GLOBAL ANALYSIS AND TEICHMÜLLER THEORY

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It is now approaching half a century since Oswald Teichmüller developed the first ideas of what is now called Teichmüller theory.

Teichmüller's work has been continued primarily through the efforts of Lars Ahlfors, Lipman Bers and their students. In this article we wish to review another new approach to this subject, one based on the ideas of Riemannian geometry and global non-linear analysis. Thus we shall not attempt to review all the achievements of this well known school but instead attempt to explain and outline the fundamentals of the subject to someone familiar with basic ideas in geometry and analysis, but not familiar with Teichmüllers theory. The present notes are based on lectures given at the Max-Planck-Institut für Mathematik, Bonn, April 1984, and is based on the joint research of the author and A.E. Fischer.

The paper is broken up as follows:

§1. THE BASIC PROBLEM

Let M be an oriented compact C^∞ surface without boundary. Such surfaces are classified by their genus, and we shall henceforth always assume that M has a fixed genus greater than one.

Surfaces of a fixed genus are all diffeomorphic. Therefore if one has a complex structure they all have a complex structure.

DEFINITION 1.1. A complex structure c for M is a coordinate atlas for M, $\{(\varphi_i, U_i)\}$, $\cup U_i = M$, such that when defined the transition mappings $\varphi_i \circ \varphi_i^{-1}$ are holomorphic.

Given one such complex structure and a C^∞ diffeomorphism $f : M \circlearrowleft$ we can produce a new complex structure $f^*c = \{(\varphi_i \circ f, f^{-1}(U_i))\}$.

Let (M,c) denote M with the given complex structure c . Then $f : (M,f^*c) \longrightarrow (M,c)$ is a holomorphic map. We want to identify these two complex structures. So let $\mathcal{C}$ be the set of all such structures and let $\mathcal{D}$ be the set of all C difeomorphisms. Then $\mathcal{D}$ acts on $\mathcal{C}$ by $c \rightsquigarrow f^*c$. Denote by $\mathcal{R}(M)$ the quotient space $\mathcal{C}/\mathcal{D}$. This is known as the Riemann space of moduli. Let $\mathcal{D}_0$ be those diffeomorphisms which are homotopic to the identity and denote by $\mathcal{T}(M)$ the quotient space $\mathcal{C}/\mathcal{D}_0$. This is the Teichmüller moduli space. Our main goal is to outline a proof that $\mathcal{T}(M)$ is a smooth finite dimensional manifold diffeomorphic to Euclidean space of dimension $6(\text{genus } M)-6$.

§2. THE SPACE OF ALMOST COMPLEX STRUCTURES

As we shall later see the space of almost complex structures $\mathcal{A}$ on M is in one to one correspondence with the space of complex structures $\mathcal{C}$. An almost complex structure J is a C^∞ 1:1 tensor; i.e. for each $x \in M$ there is a linear map $J_x : T_xM \circlearrowleft$ such that $J_x^2 = -id_x$, the identity map on the tangent space to M at x . Moreover we require that $x \longrightarrow J_x$ is C^∞ , and that for each vector $X_x \in T_xM$ (X_x, J_xX_x) forms an oriented basis for T_xM . The first theorem in this direction is

THEOREM 2.1. The space A is a "manifold" and its tangent space at $J \in A$, $T_J A$ can be identified with those 1:1 tensors $H\{H_x : T_xM \circlearrowleft$ is linear and $x \longrightarrow H_x$ is C^∞ } such that

$$H_x J_x = -J_x H_x$$

for all $x \in M$.

REMARK 2.2. The relation $HJ = -JH$ implies that each such H is trace free. To see this note that

$$-\operatorname{tr}(H) = \operatorname{tr}(JJH) = -\operatorname{tr}(JHJ) = \operatorname{tr}(H) \quad .$$

The group $\mathcal{D}$ and therefore $\mathcal{D}_0$ acts on A as follows. If $f \in \mathcal{D}$

$$(f^*J)_x = df_x^{-1} \cdot J_{f(x)} \cdot df_x \quad .$$

Clearly $(f^*J)^2 = -\mathrm{id}$ if $J^2 = -\mathrm{id}$.

The bijective correspondence between C and A is $\mathcal{D}$-equivariant so that if $c \rightsquigarrow J$ then $f^*c \rightsquigarrow f^*J$. Therefore this correspondence induces a bijective correspondence between $C/\mathcal{D}_0$ and $A/\mathcal{D}_0$. Thus we now restrict our attention to the study of the space $A/\mathcal{D}_0$.

§3. THE SPACE OF RIEMANNIAN METRICS M

Let S_2 be the space of C^∞ symmetric (0,2) tensors on M ; i.e $h \in S_2$ iff for each $x \in M$

$$h_x : T_xM \times T_xM \longrightarrow R$$

is symmetric bilinear and $x \longrightarrow h_x$ is C^∞. The space of C^∞-Riemannian metrics is the subset $M \subset S_2$ consisting of those symmetric tensors which for each $x \in M$ is positive definite; that is for $v_x \in T_xM$, $g \in M$ means $g_x(v_x, v_x) > 0$.

Metrics g can be multiplied by positive functions, so that if $\lambda : M \longrightarrow R^+$ is a C^∞ strictly positive function then $\lambda g \in M$ if $g \in M$. Thus the space P acts on M and we can form the quotient space M/P. Since the action on M is proper and free M/P is also a manifold, but what is its natural tangent space?

$M \subset S_2$ is open so that the tangent space to M at $g \in M$, $T_gM \cong S_2$. Moreover given a fixed $g \in M$ every $h \in S_2$ can be decomposed as

$$h = h^T + \rho g$$

where $\rho \in P$ and h^T is trace free with respect to g. Trace free means the following. We can use the metric g to "convert" h to a 1:1 tensor H by the rule

$$h_x(u_x, v_x) = g_x(H_x u_x, v_x) \quad .$$

Since g is positive definite such an H clearly exists. We then define the trace of h w.r.t. g by

$$(\mathrm{tr}_g h)_x = \mathrm{trace}\, H_x \quad .$$

Thus $x \longmapsto (\mathrm{tr}_g h)_x$ is a C^∞ function on M. D acts on M via the rule $g \longrightarrow f^*g$, where

$$(f^*g)(x)(u_x, v_x) = g(f(x))(df(x)u_x, df(x)v_x) \quad .$$

THEOREM 3.1. M/P is a mainfold and its tangent space at $[g]$ can be identified with those 0.2 tensors on M which are trace free.

§4. THE CORRESPONDENCE BETWEEN M/P AND A.

There is a natural map from M to A, namely for g and $x \in M$ let J_x be the map on T_xM which is "counterclockwise rotation" by 90° . However this formulation does not give the explicit dependence of J on g . J can be defined explicitly as follows.

For every metric g on M there is a uniquely defined antisymmetric two form $\mu_g(x) : T_xM \times T_xM \longrightarrow R$, called the <u>volume element of g</u> . Define J_x by relation

$$g(x)(J_x u_x, v_x) = -\mu_g(x)(u_x, v_x)$$

for $u_x, v_x \in T_xM$. One can easily check that the map $g \longmapsto J$, call it Φ, is smooth. Moreover if $\lambda \in \mathcal{P}$ is a positive function then $\Phi(\lambda g) = \Phi(g)$.

Thus Φ induces a map (which we also call Φ) on the quotient space $\mathcal{M}/\mathcal{P}$. The following result is not difficult to prove.

<u>THEOREM 4.1</u>. The map $\Phi : \mathcal{M}/\mathcal{P} \longrightarrow \mathcal{A}$ is a diffeomrophism.

§5. THE POINCARÉ METRIC ASSOCIATED TO AN ALMOST COMPLEX STRUCTURE

In the early part of this century Poincaré observed that if the genus of M is greater than one, then for every metric g there exists a unique positive function λ such that the Gauss (or scalar) curvature of M with respect to λg is constant -1 .

