathrm{im}(H^1_f(F_v, L_\rho) \to H^1(F_v, \mathcal{R}_\rho^\vee))$. For a finite set of places Σ of F define

$$\mathrm{Sel}^\Sigma(F, \rho) = \ker \left(H^1(F, \mathcal{R}_\rho^\vee) \to \prod_{v \notin \Sigma} \frac{H^1(F_v, \mathcal{R}_\rho^\vee)}{H^1_f(F_v, \mathcal{R}_\rho^\vee)} \right).$$

We write $\mathrm{Sel}(F,\rho)$ for $\mathrm{Sel}^{\emptyset}(F,\rho)$.

If p splits in $F/\mathbf{Q}$ and $\rho=\lambda_{\mathfrak{p}}$ for a Hecke character λ of infinity type $z^a\overline{z}^b$ with $a,b\in\mathbf{Z}$ ("ordinary case") we define

$$F^+_{\mathfrak{p}}L_\rho=\begin{cases} L_\rho & \text{if } a<0 \text{ (i.e., HT-wt of } \rho>0),\\ \{0\} & \text{if } a\geq 0 \text{ (i.e., HT-wt of } \rho\leq 0)\end{cases}$$

and

$$F^+_{\overline{\mathfrak{p}}}L_\rho=\begin{cases} L_\rho & \text{if } b<0,\\ \{0\} & \text{if } b\geq 0.\end{cases}$$

In the ordinary case we have $H^1_f(F_v,L_\rho)=H^1(F_v,F^+_vL_\rho)$ for $v\mid p$ (see [Guo96] p.361, [Fla90] Lemma 2).

Lemma 2. *Let ρ be unramified at $v\nmid p$. If $\rho(\mathrm{Frob}_v)\not\equiv\epsilon(\mathrm{Frob}_v)\mod p$ then*

$$\mathrm{Sel}^{\Sigma}(F,\rho)=\mathrm{Sel}^{\Sigma\setminus\{v\}}(F,\rho).$$

Proof. By definition $\mathrm{Sel}^{\Sigma\setminus\{v\}}(F,\rho)\subset\mathrm{Sel}^{\Sigma}(F,\rho)$ for any v. For places v as in the lemma we have

$$H^1_f(F_v,\mathcal{R}^\vee_\rho)=\ker(H^1(F_v,\mathcal{R}^\vee_\rho)\to H^1(I_v,\mathcal{R}^\vee_\rho)^{g_v}).$$

It is clear that $H^1(I_v,\mathcal{R}^\vee_\rho)^{g_v}=\mathrm{Hom}_{g_v}(I_v^{\mathrm{tame}},\mathcal{R}^\vee_\rho)=\mathrm{Hom}_{g_v}(I_v^{\mathrm{tame}},\mathcal{R}^\vee_\rho[\mathfrak{m}^n_{\mathcal{R}}])$ for some n. By our assumption therefore $H^1(I_v,\mathcal{R}^\vee_\rho)^{g_v}=0$ since Frob_v acts on I_v^{tame} by $\epsilon(\mathrm{Frob}_v)$. $\square$

2.4. **Cuspidal automorphic representations.** We refer to [Urb95] §3.1 for definitions. We will be using the following notation: For $K_f=\prod_v K_v\subset G(\mathbf{A}_f)$ a compact open subgroup denote by $S_2(K_f,\mathbf{C})$ the space of cuspidal automorphic forms of $\mathrm{GL}_2(F)$ of weight 2, right-invariant under K_f. For ω a finite order Hecke character write $S_2(K_f,\omega,\mathbf{C})$ for the forms with central character ω. This is isomorphic as a $G(\mathbf{A}_f)$-module to $\bigoplus\pi_f^{K_f}$ for automorphic representations π of a certain infinity type (see Theorem 3 below) with central character ω. For $g\in G(\mathbf{A}_f)$ we have the Hecke action of $[K_fgK_f]$ on $S_2(K_f,\mathbf{C})$ and $S_2(K_f,\omega,\mathbf{C})$. For places v with $K_v=\mathrm{GL}_2(\mathcal{O}_v)$ we define $T_v=[K_f\begin{pmatrix}\pi_v & 0\\ 0 & 1\end{pmatrix}K_f]$.

2.5. **Cohomology of symmetric space.** Let $G=\mathrm{Res}_{F/\mathbf{Q}}\mathrm{GL}_2$, $K_\infty=U(2)\cdot\mathbf{C}^*\subset G(\mathbf{R})$. For an open compact subgroup $K_f\subset G(\mathbf{A}_f)$ we define the adelic symmetric space

$$S_{K_f}=G(\mathbf{Q})\backslash G(\mathbf{A})/K_\infty K_f.$$

Note that S_{K_f} has several connected components. In fact, strong approximation implies that the fibers of the determinant map

$$S_{K_f}\twoheadrightarrow\pi_0(K_f):=\mathbf{A}^*_{F,f}/\det(K_f)F^*$$

are connected. Any $\gamma\in G(\mathbf{A}_f)$ gives rise to an injection

$$G_\infty\hookrightarrow G(\mathbf{A})$$

$$g_\infty\mapsto(g_\infty,\gamma)$$

and, after taking quotients, to a component $\Gamma_\gamma\backslash G_\infty/K_\infty\to S_{K_f}$, where

$$\Gamma_\gamma:=G(\mathbf{Q})\cap\gamma K_f\gamma^{-1}.$$

This component is the fiber over $\det(\gamma)$. Choosing a system of representatives for $\pi_0(K_f)$ we therefore have

$$S_{K_f} \cong \coprod_{[\det(\gamma)]\in\pi_0(K_f)} \Gamma_\gamma\backslash\mathbf{H}_3,$$

where G_∞/K_∞ has been identified with three-dimensional hyperbolic space $\mathbf{H}_3 = \mathbf{R}_{>0}\times\mathbf{C}$.

We denote the Borel-Serre compactification of S_{K_f} by $\overline{S}_{K_f}$ and write $\partial\overline{S}_{K_f}$ for its boundary. The Borel-Serre compactification $\overline{S}_{K_f}$ is given by the union of the compactifications of its connected components. For any arithmetic subgroup $\Gamma\subset G(\mathbf{Q})$, the boundary of the Borel-Serre compactification of $\Gamma\backslash\mathbf{H}_3$, denoted by $\partial(\Gamma\backslash\overline{\mathbf{H}}_3)$, is homotopy equivalent to

$$\coprod_{[\eta]\in\mathbf{P}^1(F)/\Gamma} \Gamma_{B^\eta}\backslash\mathbf{H}_3, \tag{1}$$

where we identify $\mathbf{P}^1(F) = B(\mathbf{Q})\backslash G(\mathbf{Q})$, take $\eta\in G(\mathbf{Q})$, and put $\Gamma_{B^\eta} = \Gamma\cap\eta^{-1}B(\mathbf{Q})\eta$.

For $X\subset\overline{S}_{K_f}$ and R an $\mathcal{O}$-algebra we denote by $H^i(X,R)$ (resp. $H^i_c(X,R)$) the i-th (Betti) cohomology group (resp. with compact support), and the interior cohomology, i.e., the image of $H^i_c(X,R)$ in $H^i(X,R)$, by $H^i_!(X,R)$.

There is a Hecke action of double cosets $[K_f g K_f]$ for $g\in G(\mathbf{A}_f)$ on these cohomology groups (see [Urb98] §1.4.4 for the definition). We put $T_{\pi_v} = [K_f\begin{pmatrix}\pi_v & 0\\ 0 & 1\end{pmatrix}K_f]$ and $S_{\pi_v} = [K_f\begin{pmatrix}\pi_v & 0\\ 0 & \pi_v\end{pmatrix}K_f]$.

The connection between cohomology and cuspidal automorphic forms is given by the Eichler-Shimura-Harder isomorphism (in this special case see [Urb98] Theorem 1.5.1): For any compact open subgroup $K_f\subset G(\mathbf{A}_f)$ we have

$$S_2(K_f,\mathbf{C}) \xrightarrow{\sim} H^1_!(S_{K_f},\mathbf{C}) \tag{2}$$

and the isomorphism is Hecke-equivariant.

Recall from [Ber06a] Proposition 4 that for any $\mathcal{O}[\frac{1}{6}]$-algebra R there is a natural R-functorial isomorphism

$$H^1(\Gamma\backslash\mathbf{H}_3,R)\cong H^1(\Gamma,R), \tag{3}$$

where the group cohomology $H^1(\Gamma,R)$ is just given by $\mathrm{Hom}(\Gamma,R)$.

3. Galois representations associated to cuspforms for imaginary quadratic fields

Combining the work of Taylor, Harris, and Soudry with results of Friedberg-Hoffstein and Laumon/Weissauer, one can show the following (see [BHR]):

Theorem 3. *Given a cuspidal automorphic representation π of $\mathrm{GL}_2(\mathbf{A}_F)$ with π_∞ isomorphic to the principal series representation corresponding to*

$$\begin{pmatrix}t_1 & *\\ 0 & t_2\end{pmatrix}\mapsto\left(\frac{t_1}{|t_1|}\right)\left(\frac{|t_2|}{t_2}\right)$$

and cyclotomic central character ω (i.e., $\omega^c=\omega$), let Σ_π denote the set of places above p, the primes where π or π^c is ramified, and primes ramified in $F/\mathbf{Q}$.

Then there exists a continuous Galois representation

$$\rho_\pi : G_F \to \mathrm{GL}_2(\overline{F}_\mathfrak{p})$$

such that if $v \notin \Sigma_\pi$, then ρ_π is unramified at v and the characteristic polynomial of $\rho_\pi(\mathrm{Frob}_v)$ is $x^2 - a_v(\pi)x + \omega(\mathfrak{P}_v)\mathrm{Nm}_{F/\mathbf{Q}}(\mathfrak{P}_v)$, where $a_v(\pi)$ is the Hecke eigenvalue corresponding to T_v. The image of the Galois representation lies in $\mathrm{GL}_2(L)$ for a finite extension L of $F_\mathfrak{p}$ and the representation is absolutely irreducible. □

Remark.
(1) Taylor relates π to a low weight Siegel modular form via a theta lift and uses the Galois representation attached to this form via pseudorepresentations and the Galois representations of cohomological Siegel modular forms to find ρ_π.
(2) Taylor had some additional technical assumption in [Tay94] and only showed the equality of Hecke and Frobenius polynomial outside a set of places of zero density. For this strengthening of Taylor's result see [BHR].

Urban studied in [Urb98] the case of ordinary automorphic representations π, and together with results in [Urb05] on the Galois representations attached to ordinary Siegel modular forms shows:

Theorem 4 (Corollaire 2 of [Urb05])**.** *If π is unramified at $\mathfrak{p}$ and ordinary at $\mathfrak{p}$, i.e., $|a_\mathfrak{p}(\pi)|_p = 1$, then the Galois representation ρ_π is ordinary at $\mathfrak{p}$, i.e.,*

$$\rho_\pi|_{G_\mathfrak{p}} \cong \begin{pmatrix} \Psi_1 & * \\ 0 & \Psi_2 \end{pmatrix},$$

where $\Psi_2|_{I_\mathfrak{p}} = 1$, and $\Psi_1|_{I_\mathfrak{p}} = \det(\rho_\pi)|_{I_\mathfrak{p}} = \epsilon$.

For p inert we will need a stronger statement:

Conjecture 5. *If π is unramified at $\mathfrak{p}$ then $\rho_\pi|_{G_\mathfrak{p}}$ is crystalline.*

This conjecture extends Conjecture 3.2 in [CD06] and would follow if one could prove the corresponding statement for low weight Siegel modular forms.

