# ON THE BIRATIONAL GEOMETRY OF $(\mathbb{P}^n)^{(m)}/GL_{n+1}$


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# ON THE BIRATIONAL GEOMETRY OF $(\mathbb{P}^n)^{(m)}/GL_{n+1}$.

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ABSTRACT. We prove that $(\mathbb{P}^n)^{(m)}/GL_{n+1}$ is stable birationally isomorphic to $\mathcal{K}_d$, where $d = (m, n+1)$ and $\mathcal{K}_d = (gl_d \times gl_d)/GL_d$. We prove that $(\mathbb{P}^n)^{(m)}/GL_{n+1}$ is rational for $n = 2, m = 7$.

§0. Let $\mathbb{P}^n$ be $n$-dimensional projective space and $(\mathbb{P}^n)^{(m)}$ be $m$th symmetric degree of $\mathbb{P}^n$. The group $GL_{n+1}$ acts canonically on the space $(\mathbb{P}^n)^{(m)}$. Consider the rational factor $(\mathbb{P}^n)^{(m)}/GL_{n+1}$. Recall that if a linear algebraic group acts rationally on an irreducible algebraic variety $X$, then the rational factor $X/G$ and the rational dominant morphism

$$\pi : X \longrightarrow X/G$$

are defined uniquely up to a birational isomorphism. We have $\pi^*(\mathbb{C}(X/G)) = \mathbb{C}(X)^G$.

Here are some known facts about the variety $(\mathbb{P}^n)^{(m)}/GL_{n+1}$.

1) Evidently, $(\mathbb{P}^n)^{(m)}/GL_{n+1}$ is a point for $n \geq m-2$.

2) There is the birational isomorphism

$$(\mathbb{P}^n)^{(m)}/GL_{n+1} \approx (\mathbb{P}^{m-n-2})^{(m)}/GL_{m-n-1}$$

(association, see [1]).

3)$(\mathbb{P}^1)^{(m)}/GL_2$ is rational for all $m$ (see [2], [3], [4])

4)$(\mathbb{P}^2)^{(5)}/GL_3$ is rational by Castelnuovo's theorem.

In this article we prove the following facts.

**Theorem 0.1.** *$(\mathbb{P}^2)^{(7)}/GL_3$ is rational.*

**Theorem 0.2.** *Let $d$ be the greatest common divider of the numbers $m$ and $n+1$. Suppose $n < m-2$, then $(\mathbb{P}^n)^{(m)}/GL_{n+1}$ is stable birationally isomorphic to $\mathcal{K}_d$.*

Recall that irredicible algebraic varieties $X$ and $Y$ are called stable birationally isomorphic iff $X \times \mathbb{C}^{n_1}$ is birationally isomorphic to $Y \times \mathbb{C}^{n_2}$ for some $n_1$, $n_2$. The

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definition of $\mathcal{K}_d$ is as follows. Let $GL_d : gl_d \times gl_d$ be the direct product of two adjoint representations of the group $GL_d$. Set

$$\mathcal{K}_d = (gl_d \times gl_d)/GL_d.$$

The varieties $\mathcal{K}_d$ appear in many questions of algebra and algebraic geometry [5]. Evidently, $\mathcal{K}_1$ and $\mathcal{K}_2$ are rational. D.Formanek proved rationality of $\mathcal{K}_3$ and $\mathcal{K}_4$ [6], [7]. L.Le Bruyn and Ch.Bessenrodt proved stable rationality of $\mathcal{K}_5$ and $\mathcal{K}_7$ [8]. P.Katsylo and A.Schofield proved that if $\mathcal{K}_{d_1}$ and $\mathcal{K}_{d_2}$ are stable rational, $d_1$ and $d_2$ are coprime, then $\mathcal{K}_{d_1 d_2}$ is stable rational [9], [10]. This implies that if $d$ divide 420, then $\mathcal{K}_d$ is stable rational.

**Corollary 0.3.** *Suppose the greater common divider of the numbers $m$ and $n+1$ divides 420, then $(\mathbb{P}^n)^{(m)}/GL_{n+1}$ is stable rational.*

As far as the author knows rationality of $(\mathbb{P}^n)^{(m)}/GL_{n+1}$ is known in the following cases.

