IRREDUCIBILITY OF THE HILBERT-BLUMENTHAL MODULI SPACES WITH PARAHORIC LEVEL STRUCTURE

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Abstract. We determine the number of irreducible components of the reduction mod p of any Hilbert-Blumenthal moduli space with a parahoric level structure, where p is unramified in the totally real field.

1. Introduction

In their 1984 paper [1], Brylinski and Labesse computed the L-factors of Hilbert-Blumenthal moduli spaces for almost all good places. By that time the arithmetic minimal compactification was not known. In [2] Chai furnished the desired minimal compactification by observing that Rapoport's arithmetic toroidal compactification [23] plays the crucial role. Thus, the results of Brylinski and Labesse have been improved for all good places (see [7, p. 137]). A next task is to treat the case where p is unramified and the level group K_p at p is a standard Iwahoric subgroup. This moduli space is studied in Stamm [28], following the works of Zink [35] and of Rapoport-Zink [24]. Several local properties on geometry as well as fine global descriptions of the surface case have been obtained in loc. cit. In this paper we settle a global problem concerning the irreducibility in this moduli space.

Let p be a fixed rational prime. Let F be a totally real number field of degree g and O_F the ring of integers. Let $n \geq 3$ be a prime-to-p integer. Choose a primitive n-th root ζ_n of unity in $\overline{\mathbb{Q}} \subset \mathbb{C}$ and an embedding $\overline{\mathbb{Q}} \hookrightarrow \overline{\mathbb{Q}}_p$. Let (L, L^+) be a rank one projective O_F-module with a notion of positivity. Let $\mathcal{M}_{(L,L^+),n}$ denote the moduli space over $\mathbb{Z}_{(p)}[\zeta_n]$ that parametrizes equivalence classes of objects $\underline{A} = (A, i, \iota, \eta)$ over a locally Noetherian $\mathbb{Z}_{(p)}[\zeta_n]$-scheme S, where

- A is an abelian scheme of relative dimension g;
- $\iota : O_F \to \mathrm{End}_S(A)$ is a ring monomorphism;
- $i : (L, L^+)_S \to (\mathcal{P}(A), \mathcal{P}(A)^+)$ is a morphism of étale sheaves such that the induced morphism

$$L \otimes_{O_F} A \to A^t \tag{1.1}$$

  is an isomorphism, where $(\mathcal{P}(A), \mathcal{P}(A)^+)$ is the polarization sheaf of A (see [5]);
- $\eta : (O_F/nO_F)^2_S \simeq A[n]$ is an O_F-linear isomorphism such that the pull back of the Weil pairing $e_{i(\lambda_0)}$ is the standard pairing on $(O_F/nO_F)^2$ with respect to ζ_n, where λ_0 is any element in L^+ such that $|L/O_F\lambda_0|$ is prime to pn.

It is proved in Rapoport [23] and Deligne-Pappas [5] that

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Theorem 1.1 (Rapoport, Deligne-Pappas)**.** *The fibers of* $\mathcal{M}_{(L,L^+),n} \to \operatorname{Spec} \mathbb{Z}_{(p)}[\zeta_n]$ *are geometrically irreducible.*

In the paper we consider the Iwahoric level structure $\mathcal{M}_{(L,L^+),\Gamma_0(p),n}$ over $\mathcal{M}_{(L,L^+),n}$ where $\mathcal{M}_{(L,L^+),n}$ has good reduction at p. The goal is to determine the set $\Pi_0(\mathcal{M}_{(L,L^+),\Gamma_0(p),n} \otimes \overline{\mathbb{F}}_p)$ of the irreducible components. We write $\Pi_0(X)$ for the set of irreducible components of a Noetherian scheme X; we only consider the set $\Pi_0(X \otimes \overline{K})$ of geometrically irreducible components if X is of finite type over a field K.

Assume that p is unramified in F. Let $\mathcal{M}_{(L,L^+),\Gamma_0(p),n}$ denote the moduli space over $\mathbb{Z}_{(p)}[\zeta_n]$ that parametrizes equivalence classes of objects (A, i, ι, H, η), where

- (A, i, ι, η) is in $\mathcal{M}_{(L,L^+),n}$, and
- $H \subset A[p]$ is a finite flat rank p^g subgroup scheme which is invariant under the action of O_F and maximally isotropic with respect to the Weil pairing $e_{i(\lambda_0)}$ as above.

Write $\mathcal{M} := \mathcal{M}_{(L,L^+),n} \otimes \overline{\mathbb{F}}_p$ and $\mathcal{M}_{\Gamma_0(p)} := \mathcal{M}_{(L,L^+),\Gamma_0(p),n} \otimes \overline{\mathbb{F}}_p$ through this paper. We will state our main results concerning the number $|\Pi_0(\mathcal{M}_{\Gamma_0(p)})|$ in the next section. We describe them together with background and methods. See Theorem 2.7 and Theorem 5.1 for the precise statement.

The method in this paper is completely different from that used in [32] for the Siegel moduli spaces. In the previous paper the proof is based on the Faltings-Chai theorem on the p-adic monodromy for the ordinary locus [7] and a theorem proved by Ngô and Genestier [16] that the ordinary locus is dense in the parahoric level moduli spaces. The latter is obtained by analyzing the *Kottwitz-Rapoport* stratification introduced in [12].

For the present situation, the ordinary locus is no longer dense, as has been pointed out in Stamm [28] in the surface case. Thus Ribet's p-adic monodromy result [25] can only conclude the irreducibility for ordinary components. One may need to establish a similar result of p-adic monodromy for smaller p-adic invariant strata in $\mathcal{M}$, which is not available yet. However, even though we can prove these p-adic monodromy results, one still cannot conclude the irreducibility for non-ordinary components using the standard p-adic monodromy argument. The reason is that the forgetful morphism $f : \mathcal{M}_{\Gamma_0(p)} \to \mathcal{M}$ has fibration over smaller strata. Furthermore, we do not have yet geometric properties for Kottwitz-Rapoport strata of $\mathcal{M}_{\Gamma_0(p)}$ along the direction of work of Ngô-Genestier [16].

To overcome these new difficulties, we stratify the moduli space by a suitable p-adic invariant:

$$\mathcal{M}_{\Gamma_0(p)} = \coprod_{\alpha} \mathcal{M}_{\Gamma_0(p),\alpha}.$$

Then we study the corresponding *discrete Hecke orbit problem*, namely asking whether the prime-to-p Hecke correspondences operate transitively on the set $\Pi_0(\mathcal{M}_{\Gamma_0(p),\alpha})$. This discrete Hecke orbit problem, though itself does not have an affirmative answer, can be refined through the computation of the fibers of the stratified morphism $f_\alpha : \mathcal{M}_{\Gamma_0(p),\alpha} \to \mathcal{M}_\alpha$, and is reduced to the discrete Hecke orbit problem for the set $\Pi_0(\mathcal{M}_\alpha)$ of irreducible components for the base. The former one can be done using Dieudonné calculus, for which the present computation (see Sections 3 and 4) is largely based on the work [30].

The next crucial ingredient is Chai's monodromy theorem on Hecke invariant subvarieties. This is a global method which may be regarded as the counterpart of the p-adic monodromy method. Its original form for Siegel moduli spaces is developed by Chai [4]. Chai's method works for all modular variety of PEL-type, with the modification where the reductive group in the Shimura input data should be replaced by the simply-connected cover of its derived group [4, p. 291]. We supply the proof due to Chai in Section 6 for the reader's convenience. This ingredient enables us to confirm the irreducibility in the non-supersingular contribution (components those are not supersingular). To treat the remaining supersingular contribution, the tool is essentially the result that the Tamagawa number is one for semi-simple, simply-connected algebraic groups [11]. The present cases heavily rely on the computations for the *geometric mass formula* in [33], which is based on a work [26] of Shimura.

The paper is organized as follows. In Section 2 we describe the main theorems and provide the methods and ingredients. In Section 3 we give the proofs of the theorems. In Section 4 we treat the supersingular contribution. To make the exposition clean and more accessible, we assume p inert in F in these sections. In Sections 5 we show how to establish the analogous results in the unramified situation from the inert case. Section 6 provides a proof of Chai's result on Hecke invariant subvarieties. We attempt to write this as an independent section so that the reader can read this section alone together with Chai's well written paper [4].

2. Statements and methods

2.1. We keep the notation as in the previous section. Let k be an algebraically closed field of characteristic p. We will assume in Sections 2-4 that p is inert in F. Write $f : \mathcal{M}_{\Gamma_0(p)} \to \mathcal{M}$ the forgetful morphism; it is a proper surjective morphism.

Introduce the alpha stratification as in [8] and [30, Section 3] and decompose the moduli spaces into strata

$$\mathcal{M} = \coprod \mathcal{M}_{\underline{a}}, \quad \text{and} \quad f_{\underline{a}} : \mathcal{M}_{\Gamma_0(p),\underline{a}} \to \mathcal{M}_{\underline{a}}.$$

It is proved in Goren and Oort [8] that each stratum $\mathcal{M}_{\underline{a}}$ is smooth, quasi-affine, of pure dimension $g - |\underline{a}|$, and that the Zariski closure of $\overline{\mathcal{M}}_{\underline{a}}$ in $\mathcal{M}$ is smooth and

$$\overline{\mathcal{M}}_{\underline{a}} = \cup_{\underline{a}' \preceq \underline{a}} \mathcal{M}_{\underline{a}'}.$$

We recall the alpha type associated to objects in $\mathcal{M}(k)$. Let $W := W(k)$ be the ring of Witt vectors over k and σ the absolute Frobenius map on W. Put $\mathcal{O} := O_F \otimes \mathbb{Z}_p$ and let $\mathfrak{I} := \mathrm{Hom}(\mathcal{O}, W) = \{\sigma_i\}$ be the set of embeddings, arranged in a way that $\sigma\sigma_i = \sigma_{i+1}$ for $i \in \mathbb{Z}/g\mathbb{Z}$. Let $\underline{A} = (A, \iota)$ be an abelian O_F-variety over k. and let $\underline{M}$ be the associated covariant Dieudonné $\mathcal{O}$-module. The alpha type of $\underline{A}$ is defined to be

$$\underline{a}(\underline{A}) := \underline{a}(\underline{M}) := (a_i)_{i \in \mathbb{Z}/g\mathbb{Z}}, \quad \text{where} \;\; a_i := \dim_k (M/(F,V)M)^i,$$

Here $(M/(F,V)M)^i$ denotes the σ_i-component of the k-vector space $M/(F,V)M$.

If $\underline{A}$ is a point in $\mathcal{M}(k)$, then $\underline{a}(\underline{A})$ is an element in $\{0,1\}^{\mathcal{J}}$. For each $\underline{a} \in \{0,1\}^{\mathcal{J}}$, let $\mathcal{M}_{\underline{a}}$ denote the stratum of $\mathcal{M}$ consisting of points with alpha type $\underline{a}$. It is known [8] that every alpha stratum $\mathcal{M}_{\underline{a}}$ is non-empty. Let $\Delta := \{0,1\}^{\mathcal{J}}$ be the set of possible alpha types. The partial order on Δ is given by $\underline{a}' \preceq \underline{a}$ if and only if $a'_i \geq a_i$ for all $i \in \mathbb{Z}/g\mathbb{Z}$. In [30, Section 2] an alpha type $\underline{a} = (a_i)_i \in \Delta$ is called *generic* if $a_i a_{i+1} = 0$ for all $i \in \mathbb{Z}/g\mathbb{Z}$. This notion was first introduced by Goren and Oort [8] in which it is called *spaced*. It is proved in [30, Section 6] that the alpha type of any maximal point of a Newton stratum of $\mathcal{M}$ is generic.