4. Selmer group and Eisenstein ideal

Let ϕ_1 and ϕ_2 be two Hecke characters with infinity type z and z^{-1}, respectively. Let $\mathcal{R}$ be the ring of integers in the finite extension L of $F_\mathfrak{p}$ containing the values of the finite parts of ϕ_i and $L^{\mathrm{alg}}(0, \phi_1/\phi_2)$. Denote its maximal ideal by $\mathfrak{m}_\mathcal{R}$. Let Σ_ϕ be the finite set of places dividing the conductors of the characters ϕ_i and their complex conjugates and the places dividing pd_F. Let $K_f = \prod_v K_v \subset G(\mathbf{A}_f)$ be a compact open subgroup such that $K_v = \mathrm{GL}_2(\mathcal{O}_v)$ if $v \notin \Sigma_\phi$.

Assume that there exists a finite order character η unramified outside Σ_ϕ such that $(\phi_1\phi_2\eta^2)^c = \phi_1\phi_2\eta^2$.

Denote by $\mathbf{T}$ the $\mathcal{R}$-algebra generated by the Hecke operators $T_v, v \notin \Sigma_\phi$ acting on $S_2(K_f, \phi_1\phi_2, \mathbf{C})$. Call the ideal $\mathbf{I}_\phi \subseteq \mathbf{T}$ generated by

$$\{T_v - \phi_1(\mathfrak{P}_v)\mathrm{Nm}(\mathfrak{P}_v) - \phi_2(\mathfrak{P}_v) | v \notin \Sigma_\phi\}$$

the *Eisenstein ideal associated to $\phi = (\phi_1, \phi_2)$*.

We define Galois characters

$$\begin{aligned} \rho_1 &= \phi_{1,\mathfrak{p}}\epsilon, \\ \rho_2 &= \phi_{2,\mathfrak{p}}, \\ \rho &= \rho_1 \otimes \rho_2^{-1}. \end{aligned}$$

Let Σ_ρ be the set of places dividing p and those where ρ is ramified.

Theorem 6.
$$\mathrm{val}_p(\#\mathrm{Sel}^{\Sigma_\phi \setminus \Sigma_\rho}(F, \rho)) \geq \mathrm{val}_p(\#(\mathbf{T}/\mathbf{I}_\phi)).$$

Proof. We can assume that
$$\mathbf{T}/\mathbf{I}_\phi \neq 0.$$
Let $\mathfrak{m} \subset \mathbf{T}$ be a maximal ideal containing $\mathbf{I}_\phi$. Taking the completion with respect to $\mathfrak{m}$ we write
$$S_2(K_f, \phi_1\phi_2, \mathbf{C})_\mathfrak{m} = \bigoplus_{i=1}^n V_{\pi_{i,f}}^{K_f},$$
where V_{π_f} denotes the representation space of the (finite part) of the cuspidal automorphic representation π.

By twisting the cuspforms by the finite order character η we can ensure that their central character is cyclotomic. Hence we can apply Theorem 3 to associate Galois representations $\rho_{\pi_i \otimes \eta} : G_{\Sigma_\phi} \to \mathrm{GL}_2(L_i)$ for finite extensions $L_i/F_\mathfrak{p}$ to each $\pi_i \otimes \eta$. Taking all of them together (and untwisting by η) we obtain a continuous, absolutely irreducible Galois representation
$$\rho_T := \bigoplus_{i=1}^n \rho_{\pi_i \otimes \eta} \otimes \eta^{-1} : G_{\Sigma_\phi} \to \mathrm{GL}_2(\mathbf{T}_\mathfrak{m} \otimes_\mathcal{R} L).$$
Here we use that $\mathbf{T}_\mathfrak{m} \otimes_\mathcal{R} L = \prod_{i=1}^n L_i$, which follows from the strong multiplicity one theorem. We have an embedding
$$\mathbf{T}_\mathfrak{m} \hookrightarrow \prod_{i=1}^n L_i$$
$$T_v \mapsto ((a_v(\pi_i)),$$
where $a_v(\pi_i)$ is the T_v-eigenvalue of π_i. The coefficients of the characteristic polynomial $\mathrm{char}(\rho_T)$ therefore lie in $\mathbf{T}_\mathfrak{m}$ and by the Chebotarev density theorem
$$\mathrm{char}(\rho_T) \equiv \mathrm{char}(\rho_1 \oplus \rho_2) \mod \mathbf{I}_\phi.$$

For any finite free $\mathbf{T}_\mathfrak{m} \otimes L$-module $\mathcal{M}$ any $\mathbf{T}_\mathfrak{m}$-submodule $\mathcal{L} \subset \mathcal{M}$, finite over $\mathbf{T}_\mathfrak{m}$ such that $\mathcal{L} \otimes L = \mathcal{M}$ is called a $\mathbf{T}_\mathfrak{m}$*-lattice.*

Specializing to our situation Theorem 1 of [Urb01] (and using that the $\mathcal{R}$-algebra map surjects onto $\mathbf{T}_\mathfrak{m}/\mathbf{I}_\phi$) we get:

Theorem 7 (Urban)**.** *Given a Galois representation ρ_T as above there exists a G_{Σ_ϕ}-stable $\mathbf{T}_\mathfrak{m}$-lattice $\mathcal{L} \subset (\mathbf{T}_\mathfrak{m} \otimes L)^2$ such that G_{Σ_ϕ} acts on $\mathcal{L}/\mathbf{I}_\phi\mathcal{L}$ via*
$$0 \to \mathcal{R}_{\rho_1} \otimes_\mathcal{R} (N/\mathbf{I}_\phi) \to \mathcal{L}/\mathbf{I}_\phi\mathcal{L} \to \mathcal{R}_{\rho_2} \otimes_\mathcal{R} (\mathbf{T}_\mathfrak{m}/\mathbf{I}_\phi) \to 0,$$
where $N \subset \mathbf{T}_\mathfrak{m} \otimes L$ is a $\mathbf{T}_\mathfrak{m}$-lattice with $\mathrm{val}_p(\#\mathbf{T}_\mathfrak{m}/\mathbf{I}_\phi) \leq \mathrm{val}_p(\#N/\mathbf{I}_\phi N) < \infty$ and no quotient of $\mathcal{L}$ is isomorphic to $\overline{\rho}_1 := \rho_1 \mod \mathfrak{m}_\mathcal{R}$. □

See [Ber05] §7.3.2 for an alternative construction of such a lattice involving arguments of Wiles ([Wil86] and [Wil90]).

Using the properties of the Galois representations attached to cuspforms we can now conclude the proof of Theorem 6 by similar arguments as in [Ski04] and [Urb01]. For brevity put $\mathcal{T} := N/\mathbf{I}_\phi$, and $\Sigma := \Sigma_\phi$.

Identifying $\mathcal{R}_\rho = \mathrm{Hom}_{\mathcal{R}}(\mathcal{R}_{\rho_2}, \mathcal{R}_{\rho_1})$ and writing $s : \mathcal{R}_{\rho_2} \otimes \mathbf{T}_\mathfrak{m}/\mathbf{I}_\phi \to \mathcal{L} \otimes \mathbf{T}_\mathfrak{m}/\mathbf{I}_\phi$ for the section as $\mathbf{T}_\mathfrak{m}/\mathbf{I}_\phi$-modules we define a 1-cocycle $c : G_\Sigma \to \mathcal{R}_\rho \otimes \mathcal{T}$ by

$$c(g)(m) = \text{ the image of } s(m) - g.s(g^{-1}.m) \text{ in } \mathcal{R}_{\rho_1} \otimes \mathcal{T}.$$

Consider the $\mathcal{R}$-homomorphism

$$\varphi : \mathrm{Hom}_{\mathcal{R}}(\mathcal{T}, \mathcal{R}^\vee) \to H^1(G_\Sigma, \mathcal{R}_\rho^\vee),\ \varphi(f) = \text{ the class of } (1 \otimes f) \circ c.$$

We will show that

(i) $\mathrm{im}(\varphi) \subset \mathrm{Sel}^{\Sigma \setminus \Sigma_\rho}(F, \rho)$,

(ii) $\ker(\varphi) = 0$.

From (i) it follows that

$$\mathrm{val}_p(\#\mathrm{Sel}^{\Sigma \setminus \Sigma_\rho}(F, \rho)) \geq \mathrm{val}_p(\#\mathrm{im}(\varphi)).$$

From (ii) it follows that

$$\begin{aligned} \mathrm{val}_p(\#\mathrm{im}(\varphi)) &\geq \mathrm{val}_p(\#\mathrm{Hom}_{\mathcal{R}}(\mathcal{T}, \mathcal{R}^\vee)) \\ &= \mathrm{val}_p(\#\mathcal{T}) \\ &\geq \mathrm{val}_p(\#\mathbf{T}_\mathfrak{m}/\mathbf{I}_\phi). \end{aligned}$$

For (i) we have to show that the conditions of the Selmer group at $v \mid p$ are satisfied: For split p it suffices to prove that the extension in Theorem 7 is split when considered as an extension of $\mathbf{T}_\mathfrak{m}[G_\mathfrak{p}]$-modules because then the class in $H^1(G_\mathfrak{p}, \mathcal{R}_\rho \otimes \mathcal{T})$ determined by c is the zero class. In this case the Hecke eigenvalues $a_\mathfrak{p}(\pi_i) \equiv p \cdot \phi_1(\mathfrak{p}) + \phi_2(\mathfrak{p}) \not\equiv 0 \mod \mathfrak{m}_R$, hence the cuspforms $\pi_i \otimes \eta$ are ordinary at $\mathfrak{p}$, so Theorem 4 applies and ρ_T is ordinary. Observing that the Hodge-Tate weights at $\mathfrak{p}$ of ρ_1 and ρ_2 are 0 and 1, respectively, the splitting of the extension as $\mathbf{T}_\mathfrak{m}[G_\mathfrak{p}]$-modules follows from comparing the basis given by Theorem 7 with the one coming from ordinarity.

For inert p we observe that by Conjecture 5 the ρ_{π_i} are all crystalline which implies that the class determined by c is crystalline.

To prove (ii) we first observe that for any $f \in \mathrm{Hom}_{\mathcal{R}}(\mathcal{T}, \mathcal{R}^\vee)$, $\ker(f)$ has finite index in $\mathcal{T}$ since $\mathcal{T}$ is a finite $\mathcal{R}$-module and so $f \in \mathrm{Hom}_{\mathcal{R}}(\mathcal{T}, \mathcal{R}^\vee[\mathfrak{m}_{\mathcal{R}}^n])$ for some n. Suppose now that $f \in \ker(\varphi)$. We claim that the class of c in $H^1(G_\Sigma, \mathcal{R}_\rho \otimes_{\mathcal{R}} \mathcal{T}/\ker(f))$ is zero. To see this, let $X = \mathcal{R}^\vee/\mathrm{im}(f)$ and observe that there is an exact sequence

$$H^0(G_\Sigma, \mathcal{R}_\rho \otimes_{\mathcal{R}} X) \to H^1(G_\Sigma, \mathcal{R}_\rho \otimes_{\mathcal{R}} \mathcal{T}/\ker(f)) \to H^1(G_\Sigma, \mathcal{R}_\rho \otimes_{\mathcal{R}} \mathcal{R}^\vee).$$

Since $f \in \ker(\varphi)$ and the second arrow in the sequence comes from the inclusion $\mathcal{T}/\ker(\varphi) \hookrightarrow \mathcal{R}^\vee$ induced by f, the image in the right module of the class of c in the middle is zero. Our claim follows therefore if the module on the left is trivial. But the dual of this module is a subquotient of $\mathrm{Hom}_{\mathcal{R}}(\mathcal{R}_\rho, \mathcal{R})$ on which G_Σ acts trivially. By assumption, however, $\mathcal{R}_\rho$ is a rank one module on which G_Σ acts non-trivially.