(1) $n \geq m-2$ (see 1)).
(2) $m-n=3$ or $n=1$ (see 2), 3)).

It follows from Theorem 0.1 and 2) that $(\mathbb{P}^n)^{(m)}/GL_{n+1}$ is rational in the cases

(1) $n=2, m=7$; $n=3, m=7$.

**§1.** We prove Theorem 0.1 in this section.

Let $e_1, e_2, e_3$ be the standard basis in $\mathbb{C}^3$ and let $x_1, x_2, x_3$ be the dual basis in $\mathbb{C}^{3*}$. The group $SL_3$ acts canonically in the space $S^a\mathbb{C}^3 \otimes S^b\mathbb{C}^{3*}$, $a, b \geq 0$. The linear mapping

$$\Delta = \sum \frac{\partial}{\partial e_i} \otimes \frac{\partial}{\partial x_i} : S^a\mathbb{C}^3 \otimes S^b\mathbb{C}^{3*} \longrightarrow S^{a-1}\mathbb{C}^3 \otimes S^{b-1}\mathbb{C}^{3*}$$

is $SL_3$-mapping. The representation of the group $SL_3$ in the space $V(a,b) = Ker\Delta$ is irreducible. Set $V(a,0) = S^a\mathbb{C}^3$, $V(0,b) = S^b\mathbb{C}^{3*}$.

There is a birational $SL_3$-isomorphism

$$\phi : PV(1,2) \longrightarrow (\mathbb{P}^2)^{(7)},$$

where $PV(1,2)$ is the projectivisation of the linear space $V(1,2)$ [11]. Therefore,

$$(\mathbb{P}^2)^{(7)}/GL_3 \approx (\mathbb{P}^2)^{(7)}/SL_3 \approx PV(1,2)/SL_3.$$

Therefore, we have to prove that $PV(1,2)/SL_3$ is rational.

Note that $PV(2,1)/SL_3$ is rational [12].

Set

$$\psi_1 : V(a,b) \times V(a',b') \longrightarrow V(a+a'+1, b+b'-2),$$

$$(r,r') \mapsto \sum_{\sigma \in S_3} Sgn(\sigma) e_{\sigma(1)} \frac{\partial r}{\partial x_{\sigma(2)}} \frac{\partial r'}{\partial x_{\sigma(3)}}.$$

for $b, b' \geq 1$. It is easy to see that $\psi_1$ is bilinear $SL_3$-mapping.

Set

$$V = V(1,2) \times [V(0,2) \oplus V(1,0)] \times V(1,0).$$

The group $SL_3$ acts canonically in the space $V$. Define the following linear representation of the torus $\mathbb{C}^* \times \mathbb{C}^* \times \mathbb{C}^*$ in the space $V$ :

$$(t_1,t_2,t_3)\cdot(f,g'+g'',h) = (t_1 f, t_2(g'+g''), t_3 h).$$

The actions of the groups $SL_3$ and $\mathbb{C}^* \times \mathbb{C}^* \times \mathbb{C}^*$ in the space $V$ commute. Set

$$\begin{gathered}\phi : V(1,2) \times [V(0,2) \oplus V(1,0)] \longrightarrow V(1,1),\\ (f, g'+g'') \mapsto \Delta(\psi_1(f,g')) + \Delta(fg''),\\ \gamma : V(1,2) \times V(1,0) \longrightarrow V(1,0),\\ (f,g'') \mapsto \Delta^2(fg''^2),\\ X = \{(f,g'+g'',h) \in V(1,2) \times [V(0,2) \oplus V(1,0)] \times V(1,0)\\ \mid \phi(f,g'+g'') = 0,\ h \wedge \gamma(f,g'') = 0\}.\end{gathered}$$

Note that $SL_3 \cdot X = X$, $(\mathbb{C}^* \times \mathbb{C}^* \times \mathbb{C}^*)\cdot X = X$. Set

$$\begin{gathered}f_0 = 3e_2x_1x_3 - 2e_1x_1x_2 + 6e_3x_3x_2 - 2e_2x_2^2,\\ g_0' = x_1x_3 - x_2^2,\\ g_0'' = 2e_2,\\ h_0 = \gamma(f_0, g_0'') = -32e_2.\end{gathered}$$

