Note on $\mathbb{R}$-Hodge-Tate Sheaves

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Abstract

In this article we describe the category of $\mathbb{R}$-Hodge-Tate sheaves on an algebraic variety as the category of finite dimensional representations of a certain Lie algebra.

Introduction

The definition of a Hodge sheaf on an algebraic variety comes from the geometry and reflects certain properties of fiber-wise (co)homology groups for families of algebraic varieties over this variety. Hodge sheaves form a tensor Abelian category, so it is natural to expect that this category is the category of representations of some Lie algebra. For the case of sheaves on a point (Hodge structures) the description of this Lie algebra was given by Deligne [D1].

We describe the Lie algebra which corresponds to the sub-category of Hodge-Tate sheaves of the category of Hodge sheaves. This case is simpler for a number of reasons.

First, the semi-simple part of this category is rather small, namely it contains direct sums of pure Tate sheaves which are numbered by integers.

Second, we have good fiber functor to the category of graded vector spaces (= the category of pure Tate sheaves), namely the functor which associates to any Hodge-Tate sheaf the direct sum of its quotients with respect to the weight filtration. Any such quotient is a *constant* sheaf, we associate to it a vector space which is its fiber over some (in fact, any) point. Our Lie algebra is the Tannakian Lie algebra of this fiber functor.

In the (more general) case of the category of all Hodge sheaves we can consider either the fiber functor to the category of pure Hodge structures and get as a result some Lie algebra in this latter category, or we can choose a point and consider the fiber at this point, so the Lie algebra depends on the point.

The construction of the Lie algebra is based on the idea of Deligne that the Hodge filtration on a Hodge-Tate sheaf is a rotation of the Hodge filtration on some pure Hodge-Tate sheaf (the direct sum of its quotients with respect to the weight filtration) by some unipotent operator (see the definitions in Section 1). Deligne uses the logarithm of this rotation operator for determining the action of a Lie algebra on the pure Hodge-Tate sheaf.

In our construction the operator G which determines the action of a Lie algebra is a potential of a certain connection; the rotation operator is no more the exponential and its dependence on G is more complicated.

The restrictions on G which come from the definition of a Hodge-Tate sheaf are quadratic; this makes it possible to reinterpret them in the spirit of the representation theory of certain Lie algebras. These Lie algebras are defined by generators and relation. Moreover, the space of generators is dual to the first component of a certain commutative associative dg-algebra and the space of relations is dual to the second component of this dg-algebra. The corresponding dg-algebra in our case is the direct sum of (some version of) Deligne complexes [RSS]; we construct a commutative and associative multiplication on them.

This relationship between Hodge-Tate sheaves and Deligne complexes is not so surprising, as the extension groups between *perverse* Hodge-Tate sheaves can be calculated in terms of Deligne complexes. This follows from some global (with respect to the variety on which Hodge-Tate sheaves are considered) reasons. In contrast with this, our approach is local.

We also want to mention a parallel between our construction and work of Bloch and Kriz [BlK]; but we go in the opposite direction: Bloch and Kriz started from a dg-algebra and pass to representations, we start from representations and try to recognize the corresponding dg-algebra.

The structure of the article. In the first section we give the basic definitions and formulate the main results. The second one is devoted to the main technical construction, in this section we rewright the definition of the Hodge-Tate sheaf as a system of differential equations for some operator-valued function. In the third section we recall some standard constructions from homological algebra; this makes it possible to reformulate restrictions on the operator-valued function in the spirit of the representation theory. In the last section we consider an example; we describe the well-known *polylogarithmic* Hodge sheaf in our terms.

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1 The definitions and the results

Notations. We consider some smooth algebraic variety X over $\mathbb{C}$. We fix its representation as the complement of a divisor X^∞ with normal crossings in a smooth compact variety $\overline{X}$.

The sheaf of smooth (p,q)-forms will be denoted by $\mathcal{A}^{pq}$, while $\mathcal{Z}^{pq}$ denotes the sheaf of closed forms; the sheaf of real-valued k-forms will be denoted by S^k and d^c denotes the operator $\partial - \overline{\partial}$.

A shuffle of n and m elements is a permutation σ of the segment $[1, n+m]$ which preserves the order on the segments $[1,n]$ and $[n+1, n+m]$: so if $\lambda < \mu \leq n$ or $n < \lambda < \mu$, then $\sigma(\lambda) < \sigma(\mu)$. The set of all shuffles of n and m elements will be denoted by $\mathrm{Shuffles}(n,m) \subset S_{n+m}$.

1.1. Deligne's description of $\mathbb{R}$-Hodge-Tate structures.

1.1.1. Definition. An $\mathbb{R}$-Hodge-Tate structure is an $\mathbb{R}$-vector space V, filtered by vector spaces $W_l(V)$ (the weight filtration), $\cdots \subset W_{2k} = W_{2k+1} \subset W_{2k+2} \subset \cdots$, equipped with a filtration $F^\bullet(V)$ (the Hodge filtration) of the complexification $V^{\mathbb{C}} = V \otimes_{\mathbb{R}} \mathbb{C}$ of V : $\cdots \subset F^p \subset F^{p-1} \subset \cdots$ opposite to $W_\bullet$, $V^{\mathbb{C}} = \oplus W^{\mathbb{C}}_{2i} \cap F^i$.

As we study in this article only $\mathbb{R}$-Hodge-Tate sheaves, we will often omit the symbol $\mathbb{R}$ in notation.

1.1.2. The pure Hodge-Tate structure $\mathbb{R}(l)$. Let l be an integer. Consider the trivial one-dimensional $\mathbb{C}$-vector space $\mathbb{C}$. The $\mathbb{R}$-vector space $\mathbb{R}(-l)$ is its $\mathbb{R}$-subspace, generated by $(2\pi\sqrt{-1})^l$, the weight filtration is given by $W_{2l-1} = 0$, $W_{2l} = \mathbb{R}(-l)$, and the Hodge filtration is given by $F^{l+1} = 0$, $F^l = \mathbb{C}$.

1.1.3. A morphism of Hodge-Tate structures is a linear map, compatible with filtrations. A tensor product of Hodge-Tate structures is the tensor product of $\mathbb{R}$-vector spaces equipped with the tensor product of filtrations. The Hodge-Tate structures form an Abelian tensor category $\mathbb{R}\mathfrak{H}\mathfrak{T}$. Due to Deligne [D1], we have the following description of this category:

1.1.4. Theorem. Consider the free pro-nilpotent graded Lie algebra $\mathcal{L}_{HT}$ which is generated by elements e_i, $i = 1, 2, \cdots$; the degree of the e_i equals i . The category $\mathbb{R}\mathfrak{H}\mathfrak{T}$ is equivalent to the category of the finite dimensional graded real representations of the Lie algebra $\mathcal{L}_{HT}$.

1.1.5. Idea of proof. i) *From the representation to the Hodge-Tate structure.* Let $V = \oplus V^k$, $\rho\colon \mathcal{L}_{HT} \to \mathrm{End}(V)$, $\rho(e_i)(V^k) \subset V^{k-i}$ be a representation. Define the weight filtration $W_\bullet$ on the space V as $W_{2l}(V) = \oplus_{i\leq l} V^i$. Consider the opposite filtration $\tilde{F}^p = \oplus_{i\geq p} V^i$, denote by g_i the operators $\sqrt{-1}\rho(e_i)$ on the complexification of V, put $g = \sum g_i$. The "true" Hodge filtration $F^\bullet$ is the "rotation" of the real filtration $\tilde{F}^\bullet$ by the exponential of the operator g: $F^p = \exp(g)\tilde{F}^p$.

ii) *From the Hodge-Tate structure to the representation.* Let $\{V, W_\bullet, F^\bullet\}$ be a Hodge-Tate structure. Put $U^i = W^{\mathbb{C}}_{2i} \cap F^i$. We have $V^{\mathbb{C}} = \oplus U^i = \oplus \overline{U^i}$. Both U^i and $\overline{U^i}$ are canonically isomorphic to the complexification $\mathrm{Gr}^{W\mathbb{C}}_{2i}(V)$ of $\mathrm{Gr}^W_{2i}(V) = W_{2i}(V)/W_{2i-1}(V)$. So we have two identifications of $V^{\mathbb{C}}$ with $\oplus \mathrm{Gr}^{W\mathbb{C}}_{2i}(V)$. They induce a unipotent automorphism Φ of V, which sends $\overline{U^i}$ to U^i. The complex conjugate $\overline{\Phi}$ of this operator is equal to its inverse Φ^{-1} as both of them map U^i to $\overline{U^i}$ isomorphically. Consider the nilpotent logarithm $\log \Phi$ of F. Put $g = \frac{1}{2}\log \Phi$, so $\overline{g} = -g$; define vector subspaces $V^{i\mathbb{C}}$ as $\exp(g)(\overline{U^i})$. The spaces $V^{i\mathbb{C}}$ are real, indeed $\overline{V^{i\mathbb{C}}} = \overline{\exp(g)(\overline{U^i})} = \exp(-g)(U^i) = \exp(-g)\exp(2g)(\overline{U^i}) = V^{i\mathbb{C}}$. Expand g into a sum of homogeneous components $g = \sum g_i$. So we have a graded vector space $V = \oplus V \cap V^{i\mathbb{C}}$ and a representation ρ, putting $\rho(e_i) = -\sqrt{-1}g_i$.

We call the operator $g = \sum g_i$ the *Green operator* [BL, 3.1].

1.2. The Hodge-Tate sheaves.

1.2.1. Any unipotent $\mathbb{C}$-local system $\mathcal{V}$ on X can be represented as the sheaf of (local) horizontal sections with respect to a nilpotent connection $\nabla_{\mathcal{V}}$ on the vector bundle $\mathcal{V}_{\mathrm{hol}} = \mathcal{V} \otimes_{\mathbb{C}} \mathcal{O}$ on X.

1.2.2. Let D be a coordinate disk $|z| < 1$ and denote by D^* the punctured coordinate disk $D^* = D \setminus 0$. Consider an immersion $f\colon D^* \to X$ of the punctured disk into X which can be continued to a map of the disk D into $\overline{X}$. The inverse image $(f^*(\mathcal{V}_{\mathrm{hol}}),\ f^*(\nabla_{\mathcal{V}}))$

of the flat bundle $(\mathcal{V}_{\mathrm{hol}}, \nabla_{\mathcal{V}})$ on the punctured disk D^* can be uniquely continued to the bundle $f^*(\mathcal{V}_{\mathrm{hol}})_D$ in such a way that the connection $f^*(\nabla_{\mathcal{V}})$ has a pole of first order and its residue is nilpotent. This continuation is known as the Deligne continuation.