In this paper we prove

Theorem 2.1. *We have* $\dim \mathcal{M}_{\Gamma_0(p),\underline{a}} = g$ *if and only if* $\underline{a}$ *is of generic type.*

Let $\Delta^{\mathrm{gen}} \subset \Delta$ denote the set of generic alpha types. Denote by $\tau(\underline{a}) \subset \mathbb{Z}/g\mathbb{Z}$, for $\underline{a} = (a_i) \in \Delta$, the subset consisting of elements i such that $a_i = 1$. The subset $\tau(\underline{a})$ is called the *alpha index* corresponding to $\underline{a}$. Write $\tau(\underline{a}) = \{n_1, \dots n_a\}$ with $0 \leq n_i < n_{i+1} < g$ and put $n_{a+1} = g + n_1$. Define the function $w : \Delta \to \mathbb{Z}$ by

$$w(\underline{a}) := w(\tau(\underline{a})) := \begin{cases} 2 & \text{if } \tau(\underline{a}) = \emptyset; \\ \prod_{j=1}^{a} (n_{j+1} - n_j - 1) & \text{otherwise.} \end{cases} \tag{2.1}$$

It is clear that $w(\underline{a}) > 0$ if and only if $\underline{a} \in \Delta^{\mathrm{gen}}$.

Theorem 2.2. *Let* $\underline{a}$ *be a generic alpha type.*

(1) *For any point* $x \in \mathcal{M}_{\underline{a}}(k)$*, the fiber* $f^{-1}(x)$ *has* $w(\underline{a})$ *irreducible components of dimension* $|\underline{a}|$.

(2) *The subscheme* $\mathcal{M}_{\Gamma_0(p),\underline{a}}$ *has* $w(\underline{a})|\Pi_0(\mathcal{M}_{\underline{a}})|$ *irreducible components of dimension* g.

Since the moduli space $\mathcal{M}_{\Gamma_0(p)}$ is equi-dimensional of dimension g [28, Theorem 1, p. 407], we have obtained

$$|\Pi_0(\mathcal{M}_{\Gamma_0(p)})| = \sum_{\underline{a} \in \Delta^{\mathrm{gen}}} w(\underline{a}) |\Pi_0(\mathcal{M}_{\underline{a}})|. \tag{2.2}$$

The next step to consider the ℓ-adic Hecke correspondences operating on the set $\Pi_0(\mathcal{M}_{\underline{a}})$ of irreducible components, where $\ell \neq p$ is a prime.

For any non-negative integer $m \geq 0$, let $\mathcal{H}_{\ell,m}$ be the moduli space over $\overline{\mathbb{F}}_p$ that parametrizes equivalence classes of objects $(\underline{A}_j = (A_j, i_j, \iota_j, \eta_j), j = 1,2,3; \varphi_1, \varphi_2)$ as the diagram

$$\underline{A}_1 \xleftarrow{\varphi_1} \underline{A}_3 \xrightarrow{\varphi_2} \underline{A}_2,$$

where

- $\underline{A}_1$ and $\underline{A}_2$ are objects in $\mathcal{M}$, and $\underline{A}_3$ is a g-dimensional abelian O_F-variety with a class of polarizations and a symplectic level-n structure as defined in Section 1 but the condition (1.1) is not required;
- the morphisms φ_1 and φ_2 are O_F-linear isogenies of degree ℓ^m such that $\varphi_j^* i_j = i_3$ and $\varphi_{j,*}\eta_3 = \eta_j$ for $j = 1,2$.

Let $\mathcal{H}_\ell := \cup_{m \geq 0} \mathcal{H}_{\ell,m}$. An ℓ-adic Hecke correspondence is given by an irreducible component $\mathcal{H}$ of $\mathcal{H}_\ell$ together with natural projections pr_1 and pr_2. A subset Z of $\mathcal{M}$ is called *ℓ-adic Hecke invariant* if $\mathrm{pr}_2(\mathrm{pr}_1^{-1}(\mathrm{Z})) \subset \mathrm{Z}$ for any ℓ-adic Hecke correspondence $(\mathcal{H}, \mathrm{pr}_1, \mathrm{pr}_2)$. If Z is an ℓ-adic Hecke invariant, locally closed subset of $\mathcal{M}$, then the ℓ-adic Hecke correspondences induce correspondences on the set $\Pi_0(Z)$ of irreducible components. We call $\Pi_0(Z)$ *ℓ-adic Hecke transitive* if the ℓ-adic Hecke correspondences operate transitively on $\Pi_0(Z)$. The discrete Hecke problem for any ℓ-adic Hecke invariant subscheme Z is asking whether $\Pi_0(Z)$ is ℓ-adic Hecke transitive.

Theorem 2.3 (Chai)**.** *Let Z be an ℓ-adic Hecke invariant subscheme of $\mathcal{M}$. If the set $\Pi_0(Z)$ is ℓ-adic Hecke transitive and maximal points of Z are not supersingular, then Z is irreducible.*

Notice that the formulation of Theorem 2.3 does not require our assumption on p and the statement of Theorem 2.3 remains valid without this assumption (see Section 6).

The following result is due to Goren and Oort [8, Corollary 4.2.4]

Theorem 2.4. *For any alpha type $\underline{a}$, the set $\Pi_0(\mathcal{M}_{\underline{a}})$ is ℓ-adic Hecke transitive.*

An alpha stratum $\mathcal{M}_{\underline{a}}$ is called *supersingular* if all of its maximal points are supersingular. This is equivalent to that any point of $\mathcal{M}_{\underline{a}}$ is supersingular, that is, the stratum $\mathcal{M}_{\underline{a}}$ is contained in the supersingular locus. Call an alpha type $\underline{a}$ *supersingular* if the corresponding stratum $\mathcal{M}_{\underline{a}}$ is so. It follows from Theorems 2.3 and 2.4 that

Corollary 2.5. *Any non-supersingular stratum $\mathcal{M}_{\underline{a}}$ is irreducible.*

2.2. It remains to treat the supersingular contribution in (2.2). For a generic alpha type $\underline{a}$, it is known that $\mathcal{M}_{\underline{a}}$ is supersingular if and only if $g = 2\mathrm{k}$ is even and $|\underline{a}| = \mathrm{k}$ (see [8, Introduction]). They correspond to alpha types $\underline{a} = (1, 0, \dots, 1, 0)$ and $\underline{a} = (0, 1, \dots, 0, 1)$. We actually describe all supersingular strata $\mathcal{M}_{\underline{a}}$, not just for generic ones. This is slightly more than what we need.

Choose and fix a non-zero element λ_0 in L^+ so that $(|L/O_F\lambda_0|, np) = 1$. Let x be any point in $\mathcal{M}_{(L,L^+),n}(\mathbb{C})$. One associates a skew-Hermitian O_F-module $H_1(A_x(\mathbb{C}), \mathbb{Z})$ to $(A_x, i_x(\lambda_0), \iota_x)$. The isomorphism class of the skew-Hermitian O_F-module $H_1(A_x(\mathbb{C}), \mathbb{Z})$ only depends on the moduli space $\mathcal{M}_{(L,L^+),n}$, which we write $(V_{\mathbb{Z}}, \langle\,,\rangle, \iota)$. Let G be the automorphism group scheme over $\mathbb{Z}$ associated to the skew-Hermitian O_F-module $(V_{\mathbb{Z}}, \langle\,,\rangle, \iota)$, and $\Gamma(n)$ be the kernel of the reduction map $G(\mathbb{Z}) \to G(\mathbb{Z}/n\mathbb{Z})$. One has the complex uniformization

$$\mathcal{M}_{(L,L^+),n}(\mathbb{C}) \simeq \Gamma(n)\backslash G(\mathbb{R})/SO_2(\mathbb{R})^g.$$

Theorem 2.6. *Let $\mathcal{M}_{\underline{a}}$ be a supersingular stratum.*

(1) *If g is odd, then $\mathcal{M}_{\underline{a}}$ consists of all superspecial points and*

$$|\mathcal{M}_{\underline{a}}(k)| = [G(\mathbb{Z}) : \Gamma(n)] \cdot \left[\frac{-1}{2}\right]^g \cdot \zeta_F(-1) \cdot (p^g - 1). \tag{2.3}$$

(2) *If g is even and $|\underline{a}| = g$, then $\mathcal{M}_{\underline{a}}$ consists of all superspecial points and*

$$|\mathcal{M}_{\underline{a}}(k)| = [G(\mathbb{Z}) : \Gamma(n)] \cdot \left[\frac{-1}{2}\right]^g \cdot \zeta_F(-1) \cdot (p^g + 1). \tag{2.4}$$

(3) *If g is even and $|\underline{a}|\neq g$, then any irreducible component of $\overline{\mathcal{M}_{\underline{a}}}$ is isomorphic to $(\mathbf{P}^1)^{g-|\underline{a}|}$ and*

$$|\Pi_0(\mathcal{M}_{\underline{a}})| = [G(\mathbb{Z}):\Gamma(n)]\cdot \left[\frac{-1}{2}\right]^g \cdot \zeta_F(-1). \tag{2.5}$$

Let $\Delta^{\rm gen}_{\rm ss}\subset \Delta^{\rm gen}$ denote the subset of supersingular generic alpha types. If g is odd, then $\Delta^{\rm gen}_{\rm ss}$ is empty; if $\underline{a}\in \Delta^{\rm gen}_{\rm ss}$, then $w(\underline{a})=1$. By Corollary 2.2, Theorem 2.6 (3) and (2.2), we get

Theorem 2.7. *Notation as before. Assume that p is inert in F. Then*

$$|\Pi_0(\mathcal{M}_{\Gamma_0(p)})| = \begin{cases} \left[\sum_{\underline{a}\in \Delta^{\rm gen}\smallsetminus\Delta^{\rm gen}_{\rm ss}} w(\underline{a})\right] + 2[G(\mathbb{Z}):\Gamma(n)]\cdot\left[\frac{-1}{2}\right]^g\cdot\zeta_F(-1) & \text{if } g \text{ is even;}\\ \sum_{\underline{a}\in\Delta^{\rm gen}} w(\underline{a}) & \text{if } g \text{ is odd.}\end{cases}$$

The following is an elementary combinatorial result.

Lemma 2.8. *For any subset τ of $\mathbb{Z}/g\mathbb{Z}$, let $w(\tau)$ be as in (2.1). One has*

$$\sum_{\tau\subset \mathbb{Z}/g\mathbb{Z}} w(\tau) = 2^g.$$

Using this fact, we rephrase Theorem 2.7 as below.