It can easily be checked that

$$\begin{gathered}(f_0, g_0'+g_0'', h_0) \in X,\\ \dim\ker\phi(f_0,\cdot) = 1,\\ \dim\ker\phi(\cdot, g_0'+g_0'') = 7,\\ \dim(\ker\phi(\cdot,g_0'+g_0'') \cap \ker\gamma(\cdot,g_0'')) = 4.\end{gathered} \tag{1.1}$$

Let $X_0$ be the (unique) irreducible component of the subvariety $X$ such that $(f_0, g_0'+g_0'', h_0) \in X_0$. We have: $SL_3 \cdot X_0 = X_0$, $(\mathbb{C}^* \times \mathbb{C}^* \times \mathbb{C}^*)\cdot X_0 = X_0$.

Consider the restriction $p_1$ of the canonical projection

$$V(1,2) \times [V(0,2) \oplus V(1,0)] \times V(1,0) \longrightarrow V(1,2).$$

on the subvariety $X_0$. It follows from (1.1) that a fiber of general position of the morphism $p_1$ is $\{1\} \times \mathbb{C}^* \times \mathbb{C}^*$-orbit. Therefore,

$$X_0/(SL_3 \times (\mathbb{C}^* \times \mathbb{C}^* \times \mathbb{C}^*)) \simeq PV(1,2)/SL_3. \tag{1.2}$$

Consider the restriction $p_2$ of the canonical projection

$$V(1,2) \times [V(0,2) \oplus V(1,0)] \times V(1,0) \longrightarrow [V(0,2) \oplus V(1,0)] \times V(1,0).$$

on the subvariety $X_0$. It follows from (1.1) that $p_1(p_2^{-1}(g'+g'',h))$ is 5-dimensional linear subspace in $V(1,2)$ for a point $(g'+g'',h) \in [V(0,2)\oplus V(1,0)] \times V(1,0)$ in general position. Therefore,

(1.3) $X_0/(SL_3\times\mathbb{C}^*\times\mathbb{C}^*\times\mathbb{C}^*)) \simeq ([V(0,2)\oplus V(1,0)]\times V(1,0))/(SL_3\times\mathbb{C}^*\times\mathbb{C}^*)\times\mathbb{C}^4$

(see [4, Lemma 2.1]).

Note that $([V(0,2)\oplus V(1,0)]\times V(1,0))/(SL_3\times\mathbb{C}^*\times\mathbb{C}^*)$ is unirational, 2-dimensional and hence rational by Castelnuovo's theorem. It follows now from (1.2) and (1.3) that $PV(1,2)/SL_3$ is rational.

§**2.** We prove Theorem 0.2 in this section.

Consider the regular action

$$GL_{n+1} : (\mathbb{P}^n)^{(m)} \tag{2.1}$$

We can assume that stabilizer of general position of the action (2.1) coinsides with the kernel of this action. Indeed, suppose $n < m-2$, then a stabilizer of general position of the action (2.1) does not coinside with the kernel of this action iff $n=1, m=4$. But Theorem 0.2 is evident in the case $n=1, m=4$.

Consider the linear algebraic group $GL_{n+1}\times GL_m$ and the linear representation

$$\begin{gathered} GL_{n+1}\times GL_m : \mathbb{C}^{n+1}\otimes\mathbb{C}^m, \\ (g,s)\cdot A = gAs^{-1} \end{gathered} \tag{2.2}$$