1.2.3. Definition. [K] An $\mathbb{R}$-Hodge-Tate sheaf on X is an $\mathbb{R}$-local system $\mathcal{V}$ on X filtered by local systems $W_l(\mathcal{V})$ (the weight filtration), $\cdots \subset W_{2k} = W_{2k+1} \subset W_{2k+2} \subset \cdots$, such that $\mathrm{Gr}^W_{2i}(\mathcal{V}) = W_{2i}(V)/W_{2i-1}(\mathcal{V})$ are constant local systems, equipped with a filtration $F^\bullet$ (the Hodge filtration) of the *bundle* $\mathcal{V}_{\mathrm{hol}}$ by *sub-bundles* opposite to $W^{hol}_\bullet(\mathcal{V}) = W_\bullet(\mathcal{V}) \otimes_{\mathbb{R}} \mathcal{O}$ such that :

a) $\nabla_{\mathcal{V}}(F^p) \subset F^{p-1} \otimes \Omega^1_{X\,\log}$ (the transversality condition);

b) for any immersion f of a punctured disk as above the Hodge filtration continues to 0 as a filtration by sub-bundles of $f^*(\mathcal{V}_{\mathrm{hol}})_D$.

We call *analytic Hodge-Tate sheaf* the above data which satisfy only condition a) of the definition. This definition works for any analytic manifold over $\mathbb{C}$.

1.2.4. The $\mathbb{R}$-Hodge-Tate sheaves on X form an Abelian tensor category $\mathbb{R}\mathfrak{H}\mathfrak{T}_X$.

1.2.5. We denote by the same symbol $\mathbb{R}(-l)$ the constant sheaf on X with fiber $\mathbb{R}(-l)$. We call it by pure Tate sheaf of the weight $2l$.

1.2.6. As any fiber $\mathcal{V}|_x$, $x \in X$ of an $\mathbb{R}$-Hodge-Tate sheaf $\mathcal{V}$ is an $\mathbb{R}$-Hodge-Tate structure, we can consider the operators g_i for any fiber, they depend smoothly on the point x. We have a system of differential equations for the g_i's which is equivalent to the holomorphic dependence of $F^\bullet$ in x together with the transversality condition a); but this system of equations is rather "complicated" (21). We improve the situation by choosing another system of generators of our Lie algebra. The equations for the images of these new generators are quadratic.

1.2.7. Let V be a graded $\mathbb{R}$-vector spaces $V = \oplus V^l$ and let G be a smooth function on X with values in nilpotent imaginary operators on V : $\sqrt{-1}G \in \Gamma(X, \mathrm{End}(V) \otimes_{\mathbb{R}} C^\infty_X)$, $G = \sum G_i, \sqrt{-1}G_i(V^j) \subset V^{j-i} \otimes_{\mathbb{R}} C^\infty_X$. Consider the following system of equations on G :

$$d\,G_l = \sum_{j=1}^{l-1} \alpha(j, l-j)[G_j, d^c G_{l-j}], \quad \alpha(j,k) = \frac{2j-1}{2(j+k)-2}; \tag{1}$$

$$\overline{\partial}\,\partial\, G_1 = 0; \tag{2}$$

and in any small vicinity in $\overline{X}$ of any point x of X^∞

$$G_1 = \sum_\rho r_\lambda \log|z_\lambda| + \phi, \tag{3}$$

where r_λ are constant, z_λ are local equations of branches of X^∞ passing through the point x and ϕ is smooth in the vicinity.

1.2.8. One can define a morphism between two solutions $(V^{(1)}, G^{(1)})$ and $(V^{(2)}, G^{(2)})$ of equations (1)-(3) as a graded map $V^{(1)} \to V^{(2)}$ which commutes with G's.

1.2.9. The tensor product of two solutions $(V^{(1)}, G^{(1)})$ and $(V^{(2)}, G^{(2)})$ of (1)-(3) is a Lie algebra-type tensor product :

$$(V^{(1)} \otimes V^{(2)}, G^{(1)} \otimes \mathrm{Id}_{V^{(2)}} + \mathrm{Id}_{V^{(1)}} \otimes G^{(2)}). \tag{4}$$

1.2.10. Theorem. The set of $\mathbb{R}$-Hodge-Tate sheaves on X is equivalent to the set of solutions (V, G) of equations (1)–(3) and this equivalence is compatible with morphisms and with tensor products.

We call the operator G the *modified Green operator* of the $\mathbb{R}$-Hodge-Tate sheaf.

We prove this theorem in Section 2.

1.3. The Hodge-Tate algebra.

We want to reinterpret the properties (1)-(3) of the modified Green operator as a property of V to be a representation of some nilpotent graded Lie algebra.

1.3.1. A nilpotent Lie algebra can be determined by generators and relations. If the relations are quadratic (and this is our case), the Lie algebra can be treated as the zeroth Harrison cohomology group of some commutative associative dg-algebra. The Harrison cohomology complex is the primitive part of the (reduced) cobar-complex, see details in Section 3.1.

In our case the dg-algebra will be the dg-algebra of sections of some sheaf of dg-algebras on $\overline{X}$. We call this sheaf the *Hodge-Tate algebra.* First we define this algebra over X.

1.3.2. Let us consider the following set of sheaves $\mathcal{HT}(i)^a$ on X, $i \geq 0$, $a = 1, \cdots, 2i$:

$$\mathcal{HT}(i)^a = \begin{cases} \bigoplus\limits_{p+q=a-1;p,q<i} \mathcal{A}^{pq} & \text{, if } a < 2i, \\ \mathcal{Z}^{ii} & \text{, if } a = 2i. \end{cases} \tag{5}$$

Define a map $\delta(i)\colon \mathcal{HT}(i)^a \to \mathcal{HT}(i)^{a+1}$ as

$$\delta(i) = \begin{cases} -\frac{2}{2i-p-q-2}\Big((i-p-1)\partial + (i-q-1)\overline{\partial}\Big) & \text{, if } p+q < 2i-2, \\ d\,d^c & \text{, if } p+q = 2i-2. \end{cases} \tag{6}$$

1.3.3. Lemma. $(\mathcal{HT}(i), \delta(i))$ is a complex of sheaves for any i. □

1.3.4. Consider the following map $\cap\colon \mathcal{HT}(i_1)^{a_1} \otimes \mathcal{HT}(i_2)^{a_2} \to \mathcal{HT}(i_1+i_2)^{a_1+a_2}$:

$$\omega_1 \cap \omega_2 = \begin{cases} \frac{1}{2(i_1+i_2)-(a_1+a_2)}\Big((2i_1-a_1)\omega_1 \wedge d^c\,\omega_2 + (-1)^{a_1 a_2} \\ \times (2i_2-a_2)\omega_2 \wedge d^c\,\omega_1\Big), \text{ if } a_1 < 2i_1 \text{ and } a_2 < 2i_2, \\ \omega_1 \wedge \omega_2 \text{ otherwise.} \end{cases} \tag{7}$$

1.3.5. Lemma. a) The map $\cap$ is a map of complexes.

b) The product $\cap$ is (graded) commutative and associative. □

Thus, the sheaf $\mathcal{HT} = \oplus \mathcal{HT}(i)^a$ equipped with the differential δ and the multiplication $\cap$ is a commutative associative dg-algebra.

1.3.6. Consider the following anti-linear map σ from $\mathcal{HT}(i)^a$ to itself:

$$\sigma = \begin{cases} -\text{complex conjugation} & \text{, if } a < 2i, \\ \text{complex conjugation} & \text{, if } a = 2i. \end{cases} \tag{8}$$

1.3.7. Lemma. a) The map σ commutes with the differential δ.

b) The map σ commutes with the multiplication $\cap$. $\square$

1.3.8. Definition. *The* $\mathbb{R}$*-Hodge-Tate algebra* is the sheaf of dg-algebras $(\mathbb{R}\mathcal{HT}, \delta, \cap)$ on X of σ-invariants of the dg-algebra $(\mathcal{HT}, \delta, \cap)$.

Now we define the Tate-Hodge algebra at the boundary $X^\infty \subset \overline{X}$. For a point $x \in \overline{X}$ fix some local equations z_λ of the branches of X^∞ passing through the point x.

Note that the sheaf $\mathcal{A}^\bullet_X$ is a sheaf of $\mathcal{A}^\bullet_{\overline{X}}$-modules.

1.3.9. Consider the algebra $\mathcal{A}^\bullet_{\log} \subset \mathcal{A}^\bullet_X$ generated by $\log|z_\lambda|$'s and $d^c \log|z_\lambda|$'s over the algebra of smooth forms $\mathcal{A}^\bullet_{\overline{X}}$. This definition does not depend on the choice of the equations z_λ for the components of X^∞. *It is not true that the differential of a form of such type is a form of such type.* We say that a form ω is the form of *log degree* less than or equal to n if it can be represented as a polynomial in $\log|z_\lambda|$'s and $d^c \log|z_\lambda|$'s with smooth coefficients of the degree less than or equal to n. This definition does not depend on the choice of the equations z_λ. Denote by $\mathcal{A}^\bullet_{\log,\leq n}$ the space of forms of log degree less than or equal to n. The algebra $\mathcal{A}^\bullet_{\log}$ is filtered by $\mathcal{A}^\bullet_{\log,\leq n}$.

1.3.10. Consider a form ω in $\mathcal{A}^\bullet_{\log}$; we can represent it as a polynomial:

$$\omega = \sum_{\{i_k\},\{\epsilon_k\}} \omega_{\{i_k\},\{\epsilon_k\}} \prod \log^{a_k}|z_k| \wedge \bigwedge (d^c \log|z_k|)^{\epsilon_k}, \quad a_k \geq 0, \epsilon_k = 0, 1.$$

Call a coefficient $\omega_{\{i_k\},\{\epsilon_k\}}$ *essential* for a branch $z_\lambda = 0$ if $i_\lambda \neq 0$ or $\epsilon_\lambda \neq 0$

The form ω has *weak* logarithmic singularities if the restriction of any essential for a branch $z_\lambda = 0$ coefficient $\omega_{\{i_k\},\{\epsilon_k\}}$ to this branch vanishes. This definition does not depend on the choice of the representation of the form as a polynomial and does not depend on the choice of the equations z_λ. Denote by $\mathcal{A}^\bullet_{\mathrm{w}\log}$ the algebra of forms with weak logarithmic singularities; it is filtered by $\mathcal{A}^\bullet_{\mathrm{w}\log,\leq n} = \mathcal{A}^\bullet_{\mathrm{w}\log} \cap \mathcal{A}^\bullet_{\log,\leq n}$. The differential of a form with weak logarithmic singularities is a form with weak logarithmic singularities of log degree less than or equal to log degree of the initial form. Indeed, if the restriction of a form (the coefficient of the polynomial) vanishes, the restriction of its differential vanishes also. Therefore, we shall check that if the restriction $\psi|_{z=0}$ of ψ to $z = 0$ vanishes, then $\partial \log|z| \wedge \psi$ and $\overline{\partial} \log|z| \wedge \psi$ have weak logarithmic singularities. If $\psi|_{z=0} = 0$, then the form ψ belongs to the ideal generated by z, $\overline{z}$, $d\,z$ and $d\,\overline{z}$:

$$\psi = z\psi_1 + \overline{z}\psi_2 + d\,z \wedge \phi_1 + d\,\overline{z} \wedge \phi_2.$$

As we have

$$\partial \log|z| z = \frac{1}{2} d\,z, \quad \partial \log|z| \overline{z} = d^c \log|z| \overline{z} + d\,\overline{z},$$
$$\partial \log|z| \wedge d\,z = 0, \quad \partial \log|z| \wedge d\,\overline{z} = d^c \log|z| \wedge d\,\overline{z},$$

the form $\partial \log|z| \wedge \psi$ is equal to

$$d^c \log|z| \wedge (\overline{z}\psi_2 + d\,\overline{z} \wedge \phi_2) + \frac{1}{2} d\,z \wedge \psi_1 + d\,\overline{z} \wedge \psi_2$$

and has weak logarithmic singularities and the required log degree.