Theorem 2.9. *Assume that p is inert in F. Then*

$$|\Pi_0(\mathcal{M}_{\Gamma_0(p)})| = 2^g + \sum_{\underline{a}\in\Delta^{\rm gen}_{\rm ss}} \left\{[G(\mathbb{Z}):\Gamma(n)]\cdot\left[\frac{-1}{2}\right]^g\cdot\zeta_F(-1)-1\right\}. \tag{2.6}$$

See a formula for $|\Pi_0(\mathcal{M}_{\Gamma_0(p)})|$ when p is unramified in Section 5.

3. Proof of Theorems 2.1 and 2.2

3.1. Let $f:\mathcal{M}_{\Gamma_0(p)}\to\mathcal{M}$ be the forgetful morphism, and let $x=(A,i_A,\iota_A,\eta_A)$ be a point in $\mathcal{M}_{\underline{a}}(k)$. Choose a separable O_F-linear polarization $\lambda_A=i_A(\lambda_0)$ on A. Each point in $f^{-1}(x)$ is given by an O_F-invariant finite subgroup scheme H of A of rank p^g which is maximally isotropic with respect to the Weil pairing e_{λ_A}. Then there is an O_F-linear polarization λ_B, necessarily separable, on $B:=A/H$ such that the pull back $\pi^*\lambda_B$ is equal to $p\lambda_A$. Denote by $M^*(A)$ the classical contravariant Dieudonné module of A. We have an $\mathcal{O}$-invariant Dieudonné submodule $M^*(B)$ of $M^*(A)$ such that

$$M^*(A)/M^*(B)\cong k\oplus\cdots\oplus k,\quad \text{and } \langle\,,\rangle_{M^*(A)} = p\langle\,,\rangle_{M^*(B)}.$$

Note that $M^*(A)$ is canonically isomorphic to the dual $M(A)^t$ of the covariant Dieudonné module $M(A)$. We also know that $\underline{a}(M(A)^t)=\underline{a}(M(A))$ (see [30, Lemma 8.1]). Put $M_0:=M^*(A)$ and let $\tau:=\tau(\underline{a})$ be corresponding alpha index as in Section 2. Let $\mathcal{X}_\tau$ be the space of Dieudonné $\mathcal{O}$-submodules M of M_0 such that

$$M_0/M\cong k\oplus\cdots\oplus k.$$

We regard $\mathcal{X}_\tau$ as a scheme over k with reduced structure. For any point M in $\mathcal{X}_\tau$, it is clear that the pairing $\langle\,,\rangle$ is trivial on M_0/M. Therefore we have a polarized abelian O_F-variety $\underline{B}=(B,\lambda_B,\iota_B)$ and an O_F-linear isogeny $\pi:\underline{A}\to\underline{B}$ such that $\pi^*\lambda_B=p\lambda_A$ and $M^*(B)=M$. This establishes

Lemma 3.1. *There is an (non-canonical) isomorphism $\xi_x : f^{-1}(x)_{\mathrm{red}} \simeq \mathcal{X}_\tau$, where $f^{-1}(x)_{\mathrm{red}}$ is the reduced subscheme underlying the fiber $f^{-1}(x)$.*

Lemma 3.2. *The scheme $\mathcal{X}_\tau$ is isomorphic to the subscheme of $(\mathbf{P}^1)^g = \{([s_i : t_i])_{i\in\mathbb{Z}/g\mathbb{Z}}\}$ defined by the equations $t_{i-1}s_i = 0$ for $i \notin \tau$ and $t_{i-1}t_i = 0$ for $i \in \tau$.*

Proof. A point in $\mathcal{X}_\tau(k)$ is represented by a k-subspace $\overline{M}$ of $\overline{M}_0 := M_0/pM_0$ such that $F(\overline{M}) \subset \overline{M}$, $V(\overline{M}) \subset \overline{M}$, and $\dim_k \overline{M}^i = 1$ for each $i \in \mathbb{Z}/g\mathbb{Z}$. Hence it is a closed subscheme of $(\mathbf{P}^1)^g$. Choose a basis $\{X_i, Y_i\}$ for M_0 [30, Proposition 4.2] such that

$$FX_{i-1} = \begin{cases} X_i & \text{if } i \notin \tau; \\ Y_i + pc_iX_i & \text{if } i \in \tau; \end{cases} \quad FY_{i-1} = \begin{cases} pY_i & \text{if } i \notin \tau; \\ pX_i & \text{if } i \in \tau; \end{cases}$$

where c_i are some elements of $W(k)$ for $i \in \tau$. (There should be no confusion on our notation for the Frobenius map and the totally real field.) Let $P = ([s_i : t_i])_i$ be a point in $(\mathbf{P}^1)^g(k)$ and write $\overline{M}_P$ for the k-subspace of $\overline{M}_0$ generated by $s_iY_i + t_iX_i$ for $i \in \mathbb{Z}/g\mathbb{Z}$. We have

$$F(s_{i-1}Y_{i-1} + t_{i-1}X_{i-1}) = \begin{cases} t_{i-1}^p X_i & i \notin \tau; \\ t_{i-1}^p Y_i & i \in \tau. \end{cases}$$

From the closed condition $F\overline{M}_P \subset \overline{M}_P$ we get the equations

$$t_{i-1}s_i = 0 \text{ for } i \notin \tau, \quad \text{and } t_{i-1}t_i = 0 \text{ for } i \in \tau. \tag{3.1}$$

From the closed condition $V\overline{M}_P \subset \overline{M}_P$ we get the same equations as above. This finishes the computation. ∎

3.2. **Examples.** (1) If $\underline{a} = \underline{0}$, then $\mathcal{X}_\tau$ consists of two points: $([1:0],[1:0],\dots,[1:0])$ and $([0:1],[0:1],\dots,[0:1])$.

(2) If $\underline{a} = (1,0,1,0,0)$, then $\mathcal{X}_\tau$ is defined by the equations $t_4t_0, t_0s_1, t_1t_2, t_2s_3, t_3s_4$. There are four irreducible components:

$$\mathbf{P}^1 \times [0:1] \times [1:0] \times [1:0] \times [1:0], \quad [1:0] \times [1:0] \times \mathbf{P}^1 \times [0:1] \times [0:1],$$
$$[1:0] \times \mathbf{P}^1 \times [1:0] \times \mathbf{P}^1 \times [0:1], \quad [1:0] \times \mathbf{P}^1 \times [1:0] \times [1:0] \times \mathbf{P}^1.$$

Notice that for maximally dimensional components, every $\mathbf{P}^1$ is placed at a position i where $a_i = 0$.

(3) If $\underline{a} = (1,0,1,1,1,0)$, then $\mathcal{X}_\tau$ is defined by the equations $t_5t_0, t_0s_1, t_1t_2, t_2t_3, t_3t_4, t_4s_5$. There are one 3-dimensional component $[1:0] \times \mathbf{P}^1 \times [0:1] \times \mathbf{P}^1 \times [0:1] \times \mathbf{P}^1$, and four 2-dimensional components

$$\mathbf{P}^1 \times [0:1] \times [1:0] \times \mathbf{P}^1 \times [1:0] \times [1:0], \quad [1:0] \times \mathbf{P}^1 \times [1:0] \times [1:0] \times \mathbf{P}^1 \times [0:1],$$
$$[1:0] \times [1:0] \times \mathbf{P}^1 \times [1:0] \times \mathbf{P}^1 \times [0:1], \quad [1:0] \times [1:0] \times \mathbf{P}^1 \times [1:0] \times [1:0] \times \mathbf{P}^1.$$

Proposition 3.3.

(1) *We have* $\dim \mathcal{X}_\tau \leq |\underline{a}|$. *Furthermore,* $\dim \mathcal{X}_\tau = |\underline{a}|$ *if and only if* $\underline{a} \in \Delta^{\mathrm{gen}}$.

(2) *For $\underline{a} \in \Delta^{\mathrm{gen}}$, the scheme $\mathcal{X}_\tau$ has $w(\underline{a})$ irreducible components of dimension $|\underline{a}|$.*

PROOF. We may assume that $|\underline{a}|>0$, as the case $\underline{a}=\underline{0}$ is treated in Example 3.2 (1). Since the defining equations are either $s_i=0$ or $t_i=0$, any irreducible component of $\mathcal{X}_\tau$ is of the form $X=\prod_{i\in\mathbb{Z}/g\mathbb{Z}} X_i$, where

$$X_i=[1:0],\ [0:1],\quad \text{or } \mathbf{P}^1.$$

If $i\not\in\tau$, then we have $t_{i-1}s_i=0$. This tells us that there are at least $g-|\underline{a}|$ zeros for s_i or t_i in the components $[s_i:t_i]$ for $i\not\in\tau$ or $i-1\not\in\tau$. So $X_i=\mathbf{P}^1$ for at most $|\underline{a}|$ numbers of i. This shows that $\dim\mathcal{X}_\tau\le|\underline{a}|$.

If $\underline{a}\not\in\Delta^{\rm gen}$, then one can choose i such that $i-1\not\in\tau$, $i\in\tau$ and $i+1\in\tau$. It follows from the equation $t_it_{i+1}=0$ that there are at least $g-|\underline{a}|+1$ zeros for s_i or t_i in the components $[s_i:t_i]$ for $i\in\mathbb{Z}/g\mathbb{Z}$. Thus, $\dim\mathcal{X}_\tau<|\underline{a}|$. Suppose that $\underline{a}\in\Delta^{\rm gen}$. Put $s_i=1$ for all $i\in\mathbb{Z}/g\mathbb{Z}$, then the defining equations become $t_{i-1}=0$ for $i\not\in\tau$. Thus, $\dim\mathcal{X}_\tau=|\underline{a}|$. This proves the statement (1).

(2) Let $\underline{a}\in\Delta^{\rm gen}$ and $X=\prod_{i\in\mathbb{Z}/g\mathbb{Z}}X_i$ be an irreducible component of $\mathcal{X}_\tau$. Write $\tau=\{n_1,\dots,n_a\}$. First notice that

(i) If $X_{i_0}=\mathbf{P}^1$ for some $n_j\le i_0\le n_{j+1}$, then $X_i=[0:1]$ for $i_0<i<n_{j+1}$, and $X_i=[1:0]$ for $n_j\le i<i_0$ or $i_0<i=n_{j+1}$.

It follows that

(ii) There is at most one $i\in\mathbb{Z}$ in each interval $[n_j,n_{j+1}]$ such that $X_i=\mathbf{P}^1$.
(iii) If $X_i=\mathbf{P}^1$ for some $i\in\tau$, then $\dim X<|\underline{a}|$.

If $\dim X=|\underline{a}|$, then $X_{i_j}=\mathbf{P}^1$ for one i_j in each interval $n_j<i_j<n_{j+1}$. Conversely, choose i_j in each interval $n_j<i_j<n_{j+1}$. Then there is unique irreducible component X such that $X_{i_j}=\mathbf{P}^1$ for each j; this follows from (i). There are $\prod_j(n_{j+1}-n_j-1)$ such choices. Thus, the scheme $\mathcal{X}_\tau$ has $w(\underline{a})$ irreducible components of dimension $|\underline{a}|$. ∎

Theorem 2.1 follows from Lemma 3.1 and Proposition 3.3 (1).