(we interpret $\mathbb{C}^{n+1}\otimes\mathbb{C}^m$ as a linear space of matrices of size $(n+1)\times m$). Let $T$ be the subgroup of diagonal matrices of the group $GL_m$ and $N(T)$ be the normalizer of the torus $T$ in the group $GL_m$. We have: $N(T)/T \simeq S_m$. Consider the restriction

$$GL_{n+1}\times N(T) : \mathbb{C}^{n+1}\otimes\mathbb{C}^m \tag{2.3}$$

of the linear representation (2.2) on the subgroup $GL_{n+1}\times N(T) \subset GL_{n+1}\times GL_m$ and the restriction

$$\{1\}\times N(T) : \mathbb{C}^{n+1}\otimes\mathbb{C}^m \tag{2.4}$$

of the linear representation (2.3) on the subgroup $\{1\}\times N(T) \subset GL_{n+1}\times N(T)$. Consider the algebraic variety $(\mathbb{C}^{n+1}\otimes\mathbb{C}^m)/(\{1\}\times N(T))$ and the canonical rational action

$$GL_{n+1} : (\mathbb{C}^{n+1}\otimes\mathbb{C}^m)/(\{1\}\times N(T)). \tag{2.5}$$

We have the birational isomorphisms of the algebraic varieties:

$$\begin{gathered} (\mathbb{C}^{n+1}\otimes\mathbb{C}^m)/(\{1\}\times N(T)) \\ \approx ((\mathbb{C}^{n+1}\otimes\mathbb{C}^m)/(\{1\}\times T))/((\{1\}\times N(T))/(\{1\}\times T)) \\ \approx (\underbrace{P\mathbb{C}^{n+1}\times\cdots\times P\mathbb{C}^{n+1}}_{m\ times})/S_m \approx (\mathbb{P}^n)^{(m)}. \end{gathered} \tag{2.6}$$

Rational action (2.5) correspond to rational action (2.1) under the isomorphisms (2.6). The first corollary of this fact is the birational isomorphism

$$\begin{gathered} (\mathbb{P}^n)^{(m)}/GL_{n+1} \approx ((\mathbb{C}^{n+1}\otimes\mathbb{C}^m)/(\{1\}\times N(T)))/GL_{n+1} \\ \approx (\mathbb{C}^{n+1}\otimes\mathbb{C}^m)/(GL_{n+1}\times N(T)) \end{gathered} \tag{2.7}$$

The second corollary of this fact is that stabilizer of general position of the action (2.5) coinsides with the kernel of this action. This implies the following fact.

**Lemma 2.1.** *Stabilizer of the general position of the action* (2.3) *coinsides with the kernel of this action.*

Consider the adjoint representation $GL_m : gl_m$. Let $\eta$ be linear subspace of $gl_m$ of diagonal matrices and $\eta'$ be the open subspace of $\eta$ of diagonal matrices with distant diagonal elements. Note that $\eta'$ is $(GL_m, N(T))$-section of the variety $gl_m$. Consider the linear representation

$$(2.8)\qquad \begin{aligned} &GL_{n+1} \times GL_m : \mathbb{C}^{n+1} \otimes \mathbb{C}^m \times gl_m, \\ &(g,s)\cdot(A,B) = (gAs^{-1}, sBs^{-1}). \end{aligned}$$

Set

$$R' = \{(A,B) \in \mathbb{C}^{n+1} \otimes \mathbb{C}^m \times gl_m \mid B \in \eta'\}.$$

It follows from previous considerations that $R'$ is $(GL_{n+1} \times GL_m, GL_{n+1} \times N(T))$-section of the variety $\mathbb{C}^{n+1} \otimes \mathbb{C}^m \times gl_m$. Thus

$$(2.9)\qquad \begin{aligned} (\mathbb{C}^{n+1} \otimes \mathbb{C}^m \times gl_m)/(GL_{n+1} \times GL_m) &\approx R'/(GL_{n+1} \times N(T)) \\ &\approx R/(GL_{n+1} \times N(T)), \end{aligned}$$

where $R = \overline{R'} = \mathbb{C}^{n+1} \otimes \mathbb{C}^m \times \eta$. Stabilizer of general position of the action (2.3) coinsides with the kernel of this action (Lemma 2.1).It is obvious that the kernel of the action (2.3) acts trivially on $\eta$. By Noname Lemma we have the birational isomorphism

$$(2.10)\qquad R/(GL_{n+1} \times N(T)) \approx (\mathbb{C}^{n+1} \otimes \mathbb{C}^m)/(GL_{n+1} \times N(T)) \times \eta.$$

It follows from (2.7), (2.9), and (2.10) that $(\mathbb{P}^n)^{(m)}/GL_{n+1}$ stable birationally isomorphic to $(\mathbb{C}^{n+1} \otimes \mathbb{C}^m)/(GL_{n+1} \times GL_m)$. Let us prove that $(\mathbb{C}^{n+1} \otimes \mathbb{C}^m)/(GL_{n+1} \times GL_m)$ stable birationaly isomorphic to $\mathcal{K}_d$, where $d = (n+1, m)$.

