Real analytic Eisenstein series for the Jacobi group

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§ 0. *Introduction.*

0.1. In their monograph [E-Z], Eichler-Zagier defined holomolphic Eisenstein series for the Jacobi group, studied basic properties of them and in particular obtained significant expressions for the Fourier coefficients. The aim of this paper is to study real analytic Eisenstein series for the Jacobi group which are natural generalization of holomorphic Eisenstein series of Eichler-Zagier. Our goal is to obtain the analytic continuation of the real analytic Eisenstein series and prove the functional equation. The key to a proof is to relate the real analytic Eisenstein series for the Jacobi group with those on the upper half plane associated with theta multiplier systems of $SL_2(\mathbb{Z})$, and then to make use of a general theory for real analytic Eisenstein series of $SL_2(\mathbb{R})$ due originally to Selberg [Se] and to Roelcke [Ro 1, 2] in the case with unitary multiplier systems. Here we follow Roelcke [Ro 2] and Kubota [Ku].

To make concrete the functional equation of the real analytic Eisenstein series for the Jacobi group is of some importance in further development of spectral theory for the Jacobi group (in this connection we refer to Berndt [Be]).

0.2. To be more precise, let m be a positive integer and k an integer. Set $\kappa = (k-1/2)/2$. Denote by $\mathfrak{H}$ the upper half plane. Let $\Gamma = SL_2(\mathbb{Z})$ and Γ^J denote the Jacobi group:

$$\Gamma^J = \{(M,(\lambda,\mu),\rho) \mid M \in \Gamma,\ \lambda,\ \mu,\ \rho \in \mathbb{Z}\}$$

with the multiplication law (1.4) in § 1. Denote by $\Gamma^J_{\infty,+}$ the subgroup of Γ^J consisting of elements $(\begin{pmatrix}1 & n\\ 0 & 1\end{pmatrix},(0,\mu),\rho)$ with $n,\ \mu,\ \rho \in \mathbb{Z}$. For each integer r with $r^2 \equiv 0 \bmod 4m$ and $s \in \mathbb{C}$, define a function $\phi_{r,s}\colon \mathfrak{H}\times\mathbb{C} \longrightarrow \mathbb{C}$ by

$$(0.1)\qquad \phi_{r,s}(\tau,z) = e^m\left(\frac{r^2\tau}{4m^2} + \frac{rz}{m}\right)(\mathrm{Im}(\tau))^{s-\kappa},$$

where $e^m(\alpha) = \exp(2\pi i m\alpha)$. The real analytic Eisenstein series $E_{k,m,r}((\tau,z),s)$ ($\tau \in \mathfrak{H}$, $z \in \mathbb{C}$) of weight k and index m with respect to Γ^J is defined as follows:

$$E_{k,m,r}((\tau,z),s)$$

$$= \sum_{\gamma \in \Gamma^J_{\infty,+}\backslash\Gamma^J} e^m\left(\lambda^2\tau+2\lambda z - \frac{c(z+\lambda\tau+\mu)^2}{c\tau+d}\right)\times(c\tau+d)^{-k}\phi_{r,s}\left(M\tau,\frac{z+\lambda\tau+\mu}{c\tau+d}\right).$$

where $\gamma = (M,(\lambda,\mu),\rho) \in \Gamma^J$ with $M = \begin{pmatrix}a & b\\ c & d\end{pmatrix}$. The Eisenstein series $E_{k,m,r}((\tau,z),s)$ is absolutely convergent for $\mathrm{Re}(s) > 5/4$ and, if $k>3$ and s is evaluated at $s=\kappa$, $E_{k,m,r}((\tau,z),\kappa)$ coincides with the holomorphic Eisenstein series of Eichler-Zagier. Our main result is that $E_{k,m,r}((\tau,z),s)$ are analytically continued to meromorphic functions of s in the whole complex plane which satisfy a certain functional equation (see § 3 for the explicit form of the functional equation). In particular if m is a square free positive integer and k is even, there exists the unique Eisenstein series $E_{k,m,0}((\tau,z),s)$ and it satisfies the simple functional equation:

$$E_{k,m,0}((\tau,z),1-s)=\Phi(1-s)E_{k,m,0}((\tau,z),s)$$

with

$$\Phi(s)=\frac{e^{-\pi ik/2}}{\sqrt{2m}}\cdot\frac{2^{2-2s}\pi\Gamma(2s-1)}{\Gamma(s+\kappa)\Gamma(s-\kappa)}\cdot\frac{\zeta(4s-2)}{\zeta(4s-1)}\prod_{p|m}\frac{1+p^{3/2-2s}}{1+p^{1/2-2s}}.$$

In this paper we discussed only the case of $\Gamma=SL_2(\mathbb{Z})$ for the sake of simplicity. However it can be shown that for arbitrary subgroup Γ of $SL_2(\mathbb{Z})$ with finite index, certain real analytic Eisenstein series with respect to the Jacobi group Γ^J is well-defined, and moreover that such real analytic Eisenstein series are analytically continued to meromorphic functions in the whole complex plane which satisfy a functional equation similar to that in the case of $\Gamma=SL_2(\mathbb{Z})$.

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§ 1. *Theta multiplier system*

First we recall classical theta series. Denote by $\mathfrak{H}$ the upper half plane. Let m be a fixed positive integer throughout the whole paper and let R denote the $\mathbb{Z}$-module $\mathbb{Z}/2m\mathbb{Z}$ of residue classes mod 2m. We write $e^m(\alpha)$ (resp. $e(\alpha)$) as an abbreviation of $\exp(2\pi im\alpha)$ (resp. $\exp(2\pi i\alpha)$) for any $\alpha\in\mathbb{C}$. Set, for each $r\in R$,

$$\theta_r(\tau,z)=\sum_{q\in\mathbb{Z}} e^m\left(\tau\left(q+\frac{r}{2m}\right)^2+2z\left(q+\frac{r}{2m}\right)\right)\qquad(\tau\in\mathfrak{H},\quad z\in\mathbb{C}).$$

Denote by V the $\mathbb{C}$-vector space of column vectors $(x_r)_{r\in R}$ $(x_r\in\mathbb{C})$

indexed by the set R. We define the positive definite hermitian scalar product (x,y) $(x=(x_r)_{r\in R},\ y=(y_r)_{r\in R} \in V)$ to be the sum

$$\sum_{r\in R} x_r\bar{y}_r.$$

We line up the theta series $\theta_r(\tau,z)$ as a column vector:

$$\Theta(\tau,z)= (\theta_r(\tau,z))_{r\in R} \in V.$$

In the sequel we take the branch of z^{α} $(z\neq 0,\ \alpha\in\mathbb{R})$ with $-\pi<\arg z\leq\pi$. For $M\in SL_2(\mathbb{R})$, we define the standard automorphic factor $J(M,\tau)$ by

$$J(M,\tau)= c\tau+d \quad \text{with } M=\begin{pmatrix} a & b\\ c & d\end{pmatrix},\ \tau\in\mathfrak{H}.$$