Suppose in addition that f is non-trivial, i.e., $\ker(f) \subsetneq \mathcal{T}$. Note that any $\mathcal{R}$-submodule of $\mathcal{T}$ is actually a $\mathbf{T}_\mathfrak{m}$-submodule since $\mathcal{R} \twoheadrightarrow \mathbf{T}_\mathfrak{m}/\mathbf{I}_\phi$. Hence there exists a $\mathbf{T}_\mathfrak{m}$-module A with $\ker(f) \subset A \subset \mathcal{T}$ such that $\mathcal{T}/A \cong \mathcal{R}/\mathfrak{m}_{\mathcal{R}}$. From our claim it follows that the $\mathbf{T}_\mathfrak{m}[G_\Sigma]$-extension

$$0 \to \mathcal{R}_{\rho_1} \otimes_{\mathcal{R}} \mathcal{R}/\mathfrak{m}_{\mathcal{R}} \cong \mathcal{R}_{\rho_1} \otimes_{\mathcal{R}} \mathcal{T}/A \to \mathcal{L}/(\mathcal{R}_{\rho_1} \otimes_{\mathcal{R}} A) \to \mathcal{L}/\mathcal{L}_1 \to 0$$

is split. But this would give a $\mathbf{T}_{\mathfrak{m}}[G_\Sigma]$-quotient of $\mathcal{L}$ isomorphic to $\overline{\rho}_1$, which contradicts one of the properties of the lattice constructed by Urban. Hence $\ker(\varphi)$ is trivial. □

Under conditions which will be satisfied in our later application the following Lemma will provide us with the finite order character η used in the twisting above.

Lemma 8. *If $\chi = \phi_1/\phi_2$ satisfies $\chi^c = \overline{\chi}$ then there exists a finite order character η unramified outside Σ_ϕ such that $(\phi_1\phi_2\eta^2)^c = \phi_1\phi_2\eta^2$.*

Proof. We claim that there exists a Hecke character μ unramified outside Σ_ϕ such that

$$\chi = \mu\overline{\mu}^c.$$

Given such a character μ we then define $\eta = (\mu\phi_2)^{-1}$.

In the Lemma on p.81 of [Gre83] Greenberg defines a Hecke character $\mu_G : F^*\backslash\mathbf{A}_F^* \to \mathbf{C}^*$ of infinity type z^{-1} such that $\mu_G^c = \overline{\mu}_G$ and μ_G is ramified exactly at the primes ramified in $F/\mathbf{Q}$. It therefore suffices to prove the claim for the finite order character

$$\chi' := \chi\mu_G^2 = \chi\mu_G(\overline{\mu}_G^c).$$

By assumption we have that

$$\chi' \equiv 1 \text{ on } \mathrm{Nm}_{F/\mathbf{Q}}(\mathbf{A}_F^*) \subset \mathbf{A}_\mathbf{Q}^* \subset \mathbf{A}_F^*.$$

Thus χ' restricted to $\mathbf{Q}^*\backslash\mathbf{A}_\mathbf{Q}^*$ is either the quadratic character of $F/\mathbf{Q}$ or trivial. Since our finite order character has trivial infinite component, χ' has to be trivial on $\mathbf{Q}^*\backslash\mathbf{A}_\mathbf{Q}^*$. Hilbert's Theorem 90 then implies that there exists μ such that $\chi' = \mu/\mu^c$.

To control the ramification we analyze this last step closer: χ' factors through $\mathbf{A}_F^* \to A$, where A is the subset of $\mathbf{A}_F^*$ of elements of the form x/x^c and the map is $x \mapsto x/x^c$. If $y \in A \cap F^*$ then y has trivial norm and so by Hilbert's Theorem 90, $y = x/x^c$ for some $x \in F^*$. Thus the induced character $A \to \mathbf{C}^*$ vanishes on $A \cap F^*$. This implies that there is a continuous finite order character $\mu : F^*\backslash\mathbf{A}_F^* \to \mathbf{C}^*$ which restricts to this character on A and $\chi' = \mu/\mu^c$ (this argument is taken from the proof of Lemma 1 in [Tay94]).

By the following argument we can further conclude that the induced character vanishes on $A \cap \prod_{v\notin\Sigma_\phi}\mathcal{O}_v^*$ and therefore find μ on $F^*\backslash\mathbf{A}_F^*/\mathbf{C}^*\prod_{v\notin\Sigma_\phi}\mathcal{O}_v^*$ restricting to the character $A \to \mathbf{C}^*$: Writing $U_{F,\ell} = \prod_{v|\ell}\mathcal{O}_v^*$ for a prime ℓ in $\mathbf{Q}$ we have

$$H^1(\mathrm{Gal}(F/\mathbf{Q}), \prod_{v\notin\Sigma_\phi}\mathcal{O}_v^*) \cong \prod_{\ell\notin\Sigma_\phi} H^1(\mathrm{Gal}(F/\mathbf{Q}), U_{F,\ell}),$$

where "$\ell \notin \Sigma_\phi$" denotes those $\ell \in \mathbf{Z}$ such that $v \mid \ell \Rightarrow v \notin \Sigma_\phi$. For the isomorphism we use that

$$v \in \Sigma_\phi \Rightarrow \overline{v} \in \Sigma_\phi.$$

In fact, all these groups are trivial since all $\ell \notin \Sigma_\phi$ are unramified in $F/\mathbf{Q}$ and so

$$H^1(\mathrm{Gal}(F/\mathbf{Q}), U_{F,\ell}) \cong H^1(G_v, \mathcal{O}_v^*) = 1.$$

If $y \in A \cap \prod_{v\notin\Sigma_\phi}\mathcal{O}_v^*$ then y has trivial norm in $\prod_{v\notin\Sigma_\phi}\mathcal{O}_v^*$. But as shown, its first Galois cohomology group is trivial so there exists $x \in \prod_{v\notin\Sigma_\phi}\mathcal{O}_v^* \cap \mathbf{A}_F^*$ such that $y = x/x^c$. Since χ' is unramified outside Σ_ϕ the image of y under the induced character therefore equals $\chi'(x) = 1$, as claimed above. □

5. Bounding the Eisenstein ideal

From now on we impose the following assumptions on the prime p: Let $p > 3$ be a prime of $\mathbf{Z}$ that does not ramify in F and does not divide $\#\mathrm{Cl}(F)$. Assume in addition that $\ell \not\equiv \pm 1 \mod p$ for $\ell \mid d_F$.

Recall the definitions and notations introduced in Section 2.5. Following Harder we constructed in [Ber06a] *Eisenstein cohomology classes* in the Betti cohomology group $H^1(S_{K_f}, \mathbf{C})$. Given a pair of Hecke characters $\phi = (\phi_1, \phi_2)$ with $\phi_{1,\infty}(z) = z$ and $\phi_{2,\infty}(z) = z^{-1}$ these depend on a choice of a function Ψ_{ϕ_f} in the induced representation

$$V_{\phi_f,\mathbf{C}}^{K_f} = \{\Psi : G(\mathbf{A}_f) \to \mathbf{C} | \Psi(bg) = \phi_f(b)\Psi(g) \forall b \in B(\mathbf{A}_f), \Psi(gk) = \Psi(g) \forall k \in K_f\}.$$

Refering to notation in [Ber06a] we choose $K_f = K_f^S$ and $\Psi_{\phi_f} = \Psi_\phi^0$. We recall the definition of the compact open K_f: Denote by S the finite set of places where both ϕ_i are ramified, but ϕ_1/ϕ_2 is unramified. Write $\mathfrak{M}_i$ for the conductor of ϕ_i. For an ideal $\mathfrak{N}$ in $\mathcal{O}$ and a finite place v of F put $\mathfrak{N}_v = \mathfrak{N}\mathcal{O}_v$. We define

$$K^1(\mathfrak{N}_v) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2(\mathcal{O}_v), a-1, c \equiv 0 \mod \mathfrak{N}_v \right\},$$

and

$$U^1(\mathfrak{N}_v) = \{k \in \mathrm{GL}_2(\mathcal{O}_v) : \det(k) \equiv 1 \mod \mathfrak{N}_v\}.$$

Now put

$$K_f := \prod_{v \in S} U^1(\mathfrak{M}_{1,v}) \prod_{v \notin S} K^1((\mathfrak{M}_1\mathfrak{M}_2)_v).$$

The exact definition of Ψ_{ϕ_f} will not be required in the following; we refer the interested reader to [Ber06a]. For brevity we write $\mathrm{Eis}(\phi)$ for the cohomology class denoted by $[\mathrm{Eis}(\Psi^0_{(\phi_1,\phi_2)_f})]$ in [Ber06a].

Let $\mathcal{R}$ again denote the ring of integers in the finite extension L of $F_\mathfrak{p}$ obtained by adjoining the values of the finite part of both ϕ_i and $L^{\mathrm{alg}}(0, \phi_1/\phi_2)$. We write

$$\widetilde{H}^1(X, \mathcal{R}) := H^1(X, \mathcal{R})_{\mathrm{free}} = \mathrm{im}(H^1(X, \mathcal{R}) \to H^1(X, L))$$

for $X = S_{K_f}$ or $\partial \overline{S}_{K_f}$. Also put

$$\widetilde{H}^1_!(S_{K_f}, \mathcal{R}) = H^1_!(S_{K_f}, L) \cap \widetilde{H}^1(S_{K_f}, \mathcal{R}).$$

We recall the following properties of $\mathrm{Eis}(\phi)$ proven in [Ber06a]:

(E1) $\mathrm{Eis}(\phi) \in H^1(S_{K_f}, L)$ ([Ber06a] Proposition 13)

(E2) If $\overline{\phi_1/\phi_2} = (\phi_1/\phi_2)^c$ and the conductors of the ϕ_i are coprime to (p) then the image of $\mathrm{Eis}(\phi)$ under $\mathrm{res} : H^1(S_{K_f}, \mathbf{C}) \to H^1(\partial \overline{S}_{K_f}, \mathbf{C})$ lies in $\widetilde{H}^1(\partial \overline{S}_{K_f}, \mathcal{R})$ ([Ber06a] Proposition 16).

(E3) For all places v outside the conductors of the ϕ_i the class $\mathrm{Eis}(\phi)$ is an eigenvector for the Hecke operator $T_{\pi_v} = [K_f \begin{pmatrix} \pi_v & 0 \\ 0 & 1 \end{pmatrix} K_f]$ with eigenvalue

$$\phi_2(\mathfrak{P}_v) + \mathrm{Nm}(\mathfrak{P}_v)\phi_1(\mathfrak{P}_v)$$

([Ber06a] Lemma 9).

(E4) The central character of $\mathrm{Eis}(\phi)$ is given by $\phi_1\phi_2$, i.e., the Hecke operators $S_{\pi_v} = [K_f \begin{pmatrix} \pi_v & 0 \\ 0 & \pi_v \end{pmatrix} K_f]$ act on it by multiplication by $(\phi_1\phi_2)(\mathfrak{P}_v)$.

Property (E1) allows us to define the denominator (ideal) of $\mathrm{Eis}(\phi)$, given by

$$\delta(\mathrm{Eis}(\phi)) := \{a \in \mathcal{R} : a \cdot \mathrm{Eis}(\phi) \in \widetilde{H}^1(S_{K_f}, \mathcal{R})\}.$$

Under certain conditions we prove in [Ber06a] that $\delta(\mathrm{Eis}(\phi)) \subset (L^{\mathrm{alg}}(0, \phi_1/\phi_2))$, i.e., that the denominator is bounded from below by the special L-value.

Suppose now that we are given a Hecke character χ of infinity type z^2 such that $\chi^c = \overline{\chi}$. Assume that the conductor $\mathfrak{M}$ of χ is coprime to (p). We would like to find a pair of characters $\phi = (\phi_1, \phi_2)$ with $\chi = \phi_1/\phi_2$ such that

($\phi1$) the conductor $\mathfrak{M}_1$ of ϕ_1 is coprime to $(p)\mathfrak{M}$, and $p \nmid \#(\mathcal{O}/\mathfrak{M}_1)^*$,
($\phi2$) $d_F \mid \#\mathcal{O}/\mathfrak{M}_1$ and

$$v \mid \mathfrak{M}_1 \Rightarrow v = \overline{v} \text{ and } \#\mathcal{O}_v/\mathfrak{P}_v \not\equiv \pm 1 \mod p,$$

and such that the class $\mathrm{Eis}(\phi)$ satisfies

(E5) $\delta(\mathrm{Eis}(\phi)) \subseteq (L^{\mathrm{alg}}(0, \chi))$.