3.3. **Proof of Theorem 2.2.** Part (1) follows from Lemma 3.1 and Proposition 3.3 (2). We prove the statement (2). We prove that irreducible components of $\mathcal{X}_\tau$ give rise to well-defined closed subvariety in $\mathcal{M}_{\Gamma_0(p),\underline{a}}$. Notice two isomorphisms between $f^{-1}(x)_{\rm red}$ and $\mathcal{X}_\tau$ are differed by an automorphism β of $\overline{M}_0$, which sends each factor of $(\mathbf{P}^1)^g$ to itself. If $X=\prod_i X_i$ is an irreducible component of $\mathcal{X}_\tau$, then $\beta(X)_i$ is equal to $\mathbf{P}^1$ whenever $X_i=\mathbf{P}^1$. By property (i) in the proof of Proposition 3.3, we have showed that $\beta(X)=X$. Therefore,

$$\mathcal{M}_X:=\{y\in\mathcal{M}_{\Gamma_0(p),\underline{a}}\,|\,\xi_{f(y)}(y)\in X\,\}$$

is a well-defined closed subvariety of $\mathcal{M}_{\Gamma_0(p),\underline{a}}$. One has $\mathcal{M}_{\Gamma_0(p),\underline{a}}=\cup_X\mathcal{M}_X$ as a union of components; any irreducible component of $\mathcal{M}_{\Gamma_0(p),\underline{a}}$ is contained in $\mathcal{M}_X$ for one X. The morphism $f_{\underline{a}}:\mathcal{M}_X\to\mathcal{M}_{\underline{a}}$ is proper and surjective with fibers isomorphic to X. Thus, $\Pi_0(\mathcal{M}_X)\simeq\Pi_0(\mathcal{M}_{\underline{a}})$ and $\dim\mathcal{M}_X=\dim\mathcal{M}_{\underline{a}}+\dim X$. From this and Proposition 3.3 (2) the statement (2) then follows. ∎

3.4. **Proof of Lemma 2.8.** If $|\underline{a}|=j>0$, then $w(\underline{a})$ is the number of ways replacing a zero by 2 in $\underline{a}$ on each interval $[n_j,n_{j+1}]$. In other words, $\sum_{|\underline{a}|=j}w(\underline{a})$

is the number of ways of choosing $2j$ positions from $\mathbb{Z}/g\mathbb{Z}$ and filling them with 1 and 2 alternatively. This gives $\sum_{|\underline{a}|=j} w(\underline{a}) = 2\binom{g}{2j}$. Thus

$$\sum_{\underline{a}\in\Delta} w(\underline{a}) = 2 + \sum_{j>0} 2\binom{g}{2j} = 2^g.$$

This completes the proof. ∎

4. Supersingular contribution

Keep the notation and the assumption of p as before.

4.1. Let $x_0 = \underline{A}_0 = (A_0, \lambda_0, \iota_0, \eta_0)$ be a superspecial (not necessarily separably) polarized abelian O_F-variety over k of dimension g with symplectic level-n structure with respect to ζ_n. Let $\underline{M}_0 = (M_0, \langle\, ,\rangle, \iota)$ be its covariant Dieudonné module with additional structures. As M_0 is superspecial, the alpha type $\underline{a}$ of $\underline{M}_0$ has the form

$$(e_1 + e_2,\, 2 - (e_1 + e_2),\, e_1 + e_2,\, \dots)$$

for some integers e_1, e_2 with $0 \le e_1 \le e_2 \le 1$; see [31, Section 2]. When g is odd, it satisfies an additional condition $e_1 + e_2 = 1$. We say that $\underline{M}_0$ is of *superspecial type* (e_1, e_2) if its alpha type is as above.

Let G_{x_0} denote the automorphism group scheme over $\operatorname{Spec}\mathbb{Z}$ associated to $(A_0, \lambda_0, \iota_0)$; for any commutative ring R, its group of R-points is

$$G_{x_0}(R) = \{\phi \in (\operatorname{End}_{O_F}(A_0) \otimes R)^\times; \phi'\phi = 1\},$$

where the map $\phi \mapsto \phi'$ is the Rosati involution induced by λ_0.

Let $\Lambda_{x_0,n}$ denote the set of isomorphism classes of polarized abelian O_F-varieties $\underline{A} = (A, \lambda, \iota, \eta)$ with level-n structure (w.r.t. ζ_n) over k such that (c.f. (2.4) of [33])

(i) the Dieudonné module $M(\underline{A})$ is isomorphic to $M(\underline{A}_0)$, compatible with $O_F \otimes \mathbb{Z}_p$-actions and quasi-polarizations, and
(ii) the Tate module $T_\ell(\underline{A})$ is isomorphic to $T_\ell(\underline{A}_0)$, compatible with $O_F \otimes \mathbb{Z}_\ell$-actions and the Weil pairings, for all $\ell \ne p$.

The condition (i) implies that A is superspecial and $\dim A = g$. Let K_n be the kernel of the reduction map $G_{x_0}(\hat{\mathbb{Z}}) \to G_{x_0}(\hat{\mathbb{Z}}/n\hat{\mathbb{Z}})$. There is a natural isomorphism

$$\Lambda_{x_0,n} \simeq G_{x_0}(\mathbb{Q})\backslash G_{x_0}(\mathbb{A}_f)/K_n; \tag{4.1}$$

see [30, Theorem 10.5] and [33, Theorem 2.1 and Subsection 4.6]. It is proved in [33, Theorem 3.7 and Subsection 4.6] that

$$|\Lambda_{x_0,n}| = [G_{x_0}(\hat{\mathbb{Z}}) : K_n]\left[\frac{-1}{2}\right]^g \zeta_F(-1)c_p, \tag{4.2}$$

where

$$c_p := \begin{cases} 1 & g \text{ is even and } e_1 = e_2, \\ p^g + 1 & g \text{ is even and } e_1 < e_2, \\ p^g - 1 & g \text{ is odd}, \end{cases} \tag{4.3}$$

and (e_1, e_2) is the superspecial type of M_0.

If $T_\ell(\underline{A}_0) \simeq (V\otimes \mathbb{Z}_\ell, \langle\,,\rangle, \iota)$ (Subsection 2.2) for all $\ell\neq p$, then it is easy to see that $[G_{x_0}(\hat{\mathbb{Z}}):K_n]=[G(\mathbb{Z}):\Gamma(n)]$. In this case, the formula (4.2) becomes

$$|\Lambda_{x_0,n}| = [G(\mathbb{Z}):\Gamma(n)]\left[\frac{-1}{2}\right]^g \zeta_F(-1)c_p, \tag{4.4}$$

where c_p is as above.

4.2. If g is odd, then it follows from [8, Theorem 5.4.11] that $\mathcal{M}_{\underline{a}}$ is supersingular if and only if $|\underline{a}|=g$, that is, $\mathcal{M}_{\underline{a}}$ consists of all superspecial points in $\mathcal{M}$. By the formula (4.4), we get the equation (2.3).

If g is even, then it follows from [8, Theorem 5.4.11] that $\mathcal{M}_{\underline{a}}$ is supersingular if and only if $\underline{a}\preceq(1,0,\dots,1,0)$ or $\underline{a}\preceq(0,1,\dots,0,1)$. If $|\underline{a}|=g$, then $\mathcal{M}_{\underline{a}}$ consists of all superspecial points in $\mathcal{M}$. By the formula (4.4), we get the equation (2.4). This proves the statements (1) and (2) of Theorem 2.6.

4.3. Suppose $g=2\mathrm{k}$ is even and $|\underline{a}|\neq g$. Put $\underline{a}_0:=(1,0,\dots,1,0)$. We may assume that $\underline{a}\preceq\underline{a}_0$ due to symmetry. Let $\mathcal{M}^{(p)}$ be the moduli space over $\overline{\mathbb{F}}_p$ of g-dimensional separably polarized abelian O_F-varieties with a symplectic level-n structure with respect to ζ_n. We may identify the moduli space $\mathcal{M}$ with an irreducible component of $\mathcal{M}^{(p)}$ by choosing an suitable element $\lambda_0\in L^+$; see [33, Proposition 4.1].

Choose any point $\underline{A}_0$ in $\mathcal{M}_{\underline{a}_0}(k)$. Let $\underline{M}_0$ be the covariant Dieudonné module of $\underline{A}_0$. Let $\underline{N}:=(F,V)M_0$, a Dieudonné $\mathcal{O}$-submodule with the induced quasi-polarization. Then there is a tuple $\underline{B}=(B,\lambda_B,\iota_B,\eta_B)$ and an O_F-linear isogeny $\varphi:\underline{B}\to\underline{A}_0$ of a p-power degree, compatible with additional structures, such that $M(B)=N\subset M_0$.

One easily computes that $\underline{N}$ has alpha type $(0,2,\dots,0,2)$. Then one can find a basis $\{X_i,Y_i\}$ for N^i [30, Lemma 4.4] such that

$$\begin{array}{lll} FX_i=-pY_{i+1}, & FY_i=pX_{i+1}, & \text{if } i \text{ is even},\\ FX_i=-Y_{i+1}, & FY_i=X_{i+1}, & \text{if } i \text{ is odd}. \end{array} \tag{4.5}$$

Let $N_{-1}:=(F,V)^{-1}N$; it is spanned by elements

$$\frac{1}{p}X_{2i},\ \frac{1}{p}Y_{2i},\ X_{2i+1},\ Y_{2i+1},\quad i=0,\cdots,\mathrm{k}-1.$$

We have $N_{-1}/N\cong k^2\oplus0\oplus k^2\oplus\cdots k^2\oplus0$ as $\mathcal{O}\otimes_{\mathbb{Z}_p}k$-modules. Let $\mathcal{X}$ be the space of Dieudonné $\mathcal{O}$-modules M such that

$$N\subset M\subset N_{-1},\ M/N\cong k\oplus0\oplus k\oplus\cdots k\oplus0.$$

It is clear that $\mathcal{X}\cong(\mathbf{P}^1)^{\mathrm{k}}$.

Let Λ denote the set of isomorphism classes of objects $\underline{B}'=(B',\lambda',\iota',\eta')$ (with respect to ζ_n) such that (cf. Subsection 4.1)

- the Dieudonné module $M(\underline{B}')$ is isomorphic to $M(\underline{B})$, compatible with additional structures, and
- the Tate module $T_\ell(\underline{B}')$ is isomorphic to $T_\ell(\underline{B})$, compatible with additional structures, for all $\ell\neq p$.

Proposition 4.1. *There is an isomorphism* $\mathrm{pr}:\coprod_{\xi\in\Lambda}\mathcal{X}\to\overline{\mathcal{M}_{\underline{a}_0}}$.