Moreover, for any real number μ, we define the cocycle $\sigma_{\mu}(A,B)$ as in [Fi, Definition 1.3.1]:

$$\sigma_{\mu}(A,B)= e(\mu w(A,B)) \quad \text{with}$$

$$2\pi w(A,B)= \arg J(A,B\tau)+\arg J(B,\tau)-\arg J(AB,\tau),$$

where $\arg z$ is taken so that $-\pi<\arg z\leq\pi$. The number $w(A,B)$ is independent of the choice of τ and takes the values 0, ±1. The theta series $\theta_r(\tau,z)$ satisfies the well-known theta transformation formula:

$$(1.1)\qquad \Theta(M(\tau,z))= e^{m}\left(\frac{cz^2}{c\tau+d}\right)J(M,\tau)^{1/2}U(M)\Theta(\tau,z) \qquad \left(M=\begin{pmatrix} a & b\\ c & d\end{pmatrix}\in SL_2(\mathbb{Z})\right),$$

where $M(\tau,z)=\left(M\tau,\ \frac{z}{c\tau+d}\right)$, and where $U(M)$ is a certain unitary matrix of size $2m$ with respect to the scalar product $(\ ,\ )$. We set

$$\chi(M)= \overline{U(M)},\quad \text{the complex conjugate of } U(M).$$

Via the transformation formula (1.1), $\chi(M)$ satisfies the following property as a multiplier system, (1.2) $\quad \chi(M_1M_2)=\sigma_{1/2}(M_1,M_2)\chi(M_1)\chi(M_2) \quad$ for any $M_1,\ M_2 \in SL_2(\mathbb{Z})$.

For each $r \in R$, we denote by e_r the column vector of V whose ℓ-th component ($\ell \in R$) is one or zero according as $\ell = r$ or not. Then $\{e_r\}_{r\in R}$ forms an orthonormal basis of V. Let L denote the matrix of size $2m$ characterized by $Le_r = e_{-r}$ for any $r \in R$. Since $\Theta(\tau,z)= \sum_{r \in R} \theta_r(\tau,z)e_r$, it is easy to see from (1.1) that

(1.3) $\quad \chi(-1_2)= e^{\pi i/2}L.$

We note here that the theta multiplier system χ of $SL_2(\mathbb{Z})$ does not satisfy the condition a) of Definition 1.3.4 of [Fi].

Denote by $G^J = \{(M,(\lambda,\mu),\rho) \mid M \in SL_2(\mathbb{R}),\ \lambda,\ \mu,\ \rho \in \mathbb{R}\}$ the Jacobi group of degree one. For two elements $g_j=(M_j,(\lambda_j,\mu_j),\rho_j)$ $(j=1,2)$ of G^J, the multiplication g_1g_2 is given by

(1.4) $\quad g_1g_2= \left(M_1M_2,(\lambda_1,\mu_1)M_2+(\lambda_2,\mu_2),\rho_1+\rho_2+(\lambda_1,\mu_1)M_2\begin{pmatrix}\mu_2\\-\lambda_2\end{pmatrix}\right).$

The Jacobi group G^J acts on the product space $\mathfrak{H}\times\mathbb{C}$ in the following manner; for $g=(M,(\lambda,\mu),\rho) \in G^J$ and $(\tau,z) \in \mathfrak{H}\times\mathbb{C}$,

$$g(\tau,z)= \left(M\tau,\ \frac{z+\lambda\tau+\mu}{c\tau+d}\right) \qquad \text{with } M \text{ being } \begin{pmatrix}a&b\\c&d\end{pmatrix} \in SL_2(\mathbb{R}).$$

Let k be a fixed integer. For any function ϕ: $\mathfrak{H}\times\mathbb{C} \longrightarrow \mathbb{C}$ and $g=(M,(\lambda,\mu),\rho) \in G^J$, we set

$$(\phi|_{k,m}g)(\tau,z)= e^{m}\left(\rho+\lambda^2\tau+2\lambda z+\lambda\mu-\frac{c(z+\lambda\tau+\mu)^2}{c\tau+d}\right)(c\tau+d)^{-k}\phi(g(\tau,z)).$$

Then,

(1.5) $\quad (\phi|_{k,m}g_1)|_{k,m}g_2= \phi|_{k,m}g_1g_2 \quad$ for any $g_1,\ g_2 \in G^J$.

We recall the definition of Jacobi forms. For simplicity we assume that $\Gamma = SL_2(\mathbb{Z})$. We set

$$\Gamma^J = \{(M,(\lambda,\mu),\rho) \mid M \in \Gamma, \quad \lambda, \ \mu, \ \rho \in \mathbb{Z}\},$$

which is a discrete subgroup of G^J. A holomorphic function $\phi(\tau,z)$ on $\mathfrak{H}\times\mathbb{C}$ is said to be a Jacobi form of weight k and index m with respect to Γ^J, if ϕ satisfies the following two conditions:

(i) $\phi|_{k,m}\gamma = \phi$ for any $\gamma \in \Gamma^J$.

(ii) The function $\phi(\tau,z)$ has a Fourier expansion of the form:

$$\phi(\tau,z) = \sum_{n,r \in \mathbb{Z}} c(n,r)e(n\tau+rz)$$

with $c(n,r)=0$ unless $4mn-r^2 \geq 0$.

We denote by $J_{k,m}(\Gamma)$ the space of Jacobi forms of weight k and index m with respect to Γ^J. Set, for simplicity,

$$\kappa = \frac{1}{2}\left(k - \frac{1}{2}\right). \tag{1.6}$$

We define an automorphic factor $j_M(\tau) = j_M(\tau,\kappa)$ $(M \in SL_2(\mathbb{R}))$ by

$$j_M(\tau) = \exp(2i\kappa\cdot\arg J(M,\tau)). \tag{1.7}$$

Then this automorphic factor has the property

$$j_A(B\tau)j_B(\tau) = \sigma_{2\kappa}(A,B)j_{AB}(\tau) \quad (A,\ B \in SL_2(\mathbb{R})). \tag{1.8}$$

Let $\mathfrak{M}_{k-1/2}(\Gamma,\chi)$ be the space consisting of V-valued functions f on $\mathfrak{H}$ which satisfy the following two conditions:

(i) $(\mathrm{Im}(\tau))^{-\kappa}f(\tau)$ is holomorphic on $\mathfrak{H}$ and also at the cusp ∞.

(ii) $f(M\tau) = \chi(M)j_M(\tau)f(\tau)$ for all $M \in \Gamma$.

Each Jacobi form $\phi(\tau,z) \in J_{k,m}(\Gamma)$ has the unique expression as a linear combination of theta series:

(1.9) $\phi(\tau,z) = (\mathrm{Im}(\tau))^{-\kappa} \sum_{r \in R} f_r(\tau)\theta_r(\tau,z)$ (see [E-Z, p.58, (5)]).

Then the collection $f=(f_r)_{r\in R}$ of f_r's becomes an automorphic form of $\mathfrak{M}_{k-1/2}(\Gamma,\chi)$ by the transformation formula (1.1). As is shown in [E-Z, Theorem 5.1], the space $J_{k,m}(\Gamma)$ of Jacobi forms is isomorphic to $\mathfrak{M}_{k-1/2}(\Gamma,\chi)$ via the correspondence

$$\phi \longrightarrow f=(f_r)_{r\in R}.$$

§ 2. *Real analytic Eisenstein series with theta multiplier sestem*

In § 2, § 3, we assume that k is an integer (not necessarily positive). Let κ be the number given by (1.6). We first note that the following identity for the theta multiplier system χ given in § 1

(2.1) $\chi(M_1M_2) = \sigma_{2\kappa}(M_1,M_2)\chi(M_1)\chi(M_2)$ for M_1, $M_2 \in SL_2(\mathbb{Z})$

is immediately verified by (1.2). Let $\Gamma=SL_2(\mathbb{Z})$ and let Γ_∞ be the subgroup of Γ generated by $\begin{pmatrix}1&1\\0&1\end{pmatrix}$ and -1_2. It is easy to see from the theta transformation formula (1.1) that

(2.2) $\chi\left(\begin{pmatrix}1&n\\0&1\end{pmatrix}\right)e_r = e\left(\frac{-r^2n}{4m}\right)e_r$ for any $r \in R$.