Recall that the Gauss curvature can be thought of as a function $R : \mathcal{M} \longrightarrow \mathcal{F}$, where $\mathcal{F}$ is the space of C^∞ functions on M . Thus, paraphrasing Poincarés result we know that given g there exists a unique λ so that $R(\lambda g) = -1$.

Let $\mathcal{M}_{-1}$ be all those metrics of constant curvature -1 .

<u>THEOREM 5.1</u>. $\mathcal{M}_{-1}$ is a manifold. Since $\mathcal{M}_{-1} = R^{-1}(-1)$ and -1 is a regular value for R , the tangent space to $\mathcal{M}_{-1}$ at g , $T_g\mathcal{M}_{-1}$ consists of all those $h \in S_2$ such that $DR(g)h = 0$ $\mathcal{D}$ acts on $\mathcal{M}_{-1}$, i.e. $g \in \mathcal{M}_{-1}$ implies $f^*g \in \mathcal{M}_{-1}$

Consider a metric g and its orbit $\mathcal{P}_g$ under the action of $\mathcal{P}$ on $\mathcal{M}$. Poincarés result implies that we can attach to each such orbit a unique metric which motivates the following

THEOREM 5.2. The manifolds $\mathcal{M}_{-1}$ and $\mathcal{M}/\mathcal{P}$, and hence also $\mathcal{M}_{-1}$ and $\mathcal{A}$ are diffeomorphic. Moreover the correspondence $\Theta : \mathcal{A} \longrightarrow$ between $\mathcal{A}$ and $\mathcal{M}_{-1}$ is $\mathcal{D}$-equivariant and hence establishes a bijection between $\mathcal{M}_{-1}/\mathcal{D}_0$ and $\mathcal{A}/\mathcal{D}_0$.

For each $J \in \mathcal{A}$ we shall denote by $g(J)$ the Poincaré metric associated to J.

§6. THE NATURAL L_2-METRIC ON THE SPACES $\mathcal{M}$, $\mathcal{M}_{-1}$ AND $\mathcal{A}$.

In this section we introduce a Riemannian structure on $\mathcal{M}$ and $\mathcal{A}$ (and hence by restriction to $\mathcal{M}_{-1}$) which have the property that the diffeomorphism group $\mathcal{D}$ acts as a group of isometries.

We begin with the metric on $\mathcal{A}$. Let $H,K \in T_J\mathcal{A}$. Then $HJ = -JH$ and similarly for K. This implies that H and K are symmetric w.r.t. $g(J)$. In fact the relation $HJ = -JH$ can be uniquely characterized by the two relations $\mathrm{tr}(H) = 0$ and H is symmetric w.r.t. $g(J)$. Our Riemannian structure $\langle\langle,\rangle\rangle : T_J\mathcal{A} \times T_J\mathcal{A} \longrightarrow \mathbb{R}$ is defined by

$$(1) \qquad \langle\langle H,K \rangle\rangle_J = \frac{1}{2} \int_M \mathrm{tr}\,(HK)\,d\mu_{g(J)}$$

where $g(J)$ is the Poincaré metric associated to J. An easy application of the change of variables formula implies that $\mathcal{D}$ acts as a group of isometries on $\mathcal{A}$ w.r.t. $\langle\langle,\rangle\rangle$.

We define the Riemannian structure on $\mathcal{M}$, also denoted by $\langle\langle,\rangle\rangle$ by

$$(2) \qquad \langle\langle h,k \rangle\rangle = \frac{1}{2} \int_M \mathrm{tr}\,(HK)\,d\mu_{g(J)}$$

where $h,k \in S_2 \cong T_g M$ and H (and similarly K) is defined again by the relation

$$g(x)(H_x u_x, v_x) = h_x(u_x, v_x) \quad .$$

REMARK. Let $\Theta : A \longrightarrow M_{-1}$ be the diffeomorphism given by theorem 5.2. Then it is not hard to see that Θ is not an isometry. We shall return to this point later when we discuss the Weil-Peterssen metric on $A/\mathcal{D}_0$.

§7. THE L_2-Splitting of $T_J A$

We have already observed that $\mathcal{D}$ acts on A. What is the tangent space to this action? Let $f_t, -\varepsilon < t < \varepsilon$, be a one parameter family of diffeomorphisms, $f_0 = \text{id}$, and $\left.\frac{df}{dt}\right|_{t=0} = \beta$ a vector field on M . A tangent vector to A at J is given by the derivative $\frac{d}{dt}\left\{ f_t J \right\}_{t=0}$. But this is a well known object in geometry, it is the Lie derivative of the $1:1$ tensor J with respect to the vector field β and is denoted by $L_\beta J$. In local coordinates this tensor is represented by the matrix

$$(L_\beta J)^i_j = \frac{\partial J^i_j}{\partial x^k} \beta^k + J^i_k \frac{\partial \beta^k}{\partial x^j} + J^j_k \frac{\partial \beta^k}{\partial x^i}$$

where here and throughout we adopt the Einstein convention of suming over repeated indices.

Thus tangent vectors to the orbits of $\mathcal{D}$ on A are given by Lie derivatives $L_\beta J$.

Teichmüller space $A/\mathcal{D}_0$ is the quotient space arising from the collapse of all the orbits of $\mathcal{D}_0$. Therefore the tangent space to Teichmüller space would "infinitesimally" be complimentary to $L_\beta J$. How can we define a natural complement? Well we can take a complement with respect to the L_2-metric we have introduced in §6. We then have the following result.

THEOREM 7.1. Every $H \in T_J A$ is trace free and can be decomposed uniquely and orthogonally as

(1) $$H = H^{TT} + L_\beta J$$

where H^{TT} is a trace free divergence free (with respect to the Poincaré metric $g(J)$). What does this mean? With respect to a given metric g one can take the divergence of a symmetric $1 : 1$ tensor, a $0-2$ tensor as well as that of a vector field. The divergence of a symmetric $1:1$ tensor T with respect to g is the 1-form $b_i dx^i$ where

(2) $$b_i = \left(\mathrm{div}_{g(J)} T\right)_i = \frac{1}{\sqrt{g}} \frac{\partial}{\partial x^j} \left(t_i^j \sqrt{g}\right) - \frac{1}{2} g^{kr} t_r^j \frac{\partial g_{jk}}{\partial x^i}$$

where t_i^j is the local expression for the tensor T, g_{jk} the local representation of the metric g, and $\sqrt{g} = \det g_{jk}$ and g^{kr} the inverse matrix to g_{ik}.

The trace free divergence free symmetric $1 : 1$ tensors are infinitesimally the tangent space to the quotient space $A/\mathcal{D}_0$. If $A/\mathcal{D}_0$ is a connected manifold, as indeed it is, it would follow that the dimension of the space of H^{TT}'s is constant.

Can one conclude this last fact from what we have already done?

§8. CONFORMAL COORDINATES AND THE INTERPRETATION OF TRACE FREE-DIVERGENCE FREE SYMMETRIC TENSORS IN TWO DIMENSIONS

THEOREM 8.1. (Existence of Conformal coordinates).

Let $g \in \mathcal{M}$ be any C^∞ metric on M. Then about each point $x \in M$ there exists a coordinate system $\{\varphi, U\}$ so that in this system the matrix representation of g is

$$g_{ij} = p\,\delta_{ij}$$

where $p : M \longrightarrow R^+$ is a strictly positive C^∞ function and δ_{ij} is Kronecker's δ. The pair $\{\varphi,U\}$ is called a conformal coordinate system about x.

This theorem permits us to prove the bijective relationship between complex structures $\mathcal{C}$ and almost complex structures $\mathcal{A}$. First note that by (8.1) we can cover M by orientation preserving condinate atlas $\{\varphi_i,U_i\}$, $\cup U_i = M$ so that each φ_i is a conformal coordinate system.

The transition maps $\varphi_i \circ \varphi_j^{-1}$ will then necessarily be local conformal maps of open subsets of $\mathbf{R}^2$ to $\mathbf{R}^2$ which preserve orientation and are thus holomorphic.