Proof. We write the map *set-theoretically* first. For any member $\xi \in \Lambda$ and any point $x \in \mathcal{X}_\xi(k) := \mathcal{X}(k)$, we have $M(\underline{B_\xi}) = \underline{N} \subset \underline{M}_x$. Then one gets a point $\underline{A}_x$ together with a polarized O_F-linear isogeny $\varphi : \underline{B}_\xi \to \underline{A}_x$ of p-power degree such that $M(\underline{A}_x) = \underline{M}_x$. Define $\mathrm{pr}(\underline{M}_x) := \underline{A}_x$. Then one can show that it gives a bijective map from $\coprod_{\xi\in\Lambda} \mathcal{X}_\xi(k)$ onto $\overline{\mathcal{M}}_{\underline{a}_0}(k)$. To see this map comes from a morphism of schemes, we need to construct a moduli space with a prescribed isogeny type a priori, and show that this map agrees with the natural projection. Since the construction is lengthy and is the same as [30, Lemma 9.1], we refer the reader to loc. cit. and omit the details here. Finally using the tangent space calculation, we prove that the morphism pr is étale and particularly separable; see the computation in Lemma 9.2 of [30]. Thus the morphism pr is isomorphism and the proof is complete. ∎

By the formula (4.4), we get

Lemma 4.2. $|\Lambda| = [G(\mathbb{Z}) : \Gamma(n)] \left[\frac{-1}{2}\right]^g \zeta_F(-1)$.

Denote by $\overline{\mathcal{M}}_{\underline{a}_0,\xi}$ the irreducible component corresponding to ξ and write $\mathrm{pr} : \mathcal{X} \to \overline{\mathcal{M}}_{\underline{a}_0,\xi}$. Let $\mathcal{M}_{\preceq\underline{a},\xi} \subset \overline{\mathcal{M}}_{\underline{a}_0,\xi}$ be the closed subscheme consisting of points with alpha type $\preceq \underline{a}$.

Lemma 4.3. *The scheme $\mathcal{M}_{\preceq\underline{a},\xi}$ is isomorphic to $(\mathbf{P}^1)^{g-|\underline{a}|}$*

Proof. For a point $P = ([x_0 : y_0], [x_2 : y_2], \cdots, [x_{2\mathrm{k}-2} : y_{2\mathrm{k}-2}]) \in (\mathbf{P}^1(k))^{\mathrm{k}}$, the representing Dieudonné module is given by

$$M_P = N + < \tilde{x}_{2i}\frac{1}{p}X_{2i} + \tilde{y}_{2i}\frac{1}{p}Y_{2i} >_{i=0,\cdots,\mathrm{k}-1},$$

where $\tilde{x}_{2i}, \tilde{y}_{2i}$ are any liftings of x_{2i}, y_{2i} in W, respectively.

We compute the defining equations for $\mathcal{M}_{\preceq\underline{a},\xi}$ on an affine open subset. Let $V_{2i} := \tilde{x}_{2i}\frac{1}{p}X_{2i} + \frac{1}{p}Y_{2i}$, then

$$M_P =< X_{2i}, V_{2i}, X_{2i+1}, Y_{2i+1} >_{i=0,\cdots,\mathrm{k}-1}, \text{ and } M_P^{2i+1} =< X_{2i+1}, Y_{2i+1} >$$

One computes that

$$((F,V)M_P)^{2i+1} \pmod{pM_P^{2i+1}} =< \overline{X}_{2i+1} - x_{2i}^p\overline{Y}_{2i+1},\ -\overline{X}_{2i+1} + x_{2i+2}^{p^{-1}}\overline{Y}_{2i+1} > .$$

Therefore, $a_{2i+1}(M_P) = 1$ if and only if $x_{2i}^{p^2} = x_{2i+2}$.

Let $\tau = \tau(\underline{a}) \subset \mathbb{Z}/g\mathbb{Z}$. We have showed that the subscheme $\mathcal{M}_{\preceq\underline{a},\xi}$ of $\overline{\mathcal{M}}_{\underline{a}_0,\xi} = (\mathbf{P}^1)^{\mathrm{k}} = \{(x_2, \cdots, x_{2\mathrm{k}})\}$ defined by the equations $x_{j-1}^{p^2} = x_{j+1}$ for all odd $j \in \tau$, and thus it is isomorphic to $(\mathbf{P}^1)^{g-|\underline{a}|}$. This completes the proof. ∎

By Proposition 4.1 and Lemmas 4.2 and 4.3, the statement (3) of Theorem 2.6 is proved.

5. Unramified setting

In this section we only assume that p is unramified in F.

5.1. Let

$$\mathcal{O} := O_F \otimes \mathbb{Z}_p, \quad \mathfrak{I} := \mathrm{Hom}(\mathcal{O}, W), \quad \Delta := \{0,1\}^{\mathfrak{I}}$$

be the same as in Section 2. Let $\mathbb{P}$ be the set of primes of O_F lying over p. For $v \in \mathbb{P}$, let $\mathcal{O}_v$ be the completion of O_F at v, f_v its residue degree, $\mathfrak{I}_v := \mathrm{Hom}(\mathcal{O}_v, W)$ and $\Delta_v := \{0,1\}^{\mathfrak{I}_v}$. One has

$$\mathcal{O} = \oplus_{v\in\mathbb{P}} \mathcal{O}_v, \quad \sum_{v\in\mathbb{P}} f_v = g,$$

$$\mathfrak{I} = \coprod_{v\in\mathbb{P}} \mathfrak{I}_v, \quad \mathfrak{I}_v \simeq \mathbb{Z}/f_v\mathbb{Z}, \quad \text{and } \Delta = \prod_{v\in\mathbb{P}} \Delta_v.$$

An alpha type $\underline{a} = (\underline{a}_v) \in \Delta$ is called *generic* if every component $\underline{a}_v$ is generic; it is called *supersingular* if the associated alpha stratum $\mathcal{M}_{\underline{a}}$ is supersingular.

Let $\Delta^{\mathrm{gen}} \subset \Delta$ be the subset of generic alpha types, and $\Delta^{\mathrm{gen}}_{\mathrm{ss}} \subset \Delta^{\mathrm{gen}}$ the subset of supersingular alpha types. The set $\Delta^{\mathrm{gen}}_{\mathrm{ss}}$ is empty if and only if f_v is odd for some v.

Theorem 5.1. *For any alpha type $\underline{a}$, the set $\Pi_0(\mathcal{M}_{\underline{a}})$ is ℓ-adic Hecke transitive.*

This is essentially due to Goren and Oort (cf. [8, Corollary 4.2.4]). We provide suitable details to fit the present situation: p is unramified and the objects $(A, \lambda, \iota, \eta)$ that $\mathcal{M}$ parametrizes may not be principally polarized abelian O_F-varieties.

Proposition 5.2.
(1) *Every alpha stratum $\mathcal{M}_{\underline{a}}$ is non-empty.*
(2) *Every alpha stratum $\mathcal{M}_{\underline{a}}$ is quasi-affine.*
(3) *The non-ordinary locus of $\mathcal{M}$ is proper.*
(4) *The Zariski closure $\overline{\mathcal{M}_{\underline{a}}}$ of each stratum $\mathcal{M}_{\underline{a}}$ in $\mathcal{M}$ is smooth.*
(5) *The set $\mathcal{M}_{\underline{0}}$ of superspecial points is ℓ-adic Hecke transitive.*

PROOF. (1) It is easy to construct a superspecial point in $\mathcal{M}$. Indeed, one constructs a separably polarized superspecial abelian O_F-variety, then one chooses a point within its prime-to-p isogeny class so that it lies in $\mathcal{M}$. Then one constructs a deformation of this point so that the generic point has the given alpha type $\underline{a}$. Such a construction is a local problem, and it reduces to inert cases. This proves the statement (1)

(2) This is a global property; it does not follow directly from the result of inert cases. One can slightly modify the proof in loc. cit to make it work. Alternatively, consider the forgetful morphism $b : \mathcal{M} \to \mathcal{A}_{g,d,n} \otimes \overline{\mathbb{F}}_p$, for some positive integer d with $(d,p) = 1$. Then the image $b(\mathcal{M}_{\underline{a}})$ is contained in an Ekedahl-Oort stratum S_φ of $\mathcal{A}_{g,d,n} \otimes \overline{\mathbb{F}}_p$. Since S_φ is quasi-affine [18], the image $b(\mathcal{M}_{\underline{a}})$ is also quasi-affine. Since the morphism b is finite, the stratum $\mathcal{M}_{\underline{a}}$ is quasi-affine.

(3) This follows from the semi-stable reduction theorem for abelian varieties due to Grothendieck [9].

(4) This is a local property, and hence follows directly from the results of inert cases.

(5) Let $x_0 = (A_0, \lambda_0, \iota_0, \eta_0)$ be a superspecial point in $\mathcal{M}$. Define $\Lambda_{x_0,n}$, G_{x_0}, K_n as in Subsection 4.1. Note that any point in $\mathcal{M}_{\underline{0}}$ satisfies the conditions (i) and

(ii) in Subsection 4.1 (see [30, Lemma 4.3]). Therefore, we have $\Lambda_{x_0,n} = \mathcal{M}_{\underline{0}}$ and have the double coset description

$$\mathcal{M}_{\underline{0}} \simeq G_{x_0}(\mathbb{Q})\backslash G_{x_0}(\mathbb{A}_f)/K_n \tag{5.1}$$

as (4.1). By the strong approximation [21, Theorem 7.12, p. 427], the natural map $G_{x_0}(\mathbb{Q}_\ell) \to G_{x_0}(\mathbb{Q})\backslash G_{x_0}(\mathbb{A}_f)/K_n$ is surjective. This shows that the ℓ-adic Hecke orbit $\mathcal{H}_\ell(\underline{A}_0)$ is equal to $\mathcal{M}_{\underline{0}}$, and hence that the action of ℓ-adic Hecke correspondences on the set $\mathcal{M}_{\underline{0}}$ is transitive. ∎

5.2. **Proof of Theorem 5.1.** Using (1), (2) and (3) of Proposition 5.2, one shows that the closure of any irreducible component W of $\mathcal{M}_{\underline{a}}$ contains a point in $\mathcal{M}_{\underline{0}}$. By Proposition 5.2 (4), any point in $\mathcal{M}_{\underline{0}}$ is contained in $\overline{W}$ for a unique irreducible component W of $\mathcal{M}_{\underline{a}}$. This shows that there is a surjective ℓ-adic Hecke equivalent map

$$i : \mathcal{M}_{\underline{0}} \to \Pi_0(\overline{\mathcal{M}_{\underline{a}}}) = \Pi_0(\mathcal{M}_{\underline{a}}).$$

By Proposition 5.2 (5), the set $\Pi_0(\mathcal{M}_{\underline{a}})$ is ℓ-adic Hecke transitive. ∎

An immediate consequence of Theorems 2.3 and 5.1 is the following

Corollary 5.3. *Any non-supersingular stratum $\mathcal{M}_{\underline{a}}$ is irreducible.*

5.3. Let $\underline{a} = (\underline{a}_v)_v \in \Delta$ be a supersingular alpha type. If f_v is odd, then $|\underline{a}_v| = f_v$. If f_v is even, then either $\underline{a}_v \preceq (1, 0, \dots, 1, 0)$ or $\underline{a}_v \preceq (0, 1, \dots, 0, 1)$. Define

$$\begin{aligned} \mathbb{P}_1 &:= \{v \in \mathbb{P} \,|\, f_v \text{ is odd}\,\} \\ \mathbb{P}_2(\underline{a}) &:= \{v \in \mathbb{P} \,|\, f_v \text{ is even and } |\underline{a}_v| = f_v\,\} \\ \mathbb{P}_3(\underline{a}) &:= \{v \in \mathbb{P} \,|\, f_v \text{ is even and } |\underline{a}_v| < f_v\,\} \end{aligned}$$