1.3.11. This definition can be motivated by the following consideration. Any form with weak logarithmic singularities is locally integrable, and so determines some current. Moreover, the differential of this current does not have the singular part supported on the singular locus of the initial form; hence, it is equal to the current determined by the differential of the initial form. Finally, the complex of forms with weak logarithmic singularities is quasiisomorphic to the complex of smooth forms.

1.3.12.

$$\mathcal{HT}(i)^a_{\text{weak}} = \begin{cases} \left(\bigoplus_{p+q=a-1;p,q<i} \mathcal{A}^{pq}_X \right) \bigcap \mathcal{A}^\bullet_{\text{w}\log,<i} & \text{, if } a < 2i, \\ \mathcal{Z}^{ii}_X \bigcap \mathcal{A}^\bullet_{\text{w}\log,<i} & \text{, if } a = 2i. \end{cases} \tag{9}$$

The differential δ, the $\cap$-product and the anti-involution σ preserve $\mathcal{HT}(i)^a_{\text{weak}}$.

1.3.13. Definition. The algebra $\mathcal{HT}_{(X,\overline{X})}$ is generated over $\mathcal{HT}_{\text{weak}}$ by $\log|z_\lambda|$'s with respect to the $\cap$-product. Here $\log|z_\lambda|$'s lies in the first component $\mathcal{HT}(1)^1_{(X,\overline{X})}$ of the part of weight one. We put formally $d\,d^c \log|z_\lambda| = 0$, so we ignore the delta-current on the rhs.

The differential δ, the $\cap$-product and the anti-involution σ preserve $\mathcal{HT}(i)^a_{X,\overline{X}}$. The algebra of σ-invariants is denoted by $\mathbb{R}\mathcal{HT}_{(X,\overline{X})}$.

For a commutative associative dg-algebra $\mathcal{C}$ denote by $H^i_{\mathbf{Har}}(\mathcal{C})$ its i-th Harrison cohomology group (see Section 3.1.).

1.3.14. Theorem. The category of $\mathbb{R}$-Hodge-Tate sheaves $\mathbb{R}\mathfrak{H}\mathfrak{T}_X$ on X is equivalent to the category of graded representations of the zeroth Harrison cohomology group $H^0_{\mathbf{Har}}(\Gamma(\overline{X}, \mathbb{R}\mathcal{HT}_{(X,\overline{X})}(>0)))$ of the algebra of sections of the augmented Hodge-Tate algebra.

2 The modified Green operator of a Hodge-Tate sheaf

2.1. From the modified Green operator to a Hodge-Tate sheaf

Let V be a graded vector space; let G be a function on an analytic manifold X over $\mathbb{C}$ with values in nilpotent operators on V, assume that G is subject to the equations (1), (2). We associate to $(V,\, G)$ an analytic Hodge-Tate sheaf on X.

2.1.1. First we construct a unipotent $\mathbb{R}$-local system. Consider a trivial bundle $\mathcal{V}_{C^\infty} = V \otimes \mathcal{A}^0$. Define a connection $\nabla = d + d^c\, G$ on this bundle. As the operator G is imaginary, the connection is real.

This connection is flat. Indeed, the l-th component ($l > 1$) of its curvature is equal to :

$$(d\,d^c\,G + \frac{1}{2}[d^c\,G, d^c\,G])_l = -d^c\,d\,G_l + \frac{1}{2}\sum_{i+j=l}[d^c\,G_i, d^c\,G_j])$$

$$= -d^c \frac{1}{2l-2}\sum_{j=1}^{l-1}(2j-1)[G_j, d^c G_{l-j}] + \frac{1}{2}\sum_{i+j=l}[d^c\,G_i, d^c\,G_j]) = 0,$$

as $\frac{2j-1}{2l-2} + \frac{2(l-j)-1}{2l-2} = 1 = \frac{1}{2} + \frac{1}{2}$. The first component of the curvature is equal to $d\,d^c\,G_1 = 0$.

The $\mathbb{R}$-local system $\mathcal{V}$ is the local system of the real horizontal (with respect to ∇) sections of $\mathcal{V}_{C^\infty}$.

2.1.2. The grading V^i on V determines a filtration $W_{2l} = \bigoplus_{i\leq l} V^i$. This filtration defines the weight filtration $W_\bullet(\mathcal{V}_{C^\infty}) = W_\bullet(V)\otimes\mathcal{A}^0$. As the operator G preserves the filtration $W_\bullet(V)$, the connection preserves the weight filtration $W_\bullet(\mathcal{V}_{C^\infty})$. The weight filtration on $\mathcal{V}$ is induced by the weight filtration on $\mathcal{V}_{C^\infty}$: $W_\bullet(\mathcal{V}) = \mathcal{V}\cap W_\bullet(\mathcal{V}_{C^\infty})$.

2.1.3. The Hodge filtration $F^\bullet$ is a "rotation" of the filtration $\tilde{F}^k = \bigoplus_{i\geq k} V^i$ by some unipotent operator $\Psi = 1 + \sum_{i\geq 1}\Psi_i$; $F^\bullet = \Psi(\tilde{F}^\bullet)$.

2.1.4. The Hodge filtration must be holomorphic and must be subject to the transversality condition. This means that the gauge transformation of the connection ∇ by the operator Ψ must be holomorphic :

$$\Psi^{-1}\overline{\partial}\Psi - \Psi^{-1}\overline{\partial}G\Psi = 0,$$

and has nontrivial component only in degree -1 :

$$(\Psi^{-1}\partial\Psi + \Psi^{-1}\partial G\Psi)_i = 0, i > 1.$$

From the first equation we have $\overline{\partial}\Psi_1 - \overline{\partial}G_1 = 0$. Put $\Psi_1 = G_1$; then the second equation can be rewritten as

$$\Psi^{-1}\partial\Psi + \Psi^{-1}\partial G\Psi = 2\partial G_1.$$

2.1.5. We construct an operator Ψ as a noncommutative formal power series in the G's :

$$\Psi = \sum_I a_I G_I, \text{ where } I = (i_1,\cdots,i_n) \text{ is a multi-index and } G_I = G_{i_1}\cdots G_{i_n}. \tag{10}$$

Denote by

$$\int_a^b \omega_1(t_1)\circ\omega_2(t_2)\circ\cdots\circ\omega_n(t_n)$$

the following iterated integral

$$\int\cdots\int_{a\leq t_1\leq\cdots\leq t_n\leq b} \omega_1(t_1)\wedge\omega_2(t_2)\wedge\cdots\wedge\omega_n(t_n) \quad \omega_i, \text{ where is a one-form in } t_i. \tag{11}$$

2.1.6. Proposition. Let the collection $G_\bullet$ be subject to (1),(2). Then the operator $\Psi = \sum a_I G_I$, where a_I are defined as the following iterated integrals

$$a_I = \int_0^1 \phi_{i_1}\circ\phi_{i_2}\circ\cdots\circ\phi_{i_n}, \qquad \phi_k(t) = (2k-1)(1-t^2)^{k-1}dt, \tag{12}$$

is a solution of the system

$$\partial\Psi + \partial G\Psi - 2\Psi\partial G_1 = 0 \tag{13}$$

$$\overline{\partial}\Psi - \overline{\partial}G\Psi = 0. \tag{14}$$

2.1.7. Proof. The proof is based on two rather simple general facts.

Let $F(t)$ be a one-form in the variable t which takes values in some Lie algebra $\mathcal{L}$. Denote by $A(s,u)$ the following sum in the universal enveloping algebra of the Lie algebra :

$$A(s,u) = \sum_{n=0}^{\infty}\int_s^u \underbrace{F(t_1)\circ\cdots\circ F(t_n)}_{n\text{ times}}.$$

Evidently, our operator Ψ becomes $A(0,1)$, if we put $F(t)=\sum\phi_i G_i$.

2.1.8. Lemma. Let D be a derivation of a Lie algebra. Denote by the same symbol its continuation to the universal enveloping algebra. Then

$$D\,A(s,u)=\int_s^u A(s,v)(D\,F(v))A(v,u). \tag{15}$$

2.1.9. Proof.

$$D\,A(s,u)=\sum_{n=0}^{\infty}\sum_{i=1}^{n-1}\int_s^u \underbrace{F(t_1)\circ\cdots\circ F(t_{i-1})}_{(i-1)\ \text{times}}\circ(D\,F(t_i))\circ\underbrace{F(t_{i+1})\circ\cdots\circ F(t_n)}_{(n-i)\ \text{times}}$$

$$=\int_s^u A(s,v)(D\,F(v))A(v,u);$$

the last equality follows from the following evident formula :

$$\int_a^b \Omega_1(t_1)\circ\Omega_2(t_2)\circ\cdots\circ\Omega_n(t_n)$$

$$=\int_a^b\left(\int_a^s \Omega_1(t_1)\circ\cdots\circ\Omega_{i-1}(t_{i-1})\right)\Omega_i(s)\left(\int_s^b \Omega_{i+1}(t_{i+1})\circ\cdots\circ\Omega_n(t_n)\right). \qquad \square$$

2.1.10. Lemma. Let $P(t)$ be a function in one variable which takes values in $\mathcal{L}$. Then :

$$\begin{aligned}\int_s^u A(s,v)d\,P(v)A(v,u) &= \int_s^u A(s,v)[P(v),F(v)]A(v,u)\\ &+ \quad A(s,u)P(u)-P(s)A(s,u).\end{aligned} \tag{16}$$

2.1.11. Proof. Note that

$$\frac{d}{d\,s}A(s,u)d\,s=-F(s)A(s,u);\quad \frac{d}{d\,u}A(s,u)d\,u=A(s,u)F(u).$$