Therefore <u>a conformal coordinate atlas gives a complex structure</u>. So assume we are given a $J \in \mathcal{A}$. By theorem 4.1 J determines a conformal class of metrics P_g for some g. A conformal coordinate system for g will also be a conformal coordinate system for any element in the orbit space P_g. Therefore each J induces in this way a complex structure c, and thus we have a map $J \rightsquigarrow c$.

Conversely, suppose we have a complex structure $\{\varphi_i,U_i\}$ for M. Define $J_x : T_xM \circlearrowleft$ by

$$J_x = d\varphi_i \circ J \circ d\varphi_i^{-1}$$

where J_0 is the linear map on $\mathbf{R}^2$ whose matrix with respect to the standard orthogonal basis is $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$. Clearly $J_x^2 = -\mathrm{id}_x$. The fact that the transition maps $\varphi_i \circ \varphi_j^{-1}$ are holomorphic implies that J_x is independent of this choice of φ_i. The correspondence $c \rightsquigarrow J$ is readily seen to be the inverse of $J \mapsto c$.

Conformal coordinates are very pleasant since the metric tensor is so simple in such coordinates, and other tensors determined by the metric tensor, like the divergence of a symmetric 1:1 tensor assume a particularly simple form.

First observe that in conformal coordinates, $g_{ij} = p\delta_{ij}$, formula (2) of §7 reduces to

(1) $$\left(\operatorname{div}_{g(J)} T\right)_i = \frac{1}{\rho} \frac{\partial}{\partial x^j} \left(p\, t_i^j\right)$$

in the case T is trace free.

Recall the isomorphism between 1:1 tensors and 0-2 tensors induced by a metric g. Let s_{ij} be the local representation in conformal coordinate for the 0-2 tensor S corresponding to the 1 : 1 tensor T. From the formula $g(x)(T_x u_x, v_x) = S(x)(u_x, v_x)$, we see that $s_{ij} = p\, t_i^j$ and thus $\operatorname{div}_{g(J)} T = 0$ implies

(2) $$\frac{1}{p} \frac{\partial}{\partial x^j} (s_{ij}) = 0 \quad .$$

But $\{s_{ij}\}$ is also trace free. Write S in the matrix form

$$S = \begin{pmatrix} s_{11} & s_{12} \\ s_{21} & s_{22} \end{pmatrix} = \begin{pmatrix} u & v \\ v & -u \end{pmatrix}$$ or in the classical form

$$S = u\, dx^2 - u\, dy^2 + 2\, v\, dxdy \quad *$$

where we represent the coordinates (x^1, x^2) by (x, y).

So what does (2) imply about u and v ? With this new notation (2) can be written as:

$$\frac{1}{p}\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) = 0$$

and

$$\frac{1}{p}\left(\frac{\partial u}{\partial y} - \frac{\partial v}{\partial x}\right) = 0$$

or $$\frac{\partial u}{\partial x} = -\frac{\partial v}{\partial y} \quad \text{and} \quad \frac{\partial u}{\partial y} = \frac{\partial v}{\partial x} \quad .$$

But these are the Cauchy Riemann equations for the pair $(u, -v)$ and consequently $u - iv$, $i = \sqrt{-1}$ is holomorphic. Since a conformal coordinate system is also a holomorphic

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* In conformal coordinates trace free actually means that the corresponding matrix has zero trace.

coordinate system the holomorphicity of $u - iv$ is well defined.

Let us write S as

$$S = \mathrm{Re}\{(u-iv)(dx + idy)^2\} = \mathrm{Re}\{\xi(z)dz^2\}$$

S is therefore the <u>real part</u> of a complex valued $0:2$ tensor whose coefficient in complex coordinates is holomorphic. Such an object is called a <u>holomorphic quadratic differential</u>.

This correspondence between trace free-divergence free $0:2$ tensors (and thus trace free-divergence free $1:1$ tensors) and holomorphic quadratic differentials is bijective.

The next result is an immediate consequence of the celebrated theorem of Riemann-Roch.

<u>THEOREM 8.2</u>. The dimension of the space of holomorphic quadratic differentials on a complex one-manifold of genus greater than 1 has (real) dimension $6(\text{genus } M)-6$.

We may therefore conclude that the dimension of the space of H^{TT}'s, the candidate for the tangent space to $A/\mathcal{D}_0$ always has the same fixed dimension $6(\text{genus } M)-6$, a fact which prepares us to discuss the manifold structure on $A/\mathcal{D}_0$.

§9. <u>$A/\mathcal{D}_0$ IS A C^∞ MANIFOLD</u>

Let $\Theta : A \longrightarrow M_{-1}$ be the diffeomorphism introduced in §6. The next theorem describes the image of the subspace of H^{TT} of $T_J A$ under the derivative map $D\Theta$.

<u>THEOREM 9.1</u>. The derivative map $D\Theta_J : T_J A \longrightarrow T_{\Theta(J)} M_{-1}$ maps the subspace of trace free-divergence free symmetric tensors to the space of symmetric tensors h which are representable in local conformal coordinates as

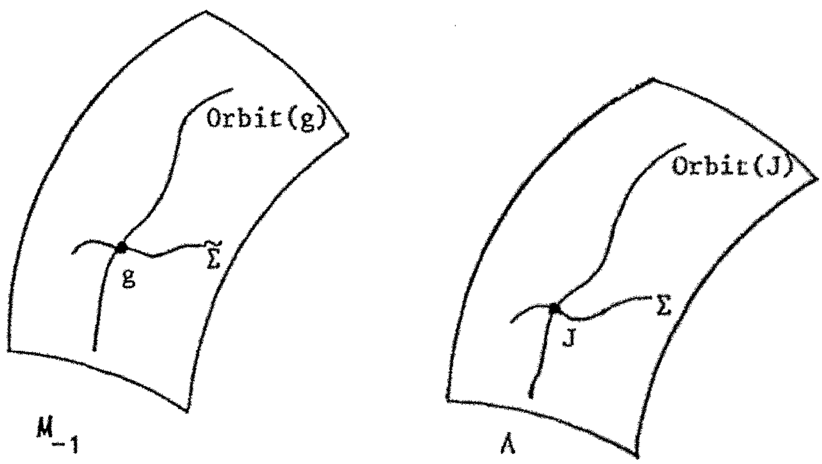
$$h = \mathrm{Re}(\xi(z)dz^2)$$

where $\xi(z)$ is holomorphic.

This space h is precisely the space of symmetric $0:2$ tensors which are trace free and divergence free w.r.t. $\Theta(J)$.

Let us denote this subspace of S_2 by $S_2^{TT}(g)$. As a consequence of (9.1) we know that $S_2^{TT}(g) \subset T_g M_{-1}$.

We will now construct a local diffeomorphism from $S_2^{TT}(g)$ to M_{-1} . The image of this diffeomorphism will be our coordinate chart for Teichmüller space $A/\mathcal{D}_0$ (see figure below)



The diffeomorphism is given by Poincarés result discussed in §5. The curvature $R(g)$ of g is -1 . Consider the family of $0:2$ tensors $g+h^{TT}$, $h^{TT} \in S_2^{TT}(g)$. For h^{TT} small enough these will also be Riemannian metrics, the curvature of $g+h^{TT}$ will not be -1 . However by Poincarés theorem we may find a unique positive function $\lambda = \lambda(h^{TT})$, so that the $0:2$ tensor $\lambda(g+h^{TT})$ has curvature -1 . The map

$$\Omega : h^{TT} \longrightarrow \lambda(g+h^{TT})$$

is C^∞ smooth and a simple computation shows that $D\Omega(g)h^{TT} = h^{TT}$. Thus a neighborhood of $S_2^{TT}(g)$ is mapped onto a submanifold $\tilde{\Sigma}$ of M_{-1}. We call $\tilde{\Sigma}$ a slice for the action of $\mathcal{D}$. This slice is our candidate for a local representation of Teichmüller space.