Theorem 5.4. *Let $\underline{a} \in \Delta$ be a supersingular alpha type. Then any irreducible component of $\overline{\mathcal{M}_{\underline{a}}}$ is isomorphic to $(\mathbf{P}^1)^{g-|\underline{a}|}$ and*

$$|\Pi_0(\mathcal{M}_{\underline{a}})| = [G(\mathbb{Z}) : \Gamma(n)] \left[\frac{-1}{2}\right]^g \zeta_F(-1) \prod_{v \in \mathbb{P}} c_v, \tag{5.2}$$

where

$$c_v := \begin{cases} p^{f_v} - 1 & \textit{if } v \in \mathbb{P}_1; \\ p^{f_v} + 1 & \textit{if } v \in \mathbb{P}_2(\underline{a}); \\ 1 & \textit{if } v \in \mathbb{P}_3(\underline{a}). \end{cases} \tag{5.3}$$

Proof. Define $\underline{a}_0 = (\underline{a}_{0,v})_v \in \Delta$ by

$$\underline{a}_{0,v} = \begin{cases} (1, 0, \dots, 1, 0) & \text{if } v \in \mathbb{P}_3(\underline{a}); \\ \underline{a}_v & \text{otherwise.} \end{cases}$$

We may assume that $\underline{a} \preceq \underline{a}_0$ due to symmetry. Choose any point $\underline{A}_0$ in $\mathcal{M}_{\underline{a}_0}(k)$. Let $\underline{M}_0$ be the covariant Dieudonné module of $\underline{A}_0$. Define $N = \oplus N_v \subset M_0 = \oplus M_{0,v}$ the Dieudonné $\mathcal{O}$-submodule with the induced quasi-polarization by

$$N_v := \begin{cases} (F, V)M_{0,v} & \text{if } v \in \mathbb{P}_3(\underline{a}); \\ M_{0,v} & \text{otherwise.} \end{cases}$$

Then there is a tuple $\underline{B} = (B, \lambda_B, \iota_B, \eta_B)$ and an O_F-linear isogeny $\varphi : \underline{B} \to \underline{A}_0$ of p-power degree, compatible with additional structures, such that $M(B) = N \subset$

M_0. Let $\mathcal{X} = \prod_{v\in\mathbb{P}_3(\underline{a})} \mathcal{X}_v$, where $\mathcal{X}_v$ is defined as $\mathcal{X}$ in Subsection 4.3. One has $\mathcal{X}_v \simeq (\mathbf{P}^1)^{f_v/2}$. Define the set Λ for $\underline{B}$ as in Subsection 4.3. By Proposition 4.1 we have an isomorphism $\coprod_{\xi\in\Lambda} \mathcal{X} \simeq \overline{\mathcal{M}}_{\underline{a}_0}$. Let $\xi \in \Lambda$ and $\overline{\mathcal{M}}_{\underline{a}_0,\xi} \simeq (\mathbf{P}^1)^{g-|\underline{a}_0|}$ be the corresponding component. By Lemma 4.3, we show that $\overline{\mathcal{M}}_{\underline{a}} \cap \overline{\mathcal{M}}_{\underline{a}_0,\xi} \simeq (\mathbf{P}^1)^{g-|\underline{a}|}$. Therefore, we have

$$\Pi_0(\overline{\mathcal{M}}_{\underline{a}}) \simeq \Pi_0(\overline{\mathcal{M}}_{\underline{a}_0}) \simeq \Lambda. \tag{5.4}$$

The alpha type of the factor N_v is $(0,2,\dots,0,2)$ if $v \in \mathbb{P}_3(\underline{a})$ and $(1,1,\dots)$ otherwise. Hence N_v has superspecial type $(e_1,e_2) = (0,0)$ if $v \in \mathbb{P}_3(\underline{a})$ and $(e_1,e_2) = (0,1)$ otherwise (Subsection 4.1). By the mass formula [33, Theorem 3.7 and Subsection 4.6] (cf. (4.4)), we get

$$|\Lambda| = [G(\mathbb{Z}) : \Gamma(n)] \left[\frac{-1}{2}\right]^g \zeta_F(-1) \prod_{v\in\mathbb{P}} c_v,$$

where c_v is as above. This completes the proof. ∎

5.4. Define the function $w' : \Delta \to \mathbb{R}$ by

$$w'(\underline{a}) := \begin{cases} [G(\mathbb{Z}) : \Gamma(n)] \left[\frac{-1}{2}\right]^g \zeta_F(-1) & \text{if } \underline{a} \in \Delta_{\rm ss}^{\rm gen}; \\ \prod_{v\in\mathbb{P}} w(\underline{a}_v) & \text{otherwise,} \end{cases} \tag{5.5}$$

where $w(\underline{a}_v)$ is the function as in (2.1). It is clear that $w'(\underline{a}) \neq 0$ if and only if $\underline{a} \in \Delta^{\rm gen}$. It is rather unclear but indeed a fact that $w'(\underline{a}) \in \mathbb{Z}_{\geq 0}$ (by (5.7)).

Theorem 5.5. *Notation as above. We have*

$$|\Pi_0(\mathcal{M}_{\Gamma_0(p)})| = \sum_{\underline{a}\in\Delta^{\rm gen}} w'(\underline{a}), \tag{5.6}$$

Proof. Suppose that $\underline{a}$ is a non-supersingular generic alpha type. It follows from local computation in Section 3 and Corollary 5.3 that $\mathcal{M}_{\Gamma_0(p),\underline{a}}$ has $w'(\underline{a})$ irreducible components of dimension g.

Suppose that $\underline{a}$ is a supersingular generic alpha type. Every fiber of the map $f_{\underline{a}}$ has one irreducible component of dimension $|\underline{a}|$ (Section 3). Thus, $\mathcal{M}_{\Gamma_0(p),\underline{a}}$ has $|\Pi_0(\mathcal{M}_{\underline{a}})|$ irreducible components of dimension g. It follows from Theorem 5.4 that

$$|\Pi_0(\mathcal{M}_{\Gamma_0(p),\underline{a}})| = |\Pi_0(\mathcal{M}_{\underline{a}})| = w'(\underline{a}). \tag{5.7}$$

This completes the proof. ∎

We can rephrase Theorem 5.5 by an elementary combinatorial result (Lemma 2.8) as follows

Theorem 5.6. *We have*

$$|\Pi_0(\mathcal{M}_{\Gamma_0(p)})| = 2^g + \sum_{\underline{a}\in\Delta_{\rm ss}^{\rm gen}} (w'(\underline{a}) - 1), \tag{5.8}$$

Remark 5.7.

(1) The connection of supersingular strata with class numbers and special zeta values becomes a standard fact now. If the moduli space $\mathcal{M}_{\Gamma_0(p)}$ contains supersingular irreducible components, then it is expect that the special zeta value $\zeta_F(-1)$ occurs in the formula for $|\Pi_0(\mathcal{M}_{\Gamma_0(p)})|$. However, the number of irreducible components of a supersingular stratum is also related to p in general. It is indeed

unexpected that the number of supersingular irreducible components of $\mathcal{M}_{\Gamma_0(p)}$ turns out to be independent of p. As a result, the number $|\Pi_0(\mathcal{M}_{\Gamma_0(p)})|$ of irreducible components is independent of p. We do not know any direct proof of this fact without knowing the explicit formula (5.8).

(2) The p-adic invariant stratification used in this paper is nothing but the Ekedahl-Oort stratification (see [18] and [8]). It is natural to consider this p-adic invariant stratification for studying moduli spaces with parahoric level structure. Indeed a parahoric level structure on an object $\underline{A}$ with a prescribed PEL-structure is a flag of finite flat subgroup schemes of $\underline{A}[p]$ that satisfy certain conditions. This structure only depends on the isomorphism class of $\underline{A}[p]$, but not on $\underline{A}$. Therefore, built on the framework of Moonen [13, 14] and Wedhorn [29], we can go a bit further on the irreducibility problem for a PEL-type moduli space $\mathcal{M}_{K_p}$ with level structure of type K_p, in the case where the defining group $G_{\mathbb{Q}_p}$ is unramified and $K_p \subset G(\mathbb{Q}_p)$ is a parahoric subgroup. Thanks to the works loc. cit., we have a group theoretic description of the set $\mathrm{EO}(G,\mu)$ of Ekedahl-Oort types, and the dimension of any Ekedahl-Oort stratum.

It may not be a very good strategy to analyze p-adic invariants in $\mathcal{M}_{K_p}$ through the forgetful morphism $f:\mathcal{M}_{K_p}\to\mathcal{M}$, where $\mathcal{M}$ is the smooth moduli space with minimal level at p. The Hilbert-Blumenthal cases are a few cases that this method can be worked out explicitly. Nevertheless, it is still interesting to know the subset $\mathrm{EO}(G,\mu,K_p)\subset\mathrm{EO}(G,\mu)$ consisting of elements φ such that $f^{-1}(S_\varphi)$ contains a maximal point of $\mathcal{M}_{K_p}$, where S_φ is the Ekedahl-Oort stratum in $\mathcal{M}$ associated to φ. And whether there is a group-theoretic meaning of this subset $\mathrm{EO}(G,\mu,K_p)$.

(3) The irreducibility problem for the moduli spaces $\mathcal{M}_{K_p}$ with parahoric level structure is, as suggested by this work, related to the same problem for Ekedahl-Oort strata in $\mathcal{M}$, which is of interest in its own right. It seems plausible to expect that in any irreducible component of $\mathcal{M}$, (i) any non-basic Ekedahl-Oort stratum is irreducible, and (ii) the number of irreducible components of a basic Ekedahl-Oort stratum is a *single* class number.

For Siegel moduli spaces, the statement (i) is confirmed in Ekedahl and van der Geer [6], and the statement (ii) is confirmed in Harashita [10].

For Hilbert-Blumenthal moduli spaces, the statement (i) is essentially due to Goren and Oort [8] and Chai [4] (Corollary 5.3), and the statement (ii) is confirmed by Theorem 5.4 (5.4).

6. ℓ-adic monodromy of Hecke invariant subvarieties

The goal of this section is to provide a proof of a theorem of Chai on Hecke invariant subvarieties for Hilbert-Blumenthal moduli spaces on which Theorem 5.6 relies. We follow the proof in Chai [4] where the Siegel case is proved. There is no novelty on the proof here and this is purely expository; the author is responsible for any inaccuracies and mistakes. We write this as an independent section; some setup and notation may be repeated and slightly modified.