Integrating by parts, we get :

$$\int_s^u A(s,v)d_v\,P(v)A(v,u)=-\int_s^u d_v\,A(s,v)P(v)A(v,u)-\int_s^u A(s,v)P(v)d_v\,A(v,u)$$

$$+A(s,v)P(v)A(v,u)\Big|_{v=s}^{v=u}$$

$$=\int_s^u A(s,v)(-F(v)P(v))A(v,u)+\int_s^u A(s,v)(P(v)F(v))A(v,u)$$

$$+A(s,u)P(u)-P(s)A(s,u)\ \ (\text{as } A(t,t)=1)$$

$$=\int_s^u A(s,v)[P(v),F(v)]A(v,u)+A(s,u)P(u)-P(s)A(s,u). \qquad \square$$

2.1.12. Now we apply these two lemmas for the proof of the proposition. Put $F(t)=\sum(2j-1)(1-t^2)^{j-1}G_j d\,t$. The equation (1) implies :

$$\partial\,F(t)=[F(t),P_1(t)]+d_t\,P_1(t),\quad P_1=\sum(1-t)^{j-1}(1+t)^j\partial\,G_j;$$

$$\overline{\partial} F(t) = [F(t), P_2(t)] + d_t\, P_2(t), \quad P_2 = -\sum (1-t)^j (1+t)^{j-1} \overline{\partial} G_j.$$

So, we get

$$\partial\, \Psi = \partial\, A(0,1) = \int_0^1 A(0,v)(\partial\, F(v))A(v,1) = \int_0^1 A(0,v)\left([F(t), P_1(t)] + d_t\, P_1(t)\right) A(v,1)$$

$$= \int_0^1 A(0,v)\left([F(t), P_1(t)] + [P_1(t), F(t)]\right) A(v,1) + A(0,1)P_1(1) - P_1(0)A(0,1)$$

$$= \Psi \times (2\partial\, G_1) - \partial\, G \times \Psi.$$

The proof of the second formula is analogous. □

2.1.13. Remark. We have two decompositions of $\mathcal{V}_{C^\infty}$ as $\oplus V_i \otimes \mathcal{A}_X^0$. The first one is the original one, it is C^∞ and the connection is $d + d^c G$; the second one is “rotated”: $\mathcal{V}_{C^\infty} = \oplus \Psi(V_i \otimes \mathcal{A}_X^0)$, this decomposition is holomorphic $\mathcal{V}_{\mathrm{hol}} = \oplus(V_i \otimes \mathcal{O}_X)$ and the ∂-part of the connection is $\nabla' = \partial + 2\partial\, G_1$.

2.1.14. Now we will discuss condition (3) near the divisor X^∞. Note that we have the trivial vector bundle $\mathcal{V}_X = \bigoplus V^i \otimes_{\mathbb{R}} \mathcal{O}_X$ on X with the connection $\nabla' = \partial + 2\partial G_1$, the Hodge filtration is $F^p = \bigoplus_{i \geq p} V^i \otimes_{\mathbb{R}} \mathcal{O}_X$. This bundle can be extended to the bundle $\mathcal{V}_{\overline{X}} = \bigoplus V^i \otimes_{\mathbb{R}} \mathcal{O}_{\overline{X}}$ on $\overline{X}$ equipped with the Hodge filtration by sub-bundles. In local coordinates z_λ we have $\nabla' = \partial + \sum 2r_\lambda \frac{dz_\lambda}{z_\lambda} + \omega$, where ω is holomorphic. Hence, the connection has a pole of order one and its residue is nilpotent. Consequently, the inverse image of the connection under a map of the disk also has a pole of order one and its residue is nilpotent. So we correspond to a solution $G_\bullet$ of (1)–(3) a Hodge-Tate sheaf.

We leave as an exercise for the reader a proof of compatibility of morphisms.

2.2. From the Hodge-Tate sheaves to the modified Green functions

Now we prove that any analytic Hodge-Tate sheaf corresponds to some filtered vector space $V = \oplus V^l$ equipped with an operator-valued function G such that (1) and (2) hold.

2.2.1. First we consider the Hodge-Tate structures which are sheaves on a point. Let G be an imaginary nilpotent operator on a filtered vector space $V = \oplus V^l$ and let $\Psi = \Psi(G)$ be the unipotent operator defined by formulas (10), (12). The operator Ψ can be represented as the product $\exp(g)\Phi$, g is imaginary nilpotent and Φ is real unipotent. Indeed, consider the product $U = \Psi \overline{\Psi}^{-1}$. Evidently, the operator U is unipotent and $\overline{U} = U^{-1}$, so the nilpotent logarithm of U is imaginary, we put $g = \frac{1}{2}\log(U)$. The operator $\Phi = \exp(-g)\Psi$ is real and unipotent; $\Psi = \exp(g)\Phi$.

2.2.2. This decomposition relates our construction of Hodge-Tate structures to the Deligne construction. Namely, the Deligne grading $V = \oplus V^i_{\mathcal{D}}$ is the rotation of our grading by $\Phi : V^i_{\mathcal{D}} = \Phi V^i$ and the operator g is half of the logarithm of $\Psi\overline{\Psi}^{-1}$.

The operator $\overline{\Psi}^{-1}$ has the shape $\sum a'_I G_I$, where $a'_\emptyset = 1$, $a'_i = a_i$, as G_i is imaginary. So we have

$$g = \frac{1}{2}\log(\Psi\overline{\Psi}^{-1}) = -\frac{1}{2}\sum \frac{1}{k}(1 - \Psi\overline{\Psi}^{-1})^k = \sum a_i G_i + \sum_{I \mid \#(I) \geq 2} c_I G_I,$$

$$g_i = a_i G_i + \sum_{I = \{i_1 \cdots i_n\} \mid n \geq 2, \sum i_k = i} c_I G_{i_1} \cdots G_{i_n} \tag{17}$$

for some rational c_I. As $a_i = \frac{(2i-2)!!}{(2i-3)!!} \neq 0$, we can invert this system of relations and get

$$G_i = \frac{(2i-3)!!}{(2i-2)!!} g_i + \sum_{I=\{i_1 \cdots i_n\} | n \geq 2, \sum i_k = i} \gamma_I g_{i_1} \cdots g_{i_n} \tag{18}$$

for some rational γ_I's.

2.2.3. For any Hodge-Tate structure consider the Green operator $g = \sum g_i$ which determines this structure and *define* the modified Green operator $G = \sum G_i$ by formula (18). It determines some Hodge-Tate structure; evidently, this structure coincides with the initial one.

For a Hodge-Tate sheaf on some variety X we can construct G fiberwise, so we shall deduce (1) and (2) from the condition on the Hodge filtration.

This is a rather simple exercise which we leave for the reader.

2.2.4. Now we deduce condition (3) near the compactification divisor X^∞ from property b) of a Hodge-Tate sheaf.

Choose a point x on the compactification divisor X^∞, consider some small vicinity $U \subset X$ of it and fix some equations z_λ of the branches X^∞_λ of X^∞, passing through x. Denote $U^* = U \cap X$. As $d\, d^c\, G_1 = 0$, the form $\nu = d^c\, G_1$ is closed. Define a multi-valued function F as an integral of ν. Denote by r_λ the period of $\nu/(2\pi\sqrt{-1})$ corresponding to a path around X^∞_λ. Then:

i) the multi-valued function $G_1 + F$ on U^* is analytic; indeed, as $d\, F = d^c\, G_1$, $d^c\, F = d\, G_1$ and $2\overline{\partial}\,(G_1 + F) = (d - d^c)\,(G_1 + F) = d\, G_1 - d^c\, F + d\, F - d^c\, G_1 = 0$;

ii) The analytic function $f = G_1 + F - \sum r_\lambda \log z_\lambda$ on U^* is single-valued; indeed, the monodromy of f corresponding to a path γ around $z_\lambda = 0$ equals to adding to f the number $\int_\gamma \nu - 2\pi\sqrt{-1} r_\lambda = 0$.

Consider the bundle $\mathcal{V}_X = \bigoplus V^i \otimes_{\mathbb{R}} \mathcal{O}_X$ on X equipped with the Hodge filtration $F^p = \bigoplus_{i \geq p} V^i \otimes_{\mathbb{R}} \mathcal{O}_X$. Note that it can be extended to the bundle $\mathcal{V}_{\overline{X}} = \bigoplus V^i \otimes_{\mathbb{R}} \mathcal{O}_{\overline{X}}$ on $\overline{X}$ and the Hodge filtration extends to a filtration by sub-bundles. Hence, its restriction to any immersed disk $\overline{f}: D \to \overline{X}$, $\overline{f}(D^*) \subset X$ is also filtered by sub-bundles.

For any filtered bundle $(\mathcal{L}, F)$ on the punctured disk its extension $(\overline{\mathcal{L}}, F)$ to the disk as a filtered bundle is unique up to tensoring by a power of the line bundle $\mathcal{O}(-0)$(here 0 denotes the point origin).

Therefore, the Deligne continuation of $f^*(\mathcal{V}_X)$ is equal to $\overline{f}^*(\mathcal{V}_{\overline{X}})(k0)$ for some integer k (here $f = \overline{f}|_{D^*}$). The connection $\nabla' = \partial + 2\partial G_1$ on $\mathcal{V}_{\overline{X}}$ is nilpotent, so $k = 0$, and the Deligne continuation is the restriction of $\mathcal{V}_{\overline{X}}$. Hence the restriction of the connection $\nabla = \partial + 2\partial\, G_1 = \partial + \partial\,(G_1 + F) = \partial + \partial(f + \sum r_\lambda \log z_\lambda)$ to the punctured disk has a pole of order less than or equal to one; consequently the restriction of f to the punctured disk is regular at the origin.

2.2.5. Choose a point y on the smooth part of the divisor X^∞_λ and consider its small vicinity U_0 such that $U_0 \cap X^\infty = U_0 \cap X^\infty_\lambda$ is smooth. For small U_0 we can represent the function f in as a Laurent series in z_λ whose coefficients are holomorphic functions in the complementary variables (the Hartogs expansion). As for any immersed disk the restriction of f is regular, any coefficient of a negative power of z_λ vanishes on a dense set of points, so it vanishes as a function. From this we conclude that f is regular at smooth points of the divisor X^∞. According to the Hartogs theorem, a function which is holomorphic on the complement of an analytic set of codimension two is holomorphic on this set also. Hence f is holomorphic on U and $G_1 = \frac{1}{2}(f+\overline{f}) + \sum r_\lambda \log|z_\lambda|$ satisfies condition (3).

2.3. The tensor product of Hodge-Tate sheaves. We want to prove that the Lie type tensor product of the operators $G = G^{(1)} \otimes \mathrm{Id}_{V(2)} + \mathrm{Id}_{V(1)} \otimes G^{(2)}$ acting on $V = V_1 \otimes V_2$ determines the tensor product of Hodge-Tate sheaves.

2.3.1. Note that the operator G is subject to the equations (1)-(3); so the pair (V, G) determines some Hodge-Tate sheaf $\mathcal{V}$. We will prove that this sheaf equals the tensor product $\mathcal{V}_1 \otimes \mathcal{V}_2$ ($\mathcal{V}_i$ corresponds to (V_i, G_i)).