Remark. In Section 7 property ($\phi2$) will allow us to apply Lemma 2 to prove a lower bound for the Selmer group $\mathrm{Sel}^{\emptyset}(F, \chi_{\mathfrak{p}}\epsilon)$ starting from Theorem 6.

Write $\omega_{F/\mathbf{Q}}$ for the quadratic Hecke character associated to the extension $F/\mathbf{Q}$ and $\tau(\tilde{\chi})$ for the Gauss sum of the unitary character $\tilde{\chi} := \chi/|\chi|$. From the proof of Theorem 29(ii) of [Ber06a] we deduce:

Theorem 9 ([Ber06a] Theorem 29)**.** *Assume that no ramified primes (or 2 if $F = \mathbf{Q}(\sqrt{-3})$) divide $\mathfrak{M}$ and no inert primes congruent to $-1 \mod p$ divide $\mathfrak{M}$ with multiplicity one, and that*

$$\omega_{F/\mathbf{Q}}(\mathfrak{M}) \frac{\tau(\tilde{\chi})}{\sqrt{\mathrm{Nm}(\mathfrak{M})}} = 1.$$

Then there exists a character $\phi = (\phi_1, \phi_2)$ satisfying ($\phi1$) and ($\phi2$) such that (E5) holds for $\mathrm{Eis}(\phi)$.

We lastly need the following assumption:

(H) Assume that there exists $c \in \widetilde{H}^1(S_{K_f}, \mathcal{R})$ with

$$\mathrm{res}(c) = \mathrm{res}(\mathrm{Eis}(\phi)) \in \widetilde{H}^1(\partial \overline{S}_{K_f}, \mathcal{R}).$$

This hypothesis is satisfied, for example, if $H^2_c(S_{K_f}, \mathcal{R})$ has no torsion. By Lefschetz duality (see [Gre67] (28.18) or [Mau80] Theorem 5.4.13)

$$H^2_c(S_{K_f}, \mathcal{R}) \cong H_1(S_{K_f}, \mathcal{R}),$$

so the occurrence of torsion reduces to the problem of torsion in Γ^{ab} for arithmetic subgroups $\Gamma \subset G(\mathbf{Q})$. This has been studied in [EGM82], [SV83], and [GS93] (see also [EGM98] §7.5). An arithmetic interpretation or explanation for the torsion has not been found yet in general (but see [EGM82] for examples in the case of $\mathbf{Q}(\sqrt{-1})$). Based on computer calculations [GS93] (2) suggests that for $\Gamma \subset \mathrm{PSL}_2(\mathcal{O})$ apart from 2 and 3 only primes less than or equal to $\frac{1}{2}[\mathrm{PSL}_2(\mathcal{O}) : \Gamma]$ occur in the torsion of Γ^{ab}. In all cases calculated so far, $\mathrm{PSL}_2(\mathcal{O})^{\mathrm{ab}}$ has only 2 or 3-torsion (see also [Swa71], [Ber06b]) but this is not known in general, hence our different approach in the following section. Even restricting to the ordinary part there can be torsion, see [Tay]§4. In the following section we will prove:

Theorem 10. *Let χ be an unramified Hecke character of infinity type z^2. Then (H) is satisfied.*

The main result of this section is the following bound on the congruence module introduced in the previous section:

Theorem 11. *Assume in addition to the assumptions on χ in Theorem 9 that $p \nmid \#(\mathcal{O}/\mathfrak{M})^*$ and that (H) holds. Then for ϕ given by Theorem 9 there is an $\mathcal{R}$-algebra surjection*

$$\mathbf{T}/\mathbf{I}_\phi \twoheadrightarrow \mathcal{R}/\left(L^{\mathrm{alg}}(0,\chi)\right).$$

Remark. The condition $p \nmid \#(\mathcal{O}/\mathfrak{M}d_F)^*$ can be weakened to the order of $\chi|_{\hat{\mathcal{O}}^*}$ being coprime to p, see [Ber05] §6.1.

By Lemma 1 unramified characters χ satisfy $\chi^c = \overline{\chi}$, so we deduce:

Corollary 12. *Let χ be an unramified Hecke character of infinity type z^2. Then for ϕ given by Theorem 9 there is an $\mathcal{R}$-algebra surjection*

$$\mathbf{T}/\mathbf{I}_\phi \twoheadrightarrow \mathcal{R}/\left(L^{\mathrm{alg}}(0,\chi)\right).$$

□

Proof of Theorem 11. By [Urb98] §1.2 and §1.4.5 we have an $\mathcal{R}$-linear action of the ray class group $\mathrm{Cl}_{\mathfrak{M}\mathfrak{M}_1}(F)$ on $H^1(S_{K_f},\mathbf{C})$ via the diamond operators S_{π_v}. Here we use that

$$K_f \supset K(\mathfrak{M}\mathfrak{M}_1) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2(\hat{\mathcal{O}}) : \begin{pmatrix} a & b \\ c & d \end{pmatrix} \equiv \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \mod \mathfrak{M}\mathfrak{M}_1 \right\}.$$

By (ϕ1) and the assumption that $p \nmid \#(\mathcal{O}/\mathfrak{M})^*$, the ray class group has order prime to p, so $\mathcal{R}[\mathrm{Cl}_{\mathfrak{M}\mathfrak{M}_1}(F)]$ is semisimple. For $\omega := \phi_1\phi_2$, which can be viewed as a character of $\mathrm{Cl}_{\mathfrak{M}\mathfrak{M}_1}(F)$, let e_ω be the idempotent associated to ω, so that $S_v e_\omega = \omega(\pi_v)e_\omega$.

Under the Eichler-Shimura-Harder isomorphism (see (2)) we have

$$e_\omega H^1_!(S_{K_f},\mathbf{C}) \cong S_2(K_f,\omega,\mathbf{C}).$$

Hence the Hecke algebra $\mathbf{T}$ from Section 4 is isomorphic to the $\mathcal{R}$-subalgebra of

$$\mathrm{End}_{\mathcal{R}}(e_\omega \widetilde{H}^1_!(S_{K_f},\mathcal{R}))$$

generated by the Hecke operators T_{π_v} for all primes $v \notin \Sigma_\phi$ and we will identify the two.

Recall the long exact sequence

$$\ldots \to H^1_c(S_{K_f},R) \to H^1(S_{K_f},R) \overset{\mathrm{res}}{\to} H^1(\partial\overline{S}_{K_f},R) \to H^2_c(S_{K_f},R) \to \ldots$$

for any $\mathcal{R}$-algebra R.

Note that for $c \in \widetilde{H}^1(S_{K_f},\mathcal{R})$ given by (H) we have

$$\mathrm{res}(e_\omega c) = e_\omega \mathrm{res}(c) = e_\omega \mathrm{res}(\mathrm{Eis}(\phi)) = \mathrm{res}(\mathrm{Eis}(\phi))$$

since $S_v(\mathrm{Eis}(\phi)) = \omega(\pi_v)\mathrm{Eis}(\phi)$ by (E4).

Without loss of generality, we can assume that $\delta(\mathrm{Eis}(\phi)) \subsetneq \mathcal{R}$; there is nothing to prove otherwise by (E5). Let δ be a generator of $\delta(\mathrm{Eis}(\phi))$. Then $\delta \cdot \mathrm{Eis}(\phi)$ is an element of an $\mathcal{R}$-basis of $e_\omega \widetilde{H}^1(S_{K_f},\mathcal{R})$. By construction, $c_0 := \delta \cdot (e_\omega c - \mathrm{Eis}(\phi)) \in$

$e_\omega H^1_!(S_{K_f}, L)$ is a nontrivial element of an $\mathcal{R}$-basis of $e_\omega \widetilde{H}^1_!(S_{K_f}, \mathcal{R})$. Extend c_0 to an $\mathcal{R}$-basis $c_0, c_1, \dots c_d$ of $e_\omega \widetilde{H}^1_!(S_{K_f}, \mathcal{R})$. For each $t \in \mathbf{T}$ write

$$t(c_0) = \sum_{i=0}^{d} a_i(t)c_i,\ a_i(t) \in \mathcal{R}.$$

Then

$$\mathbf{T} \to \mathcal{R}/(\delta),\ t \mapsto a_0(t) \mod \delta \tag{4}$$

is an $\mathcal{R}$-module surjection. We claim that it is independent of the $\mathcal{R}$-basis chosen and that it is a homomorphism of $\mathcal{R}$-algebras with the Eisenstein ideal $\mathbf{I}_\phi$ contained in its kernel. To prove this it suffices to check that each $a_0(T_{\pi_v} - \phi_2(\mathfrak{P}_v) - \mathrm{Nm}(\mathfrak{P}_v)\phi_1(\mathfrak{P}_v))$, $v \notin \Sigma_\phi$ is contained in $\delta\mathcal{R}$. This is an easy calculation using (E3). Since $\mathcal{R}/(\delta) \twoheadrightarrow \mathcal{R}/(L^{\mathrm{alg}}(0,\chi))$ by (E5), this concludes the proof of the theorem. □

6. The case of unramified characters

In this section we will prove Theorem 10, i.e., show the existence of an integral lift of the constant term of the Eisenstein cohomology class $\mathrm{Eis}(\phi)$. Our strategy is to find an involution on the boundary cohomology such that (for each connected component of $\overline{S}_{K_f}$)

$$H^1(\Gamma\backslash\overline{\mathbf{H}}_3, \mathcal{R}) \overset{\mathrm{res}}{\twoheadrightarrow} H^1(\partial(\Gamma\backslash\overline{\mathbf{H}}_3), \mathcal{R})^-,$$

where the superscript '-' indicates the -1-eigenspace of this involution. We prove the existence of such an involution for all maximal arithmetic subgroups of $\mathrm{SL}_2(F)$, extending a result of Serre for $\mathrm{SL}_2(\mathcal{O})$. Theorem 10 is then proven by showing that $\mathrm{res}(\mathrm{Eis}(\phi))$ lies in this -1-eigenspace.

6.1. **Involutions and the image of the restriction map.** In this section we work with a general arithmetic subgroup Γ. Assuming that we have an orientation-reversing involution on $\Gamma\backslash\overline{\mathbf{H}}_3$ such that

$$H^1(\Gamma\backslash\overline{\mathbf{H}}_3, \mathcal{R}) \overset{\mathrm{res}}{\to} H^1(\partial(\Gamma\backslash\overline{\mathbf{H}}_3), \mathcal{R})^-$$

we show that the map is, in fact, surjective. The existence of such an involution will be shown for maximal arithmetic subgroups in the following sections. We first recall:

Theorem 13 (Poincaré and Lefschetz duality). *Suppose $\Gamma \subset G(\mathbf{Q})$ is an arithmetic subgroup. Let R be a Dedekind domain in which 2 and 3 are invertible. Let ι be an orientation-reversing involution on $\Gamma\backslash\overline{\mathbf{H}}_3$. Denoting by a superscript + (resp. $-$) the +1-(resp. -1-) eigenspaces for the induced involutions on cohomology groups, we have perfect pairings*

$$H^r_c(\Gamma\backslash\overline{\mathbf{H}}_3, R)^\pm \times H^{3-r}(\Gamma\backslash\overline{\mathbf{H}}_3, R)^\mp \to R \ \textit{for}\ 0 \le r \le 3$$

and

$$H^r(\partial(\Gamma\backslash\overline{\mathbf{H}}_3), R)^\pm \times H^{2-r}(\partial(\Gamma\backslash\overline{\mathbf{H}}_3), R)^\mp \to R \ \textit{for}\ 0 \le r \le 2.$$

Furthermore, the maps in the exact sequence

$$H^1(\Gamma\backslash\overline{\mathbf{H}}_3, R) \xrightarrow{\mathrm{res}} H^1(\partial(\Gamma\backslash\overline{\mathbf{H}}_3), R) \xrightarrow{\partial} H^2_c(\Gamma\backslash\overline{\mathbf{H}}_3, R)$$

are adjoint, i.e.,

$$\langle \mathrm{res}(x), y\rangle = \langle x, \partial(y)\rangle.$$

References. Serre states this in the proof of Lemma 11 in [Ser70] for field coefficients, [AS86] Lemma 1.4.3 proves the perfectness for fields R and [Urb95] Theorem 1.6 for Dedekind domains as above. Other references for this Lefschetz or "relative" Poincaré duality for oriented manifolds with boundary are [May99] Chapter 21, §4 and [Gre67] (28.18). The pairings are given by the cup product and evaluation on the respective fundamental classes. We use that $\overline{\mathbf{H}}_3$ is an oriented manifold with boundary and that Γ acts on it properly discontinuously and without reversing orientation. The lemma in [Fel00] §1.1 shows that the order of any finite subgroup of $G(\mathbf{Q})$ is divisible only by 2 or 3. See also [Ber05] Theorem 5.1 and Lemma 5.2. □

Lemma 14. *Suppose in addition to the conditions of the previous theorem that R is a complete discrete valuation ring with finite residue field of characteristic $p > 2$. Suppose that we have an involution ι as in the lemma such that*

$$H^1(\Gamma\backslash\overline{\mathbf{H}}_3, R) \stackrel{\mathrm{res}}{\to} H^1(\partial(\Gamma\backslash\overline{\mathbf{H}}_3), R)^{\epsilon},$$

where $\epsilon = +1$ or -1. Then, in fact, the restriction map is surjective.