Denote by $R^{(2)}$ the subset of R consisting of r mod 2m with $r\equiv -r$ mod 2m. Immediately, $R^{(2)}=\{0 \bmod 2m,\ m \bmod 2m\}$. Set, for each $r \in R$,

$$w_r = \begin{cases} (e_r+(-1)^k e_{-r})/2 & \ldots \quad r \in R^{(2)} \\ (e_r+(-1)^k e_{-r})/\sqrt{2} & \ldots \quad r \in R-R^{(2)} \end{cases}$$

and

$$w_r^* = \begin{cases} (e_r-(-1)^k e_{-r})/2 & \ldots \quad r \in R^{(2)} \\ (e_r-(-1)^k e_{-r})/\sqrt{2} & \ldots \quad r \in R-R^{(2)}. \end{cases}$$

Then,

(2.3) $\quad Lw_r = (-1)^k w_r \quad$ and $\quad Lw_r^* = (-1)^{k-1} w_r^* \quad$ for any $r \in R$.

Let R^{null} denote the subset of R given by

$$R^{null} = \{r \in R \mid r^2 \equiv 0 \bmod 4m\}.$$

Now we define the Eisenstein series $E_r(\tau,s)$ $(r \in R^{null})$ associated with the theta multiplier system χ, following Roelcke [Ro 2, § 10]. Set, for each $r \in R^{null}$,

(2.4) $$E_r(\tau,s) = \sum_{M \in \Gamma_\infty \backslash \Gamma} j_M(\tau)^{-1}(\mathrm{Im}(M\tau))^s \chi(M)^{-1} w_r.$$

The well-definedness of the V-valued function $E_r(\tau,s)$ is easily verified by (2.1), (2.2), (2.3), (1,3), (1.7) and (1.8). The Eisenstein series $E_r(\tau,s)$ is absolutely convergent for Re(s)>1, and behaves like an automorphic form of $\mathfrak{M}_{k-1/2}(\Gamma,\chi)$. Namely,

(2.5) $\quad E_r(M\tau,s) = \chi(M) j_M(\tau) E_r(\tau,s) \quad$ for all $M \in \Gamma$.

Since $w_r = (-1)^k w_{-r}$, we get, immediately,

(2.6) $\quad E_r(\tau,s) = (-1)^k E_{-r}(\tau,s) \quad (r \in R^{null})$.

Now we discuss the Fourier expansion of $E_r(\tau,s)$ in a manner similar to [Ro1, p.301], [Ro2, p.294, Lemma 10.2].

LEMMA 2.1. *Let* $r \in R^{null}$. *Then,* $(E_r(\tau,s),w_p^*)=0$ *for any* $p \in R$.

Proof. We get, by (1.3), (1.7), (2.3) and (2.5),

$$(E_r(\tau,s),w_p^*)= (j_{-1_2}(\tau)E_r(\tau,s),\chi(-1_2)^{-1}w_p^*)$$

$$= (e^{2\pi i\kappa}E_r(\tau,s),e^{-\pi i/2}Lw_p^*)= (E_r(\tau,s),-w_p^*),$$

which completes the proof of the assertion. q.e.d.

There exists a subset $\tilde{R}$ of R with the property

$$R= R^{(2)} \cup \tilde{R} \cup \{-\tilde{R}\} \qquad \text{(disjoint union)},$$

where $-\tilde{R}= \{-r|\ r \in \tilde{R}\}$. We pick up such a subset $\tilde{R}$ and fix it once and for all. If k is even (resp. odd), then, w_r $(r \in \tilde{R} \cup R^{(2)})$, w_r^* $(r \in \tilde{R})$ (resp. w_r $(r \in \tilde{R})$, w_r^* $(r \in \tilde{R} \cup R^{(2)})$) form an orthonormal basis of V. According to Lemma 2.1, the Eisenstein series $E_r(\tau,s)$ has an expression as a linear combination of w_p's :

$$E_r(\tau,s)= \sum_{p \in \tilde{R} \cup R^{(2)}} g_p(\tau,s)w_p$$

with certain functions $g_p(\tau,s)$ $(p \in \tilde{R} \cup R^{(2)})$. We note that if k is odd, then $w_p=0$ for $p \in R^{(2)}$. Set, for each $p \in R$,

$$\beta_p= \langle -p^2/4m\rangle,$$

where $\langle x\rangle$ for $x \in \mathbb{R}$ denotes the real number with $x-\langle x\rangle \in \mathbb{Z}$ and $0\le\langle x\rangle<1$. We define the subsets R_k and R_k^{null} of R as follows:

$$R_k = \begin{cases} \tilde{R} \cup R^{(2)} & \ldots \text{ if } k \text{ is even,} \\ \tilde{R} & \ldots \text{ if } k \text{ is odd,} \end{cases}$$

and

$$R_k^{null} = R_k \cap R^{null}.$$

Then the formula (2.5) for $M=\begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix}$ $(n \in \mathbb{Z})$ and (2.2) imply that

(2.7) $$g_{rp}(\tau+n,s) = e(\beta_p n) g_{rp}(\tau,s) \quad \text{for any } p \in R_k.$$

By virtue of the identity (2.6), we need the Eisenstein series $E_r(\tau,s)$ only for $r \in R_k^{null}$. We see easily from (2.7) that, if $p \in R_k - R_k^{null}$, then $g_{rp}(\tau,s)$ has the Fourier expansion of the form:

$$g_{rp}(\tau,s) = \sum_{n=-\infty}^{\infty} q_{rp,n}(\eta,s) e((n+\beta_p)\xi) \qquad (\tau=\xi+i\eta).$$

Moreover if $p \in R_k^{null}$, then the following Fourier expansion holds:

$$g_{rp}(\tau,s) = u_{rp}(\eta,s) + \sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} q_{rp,n}(\eta,s) e(n\xi) \qquad (\tau=\xi+i\eta),$$

where the constant term $u_{rp}(\eta,s)$ is given by the integral

$$\int_0^1 (E_r(\tau,s), w_p) d\xi.$$

For r, $p \in R$, let δ_{rp} denote the Kronecker symbol. Thus using the definition (2.4) of $E_r(\tau,s)$, we have, in a usual manner,

$$u_{rp}(\eta,s) = \delta_{rp}\eta^s + \sum_{\substack{(c,d)=1 \\ c>0,\ d \bmod c}} (w_r, \chi(M) w_p) \cdot \int_{-\infty}^{\infty} j_M(\tau)^{-1} (\mathrm{Im}(M\tau))^s d\xi$$

$$(r,\ p \in R_k^{null} \text{ and } \mathrm{Re}(s)>1),$$

where in the summation c runs over all positive integers and d runs over the residue classes mod c with $(c,d)=1$, and where, for integers c, d with $(c,d)=1$, $M=\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is chosen so that $M \in \Gamma$. An elementary calculation shows that the integral on the right hand side of the above equality is equal to

$$(2.8) \qquad \frac{\eta^{1-s}}{c^{2s}} \cdot e^{-\pi i \kappa} \gamma(\kappa;s) \qquad \text{with} \qquad \gamma(\kappa;s)=\frac{2^{2-2s}\pi\Gamma(2s-1)}{\Gamma(s+\kappa)\Gamma(s-\kappa)} .$$