6.1. Let F be a totally real number field of degree g and O_F be the ring of integers in F. Let V be a 2-dimensional vector space over F and $\psi: V\times V\to\mathbb{Q}$ be a $\mathbb{Q}$-bilinear non-degenerate alternating form such that $\psi(ax,y)=\psi(x,ay)$ for all $x, y \in V$ and $a \in F$. Let p be a fixed rational prime, not necessarily unramified in F. We choose and fix an O_F-lattice $V_{\mathbb{Z}} \subset V$ so that $V_{\mathbb{Z}} \otimes \mathbb{Z}_p$ is self-dual with respect to ψ. We choose a projective system of primitive prime-to-p-th roots of unity $\zeta = (\zeta_m)_{(m,p)=1} \subset \overline{\mathbb{Q}} \subset \mathbb{C}$. We also fix an embedding $\overline{\mathbb{Q}} \hookrightarrow \overline{\mathbb{Q}_p}$. For any prime-to-$p$ integer $m \geq 1$ and any connected $\mathbb{Z}_{(p)}[\zeta_m]$-scheme S, the choice ζ determines an isomorphism $\zeta_m : \mathbb{Z}/m\mathbb{Z} \xrightarrow{\sim} \mu_m(S)$, or equivalently, a $\pi_1(S, \bar{s})$-invariant $(1+m\hat{\mathbb{Z}}^{(p)})^\times$ orbit of isomorphisms $\bar{\zeta}_m : \hat{\mathbb{Z}}^{(p)} \to \hat{\mathbb{Z}}^{(p)}(1)_{\bar{s}}$, where $\hat{\mathbb{Z}}^{(p)} := \prod_{\ell \neq p} \hat{\mathbb{Z}}_\ell$ and $\bar{s}$ is a geometric point of S.

Let G be the automorphism group scheme over $\mathbb{Z}$ associated to the pair $(V_{\mathbb{Z}}, \psi)$; for any commutative ring R, the group of R-valued points is

$$G(R) := \{g \in \mathrm{GL}_{O_F}(V_{\mathbb{Z}} \otimes_{\mathbb{Z}} R)\,;\, \psi(g(x), g(y)) = \psi(x, y),\ \forall\, x, y \in V_{\mathbb{Z}} \otimes_{\mathbb{Z}} R\,\}. \tag{6.1}$$

Let $n \geq 3$ be a prime-to-p positive integer and ℓ be a prime with $(\ell, pn) = 1$ and $(\ell, \mathrm{disc}(\psi)) = 1$, where $\mathrm{disc}(\psi)$ is the discriminant of ψ on $V_{\mathbb{Z}}$. Let $m \geq 0$ be a non-negative integer. Let $U_{n\ell^m}$ be the kernel of the reduction map $G(\hat{\mathbb{Z}}^{(p)}) \to G(\hat{\mathbb{Z}}^{(p)}/n\ell^m\hat{\mathbb{Z}}^{(p)})$; this is an open compact subgroup of $G(\hat{\mathbb{Z}}^{(p)})$.

Let $\mathcal{D} = (F, V, \psi, V_{\mathbb{Z}}, \zeta)$ be a list of data as above. Denote by $\mathcal{M}_{\mathcal{D},n\ell^m}$ the moduli space over $\mathbb{Z}_{(p)}[\zeta_{n\ell^m}]$ that parametrizes equivalence classes of objects $(A, \lambda, \iota, [\eta])_S$ over a connected locally Noetherian $\mathbb{Z}_{(p)}[\zeta_{n\ell^m}]$-scheme S, where

- (A, λ) is a p-principally polarized abelian scheme over S of relative dimension g,
- $\iota : O_F \to \mathrm{End}_S(A)$ is a ring monomorphism such that $\lambda \circ \iota(a) = \iota(a)^t \circ \lambda$ for all $a \in O_F$, and
- $[\eta]$ is a $\pi_1(S, \bar{s})$-invariant $U_{n\ell^m}$-orbit of O_F-linear isomorphisms
$$\eta : V_{\mathbb{Z}} \otimes \hat{\mathbb{Z}}^{(p)} \xrightarrow{\sim} T^{(p)}(A_{\bar{s}}) := \prod_{p' \neq p} T_{p'}(A_{\bar{s}}) \tag{6.2}$$
such that
$$e_\lambda(\eta(x), \eta(y)) = \bar{\zeta}_{n\ell^m}(\psi(x, y)) \ (\mathrm{mod}\ (1 + m\hat{\mathbb{Z}}^{(p)})^\times), \quad \forall\, x, y \in V_{\mathbb{Z}} \otimes \hat{\mathbb{Z}}^{(p)}, \tag{6.3}$$
where e_λ is the Weil pairing induced by the polarization λ and $\bar{s}$ is a geometric point of S.

We write $[\eta]_{U_{n\ell^m}}$ for $[\eta]$ in order to specify the level. Let $\mathcal{M}_{n\ell^m} := \mathcal{M}_{\mathcal{D},n\ell^m} \otimes \overline{\mathbb{F}}_p$ be the reduction modulo p of the moduli scheme $\mathcal{M}_{\mathcal{D},n\ell^m}$. We have a natural morphism $\pi_{m,m'} : \mathcal{M}_{n\ell^{m'}} \to \mathcal{M}_{n\ell^m}$, for $m < m'$, which sends $(A, \lambda, \iota, [\eta]_{U_{n\ell^{m'}}})$ to $(A, \lambda, \iota, [\eta]_{U_{n\ell^m}})$. Let $\widetilde{\mathcal{M}}_n := (\mathcal{M}_{n\ell^m})_{m\geq 0}$ be the tower of this projective system.

Let $(\mathcal{X}, \lambda, \iota, \bar{\eta}) \to \mathcal{M}_n$ be the universal family. The cover $\mathcal{M}_{n\ell^m}$ over $\mathcal{M}_n$ represents the étale sheaf

$$\mathcal{P}_m := \underline{Isom}_{\mathcal{M}_n}((V_{\mathbb{Z}}/\ell^m V_{\mathbb{Z}}, \psi), (\mathcal{X}[\ell^m], e_\lambda)\,;\, \zeta_{\ell^m}) \tag{6.4}$$

of O_F-linear symplectic level-ℓ^m structures with respect to ζ_{ℓ^m}. This is a $G(\mathbb{Z}/\ell^m\mathbb{Z})$-torsor. Let $\bar{x}$ be a geometric point in $\mathcal{M}_n$. Choose an O_F-linear isomorphism $y : V \otimes \mathbb{Z}_\ell \simeq T_\ell(\mathcal{X}_{\bar{x}})$ that is compatible with the polarizations with respect to ζ. This amounts to choose a geometric point in $\widetilde{\mathcal{M}}_n$ over the point $\bar{x}$. The action of the geometric fundamental group $\pi_1(\mathcal{M}_n, \bar{x})$ on the system of fibers $(\mathcal{X}_{\bar{x}}[\ell^m])_m$ gives rise to the monodromy representation

$$\rho_{\mathcal{M}_n,\ell} : \pi_1(\mathcal{M}_n, \bar{x}) \to \mathrm{Aut}_{O_F}(T_\ell(\mathcal{X}_{\bar{x}}), e_\lambda) \tag{6.5}$$

and to the monodromy representation (using the same notation), through the choice of y,

$$\rho_{\mathcal{M}_n,\ell} : \pi_1(\mathcal{M}_n, \bar{x}) \to G(\mathbb{Z}_\ell). \tag{6.6}$$

Lemma 6.1. *The map $\rho_{\mathcal{M}_n,\ell}$ is surjective.*

PROOF. It is well-known that $\mathcal{M}_{\mathcal{D},n\ell^m}(\mathbb{C}) \simeq \Gamma(n\ell^m)\backslash G(\mathbb{R})/SO(2,\mathbb{R})^g$, where $\Gamma(n\ell^m) := \ker G(\mathbb{Z}) \to G(\mathbb{Z}/n\ell^m\mathbb{Z})$. It follows that the geometric generic fiber $\mathcal{M}_{\mathcal{D},n\ell^m} \otimes \overline{\mathbb{Q}}$ is connected. It follows from the arithmetic toroidal compactification constructed in Rapoport [23] that the geometric special fiber $\mathcal{M}_{n\ell^m}$ is also connected. The connectedness of $\widetilde{\mathcal{M}}_n$ confirms the surjectivity of $\rho_{\mathcal{M}_n,\ell}$. ∎

6.2. The action of $G(\mathbb{Z}_\ell)$ on $\widetilde{\mathcal{M}}_n$ extends uniquely a continuous action of $G(\mathbb{Q}_\ell)$. Descending from $\widetilde{\mathcal{M}}_n$ to $\mathcal{M}_n$, elements of $G(\mathbb{Q}_\ell)$ induce algebraic correspondences on $\mathcal{M}_n$, known as the ℓ-adic Hecke correspondences on $\mathcal{M}_n$. More precisely, to each $g \in G(\mathbb{Q}_\ell)$ we associate an ℓ-adic Hecke correspondence $(\mathcal{H}_g, \mathrm{pr}_1, \mathrm{pr}_2)$ as follows. Extending isomorphisms η to isomorphisms

$$\eta' : V \otimes \mathbb{A}_f^{(p)} \to V^{(p)}(A) := T^{(p)}(A) \otimes \mathbb{A}_f^{(p)},$$

we see the class $[\eta]_{U_n}$ gives rise to a class $[\eta']_{U_n}$ in $\mathrm{Isom}(V \otimes \mathbb{A}_f^{(p)}, V^{(p)}(A))/U_n$ and $[\eta]_{U_n}$ is determined by $[\eta']_{U_n}$. The right translation $\rho_g : (A, \lambda, \iota, [\eta']_{U_n}) \mapsto (A, \lambda, \iota, [\eta' g]_{g^{-1}U_n g})$ gives rise an isomorphism $\rho_g : \mathcal{M}_n \simeq \mathcal{M}_{g^{-1}U_n g}$. Let $U_{n,g} := U_n \cap g^{-1}U_n g$ and $\mathcal{H}_g$ be the étale cover of $\mathcal{M}_n$ corresponding to the subgroup $U_{n,g} \subset U_n$. Let pr_1 be the natural projection $\mathcal{H}_g \to \mathcal{M}_n$ and $\mathrm{pr}_2 := \rho_g^{-1} \circ \mathrm{pr} : \mathcal{H}_g \to \mathcal{M}_n$ be the composition of the isomorphism ρ_g^{-1} with the natural projection $\mathrm{pr} : \mathcal{H}_g \to \mathcal{M}_{g^{-1}U_n g}$. This defines an ℓ-adic Hecke correspondence $(\mathcal{H}_g, \mathrm{pr}_1, \mathrm{pr}_2)$. For two ℓ-adic Hecke correspondences $\mathcal{H}_{g_1} = (\mathcal{H}_{g_1}, p_{11}, p_{12})$ and $\mathcal{H}_{g_2} = (\mathcal{H}_{g_2}, p_{21}, p_{22})$, one defines the composition $\mathcal{H}_{g_2} \circ \mathcal{H}_{g_1}$ by

$$(\mathcal{H}_{g_2} \circ \mathcal{H}_{g_1}, p_1, p_2),$$

where $\mathcal{H}_{g_2} \circ \mathcal{H}_{g_1} := \mathcal{H}_{g_1} \times_{p_{12}, \mathcal{M}_n, p_{21}} \mathcal{H}_{g_2}$, p_1 is the composition $\mathcal{H}_{g_2} \circ \mathcal{H}_{g_1} \to \mathcal{H}_{g_1} \stackrel{p_{11}}{\to} \mathcal{M}_n$ and p_2 is the composition $\mathcal{H}_{g_2} \circ \mathcal{H}_{g_1} \to \mathcal{H}_{g_2} \stackrel{p_{22}}{\to} \mathcal{M}_n$. A correspondence $(\mathcal{H}, \mathrm{pr}_1, \mathrm{pr}_2)$ generated by correspondences of the form $\mathcal{H}_g$ is also called an ℓ-adic Hecke correspondence.