First let us again note that $\mathcal{D}$ acts on M_{-1}, because if $R(g) \equiv -1$, $R(f^*g) = f^*R(g) = R(g) \circ f \equiv -1$. A well known lemma due to Ebin-Palais asserts that this action (on M as well as on M_{-1}) is proper.

THEOREM 9.2. (Ebin-Palais). Suppose $f_n^* g_n \longrightarrow \hat{g}$, and $g_n \longrightarrow g$. Then there exists a subsequence of $\{f_n\}$ which converges.

Why is this important? By the implicit function theorem we know that every orbit of $\mathcal{D}$ in a neighborhood of g intersects $\tilde{\Sigma}$. However each point of $\tilde{\Sigma}$ may not correspond to a unique orbit, i.e. two points could (and in some cases do) correspond to the same orbit. This is the main distinction between the $\mathcal{D}$ and $\mathcal{D}_0$ action. A classic result by Bochner on surfaces implies that the $\mathcal{D}_0$-action on M_{-1} (but not the $\mathcal{D}$-action) is free, that is has no fixed points.

THEOREM 9.3. (Bochner) Suppose $g \in M_{-1}$ and $f^*g = g$. This says that f is an isometry of (M,g). If $f \in \mathcal{D}_0$ then f must be the identity.

We can combine theorems 9.2 and 9.3 to conclude

THEOREM 9.4. For every $g \in M_{-1}$ there is a neighborhood U of g so that every point on a slice $\tilde{\Sigma} \subset U$ corresponds to a unique orbit of $\mathcal{D}_0$.

To prove 9.4 we assume the negation and use 9.2 to obtain an immediate contradiction to 9.3.

From 9.4 and some additional calculus we can summarize our results by

THEOREM 9.5. The quotient space $M_{-1}/\mathcal{D}_0$ is a C^∞ smooth manifold. The tangent space to this manifold at $[g] \in M_{-1}/\mathcal{D}_0$ consists of all symmetric 0:2 tensors which are trace free and divergent free.

Using the complex structure induced by $\Theta^{-1}[g]$ this space can be interpreted as all symmetric 0:2 tensors which are the real parts of holomorphic quadratic differentials on M with this complex structure.

Using the $\mathcal{D}$-equivariant diffeomophism $\Theta : A \longrightarrow M_{-1}$ and the fact that $D\Theta$ takes trace free-divergence free 1:1 tensors to trace free-divergence free 0:2 tensors isomorphically we obtain our first main result

THEOREM 9.6. The space $A/\mathcal{D}_0$ carries the structure of a C^∞ smooth manifold of dimension $6(\text{genus}\, M)-6$. The tangent space to $A/\mathcal{D}_0$ at $[J]$ can be identified with those 1:1 tensors which are divergent free and trace free w.r.t. $\Theta[J]$. Finally the induced map $\Theta : A/\mathcal{D}_0 \longrightarrow M_{-1}/\mathcal{D}_0$ is a diffeomorphism.

§10. TEICHMÜLLER SPACE IS A CELL

In this section we outline the proof that Teichmüller space is diffeomorphic to $\mathbf{R}^{6p-6}$, $p = \text{genus}\, M$.

To prove this it suffices to show that $M_{-1}/\mathcal{D}_0$ is diffeomorphic to $\mathbf{R}^{6p-6}$.

Let $g_0 \in M_{-1}$ and $[g_0]$ denote its class in $M_{-1}/\mathcal{D}_0$. This fixed g_0 will act as our base point. Let $g \in M_{-1}$ be any other metric and let $s : M \longrightarrow M$ be viewed as a map from (M,g) to (M,g_0). Using the metrics g and g_0 one defines Dirichlet's energy functional

$$(1) \qquad E_g(s) = \frac{1}{2}\int_M |ds|^2 d\mu(g)$$

where $|ds|^2 = \text{trace}_g\, ds^* ds$ depends on both metrics g and g_0 and again $d\mu(g)$ is the volume element induced by g.

We may assume that (M,g_0) is isometrically embedded in some Euclidean $\mathbf{R}^k$, which is possible by the Nash-Moser embedding theorem. Thus we can think of $s : (M,g) \longrightarrow (M,g_0)$

as a map into $\mathbf{R}^k$ with Dirichlet's integral having the equivalent form

(2) $$E_g(s) = \frac{1}{2} \sum_{i=1}^{k} \int g(x) < \nabla_g s^i(x), \nabla_g s^i(x) > d\mu(g) \quad .$$

For fixed g, the critical points of E are then said to be harmonic maps.

We then have the following result.

THEOREM 10.1. Given metrics g and g_0 there exists a unique harmonic map $s(g) : (M,g) \longrightarrow (M,g_0)$ which is homotopic to the identity. Moreover $s(g)$ depends differentiably on g in any H^r topology $r>2$ and $s(g)$ is a C^∞ diffeomorphism. Consider the function

$$g \longrightarrow E_g(s(g)) \quad .$$

This function on $\mathcal{M}_{-1}$ is $\mathcal{D}$-invariant and thus can be viewed as a function on Teichmüller space. To see this one must show that $E_{f^*g}(s(f^*(g))) = E_g(s(g))$. Let $c(g)$ be the complex structure associated to g given by theorem 10.1. For $f \in \mathcal{D}_0$, $f : (M, f^*c(g)) \longrightarrow (M, c(g))$ is a holomorphic map and consequently since the composition of harmonic maps and holomorphic maps is still harmonic we may conclude, by uniqueness, that

$$s(f^*g) = s(g) \circ f \quad .$$

Since Dirichlet's functional is invariant under complex holomorphic changes of coordinates it follows immediately that

$$E_{f^*(g)}(s(g) \circ f) = E_g(s(g)) \quad .$$

Consequently for $[g] \in \mathcal{M}_{-1}/\mathcal{D}_0$ define the C^∞ smooth function $\tilde{E} : \mathcal{M}_{-1}/\mathcal{D}_0 \longrightarrow R$ by

$$\tilde{E}([g]) = E_g(s(g)) \quad .$$

We wish now to outline the main theorem of this section:

<u>THEOREM 10.2</u>. Teichmüller space $M_{-1}/\mathcal{D}_0$ is C^∞ diffeomorphic to $\mathbf{R}^{6p-6}$.

To prove this result it suffices to show that $\tilde{E}$ has the following properties

(i) The inverse image of bounded sets in $\mathbf{R}$ under $\tilde{E}$ is compact in $M_{-1}/\mathcal{D}_0$

(ii) $[g_0]$ is the only critical point of $\tilde{E}$

(iii) $[g_0]$ is a non-degenerate minimum.

Once (i) through (iii) are established the result follows immediately from the application of the well known gradient deformations of Morse theory.

In the interest of space we omit a sketch of a proof of (i).

To show (ii), again let $s = s(g) : (M,g) \longrightarrow (M,g_0)$ be the unique harmonic maps. Let $N_g(z)dz^2$ be the quadratic differential defined by

$$N_g(z)dz^2 = \sum_{i=1}^{k} \frac{\partial s^i}{\partial z} \cdot \frac{\partial s^i}{\partial z}\, dz^2$$

where s^i is the i^{th} component function of $s : (M,g) \longrightarrow (M,g_0) \hookrightarrow \mathbf{R}^k$ and $z = x + iy$ are local conformal coordinates on (M,g) . We next prove

<u>THEOREM 10.3</u>. $N_g(z)dz^2$ is a holomorphic quadratic differential on $(M,c(g))$.