Proof. Let $\mathfrak{m}$ denote the maximal ideal of R. Since the cohomology modules are finitely generated (so the Mittag-Leffler condition is satisfied for $\varprojlim H^1(\cdot, R/\mathfrak{m}^r)$), it suffices to prove the surjectivity for each $r \in \mathbf{N}$ of

$$H^1(\Gamma\backslash\overline{\mathbf{H}}_3, R/\mathfrak{m}^r) \twoheadrightarrow H^1(\partial(\Gamma\backslash\overline{\mathbf{H}}_3), R/\mathfrak{m}^r)^{\epsilon}.$$

For these coefficient systems we are dealing with finite groups and can count the number of elements in the image and the eigenspace of the involution; they turn out to be the same. We observe that $H^1(\partial(\Gamma\backslash\overline{\mathbf{H}}_3), R/\mathfrak{m}^r) = H^1(\partial(\Gamma\backslash\overline{\mathbf{H}}_3), R/\mathfrak{m}^r)^+ \oplus H^1(\partial(\Gamma\backslash\overline{\mathbf{H}}_3), R/\mathfrak{m}^r)^-$ and that, by the last lemma,

$$\#H^1(\partial(\Gamma\backslash\overline{\mathbf{H}}_3), R/\mathfrak{m}^r)^+ = \#H^1(\partial(\Gamma\backslash\overline{\mathbf{H}}_3), R/\mathfrak{m}^r)^-.$$

Similarly we deduce from the adjointness of res and ∂ and the perfectness of the pairings that $\mathrm{im}(\mathrm{res})^{\perp} = \mathrm{im}(\mathrm{res})$ and so

$$\#\mathrm{im}(\mathrm{res}) = \frac{1}{2}\#H^1(\partial(\Gamma\backslash\overline{\mathbf{H}}_3), R/\mathfrak{m}^r).$$

□

6.2. **Involutions for maximal arithmetic subgroups of** $\mathrm{SL}_2(F)$**.** For $\eta \in G(\mathbf{Q})$ let B^{η} be the parabolic subgroup defined by $B^{\eta}(\mathbf{Q}) = \eta^{-1}B(\mathbf{Q})\eta$. For a general arithmetic subgroup $\Gamma \subset G(\mathbf{Q})$, the set $\{B^{\eta} : [\eta] \in B(\mathbf{Q})\backslash G(\mathbf{Q})/\Gamma\}$ is a set of representatives for the Γ-conjugacy classes of Borel subgroups. The group U^{η} is the unipotent radical of B^{η}. For $D \in \mathbf{P}^1(F)$ let $\Gamma_D = \Gamma \cap U_D$, where U_D is the unipotent subgroup of $\mathrm{SL}_2(F)$ fixing D. Note that if $D_{\eta} \in \mathbf{P}^1(F)$ corresponds to $[\eta] \in B(\mathbf{Q})\backslash G(\mathbf{Q})$ under the isomorphism of $B(\mathbf{Q})\backslash G(\mathbf{Q}) \cong \mathbf{P}^1(F)$ given by right action on $[0:1] \in \mathbf{P}^1(F)$ we have that $U_{D_{\eta}} = U^{\eta}(\mathbf{Q})$ and $\Gamma_{D_{\eta}} = \Gamma \cap U^{\eta}(\mathbf{Q}) = \Gamma_{U^{\eta}}$.

Let $U(\Gamma)$ be the direct sum $\oplus_{[D]\in\mathbf{P}^1(F)/\Gamma}\Gamma_D$. Up to canonical isomorphism this is independent of the choice of representatives $[D] \in \mathbf{P}^1(F)/\Gamma$. The inclusion $\Gamma_D \to \Gamma$ defines a homomorphism

$$\alpha : U(\Gamma) \to \Gamma^{\mathrm{ab}}.$$

We first make the following observation that links $U(\Gamma)$ to the cohomology of the boundary components:

Lemma 15. *For imaginary quadratic fields F other than $\mathbf{Q}(\sqrt{-1})$ or $\mathbf{Q}(\sqrt{-3})$, $\Gamma \subset \mathrm{SL}_2(F)$ an arithmetic subgroup, P a parabolic subgroup of $\mathrm{Res}_{F/\mathbf{Q}}(\mathrm{SL}_{2/F})$ with unipotent radical U_P, and R a ring in which 2 is invertible we have*

$$H^1(\Gamma_P, R) \cong H^1(\Gamma_{U_P}, R),$$

where $\Gamma_P = \Gamma \cap P(\mathbf{Q})$ and $\Gamma_{U_P} = \Gamma \cap U_P(\mathbf{Q})$.

Proof. Serre shows in [Ser70] Lemme 7 that $\Gamma_{U_P} \triangleleft \Gamma_P$ and that the quotient $W_P = \Gamma_P/\Gamma_{U_P}$ can be identified with a subgroup of the roots of unity of F, i.e., of $\{\pm 1\}$ since $F \neq \mathbf{Q}(\sqrt{-1}), \mathbf{Q}(\sqrt{-3})$. The Lemma follows from the Inflation-Restriction sequence. See also [Tay] p.110. $\square$

By (1), (3), and Lemma 15 we have

$$H^1(\partial(\Gamma\backslash\overline{\mathbf{H}}_3), R) \cong \coprod_{[\eta]\in\mathbf{P}^1(F)/\Gamma} H^1(\Gamma_{U^\eta}, R) = H^1(U(\Gamma), R). \tag{5}$$

We want to study the kernel of α for maximal arithmetic subgroups of $\mathrm{SL}_2(F)$. Any such is conjugate to one of the following groups (see [EGM98] Prop. 7.4.5): For $\mathfrak{b}$ be a fractional ideal let

$$H(\mathfrak{b}) := \{\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(F) | a, d \in \mathcal{O}, b \in \mathfrak{b}, c \in \mathfrak{b}^{-1}\}.$$

In order to study the structure of $U(H(\mathfrak{b}))$ we define $j : \mathbf{P}^1(F) \to \mathrm{Cl}(F)$ to be the map

$$j([z_1 : z_2]) = [z_1\mathfrak{b} + z_2\mathcal{O}].$$

Theorem 16. *For $\Gamma = H(\mathfrak{b})$, the induced map*

$$j : \mathbf{P}^1(F)/\Gamma \to \mathrm{Cl}(F)$$

is a bijection.

Proof. Let (x_1, x_2), $(y_1, y_2) \in F \times F$. It is easy to check (see [EGM98] Theorem VII 2.4 for $\mathrm{SL}_2(\mathcal{O})$, [Ber05] Lemma 5.10 for the general case) that the following are equivalent:

(1) $x_1\mathfrak{b} + x_2\mathcal{O} = y_1\mathfrak{b} + y_2\mathcal{O}$.

(2) There exists $\sigma \in H(\mathfrak{b})$ such that $(x_1, x_2) = (y_1, y_2)\sigma$.

It remains to show the surjectivity of j. Given a class in $\mathrm{Cl}(F)$ take $\mathfrak{a} \subset \mathcal{O}$ representing it. By the Chinese Remainder Theorem one can choose $z_2 \in \mathcal{O}$ such that

- $\mathrm{ord}_\wp(z_2) = \mathrm{ord}_\wp(\mathfrak{a})$ if $\wp | \mathfrak{a}$.
- $\mathrm{ord}_\wp(z_2) = 0$ if $\wp \nmid \mathfrak{a}, \mathrm{ord}_\wp(\mathfrak{b}) \neq 0$.

Then one chooses z_1 such that

- $\mathrm{ord}_\wp(z_1\mathfrak{b}) > \mathrm{ord}_\wp(z_2)$ if $\wp | \mathfrak{a}$ or $\mathrm{ord}_\wp(\mathfrak{b}) \neq 0$.
- $\mathrm{ord}_\wp(z_1\mathfrak{b}) = 0$ if $\wp | z_2, \wp \nmid \mathfrak{a}$, and $\mathrm{ord}_\wp(\mathfrak{b}) = 0$.

These choices ensure that $\mathrm{ord}_\wp(z_1\mathfrak{b} + z_2\mathcal{O}) = \mathrm{ord}_\wp(\mathfrak{a})$ for all prime ideals $\wp$. $\square$

Following Serre we now calculate explicitely $\Gamma_{[z_1:z_2]}$ for $\Gamma = H(\mathfrak{b})$ and $[z_1 : z_2] \in \mathbf{P}^1(F)$.

Lemma 17. *For $\Gamma = H(\mathfrak{b})$, $\Gamma_{[z_1:z_2]}$ is conjugate in $H(\mathfrak{b})$ to*

$$\{\theta \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} \theta^{-1} : t \in \mathfrak{a}^{-2}\mathfrak{b}\},$$

where $\mathfrak{a} = z_1\mathfrak{b} + z_2\mathcal{O}$ and θ is an isomorphism $\mathcal{O} \oplus \mathfrak{b} \xrightarrow{\sim} \mathfrak{a} \oplus \mathfrak{a}^{-1}\mathfrak{b}$ of determinant 1, i.e., such that its second exterior power

$$\Lambda^2\theta : \Lambda^2(\mathcal{O} \oplus \mathfrak{b}) = \mathfrak{b} \to \Lambda^2(\mathfrak{a} \oplus \mathfrak{a}^{-1}\mathfrak{b}) = \mathfrak{a} \otimes \mathfrak{a}^{-1}\mathfrak{b} = \mathfrak{b}$$

is the identity.

Proof. The main change to [Ser70] §3.6 is that we consider the lattice $L := \mathcal{O} \oplus \mathfrak{b}$ instead of $\mathcal{O}^2$. We claim there exists a projective rank 1 submodule E of L containing a multiple of (z_1, z_2). Let E be the kernel of the $\mathcal{O}$-homomorphism $L = \mathcal{O} \oplus \mathfrak{b} \to F$ given by $(x, y) \mapsto yz_1 - xz_2$. Since the image is $\mathfrak{a} = z_1\mathfrak{b} + z_2\mathcal{O}$, we get $L/E \cong \mathfrak{a}$, so L/E is projective of rank 1 and L decomposes as $E \oplus L/E$.