2.3.2. First, the unipotent local systems coincide. Indeed, we define a local system as a local system of real horizontal sections of some connection on some trivial C^∞-bundle. In our case the trivial bundle is $V \otimes \mathcal{A}^0 = (V_1 \otimes V_2) \otimes \mathcal{A}^0$ for $\mathcal{V}$ and the same for $\mathcal{V}_1 \otimes \mathcal{V}_2$. The connection corresponding to $\mathcal{V}$ equals $d + d^c\, G = d + d^c\, G^{(1)} \otimes \mathrm{Id}_{V(2)} + \mathrm{Id}_{V(1)} \otimes d^c\, G^{(2)}$. This expression is equal to the connection on the tensor product of the bundles $\mathcal{V}_{i,\,\mathrm{hol}}$ with the connections $d + d^c\, G_i$.

2.3.3. Second, the weight filtration $W_\bullet$ on the tensor product is the tensor product of weight filtrations on the factors, so the weight filtrations on $\mathcal{V}$ and on $\mathcal{V}_1 \otimes \mathcal{V}_2$ coincide.

2.3.4. Third, the tilde-filtration $\tilde{F}^\bullet$ on the tensor product equals the tensor product of the tilde-filtrations on the factors.

The Hodge filtration $F^\bullet$ is the rotation of $\tilde{F}^\bullet$ by the operator $\Psi(G) = \sum a_I G_I$. If the operator $\Psi(G)$ is a "group-like element" :

$$\Psi(G) = \Psi(G^{(1)}) \otimes \Psi(G^{(2)}), \tag{19}$$

then the Hodge filtration on the tensor product is the tensor product of the Hodge filtration on the factors :

$$F^p = \Psi(G)(\tilde{F}^p) = \Psi(G^{(1)}) \otimes \Psi(G^{(2)})\Big(\bigoplus_{p_1+p_2=p} \tilde{F}^{p_1} \otimes \tilde{F}^{p_2}\Big)$$

$$= \bigoplus_{p_1+p_2=p} \Psi(G^{(1)})(\tilde{F}^{p_1}) \otimes \Psi(G^{(2)})(\tilde{F}^{p_2}) = \bigoplus_{p_1+p_2=p} F^{p_1} \otimes F^{p_2}.$$

2.3.5. Lemma. The operator Ψ is a "group-like element" (19) if the a_I satisfy the shuffle relations :

$$a_I a_J = \sum_{\sigma \in \mathrm{Shuffle}(\#(I), \#(J))} a_{\sigma(I,J)}, \tag{20}$$

where $\#(k_1, \cdots k_n) = n$ denotes the length of the multiindex.

2.3.6. Proof. Consider the expression :

$$(G^{(1)} \otimes \mathrm{Id}_{V_1} + \mathrm{Id}_{V_2} \otimes G^{(2)})_K = (G^{(1)} \otimes \mathrm{Id}_{V_1} + \mathrm{Id}_{V_2} \otimes G^{(2)})_{k_1} \cdots (G^{(1)} \otimes \mathrm{Id}_{V_1} + \mathrm{Id}_{V_2} \otimes G^{(2)})_{k_n}$$

for some multiindex $K = \{k_1 \cdots k_n\}$. It is the sum of terms each of which corresponds to a choice of superscripts (1) or (2) in each factor, labeled by k's. Hence, any term corresponds to some representation of K as a union of two multi-indices I and J, here

I is the collection of indices for which we choose (1) in the corresponding factor and J corresponds to the choice of (2). The order of I and J is induced from K. Therefore, K is a shuffle of I and J. Finally we get :

$$(G^{(1)}\otimes \mathrm{Id}_{V_1}+\mathrm{Id}_{V_2}\otimes G^{(2)})_K=\sum_{(I,J)|K=\sigma(IJ),\sigma\in \mathrm{Shuffle}(\#(I),\#(J))} G_I^{(1)}\otimes G_J^{(2)}.$$

So,

$$\Psi(G)=\sum a_K G_K=\sum a_K(G^{(1)}\otimes \mathrm{Id}_{V_2}+\mathrm{Id}_{V_1}\otimes G^{(2)})_K$$

$$=\sum_{I,J}\sum_{\sigma\in \mathrm{Shuffle}(\#(I),\#(J))} a_{\sigma(I,J)}G_I^{(1)}\otimes G_J^{(2)}=\sum_{I,J} a_I a_J G_I^{(1)}\otimes G_J^{(2)}=\sum_I a_I G_I^{(1)}\otimes \sum_J a_J G_J^{(2)}$$

$$=\Psi(G^{(1)})\otimes \Psi(G^{(2)}) \qquad \square$$

2.3.7. Lemma. The numbers a_I, defined by the iterated integrals (12), are subject to the shuffle relations (20).

2.3.8. Proof. The shuffle relation is a very general property of iterated integrals. Sketch a proof: Consider two simplices $0\le t_1\le\cdots\le t_n\le 1$ and $0\le s_1\le\cdots\le s_m\le 1$ of dimensions n and m, respectively. Its product can be decomposed into the union of simplices of dimension $n+m$ by choosing an ordering of the t's and the s's. Any choice corresponds to some shuffle of n and m. The product of the iterated integrals equals the integral over the product of the two simplexes. The sum over all shuffles is equal to the sum of the integrals over the components of the decomposition. $\square$

2.4. The differential equation for the Green operator.

In this subsection we deduce a differential equation for the Green operator of a Hodge-Tate sheaf which follows from the definition of a Hodge-Tate sheaf.

2.4.1. Let $\mathcal{V}$ be a Hodge-Tate sheaf. Consider its fiberwise $\mathbb{R}$-splitting $\mathcal{V}=\oplus\mathcal{V}^l$, as in 1.1.5. The quotients Gr^W are constant, so the $\mathcal{V}^l$ are constant also. In this representation the connection ∇ has the form $d+\nu^{10}+\nu^{01}$, $\nu^{ij}\in \mathrm{End}(\mathcal{V})\otimes\mathcal{A}^{ij}$. As ∇ preserves the weight filtration, $\nu=\nu^{10}+\nu^{01}$ is a nilpotent real one-form, $\nu=\oplus\nu_i$, $\nu^{01}=\overline{\nu^{10}}$.

2.4.2. Proposition. Let g be the fiberwise Green operator for the Hodge-Tate sheaf $\mathcal{V}$. Then

$$\partial g=\frac{-2\mathrm{ad}_g}{\exp(-2\mathrm{ad}_g)-1}\partial g_1; \tag{21}$$

and

$$\nu^{10}=\frac{1}{\exp(-\mathrm{ad}_g)+1}\partial g_1 \tag{22}$$

The Hodge filtration is defined as $F^p=\exp(g)\left(\oplus_{l\ge p}\mathcal{V}^l\right)$. It must be holomorphic and satisfies the transversality condition. This means :

$$\exp(-g)\overline{\partial}\exp(g)+\exp(-g)\nu^{01}\exp(g)=0,$$
$$(\exp(-g)\partial\exp(g)+\exp(-g)\nu^{10}\exp(g))_i=0,\quad i>1.$$

From the the first equation we deduce that $\overline{\partial}\, g_1 + \nu^{01} = 0$. The function g is imaginary, so $\nu^{10} = \overline{\nu^{01}} = -\overline{\overline{\partial}\, g_1} = \partial\, g_1$. Hence the component of degree one in the second equation is equal to $\partial\, g_1 + \nu^{10} = 2\partial\, g_1$.

From 2.1.8 and 2.1.10, for $F(t) = a \times dt$ and $P(t) = -\frac{\exp((1-t)\mathrm{ad}_a)-1}{\mathrm{ad}_a} D(a)$ we get :

2.4.3. Lemma. Let D be a derivation of a Lie algebra L. Then :

$$D\,\exp(a) = \frac{\exp(\mathrm{ad}_a) - 1}{\mathrm{ad}_a} D\, a \times \exp(a)$$

for any $a \in L$. □

2.4.4. Put D equal to $\overline{\partial}$ or ∂, respectively. From the lemma we obtain:

$$\exp(-g)\frac{\exp(\mathrm{ad}_g) - 1}{\mathrm{ad}_g}(\overline{\partial} g)\exp(g) + \exp(-g)\nu^{01}\exp(g) = 0,$$

$$\exp(-g)\frac{\exp(\mathrm{ad}_g) - 1}{\mathrm{ad}_g}(\partial g)\exp(g) + \exp(-g)\nu^{10}\exp(g) = 2\partial\, g_1.$$

Therefore,

$$\nu^{01} = -\frac{\exp(\mathrm{ad}_g) - 1}{\mathrm{ad}_g}\overline{\partial} g, \quad \nu^{10} = \overline{\nu^{01}} = \frac{\exp(-\mathrm{ad}_g) - 1}{-\mathrm{ad}_g}\partial g,$$

as g is imaginary. Substitute this into the second equation :

$$\frac{\exp(\mathrm{ad}_g) - 1}{\mathrm{ad}_g}\partial g + \frac{\exp(-\mathrm{ad}_g) - 1}{-\mathrm{ad}_g}\partial g = 2\exp(\mathrm{ad}_g)\partial\, g_1.$$

Hence, we prove (21); the second equation follows from it and the expression for ν^{10} above. □

3 The Hodge-Tate algebra

3.1. The Harrison cohomology groups of commutative algebras.

3.1.1. Let $\mathcal{C} = \{\oplus \mathcal{C}^i, i > 0, \delta\ : \mathcal{C}^i \to \mathcal{C}^{i+1}\}$ be an associative dg-algebra. Consider the following complex (the bar complex) : $\mathcal{B}(\mathcal{C}) = \{\oplus_{j=0}^{\infty}\mathcal{C}^{\otimes j}, D\}$, where for homogeneous a's in $\mathcal{C}$ the differential D is defined by the formula :

$$D((a_1|a_2|\cdots|a_n)) = \sum_{j=1}^{n-1}(-1)^{j-1}(a_1|a_2|\cdots|a_j a_{j+1}|\cdots|a_n)$$

$$+(-1)^{n-1}\sum_{j=1}^{n}(-1)^{\sum_{i<j}\deg(a_i)}(a_1|a_2|\cdots|\delta(a_j)|\cdots|a_n);$$

here $(a_1|a_2|\cdots|a_n)$ denotes the tensor product $a_1 \otimes a_2 \otimes \cdots \otimes a_n$ and deg denotes the degree of an element. The degree of the homogeneous element $(a_1|a_2|\cdots|a_n)$ is $\sum\deg(a_i) - n$.