<u>PROOF</u>. Let Ω denote the second fundamental form of $(M,g_0) \subset \mathbf{R}^k$. Thus for each $p \in M$, $\Omega(p) : T_pM \times T_pM \longrightarrow T_pM^\perp$. Let Δ denote the Laplacian of maps from (M,g) to (M,g_0) and Δ_β denote the Laplace-Beltrami operator on functions. Then if s is harmonic we have

(3) $$0 = \Delta s = \Delta_\beta s + \sum_{j=1}^{2} \Omega(s)(ds(e_j),ds(e_j))$$

$e_1(p), e_2(p)$ an orthonormal basis for T_pM (w.r.t. the metric g). N_g will be holomorphic if

$$\frac{\partial}{\partial \bar{z}} \left(\sum_{i=1}^{k} \frac{\partial s^i}{\partial z} \cdot \frac{\partial s^i}{\partial z} \right) = 0 \quad .$$

But this is equal to

$$2 \sum_{i=1}^{k} \Delta_\beta s^i \cdot \frac{\partial s^i}{\partial z}$$

and by (3) we see that this in turn equals

$$- 2 \sum_{i=1}^{k} \sum_{j=1}^{2} \Omega^i(s)(ds(e_j), ds(e_j)) \cdot \frac{\partial s^i}{\partial z}$$

$$= - 2 \sum_{j=1}^{2} \left\{ \sum \Omega(s)(ds(e_j), ds(e_j) \cdot \frac{\partial s}{\partial x} + i\Omega(s)(ds(e_j), ds(e_j) \cdot \frac{\partial s}{\partial y} \right\}$$

Since $\Omega(p)$ takes values in $T_pM^\perp$ it follows that both the real and imaginary parts of this expression vanish.

∎

We have already seen that

$$\xi = \mathrm{Re}(N_g(z)dz^2)$$

is a trace free divergence free symmetric two tensor on (M,g). Let $\rho \in T_{[g]}M_{-1}/\mathcal{D}_0$. We know that we may think of ρ as a trace free divergence free symmetric two tensor. A simple calculation gives the following result:

<u>THEOREM 10.4</u>.

$$D\tilde{E}([g])\rho = - \langle\langle \xi, \rho \rangle\rangle_g$$

where $\langle\langle , \rangle\rangle$ is the Riemannien structure induced on M_{-1} introduced in §6. Thus $[g]$ is a critical point of $\tilde{E}$ iff $\xi = 0 = \mathrm{Re}(N_g(z)dz^2)$, of iff $N_g(z)dz^2 \equiv 0$.

THEOREM 10.5. $N_g(z)dz^2 = 0$ implies that $[g] = [g_0]$.

PROOF. $N_g(z)dz^2 = \{|s_x|^2 - |s_y|^2 + 2i < s_x, s_y >\} dz^2$.

Thus $N_g(z)dz^2$ implies that s is weakly conformal. Since s is a diffeomorphism it is conformal. Thus $s : (M, c(g)) \longrightarrow (M, c(g_0))$ is holomorphic, and hence $[g] = [g_0]$.

It remains to show (iii) . It is clear that since $N_{g_0}(z)dz^2 \equiv 0$ $(s(g_0) = id)$ that $[g_0]$ is a critical point.

Let $\rho, \upsilon \in T_{[g_0]}\mathcal{M}_{-1}/\mathcal{D}_0$ be trace free, and divergence free symmetric two tensors. Then a straightforward computation yields

THEOREM 10.6. The second derivative or Hessian of $\tilde{E}$ at $[g_0]$ is given by the formula

$$D^2\tilde{E}([g_0])(\rho, \upsilon) = 2 << \rho, \upsilon >>_{g_0} .$$

Thus the Hessian of $\tilde{E}$ at $[g_0]$ is essentially the natural inner product on $T_{[g_0]}\mathcal{M}_{-1}/\mathcal{D}_0$ and hence a positive definite quadratic form. This finishes the proof of our main result 10.2.

§11. THE COMPLEX STRUCTURE ON TEICHMÜLLER SPACE

Teichmüller space $\mathcal{A}/\mathcal{D}_0$ is even dimensional and it is a natural question to ask whether or not it has a natural complex structure.

To start with it would be simpler to first ask whether or not it has an almost complex structure and then second whether or not this almost complex structure is integrable, that is, comes from a complex structure.

We can attempt to simplify matters even further. Since Teichmüller space is a quotient one can ask if the space $\mathcal{A}$ of almost complex structures $\mathcal{A}$ has a natural almost complex structure.

THEOREM 11.1. The space $\mathcal{A}$ of almost complex structures has itself a natural almost complex structure Φ, where

$$\Phi_J : T_J\mathcal{A} \circlearrowleft$$

is defined by

$$\Phi_J(H) = JH \quad .$$

Since $J^2 = -\text{id}$, $\Phi_J^2 = -I$, $I : T_J\mathcal{A} \circlearrowleft$ the identity map. An easy computation shows that Φ is $\mathcal{D}$-invariant.

Let N be a finite dimensional manifold with an almost complex structure J. Let $\mathfrak{X}(N)$ denote the vector fields on N. The obstruction to the integrability of J is given by the Nijenhius tensor $N(J)$, where

$$N(J) : \mathfrak{X}(N) \times \mathfrak{X}(N) \longrightarrow \mathfrak{X}(N)$$

is bilinear and defined by

(1) $$N(J)(\beta,\gamma) = 2\{[J\beta,J\gamma] - J[\beta,J\gamma] - J[J\beta,\gamma] - [\beta,\gamma]\}$$

where $[\ ,\ ]$ denotes the Lie bracket of vector fields. Formula (1) can be rewritten as

(2) $$N(J)(\beta,\gamma) = 2\{(L_{J\beta}J - JL_{\beta}J)\gamma\} \quad .$$

The following is the theorem of Newlander-Nirenberg.

THEOREM 11.2. Let N be a finite dimensional manifold with an almost complex structure J. Then J is integrable if and only if $N(J) \equiv 0$.

In the case that the dimension of N is two it follows that $N(J) \equiv 0$ and thus almost complex structures all arise from complex structures as we already knew. However in this case $(\dim N = 2)$ formula (2) has an interesting interpretation.

Recall that on A the tangent space to the orbit of $\mathcal{D}$ through J consists of 1 : 1 tensors of the form $L_\beta J$ for some vector field β on M .

Then $N(J) \equiv 0$ implies that the almost complex structure Φ "infinitesimally" leaves orbits invariant, that is

$$\Phi_J(L_\beta J) = J \cdot L_\beta J = L_{J\beta} J \quad .$$

Formula (2) can be paraphrased in another very useful way. We can view the triple $(\pi, A, A/\mathcal{D}_0)$ as a principal $\mathcal{D}_0$ fibre bundle, π the quotient map $\pi : A \longrightarrow A/\mathcal{D}_0$. That this triple carries the structure of a C° bundle is a result originally due to Eells and Earle. However the bundle is also a C^∞ ILH principal bundle. At a point $J \in A$ we can define the <u>vertical subspace</u> $V(J)$ of $T_J A$ by

$$V(J) = \operatorname{Ker} D\pi(J)$$

where the derivative $D\pi(J) : T_J A \longrightarrow T_{\pi(J)} A/\mathcal{D}_0$.

Clearly $V(J)$ coincides with the tangent space of the orbits of $\mathcal{D}$, and hence <u>in the case</u> $N(J) \equiv 0$ <u>formula (2) implies that the induced almost complex structure</u> Φ <u>on preserves vertical subspaces</u>.

From this and the fact that Φ is $\mathcal{D}$-invariant it follows that <u>the almost complex structure</u> Φ <u>on</u> A <u>induces an almost complex structure</u> $\hat{\Phi}$ <u>on</u> $A/\mathcal{D}_0$. <u>Moreover</u> $N(\Phi) \equiv 0$ <u>implies</u> $N(\hat{\Phi}) \equiv 0$.

Thus to check that Φ is integrable it suffices to show that $N(\Phi) = 0$ and this is an easy compulation which establishes

THEOREM 11.3. Teichmüller space is a complex manifold.