Thus we obtain

PROPOSITION 2.2. *Let* $r \in R_k^{null}$. *Then the Eisenstein series* $E_r(\tau,s)$ *has the following expression*:

$$E_r(\tau,s)= \sum_{p \in R_k^{null}} (\delta_{rp}\eta^{s}+\eta^{1-s}\varphi_{rp}(s))w_p + \sum_{p \in R_k} q_{rp}(\tau,s)w_p$$

$$(\tau=\xi+i\eta, \quad Re(s)>1),$$

where each $q_{rp}(\tau,s)$ $(p \in R_k)$ *has the Fourier expansion of the form*

$$q_{rp}(\tau,s)= \begin{cases} \sum_{n=-\infty}^{\infty} q_{rp,n}(\eta,s)e((n+\beta_p)\xi) & \ldots \ p \in R_k-R_k^{null} \\ \sum_{n=-\infty,\ n\neq 0}^{\infty} q_{rp,n}(\eta,s)e(n\xi) & \ldots \ p \in R_k^{null}. \end{cases}$$

Moreover, the function $\varphi_{rp}(s)$ $(r,\ p \in R_k^{null})$ *is given as follows*:

$$\varphi_{rp}(s)= e^{-\pi i \kappa}\gamma(\kappa;s) \sum_{\substack{(c,d)=1 \\ c>0,\ d \bmod c}} \frac{(w_r,\chi(M)w_p)}{c^{2s}} \qquad (Re(s)>1),$$

where, for coprime integers c, d, $M=\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ *is chosen so that* $M \in \Gamma$.

The functions $\varphi_{rp}(s)$ $(r,\ p \in R_k^{null})$ play an important role to describe the functional equation of the Eisenstein series $E_r(\tau,s)$. Denote by t_∞ the cardinality of the set R_k^{null}. We set

$$\Phi(s) = (\varphi_{rp}(s))_{r,p \in R_k^{null}} ,$$

which is a matrix of size t_∞ whose (r,p)-component is $\varphi_{rp}(s)$.

PROPOSITION 2.3. *Let* $r, p \in R_k^{null}$. *Then,* $\overline{\varphi_{rp}(\bar{s})} = \varphi_{pr}(s)$. *In another word,* ${}^t\overline{\Phi(\bar{s})} = \Phi(s)$.

Proof. For $M=\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{R})$, we write $c(M)$ for the entry c. Let Γ_∞^+ be the subgroup of Γ generated by $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$. Denote by $B^+(\Gamma)$ the subset of Γ consisting of $M \in \Gamma$ with $c(M)>0$. The set $B^+(\Gamma)$ is bi-invariant under the left and right multiplication of elements of Γ_∞^+. Denote by $\Gamma_\infty^+\backslash B^+(\Gamma)/\Gamma_\infty^+$ a complete set of representatives of the double cosets of $B^+(\Gamma)$ by Γ_∞^+. Accoding to Proposition 2.2, $\varphi_{rp}(s)$ has a slightly modified expression:

$$\varphi_{rp}(s) = e^{-\pi i \kappa}\gamma(\kappa;s) \sum_{M \in \Gamma_\infty^+\backslash B^+(\Gamma)/\Gamma_\infty^+} \frac{(w_r, \chi(M)w_p)}{c(M)^{2s}} .$$

The set $\Gamma_\infty^+\backslash B^+(\Gamma)/\Gamma_\infty^+$ and $c(M)$ are invariant under the correspondence $M \longrightarrow -M^{-1}$. We get $\chi(M)^{-1} = \chi(M^{-1})$, since $\sigma_{2k}(M,M^{-1})=1$ ([Fi, p.18, (1.3.6)]). Thus we have

$$(2.9) \qquad \overline{\varphi_{rp}(\bar{s})} = e^{\pi i \kappa}\gamma(\kappa;s) \sum_{M \in \Gamma_\infty^+\backslash B^+(\Gamma)/\Gamma_\infty^+} \frac{(w_p, \chi(M)^{-1}w_r)}{c(M)^{2s}}$$

$$= e^{\pi i \kappa}\gamma(\kappa;s) \sum_{M \in \Gamma_\infty^+\backslash B^+(\Gamma)/\Gamma_\infty^+} \frac{(w_p, \chi(-M)w_r)}{c(M)^{2s}} .$$

Since $c(M)>0$, $\sigma_{2\kappa}(M,-1_2)=-1$. Moreover by (2.3), $\chi(-1_2)w_r = -e^{2\pi i \kappa}w_r$

for any $r \in R_k^{null}$. Thus we see immediately from the second equality of (2.9) that $\overline{\varphi_{rp}(\bar{s})} = \varphi_{pr}(s)$. q.e.d.

Let $\mathcal{F}$ be a usual fundamental domain of $\Gamma = SL_2(\mathbb{Z})$ in $\mathfrak{H}$ given by

$$\mathcal{F} = \{\tau = \xi + i\eta \in \mathbb{C} \mid -1/2 < \xi \leq 1/2, \ |\tau| \leq 1 \ \text{ and } \ \xi \geq 0 \ \text{ on } |\tau| = 1\}.$$

Let $\mathcal{H}_\kappa$ denote the space of measurable functions $f: \mathfrak{H} \longrightarrow V$ satisfying the following two conditions:

i) $f(M\tau) = \chi(M) j_M(\tau) f(\tau)$ for any $M \in \Gamma$,

ii) $\|f\|^2 = \int_{\mathcal{F}} (f(\tau), f(\tau)) d\omega(\tau) < +\infty$,

where $d\omega(\tau) = \eta^{-2} d\xi d\eta$. For two elements f, g of $\mathcal{H}_\kappa$, the scalar product (f,g) is defined by

$$(f,g) = \int_{\mathcal{F}} (f(\tau), g(\tau)) d\omega(\tau).$$

Then $\mathcal{H}_\kappa$ forms a Hilbert space with respect to the scalar product (f,g). Now we describe the Maass-Selberg relation for $E_r(\tau,s)$. Take a positive number Y sufficiently large. Set

$$\mathcal{F}^Y = \{\tau \in \mathcal{F} \mid \operatorname{Im}(\tau) > Y\} \quad \text{and} \quad \mathcal{F}_Y = \mathcal{F} - \mathcal{F}^Y.$$

We define a compact form $E_r^Y(\tau,s)$ of $E_r(\tau,s)$ as follows. Set

$$E_r^Y(\tau,s) = \begin{cases} E_r(\tau,s) - u_r(\eta,s) & \dots \text{ if } \tau \in \mathcal{F}^Y, \\ E_r(\tau,s) & \dots \text{ if } \tau \in \mathcal{F}_Y, \end{cases}$$

where $u_r(\eta,s)$ is the constant term of the Fourier expansion of $E_r(\tau,s)$:

$$u_r(\eta,s)=\eta^s+\eta^{1-s}\sum_{p\in R_k^{null}}\varphi_{rp}(s)w_p.$$

We extend $E_r^Y(\tau,s)$ to the whole upper half plane $\mathfrak{H}$ by putting

$$E_r^Y(M\tau,s)=\chi(M)j_M(\tau)E_r^Y(\tau,s)\qquad\text{for any } M\in\Gamma \text{ and } \tau\in\mathcal{F}.$$

Then, $E_r^Y(\tau,s)$ is an element of $\mathcal{H}_\kappa$.