A subset Z of $\mathcal{M}_n$ is called *ℓ-adic Hecke invariant* if $\mathrm{pr}_2(\mathrm{pr}_1^{-1}(\mathrm{Z})) \subset \mathrm{Z}$ for any ℓ-adic Hecke correspondence $(\mathcal{H}, \mathrm{pr}_1, \mathrm{pr}_2)$. If Z is an ℓ-adic Hecke invariant, locally closed subvariety of $\mathcal{M}_n$, then the ℓ-adic Hecke correspondences induce correspondences on the set $\Pi_0(Z)$ of geometrically irreducible components. We say that $\Pi_0(Z)$ is *ℓ-adic Hecke transitive* if the ℓ-adic Hecke correspondences operate transitively on $\Pi_0(Z)$, that is, for any two maximal points η_1, η_2 of Z there is an ℓ-Hecke Hecke correspondence $(\mathcal{H}, \mathrm{pr}_1, \mathrm{pr}_2)$ so that $\eta_2 \in \mathrm{pr}_2(\mathrm{pr}_1^{-1}(\eta_1))$. Let k be an algebraically closed field of characteristic p. For a geometric point $x \in \mathcal{M}_n(k)$, denote by $\mathcal{H}_\ell(x)$ the ℓ-adic Hecke orbit of x; this is the set of points generated by ℓ-adic correspondences starting from x.

Lemma 6.2.

(1) *For any point $x \in \mathcal{M}_n(k)$, the corresponding abelian variety A_x is supersingular if and only if the ℓ-adic Hecke orbit $\mathcal{H}_\ell(x)$ of x is finite.*

(2) *Any closed ℓ-adic Hecke invariant subscheme Z of $\mathcal{M}_n$ contains a supersingular point.*

PROOF. (1) This is Lemma 7 in Chai [3]. (2) This is Proposition 6 in Chai [3].

6.3. Put $G_\ell := G \otimes \mathbb{Q}_\ell$ (Subsection 6.1). One has

$$G_\ell = \prod_{\lambda|\ell} G_\lambda, \quad G_\lambda = \mathrm{Res}_{F_\lambda/\mathbb{Q}_\ell} \mathrm{SL}_{2,F_\lambda}.$$

Let $\mathrm{pr}_\lambda : G_\ell \to G_\lambda$ be the projection map. Let Z be a smooth locally closed subscheme of $\mathcal{M}_n$ that is ℓ-adic Hecke invariant. Let Z^0 be a connected component of Z, and η be the generic point of Z^0. Let

$$\rho_{Z^0,\ell} : \pi_1(Z^0, \bar\eta) \to G(\mathbb{Z}_\ell)$$

be the associated ℓ-adic monodromy representation, and $\rho_{Z^0,\lambda} := \mathrm{pr}_\lambda \circ \rho_{Z^0,\ell}$ be its projection at λ.

Lemma 6.3.
(1) *If the image $\mathrm{Im}\,\rho_{Z^0,\lambda}$ is finite for one $\lambda|\ell$, then the image $\mathrm{Im}\,\rho_{Z^0,\lambda}$ is finite for all $\lambda|\ell$.*
(2) *The abelian variety A_η is not supersingular if and only if the image $\mathrm{Im}\,\rho_{Z^0,\lambda}$ is infinite for all $\lambda|\ell$.*

PROOF. (1) Let Z^0_0 be a scheme over $\mathbb{F}_q$ such that $Z^0 = Z^0_0 \otimes \overline{\mathbb{F}}_p$, and let η_0 be the generic point of Z^0_0. Replacing by a finite surjective cover of Z^0_0 (thus of Z^0), we may assume that $\mathrm{End}^0(A_{\bar\eta}) = \mathrm{End}^0(A_{\eta_0}) := \mathrm{End}(A_{\eta_0}) \otimes \mathbb{Q}$ and that $\mathrm{Im}\,\rho_{Z^0,\lambda} = 1$ whenever it is finite. Write the Tate module $V_\ell(A_{\bar\eta}) = \prod_{\lambda|\ell} V_\lambda$ into the decomposition with respect to the action of F, and let $\rho_\lambda : \mathrm{Gal}(k(\bar\eta_0)/k(\eta_0)) \to \mathrm{Aut}(V_\lambda)$ be associated λ-adic Galois representation. Let E_λ be the F_λ-subalgebra of $\mathrm{End}_{F_\lambda}(V_\lambda)$ generated by the image $\rho_\lambda(\mathrm{Gal}(k(\bar\eta_0)/k(\eta_0)))$. By a theorem of Zarhin on endomorphisms of abelian varieties over function fields [34], the subalgebra E_λ is semi-simple and the endomorphism algebra $\mathrm{End}^0_F(A) \otimes_F F_\lambda$ is isomorphic to the commutant of E_λ in $\mathrm{End}_{F_\lambda}(V_\lambda)$. If $\mathrm{Im}\,\rho_{Z^0,\lambda} = 1$ for some λ, then ρ_λ factors through the quotient $\mathrm{Gal}(\overline{\mathbb{F}}_p/\mathbb{F}_q)$, and thus E_λ is commutative. In this case, $\dim_{F_\lambda} \mathrm{End}^0_F(A) \otimes F_\lambda$ is 2 or 4, and the same that $\dim_F \mathrm{End}^0_F(A)$ is 2 or 4. This shows that the abelian variety A_{η_0} is of CM-type. By a theorem of Grothendieck on CM abelian varieties in characteristic p ([15, p. 220] and [17, Theorem 1.1]), A_{η_0} is isogenous to, over a finite extension of $k(\eta_0)$, an abelian variety that is defined over a finite field. This shows the image $\mathrm{Im}\,\rho_{Z^0,\ell}$ is finite. Therefore, $\mathrm{Im}\,\rho_{Z^0,\lambda}$ is finite for all $\lambda|\ell$.

(2) It is proved in [4, Corollary 3.5] that A_η is not supersingular if and only if the image $\mathrm{Im}\,\rho_{Z^0,\ell}$ is infinite. The statement then follows from (1). ∎

Lemma 6.4. *Let H be a connected normal subgroup of an algebraic group $G_1 \times \cdots \times G_r$ over a field of characteristic zero, where G_i is a connected simple algebraic group. Then H is of the form $H_1 \times \cdots \times H_r$ with H_i is $\{1\}$ or G_i.*

PROOF. See Section 9.4 in [27]. ∎

Lemma 6.5. *Notation as in Subsection 6.3, if the abelian variety A_η is not supersingular, then the image $\mathrm{Im}\,\rho_{Z^0,\ell}$ is an open subgroup of $G(\mathbb{Z}_\ell)$.*

PROOF. Replacing Z by the orbit of the component Z^0 under all ℓ-adic Hecke correspondences, we may assume that the set $\pi_0(Z)$ of connected components is ℓ-adic Hecke transitive. Put $M := \operatorname{Im} \rho_{Z^0,\ell}$ and let H be the neutral component of the algebraic envelope of M. It is proved in [4, Proposition 4.1] that M is open in $H(\mathbb{Q}_\ell)$ and H is a connected normal subgroup of G_ℓ. By Lemma 6.4, the group H has the form $\prod_{\lambda|\ell} H_\lambda$ with $H_\lambda = \{1\}$ or G_λ. Since A_η is not supersingular, it follows from Lemma 6.3 that $H = G$. This completes the proof. ∎

Lemma 6.6. *Let G be a connected simply-connected semi-simple algebraic group over a local field K such that each simple factor of G is K-isotropic. Then $G(K)$ has no proper subgroup of finite index.*

PROOF. This follows immediately from the affirmative solution to the Kneser-Tits problem for K proved by Platonov for characteristic zero cases and by Prasad and Raghunathan for arbitrary characteristic cases (see [20] and [22]).

Theorem 6.7 (Chai)**.** *Let Z be an ℓ-adic Hecke invariant, smooth locally closed subscheme of M_n. Let $\bar\eta$ be a geometric generic point of an irreducible component Z^0 of Z. Suppose that the abelian variety $A_{\bar\eta}$ corresponding to the point $\bar\eta$ is not supersingular, and that the set $\pi_0(Z)$ of connected components is ℓ-adic Hecke transitive. Then the monodromy representation*

$$\rho_{Z^0,\ell} : \pi_1(Z^0, \bar\eta) \to G(\mathbb{Z}_\ell)$$

is surjective and Z is irreducible.

PROOF. Let $\widetilde{Z}^0$ and $\widetilde{Z}$ be the preimage in $\widetilde{\mathcal{M}}_n$ of the subschemes Z^0 and Z, respectively, under the morphism $\pi : \widetilde{\mathcal{M}}_n \to \mathcal{M}_n$. Let Y be a connected component of $\widetilde{Z}^0$ and M be the image $\operatorname{Im} \rho_{Z^0,\ell}$. The group $\operatorname{Aut}(Y/Z^0)$ of deck transformations is equal to M. Since the group $G(\mathbb{Z}_\ell)$ acts transitively on the fiber $\pi^{-1}(x)$ for any $x \in Z$ and $G(\mathbb{Q}_\ell)$ acts transitively on the set $\pi_0(Z)$, the group $G(\mathbb{Q}_\ell)$ acts transitively on the set $\pi_0(\widetilde{Z})$. This gives a homeomorphism (see [4, Lemma 2.8])

$$Q\backslash G(\mathbb{Q}_\ell) \xrightarrow{\sim} \pi_0(\widetilde{Z}), \quad g \mapsto g[Y],$$

where Q is the stabilizer of the class $[Y]$ (in $\pi_0(\widetilde{Z})$). Clearly $Q \cap G(\mathbb{Z}_\ell) = M$ and we have $M\backslash G(\mathbb{Z}_\ell) \simeq \pi_0(\widetilde{Z}^0)$. It follows from Lemma 6.5 that $\pi_0(\widetilde{Z}^0) = M\backslash G(\mathbb{Z}_\ell)$ is finite. Write $Z = \coprod_{i=0}^r Z_i$ as a disjoint union of connected components. Since $G(\mathbb{Q}_\ell)$ acts transitively on $\pi_0(Z)$ and $\pi_0(\widetilde{Z}^0)$ is finite, each $\pi_0(\widetilde{Z}_i)$ is finite. We have

$$\#\pi_0(\widetilde{Z}) < \infty \implies \#Q\backslash G(\mathbb{Q}_\ell) < \infty \implies Q = G(\mathbb{Q}_\ell) \text{ (Lemma 6.6)} \implies M = G(\mathbb{Z}_\ell).$$

This shows the connectedness of $\widetilde{Z}$ and hence that of Z. This completes the proof. ∎

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