§12. THE WEIL-PETERSSEN METRIC

In §6 we introduced the L_2-Riemannian structure $\langle\langle , \rangle\rangle$ on A given by

$$\langle\langle H,K\rangle\rangle_J = \frac{1}{2}\int_M \mathrm{tr}(HK)\,d\mu_{g(J)} \quad .$$

Since the group $\mathcal{D}$ acts on A as a group of isometries w.r.t. the structure $\langle\langle , \rangle\rangle$, this structure then induces a Riemannian structure $\langle , \rangle$ on the quotient space $A/\mathcal{D}_0$. This is called the Weil-Peterssen Riemannian Structure or Weil-Peterssen metric on $A/\mathcal{D}_0$.

In the next section we will show how to determine the curvature of this metric. However, we shall concern ourselves here with the outline of the proof that the metric is Kähler, a result originally due to Ahlfors.

Consider again the principal bundle $(\pi, A, A/\mathcal{D}_0)$. The map π as a map of Riemannian manifolds is a Riemannian submersion. Let $T = A/\mathcal{D}_0$. Define the Kähler two form

$$\Omega_{[J]} : T_{[J]}T \times T_{[J]}T \longrightarrow R$$

by

$$\Omega_{[J]}(X_{[J]}, Y_{[J]}) = \langle \hat{\Phi}_{[J]} X_{[J]}, Y_{[J]} \rangle$$

where $\hat{\Phi}_{[J]} : T_{[J]}T \circlearrowleft$ is the almost complex structure on T introduced in the last section.

The metric $\langle , \rangle$ is Kähler if $\Omega\ TT \times TT \longrightarrow R$ is closed, that is if $d\Omega = 0$.

Our main tool to show this will again be the exploitation of the principal bundle structure $(\pi, A, A/\mathcal{D}_0)$. The Kähler form Ω is related to a Kähler form Ω_A on the principal bundle A defined by

(1) $$\Omega_A(Z_J,W_J) = \langle\langle \Phi_J Z_J, W_J \rangle\rangle = \frac{1}{2}\int_M \operatorname{tr}(JZ_J W_J)\, d\mu_{g(J)}$$

when $Z_J, W_J \in T_J A$ and $\Phi_J : T_J A \circlearrowleft$ is the almost complex structure $W \longrightarrow JW$.

Vector fiels Z on A which are everywhere perpendicular to the orbits of $\mathcal{D}$ are called horizontal fields (those which are tangent are the vertical fields). Thus Z is horizontal if for all $J \in A$, $Z(J)$ is a trace free divergence free symmetric (w.r.t. $g(J)$) $1:1$ tensor on the surface M .

If X is a vector field on the quotient $A/\mathcal{D}_0$ then there is a unique horizontal vector field $\tilde{X}$ on A such that $D\pi(\tilde{X}) = X \circ \pi$. $\tilde{X}$ is called the horizontal lift of X . The following straight forward calculation shows how we can determine whether or not Ω is Kähler by working on the bundle A , rather then on the quotient $A/\mathcal{D}_0$.

THEOREM 12.1. Let X,Y,Z be vector fields on $A/\mathcal{D}_0$ and $\tilde{X},\tilde{Y},\tilde{Z}$ be their unique horizontal lifts. Then

$$d\Omega(X,Y,Z) = d\Omega_A(\tilde{X},\tilde{Y},\tilde{Z}) \quad .$$

Thus if $d\Omega_A$ vanishes on horizontal fields it follows that Ω is Kähler.

The next result shows that $d\Omega_A$ evaluated on horizontal fields is indeed simple.

THEOREM 12.2. The differential of the map $J \longrightarrow \mu_{g(J)}$ vanishes on horizontal fields.

PROOF. The derivative of $g \rightsquigarrow \mu_g$ is the map $h \rightsquigarrow \frac{1}{2}(\operatorname{tr}_g h)\mu_g$ where $\operatorname{tr}_g h$ is the trace of h w.r.t. g. On the other hand the derivative of the map $J \longrightarrow g(J)$ takes trace free divergence free $1:1$ tensor H to trace free divergence free two tensors h . The result then follows from the chain rule.

Now let us consider formula (1) for Ω_A . Ω_A is biliniar in Z and W and the non-linearity of Ω_A (in the variable J) comes only from the term $J \longrightarrow \mu_{g(J)}$.

The formula for $d\Omega_A$ is given by

$$(2) \qquad 3 \cdot d\Omega_A(\tilde{X},\tilde{Y},\tilde{Z}) = \tilde{X}(\Omega_A(\tilde{Y},\tilde{Z})) + \tilde{Y}(\Omega_A(\tilde{Z},\tilde{X})) + \tilde{Z}(\Omega_A(\tilde{X},\tilde{Y})) - \Omega_A([\tilde{X},\tilde{Y}],\tilde{Z}) - \Omega_A([\tilde{Y},\tilde{Z}],\tilde{X}) - \Omega_A([\tilde{Z},\tilde{X}],\tilde{Y})$$

where [,] denotes the Lie bracket of vector fields. If $\tilde{X},\tilde{Y},\tilde{Z}$ are horizontal the first three terms are easily calculated. For example

$$\tilde{X}(\Omega_A(\tilde{Y},\tilde{Z})) = \Omega_A(D\tilde{Y}(\tilde{X}),\tilde{Z}) + \Omega_A(\tilde{Y},D\tilde{Z}(\tilde{X})) \quad .$$

Collecting terms if follows immediately that $d\Omega_A$ vanishes on horizontal fields and we have proved

<u>THEOREM 12.3</u>. Teichmüller space is a complex Kähler manifold with respect to the Kähler form induced by the Weil-Peterssen metric.

§13. <u>ON THE CURVATURE OF THE WEIL-PETERSSEN METRIC</u>

Some time ago Ahlfors showed that the holomorphic sectional curvature and the Ricci curvature of the Weil-Peterssen metric is negative. However the question of obtaining an exact formula for this curvature remained open for some time. In this section we show how the methods of the previous sections enables one to compute this curvature. We define a natural symmetric connection ∇ on A by

$$(1) \qquad \nabla_Y X = DX(Y) - \frac{1}{2} J\{XY + YX\}$$

where D denotes derivative and where X and Y are vector fields on A. One can easily show that $\nabla_Y X \in T_J A$ if X, Y $T_J A$. To see this one differentiates the relation $XJ = -JX$ in the direction Y obtaining the relation

$$JDX(Y) + YX = -XY - DX(Y)J \quad .$$

Then

$$J \cdot \nabla_Y X = JDX(Y) + \frac{1}{2}(XY + YX) = -\frac{1}{2}(XY + YX) - DX(Y)J = -\nabla_Y X \cdot J \quad .$$

The computation of the curvature will involve a study of the properties of the bundle A, the map $J \longmapsto g(J)$ and the connection ∇.

The next result follows immediately from the definition of the Levi Civita connection.

<u>THEOREM 13.1</u>. If $\tilde{\nabla}$ denotes the Levi-Civita connection of $\langle\langle , \rangle\rangle$ then the horizontal components of $\nabla_Y X$ and $\tilde{\nabla}_Y X$ agree if X and Y are horizontal.

We know that the Levi Civita connection $\tilde{\nabla}$ is characterized uniquely by the relations

$$(2) \qquad X\langle\langle V, W\rangle\rangle = \langle\langle\tilde{\nabla}_X V, W\rangle\rangle + \langle\langle V, \tilde{\nabla}_X W\rangle\rangle$$

and

$$\tilde{\nabla}_V W - \tilde{\nabla}_W V = [V, W]$$

where [,] denotes the Lie bracket of vector fields. A trivial calculation shows that ∇ satisfies these relations if X, V, and W are horizontal.

There is another way one can view this connection. If X

and Y are vector fields on $\mathcal{A}$, then for each $J \in \mathcal{A}$, $X(J)$ and $Y(J)$ are trace free 1:1 tensors on M which are symmetric with respect to $g(J)$. Then $DY(X)(J)$ will be trace free but not symmetric. Define the projection map π by

$$\pi(Z) = \frac{1}{2}(Z + Z^*)$$

where * denotes the adjoint of the 1:1 tensor Z with respect to $g(J)$.