PROPOSITION 2.4 (Maass-Selberg relation). *Let* r, q $\in R_k^{null}$ *and take* Y *sufficiently large. If* Re(s), Re(s')>1, *then,*

$$(2.10)\qquad (E_r^Y(\ ,s),E_q^Y(\ ,s'))=\varphi_{rq}(\bar{s}')\frac{Y^{s-\bar{s}'}}{s-\bar{s}'}-\varphi_{rq}(s)\frac{Y^{-s+\bar{s}'}}{s-\bar{s}'}$$

$$+\frac{1}{s+\bar{s}'-1}\left(\delta_{rq}Y^{s+\bar{s}'-1}-\sum_{p\in R_k^{null}}\varphi_{rp}(s)\varphi_{pq}(\bar{s}')Y^{-s-\bar{s}'+1}\right)$$

Proof. First assume that Re(s)>Re(s')>1. By the definition of a compact form, the scalar product $(E_r^Y(\ ,s),E_q^Y(\ ,s'))$ becomes the sum of the following three integrals I_1, I_2 and I_3:

$$I_1=\int_{\mathcal{F}_Y}(E_r(\tau,s),E_q(\tau,s'))d\omega(\tau),$$

$$I_2=\int_{\mathcal{F}^Y}(E_r(\tau,s)-u_r(\eta,s),E_q(\tau,s'))d\omega(\tau),$$

$$I_3=\int_{\mathcal{F}^Y}(E_r(\tau,s)-u_r(\eta,s),-u_q(\eta,s'))d\omega(\tau).$$

Immediately, $I_3=0$. By a usual calculation with the use of (2.5), we get

$$I_1 = \sum_{M \in \Gamma_\infty \backslash \Gamma} \int_{M\mathcal{F}_Y} \eta^s (w_r, E_q(\tau, s'))d\omega(\tau),$$

$$I_2 = \sum_{M \in \Gamma_\infty \backslash (\Gamma - \Gamma_\infty)} \int_{M\mathcal{F}^Y} \eta^s (w_r, E_q(\tau, s'))d\omega(\tau)$$

$$- \sum_{p \in R_k^{null}} \varphi_{rp}(s) \int_{\mathcal{F}^Y} \eta^{1-s}(w_p, E_q(\tau, s'))d\omega(\tau).$$

We may take the set $\{\tau = \xi + i\eta \in \mathfrak{H} \mid -1/2 < \xi \leq 1/2,\ 0 < \eta \leq Y\}$ as

$$\bigcup_{M \in \Gamma_\infty \backslash \Gamma} M\mathcal{F} - \mathcal{F}^Y.$$

Therefore using the Fourier expansion of $E_q(\tau, s')$ (Proposition 2.2), we get

$$I_1 + I_2 = \int_0^Y (\delta_{rq}\eta^{s+\bar{s}'-1} + \overline{\varphi_{qr}(s')} \cdot \eta^{s-\bar{s}'}) \frac{d\eta}{\eta}$$

$$- \sum_{p \in R_k^{null}} \varphi_{rp}(s) \int_Y^\infty \eta^{-s}(\delta_{pq}\eta^{\bar{s}'} + \overline{\varphi_{qp}(s')} \cdot \eta^{1-\bar{s}'}) \frac{d\eta}{\eta}.$$

Thus with the help of Propositio 2.3, the relation (2.10) holds under the condition $Re(s) > Re(s') > 1$. By the analytic continuation as functions of s, the relation (2.10) holds also for $Re(s)$, $Re(s') > 1$.

q.e.d.

As is explained in [Ro, p.293], it can be shown that by means of Selberg [Se] the Eisenstein series $E_r(\tau, s)$ and also $\Phi(s)$ have analytic continuations to meromorphic functions in the whole s-plane. Then we may follow Roelcke [Ro2] and prove the functional equations of $E_r(\tau, s)$ and $\Phi(s)$ likewise ([Ro2, Satz 10.1, 10.2, 10.3 and 10.4]).

Or equivalently, we may proceed similarly as in Kubota's book [Ku].

We can prove in a manner similar to [Ku, Ch.III, IV] that $\Phi(s)$ and then $E_r(\tau,s)$ are analytically continued to holomorphic functions of s in the region $\mathrm{Re}(s)\geq 1/2$ except on the interval $(1/2,1]$, and moreover that the Maass-Selberg relation in Proposition 2.4 holds if $\mathrm{Re}(s), \mathrm{Re}(s')\geq 1/2$. In the relation (2.10) taking the limit of $s \longrightarrow 1/2+it$, $s' \longrightarrow 1/2+it$ $(t \in \mathbb{R}, t\neq 0)$ with the condition $\mathrm{Re}(s)$, $\mathrm{Re}(s')>1/2$ being kept, we see that

$$\sum_{p \in R_k^{null}} \varphi_{rp}(s)\varphi_{pq}(\bar{s})= \delta_{rq} \quad \text{on the line} \quad s= \frac{1}{2} +it \ (t \in \mathbb{R},\ t\neq 0),$$

since the left hand side of (2.10) is bounded. In another word,

$$(2.11) \qquad \Phi(s)\Phi(\bar{s})= 1_{t_\infty} \quad \text{on} \quad \mathrm{Re}(s)=1/2,\ s\neq 1/2.$$

Thus by virtue of the reflection principle, $\Phi(s)$ is analytically continued to a meromorphic function in the whole s-plane satisfying the functional equation

$$\Phi(s)\Phi(1-s)= 1_{t_\infty}.$$

We line up the Eisenstein series $E_r(\tau,s)$ as follows:

$$E(\tau,s)= (\ldots,\ E_r(\tau,s),\ldots)_{r\in R_k^{null}},$$

which is a $2m\times t_\infty$ matrix. We consider the difference

$$(2.12) \qquad E_r(\tau,\bar{s}) - \sum_{q \in R_k^{null}} \varphi_{rq}(\bar{s})E_q(\tau,s) \quad \text{on the line} \quad \mathrm{Re}(s)=1/2.$$

The constant term of the Fourier expansion of (2.12) is identically zero by Proposition 2.2 and (2.11). Using a procedure similar to that in Theorem 4.1.2 of [Ku], we see that the difference (2.12) is also identically zero. Thus,

$$E(\tau,\bar{s})= E(\tau,s)^{t}\Phi(\bar{s}).$$

Therefore again by the reflection principle, each $E_r(\tau,s)$ ($r \in R_k^{null}$) is analytically continued to a meromorphic function in the whole s-plane satisfying the functional equation

$$E(\tau,1-s)= E(\tau,s)^{t}\Phi(1-s).$$

Furthermore using a little more precise argument (for instance [Ro2, Satz 10.3, 10.4]), we obtain the following main result for the Eisenstein series.