By definition $\Gamma_{[z_1:z_2]}$ fixes $L \cap \{\lambda(z_1, z_2), \lambda \in F\}$, but this is exactly E. Since $\Gamma_{[z_1:z_2]}$ is unipotent it can therefore be identified with $\mathrm{Hom}_{\mathcal{O}}(L/E, E)$. For any fractional ideal $\mathfrak{a}$, $\Lambda^2(\mathfrak{a}) = 0$ and so $\mathfrak{b} = \Lambda^2(L) = \Lambda^2(E \oplus L/E) = E \otimes_{\mathcal{O}} L/E$ so E is isomorphic to $(L/E)^{-1} \otimes \mathfrak{b}$. This implies an isomorphism $\mathrm{Hom}_{\mathcal{O}}(L/E, E) = (L/E)^{-1} \otimes E \cong (L/E)^{-1} \otimes (L/E)^{-1} \otimes \mathfrak{b} \cong \mathfrak{a}^{-2}\mathfrak{b}$. Choosing an isomorphism $\theta : L \to L/E \oplus E \cong \mathfrak{a} \oplus \mathfrak{a}^{-1}\mathfrak{b}$ of determinant 1 we can represent $\Gamma_{[z_1:z_2]}$ as stated above. □

For $\Gamma = \mathrm{SL}_2(\mathcal{O})$ [Ser70] shows (by choosing an appropriate set of representatives of $\mathbf{P}^1(F)/\mathrm{SL}_2(\mathcal{O}) \cong \mathrm{Cl}(F)$) that there is a well-defined action of complex conjugation on $U(\mathrm{SL}_2(\mathcal{O}))$ induced by the complex conjugation action on the matrix entries of $G_\infty = \mathrm{GL}_2(\mathbf{C})$. Denoting by U^+ the set of elements of $U(\mathrm{SL}_2(\mathcal{O}))$ invariant under the involution and by U' the set of elements $u + \overline{u}$ for $u \in U(\mathrm{SL}_2(\mathcal{O}))$, the result is as follows:

Theorem 18 (Serre [Ser70] Théorème 9)**.** *For imaginary quadratic fields F other than $\mathbf{Q}(\sqrt{-1})$ or $\mathbf{Q}(\sqrt{-3})$ the kernel of the homomorphism $\alpha : U(\mathrm{SL}_2(\mathcal{O})) \to \mathrm{SL}_2(\mathcal{O})^{\mathrm{ab}}$ satisfies the inclusions*

$$6U' \subseteq \ker(\alpha) \subseteq U^+.$$

It is this theorem that we want to generalize to $H(\mathfrak{b})$. After we had discovered this generalization we found out that it had already been stated in [BN92], but for our application we need more detail than is provided there.

Note that since $H(\mathfrak{b})$ is the stabilizer of any lattice $\mathfrak{m} \oplus \mathfrak{n}$ with $\mathfrak{m}$ and $\mathfrak{n}$ fractional ideals of F such that $\mathfrak{m}^{-1}\mathfrak{n} = \mathfrak{b}$, one can deduce

Lemma 19. *Let $\mathfrak{a}, \mathfrak{b}$ be two fractional ideals of F. If $[\mathfrak{a}] = [\mathfrak{b}]$ in $\mathrm{Cl}(F)/\mathrm{Cl}(F)^2$, then $H(\mathfrak{a}) = H(\mathfrak{b})^\gamma$ with $\gamma \in \mathrm{GL}_2(F)$. If the fractional ideals differ by the square of an $\mathcal{O}$-ideal, then γ can be taken to be in $\mathrm{SL}_2(F)$.*

If the class of $\mathfrak{b}$ in $\mathrm{Cl}(F)$ is a square, $H(\mathfrak{b})$ is isomorphic to $\mathrm{SL}_2(\mathcal{O})$ by Lemma 19, and the involution on $U(\mathrm{SL}_2(\mathcal{O}))$ induced by complex conjugation and Serre's Théorème 9 can easily be transferred to $U(H(\mathfrak{b}))$. We therefore turn our attention to the case when

$$[\mathfrak{b}] \text{ is not a square in } \mathrm{Cl}(F).$$

Note that this implies that $[\mathfrak{b}]$ has even order, since any odd order class can be written as a square.

Define an involution on $H(\mathfrak{b})$ to be the composition of complex conjugation with an Atkin-Lehner involution, i.e., by

$$H = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto A\overline{H}A^{-1} = \begin{pmatrix} \overline{d} & -\mathrm{Nm}(\mathfrak{b})\overline{c} \\ -\overline{b}\mathrm{Nm}(\mathfrak{b})^{-1} & \overline{a} \end{pmatrix},$$

where $A = \begin{pmatrix} 0 & 1 \\ -\mathrm{Nm}(\mathfrak{b})^{-1} & 0 \end{pmatrix}$.

Like Serre, we will choose a set of representatives for the cusps $\mathbf{P}^1(F)/H(\mathfrak{b})$ on which this involution acts. For this we observe that if $\Gamma_{[z_1:z_2]}$ fixes $[z_1 : z_2]$ then $A\overline{\Gamma}_{[z_1:z_2]}A^{-1}$ fixes $[\overline{z}_1 : \overline{z}_2]A^{-1} = [\overline{z}_2 : -\mathrm{Nm}(\mathfrak{b})\overline{z}_1]$. We use the isomorphism $j : \mathbf{P}^1(F)/H(\mathfrak{b}) \to \mathrm{Cl}(F)$ to show that this action on the cusps is fixpoint-free. We observe that if $j([z_1 : z_2]) = \mathfrak{a}$ then $j([\overline{z}_1 : \overline{z}_2]A^{-1}) = [\overline{z}_2\mathfrak{b} + \mathrm{Nm}(\mathfrak{b})\overline{z}_1\mathcal{O}] = [\overline{\mathfrak{a}}\mathfrak{b}]$. Note that $[\mathfrak{a}] \neq [\overline{\mathfrak{a}}\mathfrak{b}]$ in $\mathrm{Cl}(F)$ since otherwise $[\mathfrak{a}^2] = [\mathrm{Nm}(\mathfrak{a})\mathfrak{b}] = [\mathfrak{b}]$, i.e., $[\mathfrak{b}]$ a square, contradicting our hypothesis. So $\mathrm{Cl}(F)$ can be partitioned into pairs $(\mathfrak{a}_i, \overline{\mathfrak{a}_i}\mathfrak{b})$.

Choosing $[z_1^i : z_2^i] \in \mathbf{P}^1(F)$ such that $\mathfrak{a}_i = z_1^i\mathfrak{b} + z_2^i\mathcal{O}$ we obtain

$$U(H(\mathfrak{b})) = \bigoplus_{(\mathfrak{a}_i, \overline{\mathfrak{a}_i}\mathfrak{b})} (\Gamma_{[z_1^i:z_2^i]} \oplus A\overline{\Gamma}_{[z_1^i:z_2^i]}A^{-1}).$$

Our choice of representatives of $\mathbf{P}^1(F)/H(\mathfrak{b})$ shows that the involution operates on $U(H(\mathfrak{b}))$ and, in fact, by identifying $\Gamma_{[z_1^i:z_2^i]}$ with $\{\theta \begin{pmatrix} 1 & s \\ 0 & 1 \end{pmatrix} \theta^{-1} : s \in \mathfrak{a}_i^{-2}\mathfrak{b}\}$ for $\theta : \mathcal{O} \oplus \mathfrak{b} \to \mathfrak{a}_i \oplus \mathfrak{a}_i^{-1}\mathfrak{b}$ and $A\overline{\Gamma}_{[z_1^i:z_2^i]}A^{-1}$ with $\{\theta' \begin{pmatrix} 1 & 0 \\ -t & 1 \end{pmatrix} \theta'^{-1} : t \in \overline{\mathfrak{a}_i}^{-2}\mathfrak{b}^{-1}\}$ for $\theta' = A\overline{\theta}A^{-1} : \mathcal{O} \oplus \mathfrak{b} \to \overline{\mathfrak{a}_i}^{-1} \oplus \overline{\mathfrak{a}_i}\mathfrak{b}$, we can describe the involution on each of the pairs as

$$(s, t) \in \mathfrak{a}_i^{-2}\mathfrak{b} \oplus \overline{\mathfrak{a}_i}^{-2}\mathfrak{b}^{-1} \mapsto (\overline{t}\mathrm{Nm}(\mathfrak{b}), \overline{s}\mathrm{Nm}(\mathfrak{b})^{-1}).$$

Now denote by U^+ the set of elements of $U(H(\mathfrak{b}))$ invariant under the involution $H \mapsto A\overline{H}A^{-1}$, and by U' the set of elements $u + A\overline{u}A^{-1}$ for $u \in U(H(\mathfrak{b}))$.

Theorem 20. *For $\Gamma = H(\mathfrak{b})$ with $[\mathfrak{b}]$ a non-square in $\mathrm{Cl}(F)$, the kernel N of the homomorphism*

$$\alpha : U(\Gamma) \to \Gamma^{\mathrm{ab}}$$

coming from the inclusion $\Gamma_D \hookrightarrow \Gamma$ for $D \in \mathbf{P}^1(F)$ satisfies $6U' \subset N \subset U^+$.

Proof. With small modifications, we follow Serre's proof of his Théorème 9. As in Serre's case, it suffices to prove the inclusion $6U' \subset N$, i.e., that $6(u + A\overline{u}A^{-1})$ maps to an element of the commutator $[H(\mathfrak{b}), H(\mathfrak{b})]$:

Suppose that we have $6U' \subset N$, but that there exists an element $u \in N$ not contained in U^+. Then the subgroup of N generated by $6U'$ and u has rank $\#\mathrm{Cl}(F) + 1$. This contradicts the fact that the kernel of α has rank $\#\mathrm{Cl}(F)$ (see [Ser70] Théorème 7). (The latter is proven by showing dually that the rank of the image of the restriction map $H^1(H(\mathfrak{b})\backslash\overline{\mathbf{H}}_3, R) \to H^1(\partial(H(\mathfrak{b})\backslash\overline{\mathbf{H}}_3), R)$ has half the rank of that of the boundary cohomology. This we showed in the proof of Lemma 14).

To prove $6U' \subset N$ we make use of Serre's Proposition 6:

Proposition 21 ([Ser70] Proposition 6)**.** *Let $\mathfrak{q}$ be a fractional ideal of F and let $t \in \mathfrak{q}$ and $t' = \overline{t}/\mathrm{Nm}(\mathfrak{q})$ so that $t' \in \mathfrak{q}^{-1}$. Put $x_t = \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}$ and $y_t = \begin{pmatrix} 1 & 0 \\ -t' & 1 \end{pmatrix}$. Then $(x_t y_t)^6$ lies in the commutator subgroup of $H(\mathfrak{q})$.*

Put $\mathfrak{a} := z_1\mathfrak{b} + z_2\mathcal{O}$. If $u \in \Gamma_{[z_1:z_2]}$, identify it with $\theta^{-1}\begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}\theta$ for some $t \in \mathfrak{a}^{-2}\mathfrak{b}$ and $\theta : \mathcal{O} \oplus \mathfrak{b} \to \mathfrak{a} \oplus \mathfrak{a}^{-1}\mathfrak{b}$ of determinant 1. One easily checks that $A\overline{u}A^{-1}$ then corresponds to $(A\overline{\theta}A^{-1})\begin{pmatrix} 1 & 0 \\ -\overline{t}\mathrm{Nm}(\mathfrak{b})^{-1} & 1 \end{pmatrix}(A\overline{\theta}A^{-1})$. Like Serre, we use that since $[\overline{\mathfrak{a}}] = [\mathfrak{a}^{-1}]$, $A\overline{u}A^{-1}$ is also given by Theorem 16 by $B^{-1}\theta^{-1}\begin{pmatrix} 1 & 0 \\ -t' & 1 \end{pmatrix}\theta B$ for $t' = \overline{t}\mathrm{Nm}(\mathfrak{b})^{-1}\mathrm{Nm}(\mathfrak{a})^2$ and $B \in H(\mathfrak{b})$ taking $\begin{pmatrix} \mathrm{Nm}(\mathfrak{b})\overline{z}_2 \\ \overline{z}_1 \end{pmatrix}$ to $\mathrm{Nm}(\mathfrak{a})^{-1}\begin{pmatrix} \mathrm{Nm}(\mathfrak{b})\overline{z}_2 \\ \overline{z}_1 \end{pmatrix}$.