The complex $\mathcal{B}(\mathcal{C})$ is a coassociative dg-coalgebra with respect to the following comultiplication $\nu : \mathcal{B}(\mathcal{C}) \to \mathcal{B}(\mathcal{C})^{\otimes 2}$:

$$\nu((a_1|\cdots|a_n)) = \sum_{j=0}^{n}(-1)^{(n-k)\sum_{i\le j}\deg(a_i)}(a_1|\cdots|a_j) \otimes (a_{j+1}|\cdots|a_n).$$

3.1.2. The shuffle product is defined by the following formula :

$$(a_1|a_2|\cdots|a_n)\bullet(a_{n+1}|a_{n+2}|\cdots|a_{n+m}) =$$

$$(-1)^{m\sum_{j=1}^{n}\deg a_i}\sum_{\sigma\in\mathrm{Shuffles}(n,m)}\omega(\sigma;a_1,\cdots,a_{n+m})(a_{\sigma(1)}|a_{\sigma(2)}|\cdots|a_{\sigma(n+m)}),$$

ω is the sign of the shuffle where we take the degrees of the a's into account :

$$\omega(\sigma;a_1,\cdots,a_{n+m}) = \mathrm{sign}(\sigma)\times\prod_{\lambda>\mu|\sigma(\lambda)<\sigma(\mu)}(-1)^{\deg(a_\lambda)\deg(a_\mu)}.$$

Evidently, this product is commutative.

3.1.3. The shuffle product is compatible with the comultiplication ν. If the algebra $\mathcal{C}$ is (graded) commutative, then the shuffle product is compatible with the differential. Consider the ideal $\mathcal{I}(\mathcal{C}) = \oplus_{j=1}^{\infty}\mathcal{C}^{\otimes j}$ of $\mathcal{B}(\mathcal{C})$. The complex of indecomposable elements $\mathbf{Har}_*(\mathcal{C}) = \mathcal{I}(\mathcal{C})/\mathcal{I}(\mathcal{C})\bullet\mathcal{I}(\mathcal{C})$ is a Lie dg-coalgebra with respect to the restriction of the comultiplication ν. We call this complex the homological Harrison complex of $\mathcal{C}$.

3.1.4. Definition. Let $\mathcal{C}$ be an associative commutative dg-algebra. The Lie dg-algebra $\mathbf{Har}^*(\mathcal{C})$, dual to $\mathbf{Har}_*(\mathcal{C})$ is called the *cohomological Harrison complex.* The cohomology group of this complex is known as the Harrison cohomology $H^*_{\mathbf{Har}}(\mathcal{C})$ group of $\mathcal{C}$.

3.1.5. Lemma. A representation $\rho\colon H^0_{\mathbf{Har}}(\mathcal{C})\to\mathbf{End}(V)$ of the zeroth Harrison cohomology group $H^0_{\mathbf{Har}}(\mathcal{C})$ in some vector space V is the same as an element $\tilde{\rho}\in\mathcal{C}^1\otimes\mathrm{End}(V)$ which is a solution of the Maurer-Cartan equation :

$$\delta\tilde{\rho}+\frac{1}{2}[\tilde{\rho},\tilde{\rho}] = 0\in\mathcal{C}^2\otimes\mathrm{End}(V) \tag{23}$$

3.1.6. Sketch of the proof. For any Lie algebra $\mathcal{L}$, the dual to it Lie coalgebra $\mathcal{L}^\vee$ is equipped with a *cocommutator* $\kappa\ :\mathcal{L}^\vee\to\bigwedge^2\mathcal{L}^\vee$ (dual to the commutator on $\mathcal{L}$). A representation (ρ,V) of $\mathcal{L}$ is an element $\rho\in\mathcal{L}^\vee\otimes\mathrm{End}(V)$ such that

$$(\kappa\otimes\mathrm{Id})(\rho) = (\mathrm{Id}\otimes[,])(\rho^{\otimes 2})\in\left(\bigwedge^2\mathcal{L}^\vee\right)\otimes\mathrm{End}(V). \tag{24}$$

The coalgebra $H_0^{\mathbf{Har}}(\mathcal{C})$ dual to $H^0_{\mathbf{Har}}(\mathcal{C})$ is the space of expressions

$$\{(a_1^{(1)}),\sum_\gamma(a_1^{(2),\gamma}|a_2^{(2),\gamma}),\ldots\},\quad a_i^{(n)}\in\mathcal{C}^1,$$

such that

$$\delta(a_1^{(1)})+\sum_\gamma a_1^{(2),\gamma}a_2^{(2),\gamma} = 0,\ldots$$

modulo the shuffle relations.

The cocommutator κ is induced by the comultiplication ν and is equal to :

$$\kappa\left(\{(a_1^{(1)}),\sum_\gamma(a_1^{(2),\gamma}|a_2^{(2),\gamma}),\ldots\}\right)$$

$$= \sum_{\gamma} \left(\{(a_1^{(2),\gamma}, \ldots)\} \otimes \{(a_2^{(2),\gamma}, \ldots)\} - \{(a_2^{(2),\gamma}, \ldots)\} \otimes \{(a_1^{(2),\gamma}), \ldots\} \right).$$

An element of $H^0_{\mathbf{Har}}(\mathcal{C}) \otimes \mathrm{End}(V)$ is a collection:

$$\{\sum_{\beta}(a_1^{(1),\beta}) \otimes M^{(1),\beta}, \sum_{\gamma}(a_1^{(2),\gamma}|a_2^{(2),\gamma}) \otimes M^{(2),\gamma}, \ldots\}, \quad M^{(k),\gamma} \in \mathrm{End}(V)$$

such that

$$\delta \sum_{\beta}(a_1^{(1),\beta}) \otimes M^{(1),\beta} + \sum_{\gamma} a_1^{(2),\gamma} a_2^{(2),\gamma} \otimes M^{(2),\gamma} = 0, \ldots$$

Hence, a representation ρ of $H^0_{\mathbf{Har}}(\mathcal{C})$ is an element which satisfies (24) :

$$\sum_{\beta,\gamma}(a_1^{(1),\beta} \otimes a_1^{(1),\gamma}) \otimes [M^{(1),\beta}, M^{(1),\gamma}], \ldots = (\mathrm{Id} \otimes [,])(\rho^{\otimes 2}) = (\kappa \otimes \mathrm{Id})(\rho)$$

$$= \sum_{\gamma} \left((a_1^{(2),\gamma} \otimes a_2^{(2),\gamma)}) - (a_2^{(2),\gamma} \otimes a_1^{(2),\gamma)}) \right) \otimes M^{(2),\gamma}, \ldots.$$

Apply to the first component of this identity the multiplication $\bigotimes^2 \mathcal{C}^1 \to \mathcal{C}^2$. We get :

$$[\tilde{\rho}, \tilde{\rho}] = \sum_{\beta,\gamma}(a_1^{(1),\beta} a_1^{(1),\gamma}) \otimes [M^{(1),\beta}, M^{(1),\gamma}] = 2\sum_{\gamma}(a_1^{(2),\gamma} a_2^{(2),\gamma)}) \otimes M^{(2),\gamma}$$

$$= -2\delta(a_1^{(1),\beta}) M^{(1),\beta} = -2\delta(\tilde{\rho}),$$

where $\tilde{\rho} = \sum_{\beta} a_1^{(1),\beta} M^{(1),\beta}$. □

3.2. The Hodge-Tate sheaves and the Hodge-Tate algebra

3.2.1. The Maurer-Cartan element $\tilde{\rho}$ for the algebra $\Gamma(\mathbb{R}\mathcal{HT}_{(X,\overline{X})}(>0)$ is a collection of imaginary $\mathrm{End}(V)$-valued functions $G(i)$, $i \geq 1$ on X such that :

i) $G(1)$ has logarithmic singularities of order less than or equal to one at X^{∞}; this is equivalent to (3).

ii) $-2\partial\overline{\partial}G(1) = 0$; this is equivalent to (2).

iii) $G(l)$ has weak logarithmic singularities of order less than or equal to $l-1$ at X^{∞}.

iv) For $l \geq 2$ the following equation holds :

$$-\frac{2}{2l-2}((l-1)\partial + (l-1)\overline{\partial})G(l)$$

$$+\frac{1}{2}\sum_{j+k=l}(\frac{2}{2l-2}(2j-1)[G(j), d^c\, G(k)] - (2k-1)[d^c\, G(j), G(k)]) = 0;$$

this is equivalent to (1).

Therefore, any Maurer-Cartan element defines a solution of (1)-(3) by putting $G_l = G(l)$. Hence, the equivalence follows from the following fact: for any solution $G = \sum G_l$ of (1)-(3), G_l has weak logarithmic singularities of order less than or equal to $l-1$ at X^∞ if $l \geq 2$. This can be proved by induction on l. We leave this proof as an exercise for the reader.

3.2.2. We want to show that (1)-(3) more or less determine the dg-algebra $\mathbb{R}\mathcal{HT}$.

We must construct some commutative associative dg-algebra $\mathbb{R}\mathcal{HT}$ such that (1)-(3) are equivalent to (23).

As the representation space V is a graded vector space, it is natural to expect that the Lie algebra is graded; consequently, we can expect that the commutative dg-algebra is also (extra) graded.

3.2.3. The objects which can be solutions of equations (1)-(3) are collections of functions $G(l)$ on $\overline{X}$ which take values in $\mathrm{End}(V)$. This forces us to put $\Gamma(\overline{X}, \mathbb{R}\mathcal{HT}(l)^1_{X,\overline{X})})$ equal to a subspace of the space of imaginary-valued functions on $\overline{X}$.

3.2.4. Equation (2) forces that $\delta\colon \mathbb{R}\mathcal{HT}(1)^1_{(X,\overline{X})} \to \mathbb{R}\mathcal{HT}(1)^2_{(X,\overline{X})}$ is proportional to $\partial\overline{\partial}$, and $\mathbb{R}\mathcal{HT}(1)^2$ must be a subsheaf of the sheaf $\mathcal{A}^{11}$.

3.2.5. Equation (1) means that $\mathbb{R}\mathcal{HT}(l)^2$ must be (a sub-sheaf of) $\mathcal{A}^1$, the differential δ is proportional to the de-Rham differential d and the multiplication must be proportional (with the same coefficient as the differential) to $\cap$.

3.2.6. The definition of the fiber of the algebra $\mathcal{HT}$ on X is nothing more that a continuation of both multiplication and differential from functions to differential forms in such a way that the algebra is commutative and associative.

3.2.7. Remark. The zeroth Harrison cohomology group depend on the first and second terms of the dg-algebra, therefore we can continue our dg-algebra in another way, for instance by zero. But our choice seems to be more natural.

3.2.8. As the first component G_1 of G can have logarithmic singularities, we must add logarithms to $\mathcal{HT}(1)^1$; so we add its $\cap$-products with smooth forms. If such a product is δ-exact we can add its δ-preimage. If a form with logarithmic singularities is exact, it has weak logarithmic singularities, and its preimage is a form with weak logarithmic singularities.

3.3. Further properties of the Hodge-Tate algebra.

3.3.1. Recall the notion of $\mathbb{R}$-Deligne complexes [RSS]. Denote by π_i the map $\omega \to \frac{1}{2}(\omega + (-1)^i\overline{\omega})$ from $\mathcal{A}^\bullet$ to $\sqrt{-1}^i S^\bullet$. The i-th Deligne complex $\tilde{\mathbb{R}}_{\mathcal{D}}(i)$ is the cone of the morphism $\pi_i : F^i(\Omega^\bullet) \to (\sqrt{-1})^i S^\bullet$ shifted by minus one.