PROPOSITION 2.5. *The Eisenstein series* $E_r(\tau,s)$ *and each* $\varphi_{rp}(s)$ ($r, p \in R_k^{null}$) *have the analytic continuations to meromorphic functions of* s *in the whole complex plane which are holomorphic in the region* $Re(s)\geq 1/2$ *except on the interval* $(1/2,1]$. *Each* $E_r(\tau,s)$ *has only simple poles on this interval, and has a pole at* $s_0\in (1/2,1]$, *if and only if* $\varphi_{rr}(s)$ *so does. Then* $E(\tau,s)$ *and* $\Phi(s)$ *satisfy the functional equations*

$$E(\tau,1-s)= E(\tau,s)^{t}\Phi(1-s) \quad \text{and} \quad \Phi(s)\Phi(1-s)= 1_{t_\infty}.$$

Moreover, $E_r(\tau,s)$ ($r \in R_k^{null}$) *are* $\mathbb{C}$-*linearly independent.*

§ 3. *Real analytic Eisenstein series for the Jacobi group*

The aim of this section is to obtain the analytic continuations and the functional equations for the real analytic Eisenstein series of the Jacobi group with the help of those of the real analytic

Eisenstein series $E_r(\tau,s)$.

For each integer r with $r^2 \equiv 0 \mod 4m$ and $s \in \mathbb{C}$, let $\phi_{r,s}(\tau,z)$ be the function given by (0.1) in the introduction. Let $\Gamma^J_{\infty,+}$ be the subgroup of Γ^J defined by

$$\Gamma^J_{\infty,+} = \{ (\begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix}, (0,\mu), \rho) \mid n,\ \mu,\ \rho \in \mathbb{Z} \}.$$

Following Eichler-Zagier ([E-Z, p.25, (8)]), we define the real analytic Eisenstein series $E_{k,m,r}((\tau,z),s)$ for the Jacobi group by

$$(3.1) \qquad E_{k,m,r}((\tau,z),s) = \sum_{\gamma \in \Gamma^J_{\infty,+}\backslash\Gamma^J} (\phi_{r,s}|_{k,m}\gamma)(\tau,z) \qquad ((\tau,z) \in \mathfrak{H}\times\mathbb{C}).$$

The well-definedness of $E_{k,m,r}((\tau,z),s)$ follows from the property (1.5) and the fact that

$$\phi_{r,s}|_{k,m}\gamma_1 = \phi_{r,s} \quad \text{for any } \gamma_1 \in \Gamma^J_{\infty,+}.$$

The Eisenstein series $E_{k,m,r}((\tau,z),s)$ has the following expression

$$(3.2) \qquad E_{k,m,r}((\tau,z),s) = \sum_{M \in \Gamma^+_\infty\backslash\Gamma} \sum_{q \in \mathbb{Z}} J(M,\tau)^{-k}(\mathrm{Im}(M\tau))^{s-\kappa}\cdot$$

$$e^m\left(\left(q+\frac{r}{2m}\right)^2 M\tau + 2\left(q+\frac{r}{2m}\right)\frac{z}{c\tau+d} - \frac{cz^2}{c\tau+d}\right),$$

where, for $M \in \Gamma$, $M=\begin{pmatrix} a & b \\ c & d \end{pmatrix}$. We see easily from this expression that $E_{k,m,r}((\tau,z),s)$ is absolutely convergent for $\mathrm{Re}(s)>5/4$ and depends only on r mod 2m. Therefore for each $r \in R^{null}$, we can define $E_{k,m,r}((\tau,z),s)$ by (3.1). The following properties are an immediate consequence of the definition (3.1):

i) $(E_{k,m,r}((\ ,\ ),s)|_{k,m}\gamma)(\tau,z) = E_{k,m,r}((\tau,z),s) \qquad$ for all $\gamma \in \Gamma^J$,

ii) If $k>3$ and s is evaluated at $s=\kappa$, then, $E_{k,m,r}((\tau,z),\kappa)$ coincides with the holomorphic Eisenstein series of Eichler-Zagier.

The important fact on the Eisenstein series is that $E_{k,m,r}((\tau,z),s)$ is connected with the Eisenstein series $E_r(\tau,s)$ in § 2 via the correspondence (1.9). For each $r \in R$, let ϵ_r be the symbol given by

$$\epsilon_r = \begin{cases} 2 & \ldots\ r \in R^{(2)} \\ \sqrt{2} & \ldots\ r \in R-R^{(2)}. \end{cases}$$

PROPOSITION 3.1. *Let* $r \in R^{null}$. *Then,*

$$E_{k,m,r}((\tau,z),s) = \epsilon_r \eta^{-\kappa}.{}^tE_r(\tau,s)\theta(\tau,z).$$

Proof. By (3.2), and then using the transformation formula (1.1), we have

$$E_{k,m,r}((\tau,z),s) = \sum_{M \in \Gamma_\infty^+\backslash\Gamma} J(M,\tau)^{-k}(\operatorname{Im}(M\tau))^{s-\kappa}e^m\left(\frac{-cz^2}{c\tau+d}\right)\theta_r(M(\tau,z))$$

$$= \sum_{M \in \Gamma_\infty\backslash\Gamma} J(M,\tau)^{-k}(\operatorname{Im}(M\tau))^{s-\kappa}e^m\left(\frac{-cz^2}{c\tau+d}\right)(e_r+(-1)^ke_{-r})\theta(M(\tau,z)))$$

$$= \sum_{M \in \Gamma_\infty\backslash\Gamma} J(M,\tau)^{-k+1/2}(\operatorname{Im}(M\tau))^{s-\kappa}.{}^t\chi(M)^{-1}\cdot(e_r+(-1)^ke_{-r})\theta(\tau,z),$$

from which the assertion of the proposition follows. q.e.d.

We line up $E_{k,m,r}((\tau,z),s)$ as a column vector of t_∞-components:

$$E_{k,m}((\tau,z),s) = (E_{k,m,r}((\tau,z),s))_{r \in R_k^{null}} \in \mathbb{C}^{t_\infty}.$$

Set

$$\Phi^*(s)= K\Phi(s)K^{-1},$$

where K is a matrix of size t_∞ whose (r,p)-component $(r,\ p \in R_k^{null})$ is equal to $\epsilon_r\delta_{rp}$. Then the (r,p)-component $\varphi^*_{rp}(s)$ of $\Phi^*(s)$ is given by $\epsilon_r\epsilon_p^{-1}\varphi_{rp}(s)$.

The following theorem easily follows from Propositions 2.5 and 3.1.

THEOREM 3.2. *For each* $r \in R^{null}$, *the Eisenstein series* $E_{k,m,r}((\tau,z),s)$ *is analytically continued to a meromorphic function in the whole* s*-plane. It is holomorphic in the region* $\mathrm{Re}(s)\geq 1/2$ *except on the interval* $(1/2,1]$ *and has only simple poles on this interval. The functions* $E_{k,m}((\tau,z),s)$ *and* $\Phi^*(s)$ *satisfy the functional equations*

$$E_{k,m}((\tau,z),1-s)= \Phi^*(1-s)E_{k,m}((\tau,z),s) \quad \textit{and} \quad \Phi^*(s)\Phi^*(1-s)= 1_{t_\infty}.$$

Moreover, $E_{k,m,r}((\tau,z),s)$ $(r \in R_k^{null})$ *are* $\mathbb{C}$*-linearly independent.*

Now we calculate the "constant term" of the Fourier expansion

$$E_{k,m,r}((\tau,z),s)= \sum_{n,q\,\in\,\mathbb{Z}} c_{n,q}(\eta;s)e(n\xi+qz) \qquad (r \in R^{null},\ \tau=\xi+i\eta),$$

where the "constant term" means the partial sum

$$\sum_{n,q\,\in\,\mathbb{Z},\ 4mn=q^2} c_{n,q}(\eta;s)e(n\xi+qz).$$

For the caluculation we follow the method of Eichler-Zagier [E-Z, Ch.I, § 2].