Consider the linear representation (2.8). It follows from Lemma 2.1 that stabilizer of general position of the representation (2.8) coinsides with the kernel of this representation. It can easily be checked that the kernel of the representation (2.8) is

$$H = \{(\lambda E_{n+1}, \mu E_m) \mid \lambda\mu = 1\},$$

where $E_k$ is the unit matrix of size $k \times k$. Fix the imbeding

$$\begin{aligned} &\phi : GL_{n+1} \times GL_m \hookrightarrow GL_{n+1+m}, \\ &(g,s) \mapsto \begin{pmatrix} g & 0 \\ 0 & s \end{pmatrix} \end{aligned}$$

and let $\gamma : GL_{n+1+m} \longrightarrow GL(gl_{n+1+m})$ be the adjoint representation of the group $GL_{n+1+m}$. Consider the representation

$$\gamma \circ \phi : GL_{n+1} \times GL_m \longrightarrow GL(gl_{n+1+m}).$$

Note that the kernel of this representation is $H$. Therefore, algebraic varieties

$$(\mathbb{C}^{n+1} \otimes \mathbb{C}^m \times gl_m)/(GL_{n+1} \times GL_m), \quad gl_{n+1+m}/\gamma(\phi(GL_{n+1} \times GL_m))$$

are stable birationally isomorphic (Noname Lemma). The last remark in the proof is that $gl_{n+1+m}/\gamma(\phi(GL_{n+1} \times GL_m))$ stable birationaly isomorphic to $\mathcal{K}_d$, where $d = (n+1, m)$ (see [13, Lemma 2.4]).

## REFERENCES


1. I.Dolgachev and D.Ortland, *Point set in projective spaces and theta function*, Asterisque **165** (1988).
2. P.I.Katsylo, *Rationality of the fields of invariants of reducible representations of the group* $SL_2$, Vestnik Moskov. Univ. **5** (1984), 77–79.
3. P.I.Katsylo, *The rationality of moduli spaces of hyperelliptic curves*, Math. USSR-Izv. **25** (1984), 45–50.
4. P.I.Katsylo and F.A.Bogomolov, *Rationality of some quotient varieties*, Math. USSR-Sb. **126** (1985), 584–589.
5. L.Le Bruyn, *Some remarks on rational matrix invariants*, J. Algebra **118** (1988), 587–493.
6. E.Formanek, *The center of the ring of* $3 \times 3$ *generic matrices*, Linear and Multilinear Algebra **7** (1979), 203–212.
7. E.Formanek, *The center of the ring of* $4 \times 4$ *generic matrices*, J. Algebra **62** (1980), 304–319.
8. Ch.Bessenrodt and L.Le Bruyn, *Stable rationality of certain* $PGL_n$*-quotients*, Invent. Math. **104** (1991), 179–199.
9. A.Schofield, *Matrix invariants of composite size*, J. Algebra **147** (1992), 345–349.
10. P.I.Katsylo, *Stable rationality invariant fields of linear representations of the groups* $PSL_6$ *and* $PSL_{12}$, Math. USSR-Zametki. **48** (1990), 49–52.
11. P.I.Katsylo and D.I.Mikhailov, *Ternary quartics and 3-dimensional commutative algebras*, to appear (1994).
12. T.Maeda, *On the invariant field under the action of* $SL(3, \mathbb{C})$ *with signature* $(2, 1)$, Ryukyu Math. J. **3** (1990), 7–24.
13. P.I.Katsylo, *Birational geometry of moduli varieties of vector bundles over* $\mathbf{P}^2$, Math. USSR-Izv. **55** (1991), 429–438.


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