3.3.2. Proposition. The component $(\mathbb{R}\mathcal{HT}(i), \delta)$ of the Hodge-Tate algebra of weight i is quasiisomorphic to $(\sqrt{-1})^i\tilde{\mathbb{R}}_{\mathcal{D}}(i)$.

3.3.3. Proof. The complex $(\sqrt{-1})^i\tilde{\mathbb{R}}_{\mathcal{D}}(i)$ is quasi-isomorphic to the the cone of the morphism $(\sqrt{-1})^i\pi_i : \mathcal{A}^{\geq i\bullet} \to S^\bullet$ shifted by minus one :

$$S^0 \to S^1 \to \cdots \to S^{i-1} \oplus \mathcal{A}^{i0} \to S^i \oplus \left(\mathcal{A}^{i+10} \oplus \mathcal{A}^{i1}\right) \to \cdots;$$

where S^0 is placed in degree 1 and the map

$$S^{k-1} \oplus \mathcal{A}^k \cap \mathcal{A}^{\geq i\bullet} \to S^k \oplus \mathcal{A}^{k+1} \cap \mathcal{A}^{\geq i\bullet}$$

is defined by the following formula :

$$(\omega, f) \to (d\,\omega + (-1)^{\deg(f)}(\sqrt{-1}^i f + (-\sqrt{-1})^i \overline{f}, d\,f)$$

A.Goncharov noticed [G2] that this complex is quasi-isomorphic to the complex $\tilde{\tilde{\mathbb{R}}}_{\mathcal{D}}(i)$ defined by :

$$S^{\bullet} \cap \mathcal{A}^{<i<i} \xrightarrow{\partial\overline{\partial}} \sqrt{-1} S^{\bullet} \cap \mathcal{A}^{\geq i \geq i}.$$

Indeed, we have an embedding :

$$S^{\bullet} \cap \mathcal{A}^{<i<i} \to (S^0 \to \cdots \to S^{i-1} \oplus \mathcal{A}^{i0}), \quad \psi \to (\psi, (-1)^{\deg\psi}(-\sqrt{-1})^i \partial\psi^{i-1\bullet});$$

$$\sqrt{-1} S^{\bullet} \cap \mathcal{A}^{\geq i \geq i} \to S^{\geq i} \oplus \mathcal{A}^{>i} \cap \mathcal{A}^{\geq i \bullet}, \quad \phi \to (0, (-\sqrt{-1})^i \phi).$$

The quotient is equal to the cone of the identical map $S^{(i)} \to S^{(i)}$, where $S^{(i)} = S^{\bullet} \cap (\mathcal{A}^{\geq i\bullet} + \mathcal{A}^{\bullet \geq i})$; the corresponding map is defined by the following formula:

$$(\omega, f) \to \left(\omega - \omega^{<i<i}, (-\sqrt{-1})^i f + \sqrt{-1}^i \overline{f} - (-1)^{\deg\omega}(\partial\omega^{i-1\bullet} + \overline{\partial}\omega^{\bullet i-1})\right).$$

3.3.4. On the other hand, the complex $(\mathbb{R}\mathcal{H}\mathcal{T}(i), \delta)$ can be embedded into $\tilde{\tilde{\mathbb{R}}}_{\mathcal{D}}(i)$ by the formula

$$\omega^{pq} \to (-2)^{2i-p-q-1} \frac{(i-p-1)!(i-q-1)!}{(2i-p-q-2)!} \omega^{pq}, p, q < i; \quad \omega^{ii} \to \omega^{ii}.$$

The image of $(\mathbb{R}\mathcal{H}\mathcal{T}(i), \delta)$ is the truncation of $\tilde{\tilde{\mathbb{R}}}_{\mathcal{D}}(i)$, so their cohomology sheaves coincide in dimensions less than or equal to $2i$. Now $\tilde{\mathbb{R}}(i)$, and therefore $\tilde{\tilde{\mathbb{R}}}_{\mathcal{D}}(i)$, has nontrivial sheaf cohomology groups in dimensions 1 and i, so $(\mathbb{R}\mathcal{H}\mathcal{T}(i), \delta)$ is quasi-isomorphic to the Deligne complex $\tilde{\mathbb{R}}_{\mathcal{D}}(i)$.

3.3.5. Remark. The factor $\sqrt{-1}^i$ comes from the change of the generator of $\mathbb{R}(i)$; for the Deligne complexes one chooses the "complex" generator $1 \in \mathbb{R}(i) \otimes \mathbb{C}$, while we choose the generator $(2\pi\sqrt{-1})^{-i}$ in the real space $\mathbb{R}(i)$ itself.

3.3.6. Consider a component X^{∞}_{λ} of the divisor X^{∞}, denote it by $\overline{Y}$, put $Y^{\infty} = Y \cap \bigcup_{\mu \neq \lambda} X^{\infty}_{\mu}$. Then Y^{∞} is a divisor in $\overline{Y}$ with normal crossings; denote by Y the "smooth part" $\overline{Y} \setminus Y^{\infty}$ of $\overline{Y}$. We have the natural *residue map* Res from $\mathbb{R}\mathcal{H}\mathcal{T}_{(X,\overline{X})}$ to $\mathbb{R}\mathcal{H}\mathcal{T}_{(Y,\overline{Y})}$, which shifts the grading and the degree by -1. This map is defined by the following rule: For a form of type $\log|z_{\lambda}| \cap \omega$ $\mathrm{Res}(\log|z_{\lambda}| \cap \omega) = \omega|_Y$, for a form with weak log singularities the image under this map equals zero.

3.3.7. On the other hand, the choice of an equation z_{λ} for Y determines the *specialization map* $\mathrm{Sp}_{z_{\lambda}}$ of dg-algebras from $\mathbb{R}\mathcal{H}\mathcal{T}_{(X,\overline{X})}$ to $\mathbb{R}\mathcal{H}\mathcal{T}_{(Y,\overline{Y})}$ which preserves grading and degree. This map is defined by the following rule: any form in $\mathbb{R}\mathcal{H}\mathcal{T}_{(X,\overline{X})}$ can be represented as a polynomial in $\log|z_{\lambda}|$ and $d^c \log|z_{\lambda}|$ with smooth coefficients; the image of the form is the restriction of the constant term of this polynomial.

We have the following relation between the map Res, the product $\cap$ and the map $\mathrm{Sp}_{z_{\lambda}}$:

$$\mathrm{Res}(\omega_1 \cap \omega_2) = \mathrm{Res}(\omega_1) \cap \mathrm{Sp}_{z_{\lambda}}(\omega_2) + (-1)^{\deg\omega_1} \mathrm{Sp}_{z_{\lambda}}(\omega_1) \cap \mathrm{Res}(\omega_2).$$

The map Sp_{z_λ} depends on the choice of the local equation z_λ: if z'_λ is another equation of Y, then

$$\mathrm{Sp}_{z'_\lambda} = \mathrm{Sp}_{z_\lambda} + (\log|z_\lambda| - \log|z'_\lambda|)|_Y \cap \mathrm{Res}.$$

3.3.8. Consider the following variety N. Let $\pi\colon \overline{N} \to \overline{Y}$ be a fiberwise projective compactification of the normal bundle $\mathcal{N}_{\overline{Y}\subset\overline{X}}$ of the subvariety $\overline{Y}$ in $\overline{X}$. Put $N^\infty = \pi^{-1}(Y^\infty) \cup 0(\overline{Y}) \cup \infty(\overline{Y})$, where 0 and ∞ denote zeroth and infinite sections $\overline{Y} \to \overline{N}$ of the bundle $\overline{N} \to \overline{Y}$. The variety N is the complement of N^∞ in $\overline{N}$.

3.3.9. We have the *specialization map* Sp from $\mathbb{R}\mathcal{HT}_{(X,\overline{X})}$ to $\mathbb{R}\mathcal{HT}_{(N,\overline{N})}$. The local equation z_λ of Y defines a fiberwise linear function ξ_λ on N (the differential of z_λ), and if the function ξ'_λ corresponds to another equation z'_λ, then $\xi'_\lambda = \left.\frac{z'_\lambda}{z_\lambda}\right|_Y \xi_\lambda$. So the map defined by the formula $\mathrm{Sp} = \mathrm{Sp}_{z_\lambda} + \log|\xi_\lambda| \cap \mathrm{Res}$ does not depends on the choice of the local equation.

Therefore, we have a map Sp of the corresponding Lie algebras and a functor between the categories of their representations; this functor equals the specialization functor for Hodge-Tate sheaves.

4 The modified polylogarithms.

4.1. The single-valued polylogarithms due to Zagier[Z].

4.1.1. An important example of the Hodge-Tate sheaf is the polylogarithmic sheaf [D2],[W, IV.3]. This is a sheaf on $\mathbb{CP}^1 \setminus \{0,1,\infty\}$.

The matrix elements of the Green operator for this sheaf are the single-valued versions of polylogarithms which have been constructed by Zagier. We recall some facts about them.

4.1.2. The Euler r-logarithm $Li_r(z)$, $r = 1,2,\dots$ is a multi-valued analytic function on $\mathbb{CP}^1 \setminus \{0,1,\infty\}$, defined as the analytic continuation of $Li_r(z) = \sum_{j=1}^\infty \frac{z^j}{j^r}$, $\quad |z| < 1$.

4.1.3. The single-valued version $L_r(z) = \left|\hat{L}_r(z)\right|$ of r-logarithm was constructed by Lobachevskii for $r = 2$ (and is known as the Bloch-Wigner dilogarithm) and by Zagier for general r.

$$\hat{L}_r(z) = \pi_r\left(\sum_{k=0}^{r-1} \beta_k Li_{r-k}(z)\log^k|z|\right) \tag{25}$$

where, as above, $\pi_r(a+ib) = \begin{cases} a & r \text{ odd} \\ ib & r \text{ even}\end{cases}$, and $\sum \beta_k t^k = \frac{2t}{\exp(2t)-1}$.

4.1.4. The function $\hat{L}_r(z)$ is smooth on $\mathbb{CP}^1 \setminus \{0,1,\infty\}$ and for $r \geq 2$ has week log singularity at $0,1,\infty$ of degree less than r; $\hat{L}_r(\infty) = 0$ for $r \geq 2$.