We divide the sum on the right side of the identity (3.2) in two parts according as $c=0$ or $c\neq 0$. Thus using the identity

$$X^2M\tau + 2X\frac{z}{c\tau+d} - \frac{cz^2}{c\tau+d} = -\frac{(z-X/c)^2}{\tau+d/c} + \frac{aX^2}{c} \qquad \left(M=\begin{pmatrix} a & b \\ c & d\end{pmatrix} \in SL_2(\mathbb{R}),\ c\neq 0\right),$$

we have

$$(3.3)\qquad E_{k,m,r}((\tau,z),s)= E_r^1((\tau,z),s) + E_r^2((\tau,z),s)$$

with

$$E_r^1((\tau,z),s)= \eta^{s-\kappa}(\theta_r(\tau,z)+(-1)^k\theta_{-r}(\tau,z)),$$

$$(3.4)\qquad E_r^2((\tau,z),s)= \sum_{\substack{(c,d)=1\\ c\neq 0}} \sum_{q\in\mathbb{Z}} \frac{\eta^{s-\kappa}}{(c\tau+d)^k|c\tau+d|^{2(s-\kappa)}} \times$$

$$e^m\left(-\frac{(z-(q+r/2m)/c)^2}{\tau+d/c} + \frac{a}{c}(q+r/2m)^2\right),$$

where, for coprime integers c, d $(c\neq 0)$, an integer a is chosen so that $ad\equiv 1 \bmod c$. We introduce the infinite series $F((\tau,z),s)$:

$$F((\tau,z),s)= \sum_{p,q\in\mathbb{Z}} \frac{1}{|\tau+p|^{2(s-\kappa)}(\tau+p)^k} e^m\left(-\frac{(z+q)^2}{\tau+p}\right),$$

which is absolutely convergent for $Re(s)>3/4$. The function $F((\tau,z),s)$ coincides with $F_{k,m}(\tau,z)$ of [E-Z, p.19] if $s=\kappa$. Replacing q by $\lambda-cq'$ ($q'\in\mathbb{Z}$, $\lambda \bmod c$) on the right hand side of (3.4), we get

$$(3.5)\qquad E_r^2((\tau,z),s)= \sum_{c=1}^{\infty} \sum_{\substack{d \bmod c\\ (d,c)=1}} \sum_{\lambda \bmod c} \frac{\eta^{s-\kappa}}{c^{2s+1/2}} \times$$

$$\left[e^m\left(\frac{a}{c}(\lambda+r/2m)^2\right)F((\tau+\frac{d}{c}, z-\frac{1}{c}(\lambda+r/2m)),s)\right.$$

$$\left. + (-1)^k e^m\left(\frac{a}{c}(\lambda-r/2m)^2\right)F((\tau+\frac{d}{c}, z-\frac{1}{c}(\lambda-r/2m)),s)\right].$$

The function $F((\tau,z),s)$ is periodic in τ and z with period 1 and has the Fourier expansion of the form

$$(3.6)\qquad F((\tau,z),s)=\sum_{n,q\in\mathbb{Z}}\gamma_{n,q}(\eta,s)e(n\xi+qz)\qquad(\tau=\xi+i\eta\in\mathfrak{H},\ z\in\mathbb{C})$$

with

$$\gamma_{n,q}(\eta,s)=\int_{\mathbb{R}}\int_{\mathbb{R}}\tau^{-k}|\tau|^{-2(s-\kappa)}e(-mz^2/\tau-n\xi-qz)dxd\xi\qquad(z=x+iy).$$

Integrating first with respect to x, we get

$$\gamma_{n,q}(\eta,s)=\int_{\mathbb{R}}(\tau/2im)^{1/2}\tau^{-k}|\tau|^{-2(s-\kappa)}e(q^2\tau/4m-n\xi)d\xi.$$

Changing the variable with $\xi\longrightarrow\eta\xi$, we have

$$\gamma_{n,q}(\eta,s)=\frac{\eta^{1-2s}}{\sqrt{2im}}\exp\Big(-\frac{\pi}{2m}\eta q^2\Big)\cdot\int_{\mathbb{R}}(\xi+i)^{-s-\kappa}(\xi-i)^{-s+\kappa}e\Big(\Big(\frac{q^2}{4m}-n\Big)\eta\xi\Big)d\xi.$$

Since we have the integral formula

$$\int_{\mathbb{R}}(\xi+i)^{-s-\kappa}(\xi-i)^{-s+\kappa}d\xi=e^{-\pi i\kappa}\gamma(\kappa;s)\qquad(\text{for }\gamma(\kappa;s),\text{ see (2.8)}),$$

the Fourier coefficient $\eta_{n,q}(\tau,s)$ with $4mn=q^2$ is given as follows:

$$(3.7)\qquad\gamma_{n,q}(\eta,s)=\frac{\eta^{1-2s}}{\sqrt{2m}}\cdot e^{-\pi ik/2}\gamma(\kappa;s)\exp\Big(-\frac{\pi}{2m}\eta q^2\Big)\qquad(4mn=q^2).$$

For coprime integers c, d ($c\neq0$) and integers r, q with $r^2\equiv q^2\equiv0$ mod 4m, set

$$G(r,q;c,d)=\sum_{\lambda\bmod c}e^m\Big(\frac{a}{c}(\lambda+r/2m)^2-\frac{q}{cm}(\lambda+r/2m)+\frac{dq^2}{4cm^2}\Big),$$

where an integer a is chosen so that $ad\equiv1$ mod c. The well-definedness of G(r,q;c,d) is easily checked. In an elementary manner,

$$G(r,q;c,d)=\sum_{\lambda\bmod c}e^m\Big(\frac{d}{c}(q/2m-a(\lambda+r/2m))^2+\frac{bq}{m}(\lambda+r/2m)-ab(\lambda+r/2m)^2\Big),$$

where a, b are chosen so that $\begin{pmatrix}a&b\\c&d\end{pmatrix}\in\Gamma$. Since $r^2\equiv q^2\equiv0$ mod 4m, we have

$$(3.8)\qquad G(r,q;c,d)=\sum_{\lambda \bmod c} e^{m}\Big(\frac{d}{c}((q-ar)/2m-a\lambda)^2\Big)$$