Then one easily checks that

$$\nabla_X Y = DY(X) - D\pi(X)[Y] = \pi DY(X) \quad . \tag{3}$$

The curvature tensor $R(X,Y)Z$ of ∇ is defined by

$$\{\nabla_X\nabla_Y - \nabla_Y\nabla_X - \nabla_{[X,Y]}\}Z = R(X,Y)Z \quad . \tag{4}$$

Now

$$\nabla_X\nabla_Y Z = D_Y\{DZ(Y) - \frac{1}{2}\{JZY + JYZ\}\} - \frac{1}{2} J \{X[DZ(Y) - \frac{1}{2}(JZY + JYZ)] +$$
$$+ [DZ(Y) - \frac{1}{2}(JZY + JYZ)]X\}$$
$$= D^2Z(X,Y) + DZDY(X) - \frac{1}{2} XZY - \frac{1}{2} JDZ(X)Y - \frac{1}{2} JZDY(X) -$$
$$- \frac{1}{2} XYZ - \frac{1}{2} JDY(X)Z - \frac{1}{2} JYDZ(X) - \frac{1}{2} JXDZ(Y) - \frac{1}{4} XZY -$$
$$- \frac{1}{4} XYZ - \frac{1}{2} DZ(Y)X - \frac{1}{4} ZYX - \frac{1}{4} YZX$$

$\nabla_Y\nabla_X Z$ is obtained by interchanging X and Y in the last computation. By (4) and the previous computation we see that

$$R(X,Y)Z = - ZYX + ZXY \quad .$$

Therefore

$$\langle\langle R(X,Y)Y,X \rangle\rangle_J = \frac{1}{2} \int_M \text{trace} \quad \{-Y^2X^2 + YXYX\} d\mu_{g(J)} \tag{5}$$

Thus for fixed $J \in T_J A$ let $X(J), Y(J) \in T_J A$. Furthermore for each $x \in M$ let us denote the matrices of $X(J)_x$ and $Y(J)_x$ by $\begin{pmatrix} c & d \\ d & -c \end{pmatrix}$ and $\begin{pmatrix} a & b \\ b & -a \end{pmatrix}$. Then

$$\operatorname{trace}\{-Y^2X^2 + YXYX\} = -2\{ad - bc\}^2 < 0$$

for linearly independent X and Y, and this holds whether or not X or Y is horizontal.

Let K denote the curvature of Teichmüller space $\mathcal{T} = \mathcal{A}/\mathcal{D}_0$ with respect to its Weil-Peterssen metric. If X and Y now denote vector fields on $\mathcal{T}$ <u>let $\bar{X}$ and $\bar{Y}$ denote the unique horizontal lifts with respect to the L_2-metric</u>. Then

$$(6) \qquad K(X,Y) = \langle\langle \tilde{\nabla}_{\bar{X}}(\tilde{\nabla}_{\bar{Y}}\bar{Y})^H - \tilde{\nabla}_{\bar{Y}}(\tilde{\nabla}_{\bar{X}}\bar{Y})^H - \tilde{\nabla}_{[\bar{X},\bar{Y}]^H}\bar{Y}, \bar{X} \rangle\rangle$$

where the supercripts H and V will denote horizontal and vertical component respectively.

Since $\left(\tilde{\nabla}_{\bar{X}}\bar{Y}\right)^H = \left(\nabla_{\bar{X}}\bar{Y}\right)^H$ we see that

$$(7) \qquad K(X,Y) = \langle\langle \nabla_{\bar{X}}(\nabla_{\bar{Y}}\bar{Y})^H - \nabla_{\bar{Y}}(\nabla_{\bar{X}}\bar{Y})^H - \nabla_{[\bar{X},\bar{Y}]^H}\bar{Y}, \bar{X} \rangle\rangle$$

$$= \langle\langle \nabla_{\bar{X}}\nabla_{\bar{Y}}\bar{Y} - \nabla_{\bar{Y}}\nabla_{\bar{X}}\bar{Y} - \nabla_{[\bar{X},\bar{Y}]}\bar{Y}, \bar{X} \rangle\rangle$$

$$- \langle\langle \nabla_{\bar{X}}(\nabla_{\bar{Y}}\bar{Y})^V, \bar{X} \rangle\rangle + \langle\langle \nabla_{\bar{Y}}(\nabla_{\bar{X}}\bar{Y})^V, \bar{X} \rangle\rangle + \langle\langle \nabla_{[\bar{X},\bar{Y}]^V}\bar{Y}, \bar{X} \rangle\rangle$$

In simplifying formula (7) for the curvature the next two results are important. The first determines the divergence of $D_{\bar{Y}}\bar{X}$ for horizontal $\bar{Y}, \bar{X}$ at a point J with respect to the Poincare metric $g(J)$. This will measure the deviation of $\nabla_{\bar{Y}}\bar{X}$ from being horizontal.

<u>THEOREM 13.1</u>. If $\bar{X}$ and $\bar{Y}$ are horizontal, then

$$(8) \qquad \operatorname{div}_{g(J)} [\overline{Y},\overline{X}] = d\lambda$$

$$(9) \qquad \operatorname{div}_{g(J)} (D_{\overline{X}}\overline{Y} + D_{\overline{Y}}\overline{X}) = *d\mu$$

$$(10) \qquad \operatorname{div}_{g(J)} (\nabla_{\overline{X}}\overline{Y} + \nabla_{\overline{Y}}\overline{X}) = -*d\mu$$

where $\mu, \lambda : M \longrightarrow R$ are the functions,

$$\mu(x) = (ac + bd)(x) = \frac{1}{2} \operatorname{tr}(XY + YX)$$

$$\lambda(x) = (-ad + bc)(x) = \frac{1}{2} \operatorname{tr}(YX - XY) \quad ,$$

and if $\omega = \xi dx + \eta dy$ in conformal coordinates, $*\omega = -\eta\, dx + \xi dy$.

The following formula replaces the standard formula for the Levi-Civita connection.

<u>THEOREM 13.2</u>. Let V and Z represent horizontal vector fields on A , and $W = L_\beta J$, a vertical field on A .

$$2\langle\langle\nabla_V Z, W\rangle\rangle_J = V\langle\langle Z,W\rangle\rangle_J + Z\langle\langle V,W\rangle\rangle_J$$
$$- W\langle\langle V,Z\rangle\rangle_J + \langle\langle[V,Z],W\rangle\rangle_J - \langle\langle[V,W],Z\rangle\rangle_J$$
$$- \langle\langle[Z,W],V\rangle\rangle_J + \frac{1}{2}\int_M \operatorname{tr}(ZV)(\operatorname{div}_{g(J)}\beta)\,d\mu_{g(J)} \quad .$$

These results allow us to simplify the formula (7) for the curvature, namely

$$(11) \quad K(X,Y) = \langle\langle R(\overline{X},\overline{Y})\overline{Y},\overline{X}\rangle\rangle + \langle\langle(\nabla_{\overline{Y}}\overline{Y})^V, (\nabla_{\overline{X}}\overline{X})^V\rangle\rangle$$
$$- \|(\nabla_{\overline{Y}}\overline{X})^V\|^2 + \|[\overline{Y},\overline{X}]^V\|^2 \quad .$$

We proceed further with the following fundamental

<u>LEMMA 13.3</u>. Suppose $H \in T_J A$ is vertical, $H = L_\beta J$ with $\operatorname{div} H = *d\lambda$, for some smooth function $\lambda : M \longrightarrow R$. Then $-\operatorname{div}\beta = \sigma$, where

$$\Delta\sigma - \sigma = \Delta\lambda$$

where Δ denotes the Laplace Beltrami operator on M with respect to the metric $g(J)$.

The next result gives the basic flavor of the curvature computation.