Since $\theta^{-1}x_t y_t \theta$ is a representative of $u + BA\overline{u}A^{-1}B^{-1}$, we deduce from the above Proposition with $\mathfrak{q} = \mathfrak{a}^{-2}\mathfrak{b}$ that $6(u + BA\overline{u}A^{-1}B^{-1})$ and therefore $6(u + A\overline{u}A^{-1})$ lie in $[H(\mathfrak{b}), H(\mathfrak{b})]$. $\square$

Given an involution ι on $\partial(\Gamma\backslash\overline{\mathbf{H}}_3)$ define an involution on $H^1(\partial(\Gamma\backslash\overline{\mathbf{H}}_3), R)$ via the pullback of ι on the level of singular cocycles. We now reinterpret Serre's Theorem and its generalization as follows:

Proposition 22. *For imaginary quadratic fields F other than $\mathbf{Q}(\sqrt{-1})$ or $\mathbf{Q}(\sqrt{-3})$ and R a ring in which 2 and 3 is invertible, the image of the restriction map*

$$H^1(\Gamma\backslash\overline{\mathbf{H}}_3, R) \overset{\mathrm{res}}{\to} H^1(\partial(\Gamma\backslash\overline{\mathbf{H}}_3), R)$$

is contained in the -1-eigenspace of the involution induced by

- $\iota : \mathbf{H}_3 \to \mathbf{H}_3 : (z,t) \mapsto (\overline{z}, t)$ *if* $\Gamma = \mathrm{SL}_2(\mathcal{O})$
- $\iota : \mathbf{H}_3 \to \mathbf{H}_3 : (z,t) \mapsto A.(\overline{z}, t)$ *for* $A = \begin{pmatrix} 0 & 1 \\ -\mathrm{Nm}(\mathfrak{b})^{-1} & 0 \end{pmatrix}$ *if* $\Gamma = H(\mathfrak{b})$ *with* $[\mathfrak{b}]$ *a non-square in* $\mathrm{Cl}(F)$.

and these involutions are orientation-reversing.

By Lemma 14 this immediately implies:

Corollary 23. *For imaginary quadratic fields F other than $\mathbf{Q}(\sqrt{-1})$ or $\mathbf{Q}(\sqrt{-3})$, $\Gamma = \mathrm{SL}_2(\mathcal{O})$ or $H(\mathfrak{b})$ with $[\mathfrak{b}]$ a non-square in $\mathrm{Cl}(F)$, and R a complete discrete valuation ring in which 2 and 3 are invertible and with finite residue field of characteristic $p > 2$, the restriction map*

$$H^1(\Gamma\backslash\overline{\mathbf{H}}_3, R) \overset{\mathrm{res}}{\to} H^1(\partial(\Gamma\backslash\overline{\mathbf{H}}_3), R)^-$$

surjects onto the -1-eigenspace of the involutions defined in the proposition.

Proof of Proposition. Write $I : \Gamma \to \Gamma$ for the involution

$$\begin{cases} \gamma \mapsto \overline{\gamma} & \text{if } \Gamma = \mathrm{SL}_2(\mathcal{O}), \\ \gamma \mapsto A\overline{\gamma}A^{-1} & \text{if } \Gamma = H(\mathfrak{b}). \end{cases}$$

The involutions ι extend canonically to $\overline{\mathbf{H}}_3$. One checks that for $\gamma \in \Gamma$ we have

$$\iota(\gamma.(z,t)) = I(\gamma)\iota(z,t). \tag{6}$$

This implies that the involutions operate on $\Gamma\backslash\mathbf{H}_3$ and $\Gamma\backslash\overline{\mathbf{H}}_3$, and hence on $\partial(\Gamma\backslash\overline{\mathbf{H}}_3)$. To show that they act by reversing the orientation note that complex conjugation corresponds to reflection in a half-plane of $\mathbf{H}_3$ and therefore reverses the orientation. Furthermore, $\mathrm{GL}_2(\mathbf{C})$ acts on $\mathbf{H}_3$ via $A' = (\det(A)^{-\frac{1}{2}})A \in \mathrm{SL}_2(\mathbf{C})$ and $\mathrm{SL}_2(\mathbf{C})$ acts without reversing orientation, as can be seen from the geometric definition of its action via the Poincaré extension of the action on $\mathbf{P}^1(\mathbf{C})$ (see [EGM98] pp.2-3).

Using (6) one shows that under the isomorphism

$$H^1(\partial(\Gamma\backslash\overline{\mathbf{H}}_3), R) \overset{(5)}{\cong} H^1(U(\Gamma), R)$$

ι corresponds to the involution on $H^1(U(\Gamma), R) = \mathrm{Hom}(U(\Gamma), R)$ given by $\varphi \mapsto I(\varphi)$, where $I(\varphi)(u) := \varphi(I(u))$.

We can therefore check that the image of the restriction maps is contained in the -1-eigenspace on the level of group cohomology: The restriction map is given by

$$\mathrm{Hom}(\Gamma^{\mathrm{ab}}, R) \to \mathrm{Hom}(U(\Gamma), R) : \varphi \mapsto \varphi \circ \alpha.$$

By Serre's theorem and Theorem 20, $0 = \varphi(\alpha(uI(u))) = \varphi(\alpha(u)) + \varphi(\alpha(I(u)))$, so $I(\varphi \circ \alpha)(u) = \varphi(\alpha(I(u))) = -\varphi(\alpha(u))$ for any $u \in U(\Gamma)$. □

6.3. **Integral lift of constant term.** We want to prove that if $\chi = \phi_1/\phi_2$ is an unramified character then we can lift the constant term of the Eisenstein cohomology class to an integral class, i.e., that there exists $c \in \widetilde{H}^1(S_{K_f}, \mathcal{R})$ with the same restriction to the boundary as the Eisenstein cohomology class $\mathrm{Eis}(\phi)$.

First observe that everywhere unramified characters with infinity type z^2 exist only for $F \neq \mathbf{Q}(\sqrt{-1}), \mathbf{Q}(\sqrt{-3})$. For unramified χ we have

$$K_f = \prod_{v \mid \mathfrak{M}_1} U^1(\mathfrak{M}_{1,v}) \prod_{v \nmid \mathfrak{M}_1} \mathrm{GL}_2(\mathcal{O}_v).$$

Recall that $U^1(\mathfrak{M}_{1,v}) = \{k \in \mathrm{GL}_2(\mathcal{O}_v) : \det(k) \equiv 1 \mod \mathfrak{M}_{1,v}\}$. Since furthermore $d_F \mid \mathfrak{M}_1$ by (ϕ2) and $d_F > 4$ we get $K_f \cap \mathrm{GL}_2(F) = \mathrm{SL}_2(\mathcal{O})$.

This implies that we can write S_{K_f} as a disjoint union of $\Gamma\backslash\mathbf{H}_3$ with $\Gamma = H(\mathfrak{b})$ for suitable fractional ideals $\mathfrak{b}$: For a finite idele a, denote by (a) the corresponding fractional ideal. We write

$$S_{K_f} \cong \coprod_{i=1}^{\#\pi_0(K_f)} \Gamma_{t_i}\backslash\mathbf{H}_3,$$

where $\Gamma_{t_i} = G(\mathbf{Q}) \cap t_i K_f t_i^{-1}$ and the $t_i \in G(\mathbf{A}_f)$ are given by $t_i = \begin{pmatrix} \gamma_j a_k b_m & 0 \\ 0 & b_m \end{pmatrix}$, with

- $\{\gamma_j\}$ a system of representatives of
$$\ker(\pi_0(K_f) \to \mathrm{Cl}(F)) \cong \mathcal{O}^*\backslash \prod_v \mathcal{O}_v^*/\det(K_f),$$
- $\{a_k\}$ a set of representatives of $\mathrm{Cl}(F)/(\mathrm{Cl}(F))^2$ in $\mathbf{A}_{F,f}^*$ (and we represent the principal ideals by (1)),
- $\{b_m^2\}$ a set representing $\mathrm{Cl}(F)^2$.

Note that for these choices $\Gamma_{t_i} = H((a_k))$ and either $a_k = 1$ or $[(a_k)]$ is not a square in $\mathrm{Cl}(F)$. This allows us to apply our results for maximal arithmetic subgroups from the previous sections by considering the restriction maps to the boundary separately for each connected component.

Proposition 24.

$$[\mathrm{res}(\mathrm{Eis}(\Psi_\phi))] \in (H^1(\partial \overline{S}_{K_f}, \mathcal{R})^-)_{\mathrm{free}},$$

where $H^1(\partial \overline{S}_{K_f}, \mathcal{R})^-$ is defined via the isomorphism to

$$\bigoplus_{i=1}^{\#\pi_0(K_f)} H^1(\partial(\Gamma_{t_i} \backslash \overline{\mathbf{H}}_3), \mathcal{R})^-$$

where '-' indicates the -1-eigenspace of the involutions defined in Proposition 22.

Remark. Together with Corollary 23 this shows the existence of an integral lift of the constant term and proves Theorem 10.

Proof. We will consider the restriction maps to the boundary separately for each connected component $\Gamma_{t_i} \backslash \mathbf{H}_3$:

$$H^1(\Gamma_{t_i} \backslash \mathbf{H}_3, \mathcal{R}) \overset{\mathrm{res}}{\to} H^1(\partial(\Gamma_{t_i} \backslash \overline{\mathbf{H}}_3), \mathcal{R}) \overset{(1)}{\cong} \bigoplus_{[\eta] \in \mathbf{P}^1(F)/\Gamma_{t_i}} H^1(\Gamma_{t_i, B^\eta} \backslash \mathbf{H}_3, \mathcal{R}),$$

where $\Gamma_{t_i, B^\eta} = \Gamma_{t_i} \cap \eta^{-1} B(\mathbf{Q}) \eta$.

By (3) and Lemma 15 we have $H^1(\Gamma_{t_i, B^\eta} \backslash \mathbf{H}_3, \mathcal{R}) \cong H^1(\Gamma_{t_i, U^\eta}, \mathcal{R})$. By Lemma 1, $\chi^c = \overline{\chi}$, so $L(0, \chi) = L(0, \overline{\chi})$ and we deduce from [Ber06a] Lemma 11 and Proof of Proposition 16 that $\mathrm{res}(\mathrm{Eis}(\phi))$ restricted to this boundary component is represented by

$$\eta_\infty^{-1} \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \eta_\infty \mapsto x \Psi_\phi(\eta_f t_i) - \overline{x} \Psi_{w_0.\phi}(\eta_f t_i), \tag{7}$$

where η_f and η_∞ denote the images of $\eta \in G(\mathbf{Q})$ in $G(\mathbf{A}_f)$ and $G(\mathbf{R})$, respectively, $w_0.(\phi_1, \phi_2) = (\phi_2 \cdot |\cdot|, \phi_1 \cdot |\cdot|^{-1})$, and $\Psi_\phi : G(\mathbf{A}_f) \to \mathbf{C}$ satisfies

$$\Psi_\phi(\begin{pmatrix} a & b \\ 0 & d \end{pmatrix} k) = \phi_1(a)\phi_2(d) \text{ for } \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \in B(\mathbf{A}_f), k \in \prod_v \mathrm{SL}_2(\mathcal{O}_v) \subset K_f.$$

Note that, in particular, $\Psi_\phi(bg) = \phi_\infty^{-1}(b)\Psi_\phi(g)$ for $b \in B(F) \subset G(\mathbf{A}_f)$.

We need to prove that (7) lies in the -1-eigenspace of the involution induced by $u \mapsto \overline{u}$ for $\Gamma_{t_i} = \mathrm{SL}_2(\mathcal{O})$ and by $u \mapsto A\overline{u}A^{-1}$ for $\Gamma_{t_i} = H(\mathfrak{b})$, where $A = \begin{pmatrix} 0 & 1 \\ -N^{-1} & 0 \end{pmatrix}$ with $N = \mathrm{Nm}(\mathfrak{b})$.