We see from (3.8) that $G(r,q;c,d)$ depends on $r \bmod 2m$ and $q \bmod 2m$. According to (3.5), (3.6) and (3.7), the "constant term" of the Fourier expansion of $E_r^2((\tau,z),s)$ is given explicitly by

$$(3.9)\qquad \frac{\eta^{1-s-\kappa}e^{-\pi ik/2}\gamma(\kappa;s)}{\sqrt{2m}}\times$$

$$\sum_{q\in\mathbb{Z},\ q^2\equiv 0 \bmod 4m}\Big(\psi_{rq}(s)+(-1)^k\psi_{-rq}(s)\Big)\cdot e^{m}\Big(\frac{q^2}{4m^2}\tau+\frac{qz}{m}\Big),$$

where $\psi_{rq}(s)$ ($r,\ q\in\mathbb{Z} \bmod 2m$ with $r^2\equiv q^2\equiv 0 \bmod 4m$) is the Dirichlet series defined by

$$(3.10)\qquad \psi_{rq}(s)=\sum_{c=1}^{\infty}\ \sum_{\substack{d \bmod c\\ (d,c)=1}}\frac{G(r,q;c,d)}{c^{2s+1/2}}.$$

Moreover we see immediately from (3.8) that

$$\psi_{-rq}(s)=\psi_{r,-q}(s).$$

Thus the expression (3.9) turns out

$$(3.11)\qquad \frac{\eta^{1-s-\kappa}e^{-\pi ik/2}\gamma(\kappa;s)}{\sqrt{2m}}\cdot\sum_{p\in R^{null}}(\psi_{rp}(s)+(-1)^k\psi_{r,-p}(s))\theta_p(\tau,z).$$

Accordingly by (3.3) and (3.11), the "constant term" of the Fourier expansion of $E_{k,m,r}((\tau,z),s)$ equals

$$\eta^{s-\kappa}(\theta_r(\tau,z)+(-1)^k\theta_{-r}(\tau,z))+\eta^{1-s-\kappa}.\frac{e^{-\pi ik/2}\gamma(\kappa;s)}{\sqrt{2m}}\times$$

$$\sum_{p\in R_k^{null}} 2\epsilon_p^{-2}(\psi_{rp}(s)+(-1)^k\psi_{r,-p}(s))(\theta_p(\tau,z)+(-1)^k\theta_{-p}(\tau,z)).$$

On the other hand, by Propositions 2.2 and 3.1, the "constant term" of $E_{k,m,r}((\tau,z),s)$ coincides with

$$\eta^{s-\kappa}(\theta_r(\tau,z)+(-1)^k\theta_{-r}(\tau,z))$$

$$+\ \eta^{1-s-\kappa}\sum_{p\in R_k^{null}} \epsilon_r\epsilon_p^{-1}\varphi_{rp}(s)(\theta_p(\tau,z)+(-1)^k\theta_{-p}(\tau,z))$$

We note that $\theta_p(\tau,z)+(-1)^k\theta_{-p}(\tau,z)$ $(p\in R_k^{null})$ are $\mathbb{C}$-linearly independent. Comparing these two expressions of the "constant term", we obtain the following.

PROPOSITION 3.3. *Let* $r,\ p\in R_k^{null}$. *Then,*

$$\varphi_{rp}(s)=\frac{e^{-\pi ik/2}\gamma(\kappa;s)}{\sqrt{2m}}\cdot\frac{2}{\epsilon_r\epsilon_p}\cdot(\psi_{rp}(s)+(-1)^k\psi_{r,-p}(s)).$$

COROLLARY 3.4. *Let* $r,\ p\in R_k^{null}$. *Then,*

$$\sum_{\substack{(c,d)=1\\ c>0,\ d \bmod c}}\frac{(w_r,\chi(M)w_p)}{c^{2s}}=\frac{e^{-\pi i/4}}{\sqrt{2m}}\cdot\frac{2}{\epsilon_r\epsilon_p}\cdot(\psi_{rp}(s)+(-1)^k\psi_{r,-p}(s)),$$

where $M=\begin{pmatrix}a&b\\c&d\end{pmatrix}$ *is chosen so that* $M\in\Gamma$.

The proof is immediate from Propositions 2.2 and 3.3.

Finally we give an example in which $\Phi(s)$ is explicitly determined.

We assume that m is a square free positive integer. Then, $R^{null} = \{0 \bmod 2m\}$ and hence $t_\infty = (1+(-1)^k)/2$. Therefore we assume that k is even. In this case there exists the unique Eisenstein series $E_{k,m}((\tau,z),s) = E_{k,m,0}((\tau,z),s)$, and $\Phi(s) = \varphi_{00}(s)$ is a scalar function. We get, by (3.8) and (3.10),

$$\psi_{00}(s) = \sum_{c=1}^{\infty} \sum_{\lambda \bmod c} \sum_{\substack{d \bmod c \\ (d,c)=1}} \frac{1}{c^{2s+1/2}} \cdot e\left(\frac{d}{c} m\lambda^2\right).$$

Let $\mu(n)$ $(n \in \mathbb{Z},\ n>0)$ be the Möbius function. Using a trick in [E-Z, P. 20], we have

$$\psi_{00}(s) = \sum_{c=1}^{\infty} \sum_{\lambda \bmod c} \frac{1}{c^{2s+1/2}} \sum_{a|(c,m\lambda^2)} \mu(c/a)a$$

$$= \sum_{c=1}^{\infty} \sum_{a|c} \sum_{\lambda \bmod c,\ m\lambda^2 \equiv 0 \bmod a} \frac{1}{c^{2s+1/2}} \mu(c/a)a \ .$$

Thus,

(3.12) $$\psi_{00}(s) = \frac{1}{\zeta(2s-1/2)} \cdot \left(\sum_{a=1}^{\infty} \frac{N(a)}{a^{2s-1/2}} \right),$$

where

$$N(a) = \#\{\lambda \bmod a \mid m\lambda^2 \equiv 0 \bmod a\}.$$

It is easy to see that

(3.13) $$N(ab) = N(a)N(b) \quad \text{if } (a,b)=1.$$

Therefore to simplify the summation in (3.12), it suffices to calculate the local factor

$$Z_p(s) = \sum_{n=0}^{\infty} N(p^n) p^{-n(2s-1/2)} \quad \text{for each prime integer } p.$$

Then an elementary calculation shows that

$$(3.14)\qquad Z_p(s)=\begin{cases}\dfrac{1-p^{1-4s}}{(1-p^{2-4s})(1-p^{1/2-2s})} & \ldots\ \text{if } (p,m)=1,\\[2ex] \dfrac{1+p^{3/2-2s}}{1-p^{2-4s}} & \ldots\ \text{if } p|m.\end{cases}$$

By (3.12), (3.13) and (3.14), we have

$$(3.15)\qquad \psi_{00}(s)=\frac{\zeta(4s-2)}{\zeta(4s-1)}\cdot\prod_{p|m}\frac{1+p^{3/2-2s}}{1+p^{1/2-2s}}.$$

The following is due to Theorem 3.2, Proposition 3.3 and (3.15).

PROPOSITION 3.5. *Let* m *be a square free positive integer and* k *an even integer. Then the Eisenstein series* $E_{k,m}((\tau,z),s)$ *satisfies the functional equation*

$$E_{k,m}((\tau,z),1-s)=\Phi(1-s)E_{k,m}((\tau,z),s)$$

with

$$\Phi(s)=\frac{e^{-\pi ik/2}\gamma(\kappa;s)}{\sqrt{2m}}\cdot\frac{\zeta(4s-2)}{\zeta(4s-1)}\cdot\prod_{p|m}\frac{1+p^{3/2-2s}}{1+p^{1/2-2s}}.$$

Remark. In this case the functional equation $\Phi(s)\Phi(1-s)=1$ is easily verified with the use of the above expression for $\Phi(s)$.

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