<u>THEOREM 13.4</u>. Let X and Y be vector fields on Teichmüller's space $A/\mathcal{D}_0$ and denote by $\overline{X},\overline{Y}$ their horizontal lifts. Represent $\overline{X},\overline{Y}$ in conformal coordinates by the matrices $\begin{pmatrix} a & b \\ b & -a \end{pmatrix}$ and $\begin{pmatrix} c & d \\ d & -c \end{pmatrix}$ respectively and let

$$= ad - bc = \frac{1}{2}\,\text{trace}\,\{J(XY - YX)\} \quad .$$

Then

$$\| [\overline{Y},\overline{X}]^v_J \|^2 = \int_M \lambda^2\, d\mu_{g(J)} + \int_M (L^{-1}\lambda)\lambda d\mu_{g(J)}$$

where L is the invertible elliptic operator on functions ρ given by

$$L_\rho = \Delta\rho - \rho \quad .$$

<u>PROOF</u>. Write $[\overline{Y},\overline{X}]^v = L_\beta J$. So $\| [\overline{Y},\overline{X}]^v \|^2 = \| L_\beta J \|^2 = \langle\langle L_\beta J, L_\beta J\rangle\rangle$. Let $\alpha_J(\beta) = L_\beta J$. α_J is now a map from C^∞ vector field $\mathfrak{X}(M)$ on M to C^∞ (1:1) tensors $C^\infty(T^1_1(M))$ on M. Such a map α_J has an adjoint α_J^*, namely for symmetric 1:1 tensors A, and in conformal coordinates $g_{ij} = p\delta_{ij}$,

$$\alpha_J^* : C^\infty(T^1_1(M)) \longrightarrow \mathfrak{X}(M)$$

$$\alpha_J^*(A) = (+\frac{1}{p}(\text{div}_{g(J)}A)^2, -\frac{1}{p}(\text{div}_{g(J)}A)^1) \quad .$$

Then

$\| L_\beta J \|^2 = \| \alpha_J\beta \|^2 = \langle\langle \alpha_J\beta, \alpha_J\beta \rangle\rangle = \langle \alpha_J^*\alpha_J\beta, \beta \rangle$, where $\langle , \rangle$

denotes the g innerproduct of vector fields on M .
But as in 13.1 we can compute in conformal coordinates

$$\alpha_J^*(\alpha_J\beta) = \left(+\frac{1}{p}\frac{\partial}{\partial y}(-ad+bc), -\frac{1}{p}\frac{\partial}{\partial x}(-ad+bc)\right) .$$

Therefore

$$<\alpha_J^*\alpha_J\beta,\beta> = \int_M\left[+p\beta_1\frac{\partial}{\partial y}(-ad+bc) - p\beta_2\frac{\partial}{\partial x}(-ad+bc)\right]$$

integrating by parts we see that this es equal to

$$-\int_M(-ad+bc)\frac{1}{p}\left\{\frac{\partial}{\partial y}(p\beta_1) - \frac{\partial}{\partial x}(p\beta_2)\right\}d\mu_{g(J)} =$$

(12) $$\int_M(-ad+bc)\left\{-\mathrm{div}_g(J\beta)\right\}d\mu_{g(J)} .$$

Since $JL_\beta J = L_{J\beta}J$, from lemma 13.3 we have

$$-\mathrm{div}_g(J\beta) = \rho$$

where $L_\rho = \Delta\rho - \rho = \Delta\lambda$, $\lambda = (-ad+bc)$. Thus (12) is equal to

$$\int_M \lambda\rho\, d\mu_g .$$

The operator L is clearly strictly negative and self-adjoint. So $L_\rho = (L+I)\lambda$ and hence

$$\rho = L^{-1}(L+I)\lambda = \lambda + L^{-1}\lambda$$

and

$$\int_M \rho\lambda\, d\mu_g = \int_M \lambda^2 d\mu_g + \int_M (L^{-1}\lambda)\,\lambda\, d\mu_g .$$

This concludes the proof of theorem 13.4.

Using these ideas to evaluate the second and third terms in formula (1), we obtain our main results:

THEOREM 13.5. Let X and Y be vector fields on Teichmüller's space $A/\mathcal{D}_0$ and $\overline{X},\overline{Y}$ their vertical lifts to the bundle A. Then if

$$\lambda = \frac{1}{2} \text{ trace } \{J(\overline{X}\,\overline{Y} - \overline{Y}\,\overline{X})\}, \quad \gamma = \frac{1}{2} \text{ trace } \{\overline{X}\,\overline{Y} + \overline{Y}\,\overline{X}\}$$

the sectional curvature of $A/\mathcal{D}_0$ with respect to its Weil-Peterssen metric is given by

$$K(X,Y) = -\int_M \lambda^2 d\mu_g + \frac{3}{4}\int_M (L^{-1}\lambda)\,\lambda\, d\mu_g - \frac{1}{4}\int_M (L^{-1}\gamma)\,\gamma\, d\mu_g +$$

$$+ \frac{1}{4}\int_M \{L^{-1}(a^2+b^2)\}(c^2+d^2)\,d\mu_g \quad .$$

THEOREM 13.6. The holomorphic sectional curvature of Teichmüller's space is strictly negative and bounded by $-1/4\pi(p-1)$, $p = \text{genus}(M)$.

PROOF. Let $Y = JX$. Then $\lambda = -(a^2+b^2) = -(c^2+d^2)$,

$$K(X,Y) = -\int_M \lambda^2 + \int_M (L^{-1}\lambda)\,\lambda\, d\mu_g < 0$$

since the elliptic operator L is strictly negative. The sectimal curvature of the plane spanned by X and Y is given by

$$\frac{K(X,Y)}{\|X \wedge Y\|^2}$$

where $\|X \wedge Y\|^2 = \det \begin{pmatrix} \langle\langle \overline{X},\overline{X}\rangle\rangle & \langle\langle \overline{X},\overline{Y}\rangle\rangle \\ \langle\langle \overline{X},\overline{Y}\rangle\rangle & \langle\langle \overline{Y},\overline{Y}\rangle\rangle \end{pmatrix}.$

Since for $Y = JX$, $\langle\langle \overline{X},\overline{Y}\rangle\rangle = 0$, $\|X\|^2 = \|Y\|^2$ we have that

$$\|X \wedge Y\|^2 = \|X\|^4 = \left\{\int_M |\lambda|\, d\mu_g\right\}^2 \le \int_M d\mu_g \cdot \int_M \lambda^2\, d\mu_g \quad .$$

But by the Gauss-Bonnet theorem

$$\int_M d\mu_g = 2\pi(2p-2)$$

where $p = \text{genus}\, M$.

Thus $-\int_M \lambda^2 \leq \frac{-1}{4\pi(p-1)} \;\|X\|^4$ and the holomorphic sectional curvature is bounded by $\frac{-1}{4\pi(p-1)}$.

The next results also follow from the curvature formula.

THEOREM 13.6. The biholomorphic sectional curvature is strictly negative

THEOREM 13.7. The Ricci curvature of $\mathcal{A}/\mathcal{D}_0$ with respect to its Weil-Peterssen metric is strictly negative, and

$\text{Ric}(X) \leq \frac{-1}{4\pi(p-1)} \;\|X\|^2$, where $p = \text{genus}(M)$.

Finally to see that the sectional curvature is negative we need the following lemma. Using the uniformization theorem we can represent M with a given conformal structure as U/Γ, U the hyperbolic upperhalf plane, Γ a subgroup of $SL(2,R)$. Then from the fact that the Green's function for $-L$ on a fundamental domain is positive and Hölder's inequality we obtain

LEMMA 13.4.

$$|L^{-1}(\rho\theta)| \leq |-L^{-1}\rho^2|^{\frac{1}{2}}\,|-L^{-1}\theta^2|^{\frac{1}{2}}$$

Applying this lemma and Hölder's inequality to the formula in theorem 13.5 we see that

$$-\int_M (L^{-1}\gamma)\gamma \leq -\int_M \{L^{-1}(a^2+b^2)\}\{c^2+d^2\}d\mu_g$$

This immediately implies the final result.

THEOREM 13.8. The sectional curvature of Teichmüller space is negative.

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