Case $\Gamma_{t_i} = \mathrm{SL}_2(\mathcal{O})$: Recall that in this case $t_i = \begin{pmatrix} \gamma_i b_i & 0 \\ 0 & b_i \end{pmatrix}$ for some $\gamma_i \in \hat{\mathcal{O}}^*$ and $b_i \in \mathbf{A}_{F,f}^*$.

It suffices to prove that $\Psi_\phi(\eta_f t_i) = \Psi_{w_0.\phi}(\overline{\eta}_f t_i)$. For this we use the Bruhat decomposition of matrices in $\mathrm{GL}_2(F)$ given by:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{cases} \begin{pmatrix} 1 & b/d \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix} & \text{if } c = 0, \\ \begin{pmatrix} 1 & a/c \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \frac{ad-bc}{c} & 0 \\ 0 & -c \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} 1 & d/c \\ 0 & 1 \end{pmatrix} & \text{otherwise.} \end{cases}$$

Since $\Psi_\phi(\begin{pmatrix} a & b \\ 0 & d \end{pmatrix} g) = \Psi_\phi(\begin{pmatrix} a & b \\ 0 & d \end{pmatrix})\Psi_\phi(g)$ we can consider separately the cases

(a) $\eta = \begin{pmatrix} a & b \\ 0 & d \end{pmatrix}$ for $a, b, d \in F$ and

(b) $\eta = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} 1 & e \\ 0 & 1 \end{pmatrix}$ for $e \in F$.

We check that for (a)

$$\Psi_\phi(\eta_f \begin{pmatrix} \gamma_i b_i & 0 \\ 0 & b_i \end{pmatrix}) = \phi_1(\gamma_i b_i)\phi_2(b_i)\Psi_\phi(\eta_f)$$

and

$$\Psi_{w_0.\phi}(\overline{\eta}_f \begin{pmatrix} \gamma_i b_i & 0 \\ 0 & b_i \end{pmatrix}) = \phi_2(\gamma_i)|\gamma_i|\phi_1(b_i)\phi_2(b_i)\Psi_{w_0.\phi}(\overline{\eta}_f).$$

Since $\gamma_i \in \hat{\mathcal{O}}^*$ and $\chi = \phi_1/\phi_2$ is unramified it suffices to show that $\Psi_\phi(\eta_f) = \Psi_{w_0.\phi}(\overline{\eta}_f)$. In case (b) we similarly reduce to this assertion.

In (a) we get $\Psi_\phi(\eta_f) = \phi_{1,\infty}^{-1}(a)\phi_{2,\infty}^{-1}(d) = \frac{d}{a}$. Since $w_0.\phi$ has infinity type $(\overline{z}, \overline{z}^{-1})$ this equals $\Psi_{w_0.\phi}(\overline{\eta}_f)$. In (b) we need to calculate the Iwasawa decomposition of η in $\mathrm{GL}_2(F_v)$ if $e \notin \mathcal{O}_v$ (at all other places $\Psi_\phi(\eta_v) = \Psi_{w_0.\phi}(\overline{\eta}_{\overline{v}}) = 1$). It is given by

$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} 1 & e \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} e^{-1} & 0 \\ 0 & e \end{pmatrix} \begin{pmatrix} -1 & 0 \\ -e^{-1} & -1 \end{pmatrix}.$$

So, if $e \notin \mathcal{O}_v$ then $\Psi_\phi(\eta_v) = (\phi_2/\phi_1)_v(e) = \chi_v^{-1}(e)$, which we claim matches $\Psi_{w_0.\phi}(\overline{\eta}_{\overline{v}}) = (\phi_1/\phi_2)_{\overline{v}}(\overline{e})|\overline{e}|_{\overline{v}}^{-2}$. This follows from $\chi^c = \overline{\chi}$ and $\chi\overline{\chi} = |\cdot|^2$.

Case $\Gamma_{t_i} = H(\mathfrak{b})$: The involution maps the cusp corresponding to B^η to $B^{\overline{\eta}A^{-1}}$. We therefore have to prove that

$$\Psi_\phi(\eta_f t_i) = \Psi_{w_0.\phi}(\overline{\eta}_f A^{-1} t_i). \tag{8}$$

Recall that $t_i = \begin{pmatrix} x_i b_i & 0 \\ 0 & b_i \end{pmatrix}$ for some $x_i, b_i \in \mathbf{A}_{F,f}^*$. Again making use of the Bruhat decomposition, we need to only consider η as in cases (a) and (b) above. Following the arguments used for Case (1), Case(a) reduces immediately to showing that $\Psi_\phi(t_i) = \Psi_{w_0.\phi}(A^{-1}t_i)$. The left hand side equals $\phi_{1,f}(x_i b_i)\phi_{2,f}(b_i)$, the right hand side is

$$\begin{aligned} \Psi_{w_0.\phi}(\begin{pmatrix} N & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} x_i b_i & 0 \\ 0 & b_i \end{pmatrix}) &= N^{-1}\Psi_{w_0.\phi}(\begin{pmatrix} b_i & 0 \\ 0 & x_i b_i \end{pmatrix}) \\ &= N^{-1}\phi_{1,f}(x_i b_i)\phi_{2,f}(b_i)|x_i|_f^{-1}. \end{aligned}$$

Equality follows from $|x_i|_f^{-1} = \mathrm{Nm}(\mathfrak{b})$.

For (b), one quickly checks that for $\eta = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ the two sides in (8) agree. For general $\eta = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} 1 & e \\ 0 & 1 \end{pmatrix}$ one shows that, on the one hand,

$$\eta_f \begin{pmatrix} x_i b_i & 0 \\ 0 & b_i \end{pmatrix} = \begin{pmatrix} b_i & 0 \\ 0 & x_i b_i \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} 1 & e x_i \\ 0 & 1 \end{pmatrix},$$

and on the other hand,

$$\overline{\eta}_f A^{-1} \begin{pmatrix} x_i b_i & 0 \\ 0 & b_i \end{pmatrix} = \begin{pmatrix} x_i b_i & 0 \\ 0 & b_i N \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} 1 & \overline{e} x_i/N \\ 0 & 1 \end{pmatrix}.$$

Since $(x_i\overline{x}_i) = (N)$ the valuations of $\overline{e}x_i/N$ agrees with that of $\overline{e x}_i$. Repeating the calculation for $\eta = w_0$ and then applying the argument from Case 1(b) (since χ is unramified we are only concerned about the valuation of the upper right hand entry) we also obtain equality.

□

7. Bloch-Kato Conjecture for Hecke characters

Combining the results of the previous sections we get lower bounds on the size of Selmer groups of certain Hecke characters. We want to state this application and relate it to the Bloch-Kato conjecture.

Theorem 25. *Assume that p satisfies the conditions from the beginning of Section 5. If p is inert in $F/\mathbf{Q}$ then assume Conjecture 5. Let χ be an unramified Hecke character of infinity type z^2. Then*

$$\mathrm{val}_p \# \mathrm{Sel}(F, \chi_{\mathfrak{p}}\epsilon) \geq \mathrm{val}_p(\#\mathcal{R}/L^{\mathrm{alg}}(0,\chi))$$

Proof. Put $\rho := \chi_{\mathfrak{p}}\epsilon$. Theorem 6 and Corollary 12 imply

$$\mathrm{val}_p \# \mathrm{Sel}^{\Sigma_\phi \setminus \Sigma_\rho}(F, \chi_{\mathfrak{p}}\epsilon) \geq \mathrm{val}_p(\#\mathcal{R}/L^{\mathrm{alg}}(0,\chi)),$$

where $\Sigma_\rho = \{v \mid p\}$.

Recall that by ($\phi 2$) the set $\Sigma_\phi \setminus \{v \mid p\}$ for the characters ϕ_i of Theorem 11 contains only places v such that $\overline{v} = v$ and $\#\mathcal{O}_v/\mathfrak{P}_v \not\equiv \pm 1 \mod p$. By Lemma 1 we have $\chi^c = \overline{\chi}$, which implies that ρ is anticyclotomic, and so we get $\rho(\mathrm{Frob}_v) = \rho(\mathrm{Frob}_v^c) = \rho^{-1}(\mathrm{Frob}_v)$, or $\rho(\mathrm{Frob}_v) = \pm 1$. Hence we have ensured that

$$\rho(\mathrm{Frob}_v) \not\equiv \epsilon(\mathrm{Frob}_v) \mod p$$

for all $v \in \Sigma_\phi \setminus \Sigma_\rho$ so the theorem follows from applying Lemma 2. □

Example 26. A numerical example in which the conditions of our Theorem are satisfied and a non-trivial lower bound on a Selmer group is obtained is given by the following: Let $F = \mathbf{Q}(\sqrt{-67})$ and $p = 19$. One checks that 19 splits in F. Since the class number is 1, there is only one unramified Hecke character of infinity type z^2. Up to p-adic units $L^{\mathrm{alg}}(0,\chi)$ is given by $\frac{L(0,\chi)}{\Omega^2}$ where Ω is the Neron period of the elliptic curve $y^2 + y = x^3 - 7370x + 243582$, which has conductor 67^2 and complex multiplication by $\mathcal{O}$. Using MAGMA and ComputeL [Dok04] one calculates that $L^{\mathrm{alg}}(0,\chi) \in \mathbf{Z}_{19}$ and

$$\mathrm{val}_{19}(L^{\mathrm{alg}}(0,\chi)) = 1.$$

7.1. **Comparison with other results.**

Assume from now on that $\#\mathrm{Cl}(F) = 1$. Let

$$\Psi : F^* \setminus \mathbf{A}_F^* \to \mathbf{C}^*$$

be a Hecke character of infinity type z^{-1} which satisfies $\Psi^c = \overline{\Psi}$. Then there exists an elliptic curve E over $\mathbf{Q}$ with complex multiplication by $\mathcal{O}$ and associated Grössencharacter Ψ. Consider

$$\rho = (\Psi^k \overline{\Psi^{-j}})_{\mathfrak{p}} \text{ for } k > 0, j \geq 0.$$

We now have the following proposition from [Dee99] (Proposition 4.4.3 and §5.3):

Proposition 27 (Dee). *The group $\mathrm{Sel}(F,\rho)$ is finite if and only if $\mathrm{Sel}(F,\rho^{-1}\epsilon)$ is finite. If this is the case then*

$$\#\mathrm{Sel}(F,\rho) = \#\mathrm{Sel}(F,\rho^{-1}\epsilon) < \infty.$$

Since χ can be written as Ψ^{-2} for some Ψ as above, compare therefore Theorem 25 with the following result:

Theorem 28 (Han, [Han97])**.** *Suppose $k > j+1$. For inert p also assume that ρ is non-trivial when restricted to* $\mathrm{Gal}(F(E_{\mathfrak{p}})/F)$*. Then* $\mathrm{Sel}(F,\rho)$ *is finite and*

$$\mathrm{val}_p \# \mathrm{Sel}(F,\rho) = \mathrm{val}_p(\#\mathcal{R}/L^{\mathrm{alg}}(0, \Psi^{-k}\overline{\Psi}^j)).$$

Previously, Kato proved this in the case $k > 0$ and $j = 0$, cf. [Kat93]. For a similar result in the case of split p see [Guo93]. Han claims that his method extends to general class numbers. All proofs take as input the proof of the Main Conjecture of Iwasawa theory by Rubin [Rub91].

We refer to [Guo96] §3 for the proof that these statements on the size of Selmer groups are equivalent to the (critical cases of the) p-part of the Bloch-Kato Tamagawa number conjectures for the motives associated to the Hecke characters.

For cases of the Bloch-Kato conjecture when the Selmer groups are infinite see [BC04]. Their method is similar to ours in that they use congruences between Eisenstein series and cuspforms, however, they work with p-adic families on U(3).

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