4.1.5. The differential equations for $\hat{L}_r(z)$ is given as follows :

$$d\hat{L}_2(z) = -\log|1-z| d^c \log|z| + \log|z| d^c \log|1-z| \tag{26}$$

$$\begin{aligned} d\,\hat{L}_r(z) \;&=\; \hat{L}_{r-1}(z)d^c \,\log|z| - \sum_{k=2}^{r-2} \beta_k \hat{L}_{r-k}(z) \log^{k-1}|z| d\, \log|z| \\ &-\; \beta_{r-1} \log^{r-2}|z| d^c \hat{L}_2(z), \text{ if } r > 2. \end{aligned} \tag{27}$$

4.2. The modified single-valued polylogarithms.

4.2.1. Definition. The *modified single-valued polylogarithms* are the following functions :

$$\tilde{L}_r = \frac{(2r-3)!!}{(2r-2)!!} \sum_{k|k \text{ even},\, 0\le k\le n-2} \frac{2^k(n-2)!(2n-k-3)!}{(2n-3)!(k+1)!(n-k-2)!} \hat{L}_{r-k}(z) \log^k|z|. \tag{28}$$

4.2.2. Remark. The function $\tilde{L}_r(z)$ is smooth on $\mathbb{CP}^1 \setminus \{0,1,\infty\}$ and for $r \geq 2$ has week log singularity at $0,1,\infty$ of degree less than r; $\hat{L}_r(\infty) = 0$ for $r \geq 2$.

4.2.3. Proposition. The differential equation for $\tilde{L}_r(z)$ is the following :

$$d\,\tilde{L}_r(z) = \frac{2r-3}{2r-2} \tilde{L}_{r-1}(z) d^c \,\log|z| - \frac{1}{2r-2} \log|z| d^c \,\tilde{L}_{r-1}(z). \tag{29}$$

4.2.4. Proof. Let us denote the coefficient $2^k \frac{(n-2)!(2n-k-3)!}{(2n-3)!(k+1)!(n-k-2)!}$ by $C_{k,n-k}$. A straightforward computation shows that the statement of the proposition is equivalent to the following equalities :

$$C_{k,n-k} = C_{k,n-k-1} + \frac{1}{(2n-3)(2n-5)} C_{k-2,n-k},$$

$$C_{n-2,2} = \frac{1}{(2n-3)(2n-5)} C_{k-4,2},$$

$$\sum_{j+l=k,\, l\geq 2} C_{j,n-j}\beta_l = C_{k,n} + \frac{2n-k-3}{2n-3} C_{k,n-k-1}.$$

The two first relations are evident, the last one can be proved by the following consideration. As $\beta_0 = 1$ and $\beta_1 = -1$, we can rewrite this equality as:

$$\sum_{j+l=2p,\, l\geq 0,\, j\leq n-2} 2^j \frac{(2n-j-3)!}{(j+1)!(n-j-2)!} \beta_l = 0, \quad 2p \leq n,$$

or

$$\sum_{j+l=2p,\, l\geq 0,\, j\leq n-2} 2^j \frac{(2n-2p-3+l)!}{(j+1)!(n-2p-2+l)!} \beta_l = 0, \quad 2p \leq n. \tag{30}$$

The expression $\frac{(2n-2p-3+l)!}{(n-2p-2+l)!}\beta_l$ is the $(n-2p-2+l)$-th coefficient of the power series of the function :

$$\partial^{n-1} \left(x^{2n-2p-3} \frac{2x}{\exp(2x)-1} \right)$$

($\partial = \frac{d}{dx}$), so the lhs of (30) equals the $(n-1)$-st coefficient of :

$$(\frac{1}{2}\exp(2x)) \times \partial^{n-1} \left(x^{2n-2p-3} \frac{2x}{\exp(2x)-1} \right),$$

the last expression is equal to the value at zero of :

$$\frac{1}{(n-1)!}\partial^{n-1}\left(\exp(2x)\partial^{n-1}\left(\frac{x^{2n-2p-2}}{\exp(2x)-1}\right)\right).$$

One has $\partial^k(\exp(2x)f(x)) = \sum\binom{k}{l}2^l\exp(2x)\partial^{k-l}f(x)$. Conversely, $\exp(2x)\partial^k f(x) = \sum\binom{k}{l}(-2)^l\partial^{k-l}(\exp(2x)f(x))$. So, the expression in the lhs of the last formula is equal to:

$$\frac{1}{2}\left(\sum_{l=0}^{n-1}\binom{n-1}{l}2^l\exp(2x)\partial^{2(n-1)-l}\frac{x^{2n-2p-2}}{\exp(2x)-1}\right.$$

$$\left.+\sum_{l=0}^{n-1}\binom{n-1}{l}(-2)^l\partial^{2(n-1)-l}\left(\exp(2x)\frac{x^{2n-2p-2}}{\exp(2x)-1}\right)\right)\Bigg|_{x=0} =$$

$$\frac{1}{2}\left(\sum_{l=0}^{n-1}\binom{n-1}{l}2^l\partial^{2(n-1)-l}\frac{x^{2n-2p-2}}{\exp(2x)-1}\right.$$

$$\left.+\sum_{l=0}^{n-1}\binom{n-1}{l}(-2)^l\partial^{2(n-1)-l}\left(\exp(2x)\frac{x^{2n-2p-2}}{\exp(2x)-1}\right)\right)\Bigg|_{x=0} =$$

$$\frac{1}{2}\left(\sum_{l=0, l\text{ even},}^{n-1}\binom{n-1}{l}2^l\partial^{2(n-1)-l}\frac{x^{2n-2p-2}(\exp(2x)+1)}{\exp(2x)-1}\right)\Bigg|_{x=0}$$

$$-\frac{1}{2}\left(\sum_{l=0, l\text{ even},}^{n-1}\binom{n-1}{l}2^l\partial^{2(n-1)-l}\frac{x^{2n-2p-2}(\exp(2x)-1)}{\exp(2x)-1}\right)\Bigg|_{x=0} = 0,$$

as all functions in this sum are odd. □

4.3. The polylogarithmic sheaf.

4.3.1. Consider the graded pro-space $\mathbb{R}[K]$ of the formal power series in one variable K, where $\deg K^l = -2l$. Denote by $\mathbb{J}$ the operator of multiplication by K.

4.3.2. The imaginary operator-valued function $G^{Log}(z) = G_1^{Log}(z) = \frac{\log|z|}{2\pi\sqrt{-1}}\mathbb{J}$ on $\mathbb{C}^*$ is a solution of the equations (2) and (3), so it defines some Hodge-Tate pro-sheaf.

We call it the *logarithmic* Hodge-Tate pro-sheaf.

We leave as an exercise for the reader to check that this definition gives the same object as in [W, IV.3].

4.3.3. Consider the graded pro-space $\mathbb{R}\oplus\mathbb{R}[K]$ and the following imaginary operator-valued function G^{Pol} on $\mathbb{C}^*\setminus 1$:

$$G^{Pol}(z)|_{\mathbb{R}[K]} = G^{Log}(z); \quad G^{Pol}(z)\colon 1 \mapsto \sum\left(\frac{K}{2\pi\sqrt{-1}}\right)^i\tilde{L}_i(z).$$

This operator determines some Hodge-Tate sheaf on $\mathbb{C}^*\setminus 1$. Indeed, condition (1) is equivalent to the differential equation (29), while (2) and (3) are evident.

4.3.4. Proposition. The Hodge-Tate sheaf determined by $G^{Pol}(z)$ is equal to the polylogarithmic sheaf [W, IV.3].

4.3.5. A proof follows immediately from the rigidity of the polylogarithmic sheaf [W, III.2].

4.4. The integral representation of the modified polylogarithms.

We present an integral representation of the modified single-valued polylogarithms which generalizes the integral representation for the Bloch-Wigner dilogarithm [G1].

4.4.1. Proposition. Let z be a complex number. Consider the following differential form $\nu_n(z)$ of top degree on $\mathbb{C}^n$:

$$\nu_0(z) = -\log|1-z|,$$

$$\nu_n(z) = -\log|1-t_1| \left(\bigwedge_{i=1}^{n-1} d\,\log|t_i| \wedge d\,\log|t_i - t_{i+1}| \right) \wedge d\,\log|t_n| \wedge d\,\log|t_n - z|.$$

This form has integrable singularities and its integral over $\mathbb{C}^n$ is equal to $(2\pi i)^{n-1}\tilde{L}_{n-1}$.

4.4.2. Proof. The singularities of ν are either $\log|x|\omega$ or $d\,\log|x| \wedge \omega$, where ω is smooth, so they are integrable. We calculate the value of the integral by induction on n. For $n = 0$ the statement is true : $-\log|1-z| = \tilde{L}_1(z)$.

Now assume that the statement is true for $n-1$. As $\nu_n(z) = \nu_{n-1}(t_n) \wedge d\,\log|t_n| \wedge d\,\log|t_n - z|$, after integration over the first $n-1$ variables the integral reduces to :

$$(2\pi i)^{n-1} \int_{\mathbb{C}} \tilde{L}_{n-1}(t_n) d\,\log|t_n| \wedge d\,\log|t_n - z|.$$

We can replace $\tilde{L}_{n-1}(t_n)d\,\log|t_n|$ by $\frac{2n-3}{2n-2}\tilde{L}_{n-1}(t_n)d\,\log|t_n| - \frac{1}{2n-2}\log|t_n| d\,\tilde{L}_{n-1}(t_n)$, since

$$\frac{1}{2n-2}(\log|t_n| d\,\tilde{L}_{n-1}(t_n) + \tilde{L}_{n-1}(t_n)d\,\log|t_n|) = \frac{1}{2n-2}(d\,\tilde{L}_{n-1}(t_n)\log|t_n|),$$

$\tilde{L}_{n-1}(t_n)\log|t_n|$ vanishes at ∞ and $d\,\log|t_n - z|$ is closed. Therefore, the integral is equal to

$$(2\pi i)^{n-1} \int_{\mathbb{C}} \left(\frac{2n-3}{2n-2}\tilde{L}_{n-1}(t_n)d\,\log|t_n| - \frac{1}{2n-2}\log|t_n| d\,\tilde{L}_{n-1}(t_n) \right) \wedge d\,\log|t_n - z|.$$

For any functions f and g on $\mathbb{C}$ one has $d\,f \wedge d\,g = -d^c\,f \wedge d^c\,g$, hence the integral can be rewritten as :

$$-(2\pi i)^{n-1} \int_{\mathbb{C}} \left(\frac{2n-3}{2n-2}\tilde{L}_{n-1}(t_n)d^c\,\log|t_n| - \frac{1}{2n-2}\log|t_n| d^c\,\tilde{L}_{n-1}(t_n) \right) \wedge d^c\,\log|t_n - z|.$$

The expression in parentheses is equal to $d\,\tilde{L}_n(t_n)$; so we get :

$$-(2\pi i)^n \int_{\mathbb{C}} d\,\tilde{L}_{n-1}(t_n) \wedge d^c\,\log|t_n - z| = (2\pi i)^{n-1} \int_{\mathbb{C}} \tilde{L}_n(t_n) \wedge d\,d^c\,\log|t_n - z|$$

$$= (2\pi i)^n \tilde{L}_n(z).$$

as $\tilde{L}_n(\infty) = 0$ and $d\,d^c\,\log|t_n - z| = 2\pi i \delta_{t_n = z}$. $\square$

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